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Permanence and existence of periodic solution of a discrete periodic Lotka–Volterra competition system with feedback control and time delays*

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Abstract. In this paper, we consider a discrete predator–prey system with feedback and time delays. By applying the theory of difference inequality, as well as analysis technique, sufficient conditions are obtained for the permanence of the system. And by applying Mawhin's coincidence degree theory, we obtain the existence of the positive periodic solutions.

Keywords: discrete competition system, permanence, periodic solution, feedback control, coincidence degree theory.

1 Introduction

In the past decades, predator–prey systems with feedback controls have been extensively studied by many authors (see [1, 3, 10]). Zhou and Zou [13] studied the following discrete logistic equation:

$$x(n+1) = x(n)\exp\left\{r(n)\left(1-\frac{x(n)}{K(n)}\right)\right\}, \tag{1}$$

where $r(n)$ and $K(n)$ are positive ω-periodic sequences, some sufficient conditions are obtained for the existence of a globally stable positive solution.

Li and Zhu [10] studied the following difference equations with feedback control:

$$\begin{aligned} N(n+1) &= N(n)\exp\left\{r(n)\left(1-\frac{N(n-m)}{k(n)}-c(n)\mu(n)\right)\right\}, \\ \Delta\mu(n) &= -a(n)\mu(n)+b(n)N(n-m). \end{aligned} \tag{2}$$

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By applying Mawhin's coincidence theorem, they obtained some sufficient conditions for the existence of positive solutions.

Li and Zhang [9] discussed the following delay logistic equation with feedback control:

$$\begin{aligned} x(n+1) &= x(n)\exp\Biggl\{r(n)\Biggl[1-\frac{x(n)}{K(n)}-\sum_{s_0=1}^{m_0}a_{s_0}(n)x(n-s_0)\Biggr] \\ &\qquad -\sum_{s_1=0}^{m_1}b_{s_1}(n)\mu(n-s_1)\Biggr\}, \\ \mu(n+1) &= \bigl(1-\alpha(n)\bigr)\mu(n)+\sum_{s_2=0}^{m_2}\beta_{s_2}(n)x(n-s_2). \end{aligned} \tag{3}$$

The authors used the theory of difference inequality to obtain the permanence and the almost periodic sequence solution for the system.

Chen and Zhou [4] discussed the permanence and the existence of a globally stable periodic solution of the following discrete competition system:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\biggl\{r_1(n)\biggl(1-\frac{x_1(n)}{K_1(n)}-\mu_2(n)x_2(n)\biggr)\biggr\}, \\ x_2(n+1) &= x_2(n)\exp\biggl\{r_2(n)\biggl(1-\frac{x_2(n)}{K_2(n)}-\mu_1(n)x_1(n)\biggr)\biggr\}. \end{aligned} \tag{4}$$

Li, Chen and He [8] discussed the following system with delays:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\biggl\{r_1(n)\biggl(1-\frac{x_1(n-\tau_{11})}{K_1(n)}-\mu_2(n)x_2(n-\tau_{12})\biggr)\biggr\}, \\ x_2(n+1) &= x_2(n)\exp\biggl\{r_2(n)\biggl(1-\frac{x_2(n-\tau_{22})}{K_2(n)}-\mu_1(n)x_1(n-\tau_{21})\biggr)\biggr\}, \end{aligned} \tag{5}$$

they first discussed the permanence and global attractivity of system (5). Further, by means of an almost periodic functional hull theory, they showed that the system has a unique strictly positive almost periodic solution, which is globally attractive.

Recently, Chen [2] considered the following system with feedback control:

$$\begin{aligned} x_1(n+1) &= x_1(n)\exp\biggl\{r_1(n)\biggl(1-\frac{x_1(n)}{K_1(n)}-\mu_2(n)x_2(n)-b_1(n)\mu_1(n)\biggr)\biggr\}, \\ x_2(n+1) &= x_2(n)\exp\biggl\{r_2(n)\biggl(1-\frac{x_2(n)}{K_2(n)}-\mu_1(n)x_1(n)-b_2(n)\mu_2(n)\biggr)\biggr\}, \\ \Delta\mu_1(n) &= -\alpha_1(n)\mu_1(n)+\beta_1(n)x_1(n), \\ \Delta\mu_2(n) &= -\alpha_2(n)\mu_2(n)+\beta_2(n)x_2(n). \end{aligned} \tag{6}$$

The permanence and the existence of periodic solution of system (6) were discussed with fixed point theorem. Moreover, they discussed the periodic solution of system (6), which is globally stable.

Niu and Chen [11] discussed system (6) about the almost periodic sequence solution which is uniformly asymptotically stable.

In this paper, we consider the following discrete predator–prey system with feedback control and time delays:

$$\begin{aligned}
x_1(n+1) &= x_1(n)\exp\biggl\{r_1(n)\biggl(1-\frac{x_1(n-\tau_{11})}{K_1(n)}-\mu_2(n)x_2(n-\tau_{12}) \\
&\qquad - b_1(n)\mu_1(n-\tau_{13})\biggr)\biggr\}, \\
x_2(n+1) &= x_2(n)\exp\biggl\{r_2(n)\biggl(1-\frac{x_2(n-\tau_{21})}{K_2(n)}-\mu_1(n)x_1(n-\tau_{22}) \\
&\qquad - b_2(n)\mu_2(n-\tau_{23})\biggr)\biggr\}, \\
\mu_1(n+1) &= \bigl(1-\alpha_1(n)\bigr)\mu_1(n)+\beta_1(n)x_1(n-\sigma_1), \\
\mu_2(n+1) &= \bigl(1-\alpha_2(n)\bigr)\mu_2(n)+\beta_2(n)x_2(n-\sigma_2),
\end{aligned} \tag{7}$$

where $x_i(n)$ is the density of the species at time n and $\mu_i(n)$ is the control variable at time n, $r_i(n)$ represent the intrinsic growth and $K_i(n)$ represent the carrying capacity. Here τ_{ij}, σ_i, $i=1,2$; $j=1,2,3$ are all nonnegative integers.

Let $\tau=\max\{\tau_{ij},\sigma_i,\ i=1,2;\ j=1,2,3\}$. We consider system (7) together with the following initial conditions:

$$\begin{aligned}
x_i(\theta) &= \varphi_i(\theta)\geqslant 0, \quad \varphi_i(0)>0, \quad \theta\in N[-\tau,0],\ i=1,2, \\
\mu_i(\theta) &= \psi_i(\theta)\geqslant 0, \quad \psi_i(0)>0, \quad \theta\in N[-\tau,0],\ i=1,2,
\end{aligned} \tag{8}$$

where $N[-\tau,0]=\{-\tau,-\tau+1,\dots,0\}$. It is not difficult to see that solutions of system (7)–(8) are well defined for all $n>0$ and satisfy $x_i(n)>0$, $\mu_i(n)>0$, $n\in\mathbb{Z}$, $i=1,2$.

The organization of this paper is as follows. In Section 2, we will introduce some definitions and several useful definitions and lemmas. In Section 3, by applying the theory of difference inequality, we get the permanence of system (7)–(8). In Section 4, by means of Mawhin's coincidence degree theory, we obtain the existence of the periodic solution for system (7)–(8). Finally, we give some examples and numerical simulations to verify our results.

2 Preliminaries

We first give some notations as follows:

$$x_1^*=\frac{K_1^M\exp\{r_1^M-1\}}{r_1^L\exp\{-\tau_{11}r_1^M\}}, \qquad x_2^*=\frac{K_2^M\exp\{r_2^M-1\}}{r_2^L\exp\{-\tau_{21}r_2^M\}},$$

$$\mu_1^*=\frac{\beta_1^M x_1^*}{\alpha_1^M}, \qquad \mu_2^*=\frac{\beta_2^M x_2^*}{\alpha_2^M}, \qquad \alpha^M=\sup_{n\in N}\alpha(n), \qquad \alpha^L=\inf_{n\in N}\alpha(n),$$

$$N_1 = \left(\log_{(1-\alpha_1^L)} \frac{r_1^L - \mu_2^* x_2^*}{b_1^M \mu_1^*}, +\infty\right) \cap N,$$

$$N_2 = \left(\log_{(1-\alpha_2^L)} \frac{r_2^L - \mu_1^* x_1^*}{b_2^M \mu_2^*}, +\infty\right) \cap N,$$

$$F_1^{N_1} = r_1^L - \mu_2^* x_2^* - b_1^M \bigl(1-\alpha_1^L\bigr)^{N_1} \mu_1^*, \qquad F_2^{N_2} = r_2^L - \mu_1^* x_1^* - b_2^M \bigl(1-\alpha_2^L\bigr)^{N_2} \mu_2^*,$$

$$G_1^{N_1} = \frac{r_1^M}{K_1^L} \exp\{-\tau_{11} D_1\} + b_1^M \exp\{-\tau_{13} D_1\} E_1^{N_1},$$

$$G_2^{N_2} = \frac{r_2^M}{K_2^L} \exp\{-\tau_{21} D_2\} + b_2^M \exp\{-\tau_{23} D_2\} E_2^{N_2},$$

$$D_1 = r_1^L - \frac{r_1^M}{K_1^L} x_1^* - \mu_2^* x_2^* - b_1^M \mu_1^*, \qquad D_2 = r_2^L - \frac{r_2^M}{K_2^L} x_2^* - \mu_1^* x_1^* - b_2^M \mu_2^*,$$

$$E_1^{N_1} = \sum_{j=0}^{N_1-1} \bigl(1-\alpha_1^L\bigr)^j \beta_1^M \exp\bigl\{-D_1(\sigma_1 + j + 1)\bigr\},$$

$$E_2^{N_2} = \sum_{j=0}^{N_2-1} \bigl(1-\alpha_2^L\bigr)^j \beta_2^M \exp\bigl\{-D_2(\sigma_2 + j + 1)\bigr\}.$$

Then we introduce some basic definitions and useful lemmas.

Definition 1. System (7) is said to be permanent if there exists positive constants M_{1i}, M_{2i}, m_{1i}, m_{2i}, which are independent of the solutions of the system such that any positive solution $(x_1(n), x_2(n), \mu_1(n), \mu_2(n))^{\mathrm{T}}$ of system (7) satisfies

$$m_{1i} \leqslant \lim_{n\to+\infty} \inf x_i(n) \leqslant \lim_{n\to+\infty} \sup x_i(n) \leqslant M_{1i}, \quad i = 1, 2;$$

$$m_{2i} \leqslant \lim_{n\to+\infty} \inf u_i(n) \leqslant \lim_{n\to+\infty} \sup u_i(n) \leqslant M_{2i}, \quad i = 1, 2.$$

Lemma 1. *See* [12]. *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \leqslant x(n) \exp\bigl\{a(n) - b(n)x(n)\bigr\}, \quad n \in N,$$

where $a(n)$ *and* $b(n)$ *are nonnegative sequences bounded above and below by positive constants. Then*

$$\lim_{n\to+\infty} \sup x(n) \leqslant \frac{1}{b^L} \exp\bigl\{a^M - 1\bigr\}.$$

Lemma 2. (*See* [12].) *Assume that* $\{x(n)\}$ *satisfies* $x(n) > 0$ *and*

$$x(n+1) \geqslant x(n) \exp\bigl\{a(n) - b(n)x(n)\bigr\}, \quad n \geqslant N_0,$$

$\lim_{n\to+\infty} \sup x(n) \leqslant x^*$, $(b^M/a^L)x^* > 1$ *and* $x(N_0) > 0$, *where* $a(n)$ *and* $b(n)$ *are nonnegative sequences bounded above and below by positive constants and* $N_0 \in N$. *Then*

$$\lim_{n\to+\infty} \inf x(n) \geqslant \frac{a^L}{b^M} \exp\bigl\{a^M - b^M x^*\bigr\}.$$

Lemma 3. (*See* [6].) *Assume that* $A > 0$ *and* $y(0) > 0$, *suppose that*

$$y(n+1) \leqslant Ay(n) + B(n), \quad n \in N.$$

Then for any integer $m \leqslant n$,

$$y(n) \leqslant A^m y(n-m) + \sum_{j=0}^{m-1} A^j B(n-j-1).$$

If $A < 1$ *and* B *is bounded above with respect to* U, *then*

$$\lim_{n\to+\infty} \sup y(n) \leqslant \frac{U}{1-A}.$$

Lemma 4. (*See* [6].) *Assume that* $A > 0$ *and* $y(0) > 0$. *Suppose that*

$$y(n+1) \geqslant Ay(n) + B(n), \quad n \in N.$$

Then for any integer $m \leqslant n$,

$$y(n) \geqslant A^m y(n-m) + \sum_{j=0}^{m-1} A^j B(n-j-1).$$

If $A < 1$ *and* B *is bounded below with respect to* K, *then*

$$\lim_{n\to+\infty} \inf y(n) \leqslant \frac{K}{1-A}.$$

Let X and Y be two Banach spaces, $L : \operatorname{Dom} L \subset X \to Y$ is a linear map, and $N : X \to Y$ is a continuous map. If $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L \in Y$ is closed, then we call the operator L is a Fredholm operator with index zero. And if L is a Fredholm operator with index zero and there exist continuous projects $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$, then $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : (I-P)X \to \operatorname{Im} L$ has an inverse function, we set it as K_P. Assume $\Omega \in X$ is any open set, if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega}) \in X$ is relative compact, then we say N is L-compact on $\overline{\Omega}$. Following we recall the Mawhin's coincidence theorem.

Lemma 5. (*See* [7].) *Let* X *and* Y *be both Banach spaces,* $L : \operatorname{Dom} L \subset X \to Y$ *be a Fredholm operator with index zero,* $\Omega \in Y$ *be an open bounded set, and* $N : \overline{\Omega} \to X$ *be* L*-compact on* $\overline{\Omega}$. *If all the following conditions hold*:

(C1) $Lx \neq \lambda Nx$ *for* $x \in \partial\Omega \cap \operatorname{Dom} L$, $\lambda \in (0,1)$;

(C2) $Nx \notin \operatorname{Im} L$ *for* $x \in \partial\Omega \cap \operatorname{Ker} L$;

(C3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where* $J : \operatorname{Im} Q \to \operatorname{Ker} L$ *is an isomorphism*;

then the equation $Lx = Nx$ *has at least one solution on* $\overline{\Omega} \cap \operatorname{Dom} L$.

Lemma 6. (*See* [5].) *Let* $g : \mathbb{Z} \to \mathbb{R}$ *be* ω *periodic, i.e.,* $g(k+\omega) = g(k)$; *then for any fixed* $k_1, k_2 \in I_\omega$ *and any* $k \in \mathbb{Z}$, *one has*

$$g(k) \leqslant g(k_1) + \sum_{s=0}^{\omega-1} \bigl|g(s+1) - g(s)\bigr|, \qquad g(k) \geqslant g(k_2) - \sum_{s=0}^{\omega-1} \bigl|g(s+1) - g(s)\bigr|.$$

3 Permanence

In this section, we consider some permanence results for system (7) and (8) with $r_i(n)$, $K_i(n)$, $b_i(n)$, $\alpha_i(n)$, $\beta_i(n)$, $\mu_i(n)$, $i = 1, 2$, are all bounded nonnegative sequences such that

$$0 < \alpha_i^L < \alpha_i^M < 1, \qquad r_i^L > 0.$$

Theorem 1. *Assume that*

(H1) $r_1^L - \mu_2^* x_2^* > 0$ *and* $r_2^L - \mu_1^* x_1^* > 0$,

then every solution $(x_1(n), x_2(n), \mu_1(n), \mu_2(n))^{\mathrm{T}}$ *of system* (7) *satisfies*

$$x_{i*}^{N_i} \leqslant \lim_{n\to+\infty} \inf x_i(n) \leqslant \lim_{n\to+\infty} \sup x_i(n) \leqslant x_i^*, \quad i = 1, 2,$$
$$\mu_{i*}^{N_i} \leqslant \lim_{n\to+\infty} \inf \mu_i(n) \leqslant \lim_{n\to+\infty} \sup \mu_i(n) \leqslant \mu_i^*, \quad i = 1, 2,$$

that is, system (7) *is permanent.*

Proof. Let $(x_1(n), x_2(n), \mu_1(n), \mu_2(n))^{\mathrm{T}}$ be any positive solution of system (7), from the first equation of system (7) it follows that

$$x_1(n+1) \leqslant x_1(n) \exp\bigl\{r_1(n)\bigr\} \leqslant x_1(n) \exp\bigl\{r_1^M\bigr\}. \tag{9}$$

By (9), one can easily obtain that

$$x_1(n - \tau_{11}) \geqslant \exp\bigl\{-\tau_{11} r_1^M\bigr\}. \tag{10}$$

Substituting (10) into the first equation of system (7), it follows that

$$\begin{aligned} x_1(n+1) &\leqslant x_1(n) \exp\left[r_1(n) - \frac{r_1(n) x_1(n)}{K_1(n)} \exp\bigl\{-\tau_{11} r_1^M\bigr\}\right] \\ &\leqslant x_1(n) \exp\left[r_1(n) - \frac{r_1(n)}{k_1(n)} \exp\bigl\{-\tau_{11} r_1^M\bigr\} x_1(n)\right]. \end{aligned} \tag{11}$$

Thus, as a direct corollary of Lemma 1, according to (11), one has

$$\lim_{n\to+\infty} \sup x_1(n) \leqslant \frac{K_1^M \exp\{r_1^M - 1\}}{r_1^L \exp\{-\tau_{11} r_1^M\}} := x_1^*. \tag{12}$$

In the same way, we can get

$$\lim_{n\to+\infty} \sup x_2(n) \leqslant \frac{K_2^M \exp\{r_2^M - 1\}}{r_2^L \exp\{-\tau_{21} r_2^M\}} := x_2^*. \tag{13}$$

For any positive contant ϵ_0 small enough, it follows from (12) that there exists a large enough $n_0 > 0$ such that

$$x_1(n) \leqslant x_1^* + \epsilon_0, \quad n \geqslant n_0. \tag{14}$$

Then the third equation of system (7) leads to

$$\mu_1(n+1) \leqslant \bigl(1-\alpha_1(n)\bigr)\mu_1(n) + \beta_1(n)(x_1^* + \epsilon_0), \quad n \geqslant n_0 + \tau. \tag{15}$$

By applying Lemma 3, it follows from (15) that

$$\lim_{n\to+\infty} \sup \mu_1(n) \leqslant \frac{\beta_1^M (x_1^* + \epsilon_0)}{\alpha_1^M}. \tag{16}$$

Letting $\epsilon_0 \to 0$ in the above inequality yields that

$$\lim_{n\to+\infty} \sup \mu_1(n) \leqslant \frac{\beta_1^M x_1^*}{\alpha_1^M} := \mu_1^*. \tag{17}$$

In the same way, we can get

$$\lim_{n\to+\infty} \sup \mu_2(n) \leqslant \frac{\beta_2^M x_2^*}{\alpha_2^M} := \mu_2^*. \tag{18}$$

From the definition of N_1 and (H1) there exists a positive constant ϵ_1 small enough and large enough $n_1 \geqslant \max\{n_0, N_1\}$ such that

$$r_1^L - (\mu_2^* + \epsilon_1)(x_2^* + \epsilon_1) - b_1^M \bigl(1 - \alpha_1^L\bigr)^{N_1} (\mu_1^* + \epsilon_1) \geqslant 0,$$

and

$$x_i(n) \leqslant x_i^* + \epsilon_1, \quad \mu_i(n) \leqslant \mu_i^* + \varepsilon_1, \quad i = 1, 2, \ n \geqslant n_1,$$

which imply that

$$\begin{aligned} x_1(n+1) &\geqslant x_1(n) \exp\biggl\{ r_1^L - \frac{r_1^M}{K_1^L}(x_1^* + \epsilon_1) - (\mu_2^* + \epsilon_1)(x_2^* + \epsilon_1) - b_1^M(\mu_1^* + \epsilon_1) \biggr\} \\ &:= x_1(n) \exp\{D_{1\epsilon_1}\}, \quad n \geqslant n_1 + \tau, \end{aligned} \tag{19}$$

where

$$\begin{aligned} D_{1\epsilon_1} &= r_1^L - \frac{r_1^M}{K_1^L}(x_1^* + \epsilon_1) - (\mu_2^* + \epsilon_1)(x_2^* + \epsilon_1) - b_1^M(\mu_1^* + \epsilon_1) \\ &\leqslant r_1^L - \frac{r_1^M}{K_1^L} x_1^* - \mu_2^* x_2^* - b_1^M \mu_1^* \leqslant r_1^L - \frac{r_1^M}{K_1^L} x_1^* - \frac{\beta_2^M}{\alpha_2^L} {x_2^*}^2 - \frac{b_1^M \beta_1^M}{\alpha_1^L} x_1^* \\ &\leqslant r_1^L - \biggl(\frac{r_1^M}{K_1^L} + \frac{b_1^M \beta_1^M}{\alpha_1^L}\biggr) x_1^* \leqslant r_1^L - \frac{r_1^M \alpha_1^L + b_1^M \beta_1^M K_1^L}{K_1^L \alpha_1^L} \frac{K_1^M \exp\{r_1^M - 1\}}{r_1^L \exp\{-\tau_{11} r_1^M\}} \\ &\leqslant r_1^L - \frac{K_1^M}{K_1^L} \biggl(\frac{r_1^M}{r_1^L} + \frac{b_1^M \beta_1^M K_1^L}{\alpha_1^L r_1^L}\biggr) \exp\bigl\{\tau_{11} r_1^M\bigr\} \exp\bigl\{r_1^M - 1\bigr\} \\ &\leqslant r_1^M - \exp\bigl\{r_1^M - 1\bigr\} < r_1^L - r_1^M \leqslant 0. \end{aligned}$$

Therefore, for $\eta \leqslant n$, by using (19), we have

$$x_1(\eta) \leqslant x_1(n)\exp\bigl\{-D_{1\epsilon_1}(n-\eta)\bigr\}. \tag{20}$$

From the third equation of system (7) we have

$$\mu_1(n+1) \leqslant \bigl(1-\alpha_1^L\bigr)^{N_1}\mu_1(n) + \beta_1(n)x_1(n-\sigma_1). \tag{21}$$

According to Lemma 3 and (21), for any $n \geqslant N_1$, we have

$$\begin{aligned}
\mu_1(n) &\leqslant \bigl(1-\alpha_1^L\bigr)^{N_1}\mu_1(n-N_1) + \sum_{j=0}^{N_1-1}\bigl(1-\alpha_1^L\bigr)^j\beta_1^M x_1(n-\sigma_1-j-1) \\
&\leqslant \bigl(1-\alpha_1^L\bigr)^{N_1}(\mu_1^*+\epsilon_1) + \sum_{j=0}^{N_1-1}\bigl(1-\alpha_1^L\bigr)^j\beta_1^M\exp\bigl\{-D_{1\epsilon_1}(\sigma_1+j+1)\bigr\}x_1(n) \\
&:= \bigl(1-\alpha_1^L\bigr)^{N_1}(\mu_1^*+\epsilon_1) + E_{1\epsilon_1}x_1(n), \quad n \geqslant n_1+\tau,
\end{aligned} \tag{22}$$

where

$$E_{1\epsilon_1}^{N_1} = \sum_{j=0}^{N_1-1}\bigl(1-\alpha_1^L\bigr)^j\beta_1^M\exp\bigl\{-D_{1\epsilon_1}(\sigma_1+j+1)\bigr\}.$$

Substituting (20) and (22) into the first equation of system (7), one has

$$\begin{aligned}
x_1(n+1) &\geqslant x_1(n)\exp\biggl\{r_1^L - \frac{r_1^M}{K_1^L}\exp\{-\tau_{11}D_{1\epsilon_1}\}x_1(n) - (\mu_2^*+\epsilon_1)(x_2^*+\epsilon_1) \\
&\qquad - b_1^M\bigl(1-\alpha_1^L\bigr)^{N_1}(\mu_1^*+\epsilon_1) - b_1^M E_{1\epsilon_1}^{N_1}\exp\{-\tau_{13}D_{1\epsilon_1}\}x_1(n)\biggr\} \\
&:= x_1(n)\exp\bigl\{F_{1\epsilon_1}^{N_1} - G_{1\epsilon_1}^{N_1}x_1(n)\bigr\},
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
F_{1\epsilon_1}^{N_1} &= r_1^L - (\mu_2^*+\epsilon_1)(x_2^*+\epsilon_1) - b_1^M\bigl(1-\alpha_1^L\bigr)^{N_1}(\mu_1^*+\epsilon_1), \\
G_{1\epsilon_1}^{N_1} &= \frac{r_1^M}{K_1^L}\exp\{-\tau_{11}D_{1\epsilon_1}\} + b_1^M E_{1\epsilon_1}^{N_1}\exp\{-\tau_{13}D_{1\epsilon_1}\}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{G_{1\epsilon_1}^{N_1}}{F_{1\epsilon_1}^{N_1}}x_1^* &\geqslant \frac{\frac{r_1^M}{K_1^L}\exp\{-\tau_{11}D_{1\epsilon_1}\}}{r_1^L}x_1^* \geqslant \frac{r_1^M\exp\{-\tau_{11}D_{1\epsilon_1}\}}{r_1^L K_1^L}\,\frac{K_1^M\exp\{r_1^M-1\}}{r_1^L\exp\{-\tau_{11}r_1^M\}} \\
&\geqslant \frac{r_1^M}{r_1^L}\,\frac{K_1^M}{K_1^L}\,\frac{\exp\{r_1^M-1\}}{r_1^L}\exp\bigl\{-\tau_{11}\bigl(D_{1\epsilon_1}-r_1^M\bigr)\bigr\} \geqslant \frac{\exp\{r_1^M-1\}}{r_1^L} > 1.
\end{aligned}$$

Applying Lemma 2 to (23), we obtain

$$\liminf_{n\to+\infty} x_1(n) \geqslant \frac{F_{1\epsilon_1}^{N_1}\exp\{F_{1\epsilon_1}^{N_1} - G_{1\epsilon_1}^{N_1}x_1^*\}}{G_{1\epsilon_1}^{N_1}}.$$

Letting $\epsilon_1 \to 0$ in the above inequality leads to

$$\liminf_{n\to+\infty} x_1(n) \geqslant \frac{F_1^{N_1}\exp\{F_1^{N_1} - G_1^{N_1}x_1^*\}}{G_1^{N_1}} := x_{1*}^{N_1}. \tag{24}$$

For any positive constant ϵ_2 small enough, from (24) it follows that there exists a large enough $n_2 \geqslant n_1$ such that

$$x_1(n) \geqslant x_{1*}^{N_1} - \epsilon_2, \quad n \geqslant n_2.$$

Then the third equation of system (7) leads to

$$\mu_1(n+1) \geqslant \bigl(1-\alpha_1^M\bigr)\mu_1(n) + \beta_1^L\bigl(x_{1*}^{N_1} - \epsilon_2\bigr),$$

which implies from Lemma 4 that

$$\liminf_{n\to+\infty} \mu_1(n) \geqslant \frac{\beta_1^L(x_{1*}^{N_1} - \epsilon_2)}{\alpha_1^M}.$$

Letting $\epsilon_2 \to 0$ in the above inequality leads to

$$\liminf_{n\to+\infty} \mu_1(n) \geqslant \frac{\beta_1^L x_{1*}^{N_1}}{\alpha_1^M} := \mu_{1*}^{N_1}. \tag{25}$$

In the same way, we can get

$$\liminf_{n\to+\infty} x_2(n) \geqslant \frac{F_2^{N_2}\exp\{F_2^{N_2} - G_2^{N_2}x_2^*\}}{G_2^{N_2}} := x_{2*}^{N_2}, \tag{26}$$

$$\liminf_{n\to+\infty} \mu_2(n) \geqslant \frac{\beta_2^L x_{2*}^{N_2}}{\alpha_2^M} := \mu_{2*}^{N_2}. \tag{27}$$

Then system (7) is permanence. □

4 Existence of positive periodic solutions

In this section, we consider assume that $r_i(n)$, $K_i(n)$, $b_i(n)$, $\alpha_i(n)$, $\beta_i(n)$, $\mu_i(n)$, $i = 1, 2$, are all periodic nonnegative sequences with a common period ω and satisfy

$$K_i(n) > 0, \quad r_i(n) > 0, \quad 0 < \alpha_i < 1, \quad i = 1, 2,\ n \in I_\omega.$$

For convenience and simplicity, in the following discussion, we use the notation

$$\overline{f} := \frac{1}{\omega}\sum_{s=0}^{\omega-1} f(s), \qquad I_\omega := \{0, 1, \ldots, \omega - 1\},$$

where $f(s)$ is an ω-periodic sequence of real numbers defined $s \in \mathbb{Z}$.

Theorem 2. $(x_1(n),\ x_2(n),\ \mu_1(n),\ \mu_2(n))^{\mathrm{T}}$ *is an* ω*-periodic solution of system* (7) *if and only if it is also an* ω*-periodic solution of*

$$\begin{aligned}
x_1(n+1) &= x_1(n)\exp\biggl\{r_1(n)\biggl(1-\frac{x_1(n-\tau_{11})}{K_1(n)}-\mu_2(n)x_2(n-\tau_{12})\\
&\qquad -b_1(n)\mu_1(n-\tau_{13})\biggr)\biggr\},\\
x_2(n+1) &= x_2(n)\exp\biggl\{r_2(n)\biggl(1-\frac{x_2(n-\tau_{11})}{K_2(n)}-\mu_1(n)x_1(n-\tau_{12})\\
&\qquad -b_2(n)\mu_2(n-\tau_{13})\biggr)\biggr\},\\
\mu_1(n) &= \sum_{u=n}^{n+\omega-1}G_1(n,u)\beta_1(u)x_1(u-\sigma_1) := (\varPhi_1x_1)(n),\\
\mu_2(n) &= \sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u)x_2(u-\sigma_2) := (\varPhi_2x_2)(n),\\
G_i(n,u) &= \frac{\prod_{s=u+1}^{n+\omega-1}(1-\alpha_i(s))}{1-\prod_{s=n}^{n+\omega-1}(1-\alpha_i(s))},\quad i=1,2,\ u\in\{n,n+1,\dots,n+\omega-1\}.
\end{aligned}\tag{28}$$

Proof. First, let $(x_1(n),\ x_2(n),\ \mu_1(n),\ \mu_2(n))^{\mathrm{T}}$ be an ω-periodic solution of system (7). From the third and the fourth equation of system (7) and the variation-of-constant formulas it follows that

$$\mu_i(n) = \prod_{s=0}^{n-1}\bigl(1-\alpha_i(s)\bigr)\left[\mu_i(0)+\sum_{s=0}^{n-1}\frac{\beta_i(s)x_i(s-\sigma_i)}{\prod_{j=0}^{s}(1-\alpha_i(j))}\right]. \tag{29}$$

Then

$$\mu_i(n+\omega) = \prod_{s=0}^{n+\omega-1}\bigl(1-\alpha_i(s)\bigr)\left[\mu_i(0)+\sum_{s=0}^{n+\omega-1}\frac{\beta_i(s)x_i(s-\sigma_i)}{\prod_{j=0}^{s}(1-\alpha_i(j))}\right],$$

hence, using $\mu_i(n)=\mu_i(n+\omega)$, we get

$$\mu_i(0) = \frac{\prod_{s=n}^{n+\omega-1}(1-\alpha_i(s))\sum_{s=n}^{n+\omega-1}\frac{\beta_i(s)x_i(s-\sigma_i)}{\prod_{j=0}^{s}(1-\alpha_i(j))}}{1-\prod_{s=n}^{n+\omega-1}(1-\alpha_i(s))}-\sum_{s=0}^{n-1}\frac{\beta_i(s)x_i(s-\sigma_i)}{\prod_{j=0}^{s}(1-\alpha_i(j))}. \tag{30}$$

Substituting (30) into (29), we get

$$\mu_i(n) = \sum_{u=n}^{n+\omega-1}G_i(n,u)\beta_i(u)x_i(u-\sigma_i) := (\varPhi_ix_i)(n).$$

Next let $(x_1(n),\ x_2(n),\ \mu_1(n),\ \mu_2(n))^{\mathrm{T}}$ be an ω-periodic solution of system (28). Then

$$\mu_i(n+1) = \sum_{u=n+1}^{n+\omega}G_i(n+1,u)\beta_i(u)x_i(u-\sigma_i)$$

$$\begin{aligned}
&= \sum_{u=n}^{n+\omega-1} G_i(n+1,u)\beta_i(u)x_i(u-\sigma_i) + G_i(n+1,n+\omega)\beta_i(n+\omega)x_i(u+\omega-\sigma_i)\\
&\qquad - G_i(n+1,n)\beta_i(n)x_i(n-\sigma_i)\\
&= \sum_{u=n}^{n+\omega-1} G_i(n+1,u)\beta_i(u)x_i(u-\sigma_i)\\
&\qquad + \bigl[G_i(n+1,n+\omega) - G_i(n+1,n)\bigr]\beta_i(n)x_i(n-\sigma_i)\\
&= \bigl(1-\alpha_i(n)\bigr)\mu_i(n) + \beta_i(n)x_i(n-\sigma_i), \quad i=1,2.
\end{aligned}$$

Above we use the period of α_i. Then the proof is completed. □

Theorem 3. *Assume that*:

(H2) $\overline{r_2}\omega - \mathrm{e}^{2B_1}\sum_{n=0}^{\omega-1} r_2(n)\sum_{u=n}^{n+\omega-1} G_1(n,u)\beta_1(u) > 0$,

(H3) $\overline{r_1}\omega - \mathrm{e}^{2B_2}\sum_{n=0}^{\omega-1} r_1(n)\sum_{u=n}^{n+\omega-1} G_2(n,u)\beta_2(u) > 0$, *where*

$$B_1 = 2\overline{r_1}\omega + \ln\left[\frac{\overline{r_1}}{\overline{(\frac{r_1}{K_1})}}\right], \qquad B_2 = 2\overline{r_2}\omega + \ln\left[\frac{\overline{r_2}}{\overline{(\frac{r_2}{K_2})}}\right].$$

The system (7) *has at least one positive ω-periodic solution.*

Proof. By Theorem 2, system (7) can be reformulated as

$$\begin{aligned}
x_1(n+1) &= x_1(n)\exp\biggl[r_1(n)\biggl(1-\frac{x_1(n-\tau_{11})}{K_1(n)} - (\Phi_2x_2)(n)x_2(n-\tau_{12})\\
&\qquad - b_1(n)(\Phi_1x_1)(n-\tau_{13})\biggr)\biggr],\\
x_2(n+1) &= x_2(n)\exp\biggl[r_2(n)\biggl(1-\frac{x_2(n-\tau_{11})}{K_2(n)} - (\Phi_1x_1)(n)x_1(n-\tau_{22})\\
&\qquad - b_2(n)(\Phi_2x_2)(n-\tau_{23})\biggr)\biggr].
\end{aligned} \tag{31}$$

Let $x_i(n) = \exp y_i(n)$ and according to (28), then (31) is the same as

$$\begin{aligned}
y_1(n+1)-y_1(n) &= r_1(n)\biggl[1-\frac{\exp\{y_1(n-\tau_{11})\}}{K_1(n)}\\
&\qquad - (\Phi_2^*y_2)(n)\exp\bigl\{y_2(n-\tau_{12})\bigr\} - b_1(n)(\Phi_1^*y_1)(n-\tau_{13})\biggr],\\
y_2(n+1)-y_2(n) &= r_2(n)\biggl[1-\frac{\exp\{y_2(n-\tau_{21})\}}{K_2(n)}\\
&\qquad - (\Phi_1^*y_1)(n)\exp\bigl\{y_1(n-\tau_{22})\bigr\} - b_2(n)(\Phi_2^*y_2)(n-\tau_{23})\biggr],
\end{aligned} \tag{32}$$

where

$$(\varPhi_i^* y_i)(n) = \sum_{u=n}^{n+\omega-1} G_i(n,u)\beta_i(u)\exp\bigl\{y_i(u-\sigma_i)\bigr\}, \quad i = 1, 2.$$

In order to apply Lemma 5, we take

$$X = Y = \bigl\{y(n) = \bigl(y_1(n), y_2(n)\bigr)\colon\ y(n+\omega) = y(n),\ n \in \mathbb{Z}\bigr\}.$$

Denote the subspace of all ω periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e.,

$$\|y\| = \max_{n \in I_\omega} \bigl|y_i(n)\bigr|, \quad i = 1, 2,\ y \in X.$$

It is not difficult to show that X, Y are Banach space. Let

$$X_0 = \Biggl\{y \in X\colon\ \sum_{k=0}^{\omega-1} y(n) = 0\Biggr\}, \qquad X_c = \bigl\{y \in X\colon\ y(n) = h \in \mathbb{R}^2,\ n \in \mathbb{Z}\bigr\}.$$

Then it is easy to check that X_0 and X_c are both closed linear subspaces of X and

$$X = X_0 \oplus X_c.$$

Let

$$L : \operatorname{Dom} L \cap X \to Y, \qquad (Ly)(n) = \begin{pmatrix} y_1(n+1) - y_1(n) \\ y_2(n+1) - y_2(n) \end{pmatrix},$$

$$Ny = \begin{pmatrix} r_1(n)[1 - \frac{\mathrm{e}^{y_1(n-\tau_{11})}}{K_1(n)} - (\varPhi_2^* y_2)(n)\mathrm{e}^{y_2(n-\tau_{12})} - b_1(n)(\varPhi_1^* y_1)(n-\tau_{13})] \\ r_2(n)[1 - \frac{\mathrm{e}^{y_2(n-\tau_{21})}}{K_2(n)} - (\varPhi_1^* y_1)(n)\mathrm{e}^{y_1(n-\tau_{22})} - b_2(n)(\varPhi_2^* y_2)(n-\tau_{23})] \end{pmatrix}.$$

It is not difficult to find L is a bounded linear operator with $\operatorname{Ker} L = X_c$, $\operatorname{Im} L = X_0$, $\dim \operatorname{Ker} L = 2 = \operatorname{codim} \operatorname{Im} L$, and it follows that L is a Fredholm operator with index zero. Define

$$Pz = \frac{1}{\omega} \sum_{s=0}^{\omega-1} z(s) = Qz,\ z \in X,$$

then P, Q are continuous projectors such that

$$\operatorname{Im} P = \operatorname{Ker} L, \qquad \operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q),$$

furthermore, the generalized inverse $K_P : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ exists and has the form

$$K_P(z)(n) = \sum_{s=0}^{k-1} z(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s) z(s).$$

$$
\begin{aligned}
&QNy\\
&\quad=\begin{pmatrix}\frac{1}{\omega}\sum_{n=0}^{\omega-1}\{r_1(n)[1-\frac{\mathrm{e}^{y_1(n-\tau_{11})}}{K_1(n)}-(\Phi_2^*y_2)(n)\mathrm{e}^{y_2(n-\tau_{12})}-b_1(n)(\Phi_1^*y_1)(n-\tau_{13})]\}\\ \frac{1}{\omega}\sum_{n=0}^{\omega-1}\{r_2(n)[1-\frac{\mathrm{e}^{y_2(n-\tau_{21})}}{K_2(n)}-(\Phi_1^*y_1)(n)\mathrm{e}^{y_1(n-\tau_{22})}-b_2(n)(\Phi_2^*y_2)(n-\tau_{23})]\}\end{pmatrix}\\
&\quad=\begin{pmatrix}\overline{r_1}-\frac{1}{\omega}\sum_{n=0}^{\omega-1}\frac{r_1(n)}{K_1(n)}\mathrm{e}^{y_1(n-\tau_{11})}\\ \overline{r_2}-\frac{1}{\omega}\sum_{n=0}^{\omega-1}\frac{r_2(n)}{K_2(n)}\mathrm{e}^{y_2(n-\tau_{21})}\end{pmatrix}\\
&\qquad-\begin{pmatrix}\frac{1}{\omega}\sum_{n=0}^{\omega-1}r_1(n)b_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1}G_1(n-\tau_{13},u)\beta_1(u)\mathrm{e}^{y_1(u-\sigma_1)}\\ \frac{1}{\omega}\sum_{n=0}^{\omega-1}r_2(n)b_2(n)\sum_{u=n-\tau_{23}}^{n-\tau_{23}+\omega-1}G_2(n-\tau_{23},u)\beta_2(u)\mathrm{e}^{y_2(u-\sigma_2)}\end{pmatrix}\\
&\qquad-\begin{pmatrix}\frac{1}{\omega}\sum_{n=0}^{\omega-1}r_1(n)\sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u)\mathrm{e}^{y_2(n-\tau_{22})+y_2(u-\sigma_2)}\\ \frac{1}{\omega}\sum_{n=0}^{\omega-1}r_2(n)\sum_{u=n}^{n+\omega-1}G_1(n,u)\beta_1(u)\mathrm{e}^{y_1(n-\tau_{12})+y_1(u-\sigma_1)}\end{pmatrix}.
\end{aligned}
$$

Obviously, QN and $K_P(I-Q)N$ are both continuous. Since X is Banach space, by using Arzela–Ascoli theorem, we know that operator $K_P(I-Q)N(\overline{\Omega})$ is compact and $QN(\overline{\Omega})$ is bounded for any open bounded set $\Omega\in X$. So, $N\in\Omega$ is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega\in X$.

In order to use Lemma 5, we need to find an appropriate open, bounded subset Ω. Considering the operator equation $Ly=\lambda Ny$, $\lambda\in(0,1)$, i.e.,

$$
\begin{aligned}
&\begin{pmatrix}y_1(n+1)-y_1(n)\\ y_2(n+1)-y_2(n)\end{pmatrix}\\
&\quad=\lambda\begin{pmatrix}r_1(n)[1-\frac{\mathrm{e}^{y_1(n-\tau_{11})}}{K_1(n)}-(\Phi_2^*y_2)(n)\mathrm{e}^{y_2(n-\tau_{12})}-b_1(n)(\Phi_1^*y_1)(n-\tau_{13})]\\ r_2(n)[1-\frac{\mathrm{e}^{y_2(n-\tau_{21})}}{K_2(n)}-(\Phi_1^*y_1)(n)\mathrm{e}^{y_1(n-\tau_{22})}-b_2(n)(\Phi_2^*y_2)(n-\tau_{23})]\end{pmatrix}.
\end{aligned}
$$

Suppose that $y(n)=(y_1(n),y_2(n))^{\mathrm{T}}\in X$ is an arbitrary solution of above equation for a certain $\lambda\in(0,1)$. Integrating both sides of the above equation over the interval $[0,\omega-1]$ with respect to n, we obtain

$$
\begin{aligned}
0=\lambda\sum_{n=0}^{\omega-1}\biggl[&r_1(n)-\frac{r_1(n)\exp\{y_1(n-\tau_{11})\}}{K_1(n)}\\
&-r_1(n)(\Phi_2^*y_2)(n)\exp\bigl\{y_2(n-\tau_{12})\bigr\}-b_1(n)r_1(n)\bigl(\Phi_1^*y_1\bigr)(n-\tau_{13})\biggr],
\end{aligned}\tag{33}
$$

$$
\begin{aligned}
0=\lambda\sum_{n=0}^{\omega-1}\biggl[&r_2(n)-\frac{r_2(n)\exp\{y_2(n-\tau_{21})\}}{K_2(n)}\\
&-r_2(n)(\Phi_1^*y_1)(n)\exp\{y_1(n-\tau_{22})\}-b_2(n)r_2(n)(\Phi_2^*y_2)(n-\tau_{23})\biggr],
\end{aligned}\tag{34}
$$

that is,

$$\begin{aligned}\overline{r_1}\omega = \sum_{n=0}^{\omega-1}\biggl[&\frac{r_1(n)\exp\{y_1(n-\tau_{11})\}}{K_1(n)}\\ &+ r_1(n)(\varPhi_2^* y_2)(n)\exp\bigl\{y_2(n-\tau_{12})\bigr\} + b_1(n)r_1(n)(\varPhi_1^* y_1)(n-\tau_{13})\biggr], \qquad (35)\\ \overline{r_2}\omega = \sum_{n=0}^{\omega-1}\biggl[&\frac{r_2(n)\exp\{y_2(n-\tau_{21})\}}{K_2(n)}\\ &+ r_2(n)(\varPhi_1^* y_1)(n)\exp\bigl\{y_1(n-\tau_{22})\bigr\} + b_2(n)r_2(n)(\varPhi_2^* y_2)(n-\tau_{23})\biggr]. \qquad (36)\end{aligned}$$

From (33)–(36) it follows that

$$\begin{aligned}&\sum_{n=0}^{\omega-1}\bigl|\bigl(y_1(n+1)-y_1(n)\bigr)\bigr|\\ &\quad\leqslant \lambda\Biggl\{\sum_{n=0}^{\omega-1}\bigl|r_1(n)\bigr| + \sum_{n=0}^{\omega-1}\biggl[\frac{r_1(n)\exp\{y_1(n-\tau_{11})\}}{K_1(n)}\\ &\qquad + r_1(n)(\varPhi_2^* y_2)(n)\exp\bigl\{y_2(n-\tau_{12})\bigr\} + b_1(n)r_1(n)(\varPhi_1^* y_1)(n-\tau_{13})\biggr]\Biggr\}\\ &\quad< 2\overline{r_1}\omega,\\ &\sum_{n=0}^{\omega-1}\bigl|\bigl(y_2(n+1)-y_2(n)\bigr)\bigr|\\ &\quad\leqslant \lambda\Biggl\{\sum_{n=0}^{\omega-1}\bigl|r_2(n)\bigr| + \sum_{n=0}^{\omega-1}\biggl[\frac{r_2(n)\exp\{y_2(n-\tau_{21})\}}{K_2(n)}\\ &\qquad + r_2(n)(\varPhi_1^* y_1)(n)\exp\bigl\{y_1(n-\tau_{22})\bigr\} + b_2(n)r_2(n)(\varPhi_2^* y_2)(n-\tau_{23})\biggr]\Biggr\}\\ &\quad< 2\overline{r_2}\omega.\end{aligned}$$

In the view of the fact $y = \{y(n)\} \in X$, there exist $\xi_i, \eta_i \in I_\omega$ such that

$$y_i(\xi_i) = \min_{n\in I_\omega}\bigl\{y_i(n)\bigr\}, \qquad y_i(\eta_i) = \max_{n\in I_\omega}\bigl\{y_i(n)\bigr\}, \quad i = 1, 2. \qquad (37)$$

From (35) and (37) we get

$$\begin{aligned}\overline{r_1}\omega &\geqslant \sum_{n=0}^{\omega-1}\frac{r_1(n)\exp\{y_1(n-\tau_{11})\}}{K_1(n)} \geqslant \exp\bigl\{y_1(\xi_1)\bigr\}\sum_{n=0}^{\omega-1}\frac{r_1(n)}{K_1(n)}\\ &= \exp\bigl\{y_1(\xi_1)\bigr\}\overline{\biggl(\frac{r_1}{K_1}\biggr)}\omega, \qquad (38)\end{aligned}$$

which implies

$$y_1(\xi_1) \leqslant \ln\left[\frac{\overline{r_1}}{\overline{\left(\frac{r_1}{K_1}\right)}}\right].$$

Then from Lemma 6 we obtain

$$y_1(n) \leqslant y_1(\xi_1) + \sum_{n=0}^{\omega-1}\bigl|\bigl(y_1(n+1) - y_1(n)\bigr)\bigr| < 2\overline{r_1}\omega + \ln\left[\frac{\overline{r_1}}{\overline{\left(\frac{r_1}{K_1}\right)}}\right] := B_1.$$

In the same way, we can get

$$y_2(n) < 2\overline{r_2}\omega + \ln\left[\frac{\overline{r_2}}{\overline{\left(\frac{r_2}{K_2}\right)}}\right] := B_2. \tag{39}$$

On the other hand, (35) and (37) imply that

$$\begin{aligned}
\overline{r_1}\omega &= \sum_{n=0}^{\omega-1}\left[\frac{r_1(n)\exp\{y_1(n-\tau_{11})\}}{K_1(n)} + r_1(n)(\varPhi_2^* y_2)(n)\exp\bigl\{y_2(n-\tau_{12})\bigr\}\right.\\
&\quad \left. + b_1(n)r_1(n)(\varPhi_1^* y_1)(n-\tau_{13})\right]\\
&\leqslant \exp\bigl\{y_1(\eta_1)\bigr\}\overline{\left(\frac{r_1}{K_1}\right)}\omega\\
&\quad + \exp\bigl\{y_2(\eta_2)\bigr\}\sum_{n=0}^{\omega-1} r_1(n)\left[\sum_{u=n}^{n+\omega-1} G_2(n,u)\beta_2(u)\exp\bigl\{y_2(u-\sigma_2)\bigr\}\right]\\
&\quad + \sum_{n=0}^{\omega-1} b_1(n)r_1(n)\left[\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1} G_1(n-\tau_{13},u)\beta_1(u)\exp\bigl\{y_1(u-\sigma_1)\bigr\}\right]\\
&\leqslant \exp\bigl\{y_1(\eta_1)\bigr\}\left[\overline{\left(\frac{r_1}{K_1}\right)}\omega + \sum_{n=0}^{\omega-1} b_1(n)r_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1} G_1(n-\tau_{13},u)\beta_1(u)\right]\\
&\quad + \exp\bigl\{2y_2(\eta_2)\bigr\}\left[\sum_{n=0}^{\omega-1} r_1(n)\sum_{u=n}^{n+\omega-1} G_2(n,u)\beta_2(u)\right].
\end{aligned}$$

From (39) we will get

$$\begin{aligned}
\overline{r_1}\omega &\leqslant \exp\bigl\{y_1(\eta_1)\bigr\}\left[\overline{\left(\frac{r_1}{K_1}\right)}\omega + \sum_{n=0}^{\omega-1} b_1(n)r_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1} G_1(n-\tau_{13},u)\beta_1(u)\right]\\
&\quad + \mathrm{e}^{2B_2}\sum_{n=0}^{\omega-1} r_1(n)\sum_{u=n}^{n+\omega-1} G_2(n,u)\beta_2(u),
\end{aligned}$$

namely,

$$\begin{aligned}
&\exp\{y_1(\eta_1)\}\left[\overline{\left(\frac{r_1}{K_1}\right)}\omega+\sum_{n=0}^{\omega-1}b_1(n)r_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1}G_1(n-\tau_{13},u)\beta_1(u)\right]\\
&\quad\geqslant\overline{r_1}\omega-\mathrm{e}^{2B_2}\sum_{n=0}^{\omega-1}r_1(n)\sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u).
\end{aligned}$$

From (H3) we can get

$$y_1(\eta_1)\geqslant\ln\frac{\overline{r_1}\omega-\mathrm{e}^{2B_2}\sum_{n=0}^{\omega-1}r_1(n)\sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u)}{\overline{(\frac{r_1}{K_1})}\omega+\sum_{n=0}^{\omega-1}b_1(n)r_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1}G_1(n-\tau_{13},u)\beta_1(u)}.$$

From Lemma 6 we get

$$\begin{aligned}
y_1(n)&\geqslant y_1(\eta_1)-\sum_{s=0}^{\omega-1}\left|y_1(n+1)-y_1(n)\right|\\
&>\ln\frac{\overline{r_1}\omega-\mathrm{e}^{2B_2}\sum_{n=0}^{\omega-1}r_1(n)\sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u)}{\overline{(\frac{r_1}{K_1})}\omega+\sum_{n=0}^{\omega-1}b_1(n)r_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1}G_1(n-\tau_{13},u)\beta_1(u)}-2\overline{r_1}\omega\\
&:=B_3.
\end{aligned}$$

In the same way, according to (H2), we can get

$$\begin{aligned}
y_2(n)&>\ln\frac{\overline{r_2}\omega-\mathrm{e}^{2B_1}\sum_{n=0}^{\omega-1}r_2(n)\sum_{u=n}^{n+\omega-1}G_1(n,u)\beta_1(u)}{\overline{(\frac{r_2}{K_2})}\omega+\sum_{n=0}^{\omega-1}b_2(n)r_2(n)\sum_{u=n-\tau_{23}}^{n-\tau_{23}+\omega-1}G_2(n-\tau_{23},u)\beta_2(u)}-2\overline{r_2}\omega\\
&:=B_4.
\end{aligned}$$

Clearly, B_1, B_2, B_3, B_4 are independent of λ. Set $M=M_1+M_2+M_0$, where $M_1=\max\{|B_1|,|B_3|\}$, $M_2=\max\{|B_2|,|B_4|\}$, M_0 is taken sufficiently large such that each solution (if it exists) $y^*=(y_1^*,y_2^*)$ of the algebraic equations

$$QNy=(F_1,F_2)^{\mathrm{T}}=0$$

satisfies $\|y^*\|=\|(y_1^*,y_2^*)^{\mathrm{T}}\|=|y_1^*|+|y_2^*|<M$, in which

$$\begin{aligned}
F_1=\overline{r_1}-\overline{\left(\frac{r_1(n)}{K_1(n)}\right)}\mathrm{e}^{y_1}-\overline{\left(r_1(n)b_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1}G_1(n-\tau_{13},u)\beta_1(u)\right)}\mathrm{e}^{y_1}\\
-\overline{\left(r_1(n)\sum_{u=n}^{n+\omega-1}G_2(n,u)\beta_2(u)\right)}\mathrm{e}^{2y_2},
\end{aligned}$$

$$
\begin{aligned}
F_2 = \overline{r_2} &- \overline{\left(\frac{r_2(n)}{K_2(n)}\right)}\mathrm{e}^{y_2} - \overline{\left(r_2(n)b_2(n)\sum_{u=n-\tau_{23}}^{n-\tau_{23}+\omega-1} G_2(n-\tau_{23},u)\beta_2(u)\right)}\mathrm{e}^{y_2} \\
&- \overline{\left(r_2(n)\sum_{u=n}^{n+\omega-1} G_1(n,u)\beta_1(u)\right)}\mathrm{e}^{2y_1}.
\end{aligned}
$$

We now take $\Omega = \{y = (y_1(n), y_2(n))^{\mathrm{T}}\colon\ y \in X,\ \|y\| < M\}$. This satisfies condition (C1) of Lemma 5. When $y = (y_1, y_2)^{\mathrm{T}} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2$, $y = (y_1, y_2)^{\mathrm{T}}$ is a constant vector in $\mathbb{R}^2$ with $\|y\| = M$, then we have $QNy \neq 0$. This prove that condition (C2) of Lemma 5 holds.

Now we consider homotopic, let

$$
H(y_1, y_2, \mu) = \begin{pmatrix} \overline{r_1} - \overline{(\frac{r_1(n)}{K_1(n)})}\mathrm{e}^{y_1} \\ \overline{r_2} - \overline{(\frac{r_2(n)}{K_2(n)})}\mathrm{e}^{y_2} \end{pmatrix} - \mu \begin{pmatrix} H_1(y_1, y_2) \\ H_2(y_1, y_2) \end{pmatrix}, \quad \mu \in [0, 1],
$$

where

$$
\begin{aligned}
H_1(y_1, y_2) &= \frac{1}{\omega}\sum_{n=0}^{\omega-1} r_1(n)b_1(n)\sum_{u=n-\tau_{13}}^{n-\tau_{13}+\omega-1} G_1(n-\tau_{13},u)\beta_1(u)\mathrm{e}^{y_1} \\
&\quad + \frac{1}{\omega}\sum_{n=0}^{\omega-1} r_1(n)\sum_{u=n}^{n+\omega-1} G_2(n,u)\beta_2(u)\mathrm{e}^{2y_2}, \\
H_2(y_1, y_2) &= \frac{1}{\omega}\sum_{n=0}^{\omega-1} r_2(n)b_2(n)\sum_{u=n-\tau_{23}}^{n-\tau_{23}+\omega-1} G_2(n-\tau_{23},u)\beta_2(u)\mathrm{e}^{y_2} \\
&\quad + \frac{1}{\omega}\sum_{n=0}^{\omega-1} r_2(n)\sum_{u=n}^{n+\omega-1} G_1(n,u)\beta_1(u)\mathrm{e}^{2y_1}.
\end{aligned}
$$

When $(y_1, y_2, \mu) \in \partial\Omega \cap \operatorname{Ker} L \times [0, 1]$, $H(y_1, y_2, \mu) \neq 0$. Hence, by a direct calculation we have

$$
\begin{aligned}
&\deg\{JQN,\ \Omega \cap \operatorname{Ker} L,\ 0\} \\
&\quad = \deg\bigl\{H(y_1, y_2, 1),\ \Omega \cap \operatorname{Ker} L,\ 0\bigr\} = \deg\bigl\{H(y_1, y_2, 0),\ \Omega \cap \operatorname{Ker} L,\ 0\bigr\} \\
&\quad = \deg\biggl\{\biggl(\overline{r_1} - \overline{\left(\frac{r_1(n)}{K_1(n)}\right)}\mathrm{e}^{y_1}, \overline{r_2} - \overline{\left(\frac{r_2(n)}{K_2(n)}\right)}\mathrm{e}^{y_2}\biggr)^{\mathrm{T}},\ \Omega \cap \operatorname{Ker} L,\ 0\biggr\}.
\end{aligned}
$$

Obviously, the algebraic equation

$$
\overline{r_1} - \overline{\left(\frac{r_1(n)}{K_1(n)}\right)}\mathrm{e}^{y_1} = 0, \qquad \overline{r_2} - \overline{\left(\frac{r_2(n)}{K_2(n)}\right)}\mathrm{e}^{y_2} = 0,
$$

has a unique solution $(\widetilde{y_1}, \widetilde{y_2})^{\mathrm{T}} \in \Omega \cap \operatorname{Ker} L$, thus

$$
\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} = \operatorname{sgn} \left.\begin{vmatrix} -(\overline{\frac{r_1(n)}{K_1(n)}})\mathrm{e}^{y_1} & 0 \\ 0 & -(\overline{\frac{r_2(n)}{K_2(n)}})\mathrm{e}^{y_2} \end{vmatrix}\right|_{(\widetilde{y_1}, \widetilde{y_2})} = 1.
$$

This completes the proof of condition (C3) in the Lemma 5. According to Lemma 5, system (32) has at least one positive ω-periodic solution, that is, system (7) has at least one positive ω-periodic solution. ☐

Remark 1. If we take $\tau_{11} = \tau_{12} = \tau_{13} = \tau_{21} = \tau_{22} = \tau_{23} = \sigma_1 = \sigma_2 = 0$, system (7) becomes the model in [2]. So, we generalize the main result of [2].

Remark 2. When $\alpha_1(n) = \alpha_2(n) = 1$, $\beta_1(n) = \beta_2(n) = b_1(n) = b_2(n) = 0$, system (7) becomes (5). So, we extend and improve the main results of [8].

5 Application

As application, we consider the following system:

$$\begin{aligned}
x_1(n+1) &= x_1(n)\exp\bigl[(1.15+0.05\sin n)\bigl(1-x_1(n-1)-\mu_2(n)x_2(n-1)\\
&\qquad -(0.02+0.01\sin n)\mu_1(n-1)\bigr)\bigr],\\
x_2(n+1) &= x_2(n)\exp\bigl[(1.25-0.05\cos n)\bigl(1-x_2(n-1)-\mu_1(n)x_1(n-2)\\
&\qquad -(0.015-0.005\cos n)\mu_2(n-2)\bigr)\bigr],\\
\mu_1(n+1) &= (0.075-0.025\sin n)\mu_1(n)+(0.015-0.005\cos n)x_1(n-1),\\
\mu_2(n+1) &= (0.075+0.025\cos n)\mu_2(n)+(0.025+0.005\sin n)x_2(n-2)
\end{aligned}\tag{40}$$

with the initial condition $(x_1(0), y_1(0), \mu_1(0), \mu_2(0)) = (1, 0.1, 0.1, 0.2)$.

By simple computation, we derive

$$x_1^* \approx 3.6865, \qquad \mu_1^* \approx 0.0388, \qquad x_2^* \approx 3.7347, \qquad \mu_2^* \approx 0.1179,$$
$$r_1^L - \mu_2^* x_2^* \approx 0.6597 > 0, \qquad r_2^L - \mu_1^* x_1^* \approx 1.0569 > 0.$$

So, by Theorem 1, we claim that system (40) is persistent. Its integral curves and orbits are shown in Figs. 1–4, respectively.

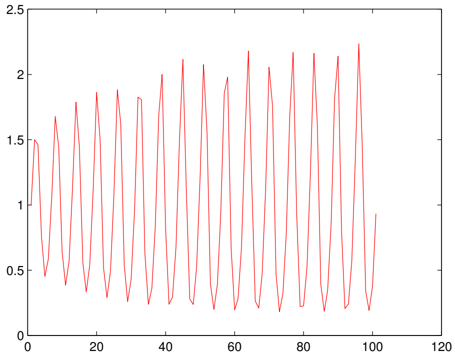

Fig. 1. The orbit of x_1-time n.

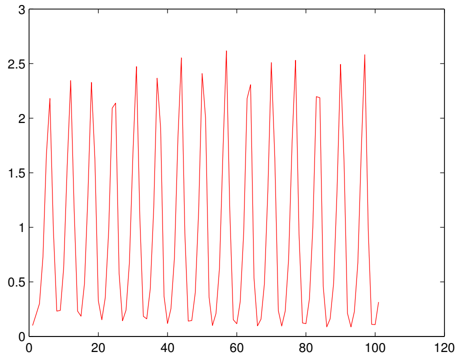

Fig. 2. The orbit of x_2-time n.

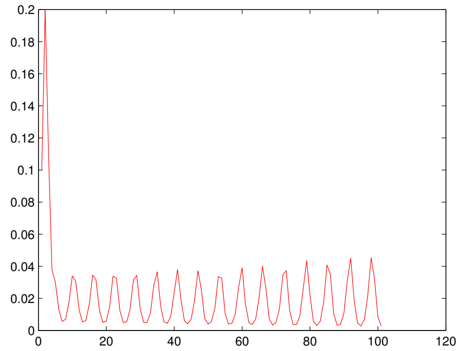

Fig. 3. The orbit of μ_1-time n.

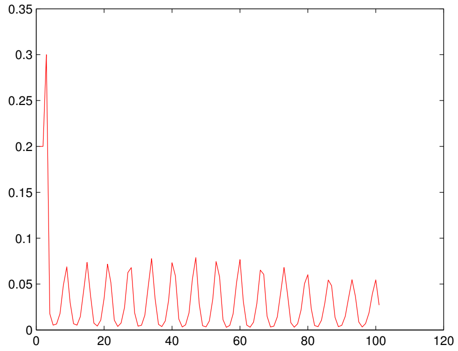

Fig. 4. The orbit of μ_2-time n.

From Figs. 1–4 we see that there is a positive periodic solution of system (40).

6 Conclusions

In this paper, we consider a discrete periodic Lotka–Volterra competition system with feedback control and time delays. By applying the theory of difference inequality and Mawhin's coincidence degree theory, which is different from that of [2, 4, 8], we show that the permanence and the existence of periodic solution of system (7). From the proof of Theorems 1 and 3 we can also consider the permanence and periodic solutions of a discrete n-species Lotka–Volterra competition system with feedback control and time delays, the similar results can be obtain. Compared with [4], we can find that the feedback control variables have no influence on the persistent properties of the system, Similarly, we can obtain the delay is harmless for the permanence and the positive periodic solution of system (7). In [11], the author considered the almost periodic solution of system (6). As for the existence of system (7), we leave this for the future work.

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