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Invariant analysis and explicit solutions of the time fractional nonlinear perturbed Burgers equation*

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Abstract. The Lie group analysis method is performed for the nonlinear perturbed Burgers equation and the time fractional nonlinear perturbed Burgers equation. All of the point symmetries of the equations are constructed. In view of the point symmetries, the vector fields of the equations are constructed. Subsequently, the symmetry reductions are investigated. In particular, some novel exact and explicit solutions are obtained.

Keywords: perturbed Burgers equation, Lie group analysis, symmetry reductions, power series method, explicit solutions.

1 Introduction

Nonlinear evolution equations (NLEEs) play an increasingly important role in mathematical modeling of physical, biological, and chemical processes, and it is also used in fractal and differential geometry and so on [1–11, 13–26, 29–32].

In general, the fractional partial differential equations (FPDEs) can be written as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F[u].$$

Here $u = u(x,t)$ represents the unknown function, $F[u]$ is a given function in regard to their variables, α is a real number. $\partial^\alpha u/\partial t^\alpha = D_t^\alpha$ represents the Riemann–Liouville (R–L) derivative,

$$D_t^\alpha u = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-\xi)^{n-\alpha-1} u(\xi,x)\,\mathrm{d}\xi, & n-1<\alpha<n,\ n\in N, \\ \frac{\partial^n u}{\partial t^n}, \quad \alpha = n \in N, \end{cases}$$

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where the Euler gamma function $\Gamma(z)$ is given by

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} \, \mathrm{d}t.$$

This paper focus on the Burgers equation (BE) that is used in the studies of cosmic rays. The study of Burgers equation plays an key role in solitary waves theory [1, 6, 9, 11, 13, 31, 32]. It was demonstrated earlier that the cosmic ray shocks can be modeled by the BE in the long wavelength and small amplitude limit [31]. Then, it was also shown that the generalized BE describes the temporal evolution of weak shocks in the context of diffusive shock accleration [32].

The perturbed Burgers equation (pBE) that is studied is given by [9, 11]

$$u_t - au^2 u_x + buu_x + cu_{xx} - duu_{xx} - e(u_x)^2 - ku_{xxx} = 0, \tag{1}$$

where $u(x,t)$ denotes the unknown function of the space variable x and time t, also indicates the profile of the shock wave. Equation (1) play an important role in gas dynamics and heated fluids. Equation (1) display in the long-wave small-amplitude limit of extended systems dominated by dissipation [9]. In different circumstances, Eq. (1) contains a lot of important nonlinear PDEs. For example, if $a = c = d = e = 0$, then (1) just is celebrated Korteweg–de Vries equation. Let $b = c = d = e = 0$, it is the mKdV equation. When $a = d = e = k = 0$, it becomes famous Burgers equation.

The time fractional nonlinear perturbed Burgers equation is

$$\frac{\partial^\alpha u}{\partial t^\alpha} - au^2 u_x + buu_x + cu_{xx} - duu_{xx} - e(u_x)^2 - ku_{xxx} = 0, \tag{2}$$

where $0 < \alpha \leqslant 1$, a, b, c, d, e, k are parameters, α shows up the order of the fractional time-derivative. If $\alpha = 1$, the fractional equation reduces to the classical perturbed Burgers equation. In [22], invariant analysis of time fractional generalized Burgers equation are investigated.

The remainder of this paper is divided as follows. In Section 2, we perform Lie group classification on the perturbed Burgers equation (pBE), all of the geometric vector fields of the pBE are presented. In Section 3, the complete symmetry classification of the time fractional nonlinear perturbed Burgers equation are established. In Section 4, some exact solutions of pBE are discussed. In Section 5, we firstly give some exact solutions of time fractional nonlinear perturbed Burgers equation. The main results of the paper are summarized in the last section.

2 Lie symmetry group analysis of Eq. (1)

In this section, we apply Lie group method to deal with Eq. (1).

Considering one parameter Lie group of point transformations

$$t^* = t + \epsilon\tau(x,t,u) + O(\epsilon^2),$$
$$x^* = x + \epsilon\xi(x,t,u) + O(\epsilon^2),$$
$$u^* = u + \epsilon\eta(x,t,u) + O(\epsilon^2)$$

and the corresponding infinitesimal generator

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}.$$

Consider the Lie's symmetry condition, one can get

$$\mathrm{pr}^{(3)}V(\Delta_1)|_{\Delta_1=0} = 0,$$

where $\Delta_1 = u_t - au^2u_x + buu_x + cu_{xx} - duu_{xx} - e(u_x)^2 - ku_{xxx}$. We employ the third prolongation $\mathrm{pr}^{(3)}V$ to Eq. (1), one can get

$$\begin{aligned}&\eta^t - k\eta^{xxx} - 2eu_x\eta^x - du\eta^{xx} - d\eta u_{xx}\\ &\quad - au^2\eta^x - 2a\eta uu_x + bu\eta^x + b\eta u_x + c\eta^{xx} = 0,\end{aligned}$$

where

$$\begin{aligned}\eta^t &= D_t(\eta) - u_xD_t(\xi) - u_tD_t(\tau)\\ &= D_t(\eta - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}\\ &= \eta_t - \xi_t u_x + (\eta_u - \tau_t)u_x - \tau_t u_t - \xi_u u_x u_t - \tau_u u_t^2,\end{aligned}$$

$$\begin{aligned}\eta^x &= D_x(\eta) - u_xD_x(\xi) - u_tD_x(\tau)\\ &= D_x(\eta - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}\\ &= \eta_x + (\eta_u - \xi_x)u_x - \tau_x u_t - \xi_u u_x^2 - \tau_u u_x u_t,\end{aligned} \tag{3}$$

$$\begin{aligned}\eta^{xx} &= D_x(\eta^x) - u_{xt}D_x(\tau) - u_{xx}D_x(\xi)\\ &= \eta_{xx} + (2\eta_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\eta_{uu} - 2\xi_{xu})u_x^2\\ &\quad - 2\tau_{xu}u_xu_t - \xi_{uu}u_x^3 - \tau_{uu}u_x^2u_t + (\eta_u - 2\xi_x)u_{xx}\\ &\quad - 2\tau_x u_{xt} - 3\xi_u u_{xx}u_x - \tau_u u_{xx}u_t - 2\tau_u u_{xt}u_x,\end{aligned} \tag{4}$$

$$\begin{aligned}\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt}D_x(\tau) - u_{xxx}D_x(\xi)\\ &= \eta_{xxx} + (3\eta_{xxu} - \xi_{xxx})u_x - \tau_{xxx}u_t + 3(\eta_{xuu} - \xi_{xxu})u_x^2\\ &\quad - 3\tau_{xxu}u_xu_t + (\eta_{uuu} - 3\xi_{xuu})u_x^3 + 3(\eta_{xu} - \xi_{xx})u_{xx}\\ &\quad - 3\tau_x u_{xxt} + (\eta_u - 3\xi_x)u_{xxx} - 3\tau_{xx}u_{xt} - 3\tau_{xuu}u_x^2u_t\\ &\quad + 3(\eta_{uu} - 3\xi_{xu})u_xu_{xx} - 3\tau_{xu}u_tu_{xx} - 3\tau_{uu}u_{xx}u_xu_t\\ &\quad - 6\tau_{xu}u_{xt}u_x - \xi_{xxx}u_x^4 - 6\xi_{uu}u_x^2u_{xx} - 3\tau_{uu}u_x^2u_{tx} - \tau_u u_{xx}u_t\\ &\quad - 4\xi_u u_{xxx}u_x - \tau_{uuu}u_x^3u_t - 3\xi_u u_{xx}^2 - 3\tau_u u_{xxt}u_x - 3\tau_u u_{xt}u_{xx}.\end{aligned} \tag{5}$$

Here $(x^1, x^2) = (t, x)$ and D_i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots, \quad i = 1, 2.$$

Then, in terms of the Lie symmetry analysis method, one can obtain the following results:

1) For the arbitrary parameters a, b, c, d, e, k, the infinitesimal generator of Eq. (1) is as follows:
$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}. \tag{6}$$
2) For the case $a = c = d = e = 0$, infinitesimal generator of Eq. (1) is given by
$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = t\frac{\partial}{\partial x} + \frac{1}{b}\frac{\partial}{\partial u},$$
$$V_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$
3) If $b = c = d = e = 0$, then the infinitesimal generator of Eq. (1) is
$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{7}$$
4) When $b = c = 0$, we obtain that the infinitesimal generator of Eq. (1) is the same as (7).
5) Let $c = 0$, $e = 0$, one can get that the infinitesimal generator of Eq. (1) is the same as (6), that is
$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}.$$
6) Let $b = d = e = k = 0$, one can have
$$V_1 = \frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = 2x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}.$$

3 Lie symmetry group analysis of Eq. (2)

In this section, we deal with all of the point symmetries of the FPDE (2), then the symmetries are presented. As previous step, under a one parameter Lie group of point transformations [3, 5, 7, 22, 24, 25, 26, 29], one can get

$$\begin{aligned}
t^* &= t + \epsilon\tau(x, t, u) + O(\epsilon^2),\\
x^* &= x + \epsilon\xi(x, t, u) + O(\epsilon^2),\\
u^* &= u + \epsilon\eta(x, t, u) + O(\epsilon^2),\\
\frac{\partial^\alpha \bar{u}}{\partial \bar{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \epsilon\eta_\alpha^0(x, t, u) + O(\epsilon^2), \quad \ldots,
\end{aligned}$$

where

$$\begin{aligned}\eta_\alpha^0 &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha\big(D_t(\tau)u\big) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u) \\ &= \frac{\partial^\alpha \eta}{\partial t^\alpha} + \big(\eta_u - \alpha D_t(\tau)\big)\frac{\partial^\alpha u}{\partial t^\alpha} - u\frac{\partial^\alpha \eta_u}{\partial t^\alpha} + \mu \\ &\quad + \sum_{n=1}^{\infty}\left[\binom{a}{n}\frac{\partial^\alpha \eta_u}{\partial t^\alpha} - \binom{a}{n+1}D_t^{n+1}(\tau)\right]D_t^{\alpha-n}(u) \\ &\quad - \sum_{n=1}^{\infty}\binom{a}{n}D_t^n(\xi)D_t^{\alpha-n}(u_x)\end{aligned}$$

with

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{a}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}[-u]^r\frac{\partial^m}{\partial t^m}\big[u^{k-r}\big]\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

The corresponding vector field is given by

$$V = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}. \tag{8}$$

We will determine the coefficient functions $\xi(x,t,u)$, $\tau(x,t,u)$, and $\eta(x,t,u)$ later.

Assume that the vector field (8) can derive a symmetry of (2), then V must satisfy the following Lie's symmetry condition:

$$\mathrm{pr}^{(n)}V(\Delta_1)|_{\Delta_1=0} = 0,$$

where

$$\Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} - au^2u_x + buu_x + cu_{xx} - duu_{xx} - e(u_x)^2 - ku_{xxx}.$$

As we have given in previous process, one can obtain

$$\begin{aligned}&\eta_\alpha^0 - k\eta^{xxx} - 2eu_x\eta^x + bu\eta^x + b\eta u_x \\ &\quad - au^2\eta^x - 2a\eta uu_x + c\eta^{xx} - du\eta^{xx} - d\eta u_{xx} = 0.\end{aligned}$$

Here η^x, η^{xx}, η^{xxx} are the same as Eqs. (3)–(5). Based on the Lie group calculation method, one obtains:

1) For the arbitrary parameters a, b, c, d, e, k, the corresponding vector fields of Eq. (2) are

$$V_1 = \frac{\partial}{\partial x}. \tag{9}$$

2) For the case $a = c = d = e = 0$, the corresponding vector fields of Eq. (2) are given by

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u}.$$

3) If $b = c = d = e = 0$, we get the corresponding vector fields of Eq. (2)

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = x\frac{\partial}{\partial x} + \frac{3t}{\alpha}\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{10}$$

4) When $b = c = 0$, we obtain that the corresponding vector fields of Eq. (2) are the same as (10).
5) Let $c = 0$, $e = 0$, one can get that the corresponding vector fields of Eq. (2) are the same as (9).

4 Symmetry reductions and exact solutions of Eq. (1)

In this section, we deal with the symmetry reductions and exact solutions of this equation. We will consider the following similarity reductions and group-invariant solutions:

4.1. V_1
The group-invariant solution corresponding to V_1 is $u = f(\xi)$, where $\xi = t$ is the group-invariant, the substitution of this solution into Eq. (1) gives the trivial solution $u(x,t) = C$, C is a constant.

4.2. V_2 (*stationary solution*)
The group-invariant solution corresponding to V_2 is $u = f(\xi)$, where $\xi = x$ is the group-invariant, the substitution of this solution into the (1) yields

$$bff' - af^2f' + cf'' - dff'' - e(f')^2 - kf''' = 0. \tag{11}$$

4.3. $V_2 + \lambda V_1$ (*travelling wave solutions*)
For the case of linear combination $V_2 + \lambda V_1$, we get

$$u = f(\xi), \tag{12}$$

where $\xi = x - \lambda t$ is the group-invariant. Putting (12) into the (1), one gets

$$\lambda f' + bff' - af^2f' + cf'' - dff'' - e(f')^2 - kf''' = 0. \tag{13}$$

According to the homogeneous balance principle, one can get

$$f(\xi) = a_0 + a_1\varphi, \tag{14}$$

where a_0, a_1 are constants to be determined, $\varphi(\xi)$ satisfies

$$\varphi' = A + B\varphi + C\varphi^2. \tag{15}$$

Plugging (14) with (15) into (13), collecting different coefficients of φ, and letting each coefficients equal to zero, one gets

$$A = A, \qquad B = B, \qquad C = C, \qquad a = a, \qquad b = b, \qquad a_0 = a_0, \qquad a_1 = a_1,$$

$$c = -\frac{BCda_1 + Baa_1{}^2 - 2C^2da_0 - 2Caa_0a_1 + Cba_1}{2C^2}, \qquad d = d,$$

$$e = -\frac{6C^2k + 2Cda_1 + aa_1^2}{Ca_1}, \qquad k = k,$$

$$\lambda = -\frac{8AC^3k + 4AC^2da_1 + 2ACaa_1^2 - 2B^2C^2k - B^2Cda_1 - B^2aa_1^2}{2C^2} + \frac{2BCaa_0a_1 - 2C^2aa_0^2 - BCba_1 + 2C^2ba_0}{2C^2}.$$

One can find many exact travelling wave solutions for (1) as follows.

Family 1. When $\Delta = B^2 - 4AC > 0$ and $BC \neq 0$ (or $AC \neq 0$),

$$u(x,t) = a_0 - a_1\frac{1}{2C}\left[B + \Delta\tanh\left(\frac{\Delta}{2}\xi\right)\right],$$

$$u(x,t) = a_0 - a_1\frac{1}{2C}\left[B + \Delta\coth\left(\frac{\Delta}{2}\xi\right)\right],$$

$$u(x,t) = a_0 - a_1\frac{1}{2C}\big[B + \Delta\big(\tanh(\Delta\xi) \pm \mathrm{i}\,\mathrm{sech}(\Delta\xi)\big)\big],$$

$$u(x,t) = a_0 - a_1\frac{1}{2C}\big[B + \Delta\big(\coth(\Delta\xi) \pm \mathrm{i}\,\mathrm{csch}(\Delta\xi)\big)\big],$$

$$u(x,t) = a_0 - a_1\frac{1}{4C}\left[2B + \Delta\left(\tanh\left(\frac{\Delta}{4}\xi\right) + \coth\left(\frac{\Delta}{4}\xi\right)\right)\right],$$

$$u(x,t) = a_0 + a_1\frac{1}{2C}\left[-B + \frac{\sqrt{(E^2+F^2)}\Delta - E\Delta\cosh(\Delta\xi)}{E\sinh(\Delta\xi) + F}\right],$$

$$u(x,t) = a_0 + a_1\frac{1}{2C}\left[-B - \frac{\sqrt{(F^2-E^2)}\Delta + E\Delta\sinh(\Delta\xi)}{E\cosh(\Delta\xi) + F}\right],$$

where E and F are two non-zero real constants and satisfies $F^2 - E^2 > 0$.

$$u(x,t) = a_0 + a_1\frac{2A\cosh(\frac{\Delta}{2}\xi)}{\Delta\sinh(\frac{\Delta}{2}\xi) - B\cosh(\frac{\Delta}{2}\xi)},$$

$$u(x,t) = a_0 + a_1\frac{-2A\sinh(\frac{\Delta}{2}\xi)}{-\Delta\cosh(\frac{\Delta}{2}\xi) + B\sinh(\frac{\Delta}{2}\xi)},$$

$$u(x,t) = a_0 + a_1\frac{2A\cosh(\Delta\xi)}{\Delta\sinh(\Delta\xi) - B\cosh(\Delta\xi) \pm \mathrm{i}\Delta},$$

$$u(x,t) = a_0 + a_1\frac{2A\sinh(\Delta\xi)}{\Delta\cosh(\Delta\xi) - B\sinh(\Delta\xi) \pm \Delta},$$

$$u(x,t) = a_0 + a_1\frac{4A\sinh(\frac{\Delta}{4}\xi)\cosh(\frac{\Delta}{4}\xi)}{-2B\sinh(\frac{\Delta}{4}\xi)\cosh(\frac{\Delta}{4}\xi) + 2\Delta\cosh^2(\frac{\Delta}{4}\xi) - \Delta}.$$

Family 2. When $\Delta = 4AC - B^2 > 0$ and $BC \neq 0$ (or $AC \neq 0$),

$$u(x,t) = a_0 + a_1 \frac{1}{2C}\left[-B + \Delta \tan\left(\frac{\Delta}{2}\xi\right)\right],$$
$$u(x,t) = a_0 - a_1 \frac{1}{2C}\left[B + \Delta \cot\left(\frac{\Delta}{2}\xi\right)\right],$$
$$u(x,t) = a_0 + a_1 \frac{1}{2C}\left[-B + \Delta\bigl(\tan(\Delta\xi) \pm \sec(\Delta\xi)\bigr)\right],$$
$$u(x,t) = a_0 - a_1 \frac{1}{2C}\left[B + \Delta\bigl(\cot(\Delta\xi) \pm \csc(\Delta\xi)\bigr)\right],$$
$$u(x,t) = a_0 - a_1 \frac{1}{4C}\left[-2B + \Delta\left(\tan\left(\frac{\Delta}{4}\xi\right) - \cot\left(\frac{\Delta}{4}\xi\right)\right)\right],$$
$$u(x,t) = a_0 + a_1 \frac{1}{2C}\left[-B + \frac{\pm\sqrt{(F^2 - E^2)}\Delta - E\Delta\cos(\Delta\xi)}{E\sin(\Delta\xi) + F}\right],$$
$$u(x,t) = a_0 + a_1 \frac{1}{2C}\left[-B + \frac{\pm\sqrt{(F^2 - E^2)}\Delta + E\Delta\sinh(\Delta\xi)}{E\cos(\Delta\xi) + F}\right],$$

where E and F are two non-zero real constants and satisfies $F^2 - E^2 > 0$.

$$u(x,t) = a_0 + a_1 \frac{-2A\cos(\frac{\Delta}{2}\xi)}{\Delta\sin(\frac{\Delta}{2}\xi) + B\cos(\frac{\Delta}{2}\xi)},$$
$$u(x,t) = a_0 + a_1 \frac{2A\sin(\frac{\Delta}{2}\xi)}{\Delta\cos(\frac{\Delta}{2}\xi) - B\sin(\frac{\Delta}{2}\xi)},$$
$$u(x,t) = a_0 + a_1 \frac{-2A\cos(\Delta\xi)}{\Delta\sin(\Delta\xi) + B\cos(\Delta) \pm \Delta},$$
$$u(x,t) = a_0 + a_1 \frac{2A\sin(\Delta\xi)}{\Delta\cos(\Delta\xi) - B\sin(\Delta) \pm \Delta},$$
$$u(x,t) = a_0 + a_1 \frac{4A\sin(\frac{\Delta}{4}\xi)\cos(\frac{\Delta}{4}\xi)}{-2B\sin(\frac{\Delta}{4}\xi)\cos(\frac{\Delta}{4}\xi) + 2\Delta\cos^2(\frac{\Delta}{4}\xi) - \Delta}.$$

Family 3. When $A = 0$ and $BC \neq 0$,

$$u(x,t) = a_0 + a_1 \frac{-Bd}{C(d + \cosh(B\xi) - \sinh(B\xi))},$$
$$u(x,t) = a_0 + a_1 - \frac{\cosh(B\xi) + \sinh(B\xi)}{C(d + \cosh(B\xi) + \sinh(B\xi))},$$

where d is an arbitrary constant.

Family 4. When $A = B = 0$ and $C \neq 0$,

$$u(x,t) = a_0 + a_1 \frac{-1}{B\xi + k},$$

where k is an arbitrary constant.

Remark 1. In fact, all of these solutions can be derived from the following equation [12]:

$$\varphi = \frac{\sqrt{4AC - B^2}}{2C} \frac{C_1 \mathrm{e}^{(\theta/2)\sqrt{4AC-B^2}} - C_2 \mathrm{e}^{-(\theta/2)\sqrt{4AC-B^2}}}{C_1 \mathrm{e}^{(\theta/2)\sqrt{4AC-B^2}} + C_2 \mathrm{e}^{-(\theta/2)\sqrt{4AC-B^2}}} - \frac{B}{2C},$$

where C_1, C_2 are arbitrary constants.

Next, we look for a solution of (13) as follows:

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{16}$$

Putting (16) into (13), one can get

$$\begin{aligned}
&\lambda c_1 + \lambda \sum_{n=1}^{\infty} (n+1) c_{n+1} \xi^n + b c_0 c_1 + b \sum_{n=1}^{\infty} \sum_{j=0}^{n} (n+1-j) c_j c_{n+1-j} \xi^n \\
&\quad - 6 c_3 k + 2 c c_2 + c \sum_{n=1}^{\infty} (n+1)(n+2) c_{n+2} \xi^n - e c_1^2 \\
&\quad - e \sum_{n=1}^{\infty} \sum_{j=0}^{n+1} j(n+2-j) c_j c_{n+2-j} \xi^n - 2 d c_0 c_2 \\
&\quad - d \sum_{n=1}^{\infty} \sum_{j=0}^{n} (n+1-j)(n+2-j) c_j c_{n+2-j} \xi^n - a c_0^2 c_1 \\
&\quad - a c_0^2 \sum_{n=1}^{\infty} (n+1) c_{n+1} \xi^n - a \sum_{n=1}^{\infty} \sum_{j=1}^{n} \sum_{i=0}^{j} (n+1-j) c_i c_{j-i} c_{n+1-j} \xi^n \\
&\quad - k \sum_{n=1}^{\infty} (n+3)(n+2)(n+1) c_{n+3} \xi^n = 0.
\end{aligned} \tag{17}$$

Comparing coefficients for $n = 0$ in (17), one can get

$$c_3 = \frac{\lambda c_1 + b c_0 c_1 + 2 c c_2 - e c_1^2 - 2 d c_0 c_2 - a c_0^2 c_1}{6k}.$$

For the general case, for $n \geqslant 1$, one can arrive at

$$\begin{aligned}
c_{n+3} = \frac{1}{k(n+1)(n+2)(n+3)} \Bigg(& b \sum_{j=0}^{n} (n+1-j) c_j c_{n+1-j} \\
& + c(n+1)(n+2) c_{n+2} \lambda (n+1) c_{n+1} + c(n+1)(n+2) c_{n+2}
\end{aligned}$$

$$- e\sum_{j=1}^{n+1} j(n+2-j)c_j c_{n+2-j} - ac_0^2(n+1)c_{n+1}$$
$$- d\sum_{j=0}^{n}(n+1-j)(n+2-j)c_j c_{n+2-j}$$
$$- a\sum_{j=1}^{n}\sum_{i=0}^{j}(n+1-j)c_i c_{j-i} c_{n+1-j}\Bigg).$$

Therefore, the power series solution of (16) can be rewritten

$$\begin{aligned}
f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \sum_{n=1}^{\infty} c_{n+3}\xi^{n+3} \\
&= c_0 + c_1\xi + c_2\xi^2 + \frac{\lambda c_1 + bc_0c_1 + 2cc_2 - ec_1^2 - 2dc_0c_2 - ac_0^2c_1}{6k}\xi^3 \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{k(n+1)(n+2)(n+3)}\Bigg(b\sum_{j=0}^{n}(n+1-j)c_j c_{n+1-j} \\
&\quad + c(n+1)(n+2)c_{n+2}\lambda(n+1)c_{n+1} + c(n+1)(n+2)c_{n+2} \\
&\quad - e\sum_{j=1}^{n+1} j(n+2-j)c_j c_{n+2-j} - ac_0^2(n+1)c_{n+1} \\
&\quad - d\sum_{j=0}^{n}(n+1-j)(n+2-j)c_j c_{n+2-j} \\
&\quad - a\sum_{j=1}^{n}\sum_{i=0}^{j}(n+1-j)c_i c_{j-i} c_{n+1-j}\Bigg)\xi^{n+3}.
\end{aligned}$$

Thus, the explicit solution of (1) is

$$\begin{aligned}
u(x,t) &= c_0 + c_1(x-\lambda t) + c_2(x-\lambda t)^2 \\
&\quad + \frac{\lambda c_1 + bc_0c_1 + 2cc_2 - ec_1^2 - 2dc_0c_2 - ac_0^2c_1}{6k}(x-\lambda t)^3 \\
&\quad + \sum_{n=1}^{\infty} c_{n+3}(x-\lambda t)^{n+3} \\
&= c_0 + c_1(x-\lambda t) + c_2(x-\lambda t)^2 \\
&\quad + \frac{\lambda c_1 + bc_0c_1 + 2cc_2 - ec_1^2 - 2dc_0c_2 - ac_0^2c_1}{6k}(x-\lambda t)^3 \\
&\quad + \sum_{n=1}^{\infty}\frac{1}{k(n+1)(n+2)(n+3)}\Bigg(b\sum_{j=0}^{n}(n+1-j)c_j c_{n+1-j} \\
&\quad + c(n+1)(n+2)c_{n+2}\lambda(n+1)c_{n+1} + c(n+1)(n+2)c_{n+2}
\end{aligned}$$

$$
\begin{aligned}
&- e\sum_{j=1}^{n+1} j(n+2-j)c_j c_{n+2-j} - ac_0^2(n+1)c_{n+1} \\
&- d\sum_{j=0}^{n}(n+1-j)(n+2-j)c_j c_{n+2-j} \\
&- a\sum_{j=1}^{n}\sum_{i=0}^{j}(n+1-j)c_i c_{j-i} c_{n+1-j}\Bigg)(x-\lambda t)^{n+3},
\end{aligned}
$$

where c_i ($i = 0, 1, 2, 3$) are arbitrary constants.

Remark 2. As far as we know, these solutions are new.

Remark 3. If $\lambda = 0$, the exact solutions of Eq. (11) can be derived.

5 Symmetry reductions and exact solutions of Eq. (2)

In this section, we will deal with the symmetry reductions and exact solutions of Eq. (2).

Recently, sub-equation method is successfully used for solving fractional differential equations [16, 27, 28].

Firstly, we employ the following transformations:

$$u(x,t) = u(\xi), \quad \xi = x + lt, \tag{18}$$

where l is an constant. Putting (18) into (2), we get the following equation:

$$l^\alpha D_\xi^\alpha u - au^2 u_\xi + buu_\xi + cu_{\xi\xi} - duu_{\xi\xi} - e(u_\xi)^2 - ku_{\xi\xi\xi} = 0. \tag{19}$$

We assume that solution of Eq. (19) can be written as follows:

$$u(\xi) = a_0 + \sum_{i=1}^{n} a_i\big(\psi(\xi)\big)^i,$$

where a_i ($i = 1, \dots, n$) are constants to be fixed later. The function $\psi(\xi)$ satisfy the following Bäklund transformation of fractional Riccati equation [16, 25, 28]:

$$\psi(\xi) = \frac{-\sigma B + D\phi(\xi)}{D + B\phi(\xi)}, \tag{20}$$

where B, D are arbitrary parameters, and $B \neq 0$. Also, $\phi(\xi)$ meets the following equation:

$$D_\xi^\alpha \phi(\xi) = \sigma + \phi^2(\xi), \tag{21}$$

where σ is a constant. Equation (21) has solutions

$$\phi(\xi) = \begin{cases} -\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi, \alpha), & \sigma < 0, \\ -\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi, \alpha), & \sigma < 0, \\ \sqrt{\sigma}\tan(\sqrt{\sigma}\xi, \alpha), & \sigma > 0, \\ -\sqrt{\sigma}\cot(\sqrt{\sigma}\xi, \alpha), & \sigma > 0, \\ -\Gamma(1+\alpha)/(\xi^\alpha + \omega), & \omega \text{ is constant}, \sigma = 0, \end{cases}$$

with

$$\sin_\alpha(\xi) = \frac{E_\alpha(\mathrm{i}\xi^\alpha) - E_\alpha(-\mathrm{i}\xi^\alpha)}{2\mathrm{i}}, \qquad \cos_\alpha(\xi) = \frac{E_\alpha(\mathrm{i}\xi^\alpha) + E_\alpha(-\mathrm{i}\xi^\alpha)}{2\mathrm{i}},$$

$$\sinh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2}, \qquad \cosh_\alpha(\xi) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2},$$

$$\tan_\alpha(\xi) = \frac{\sin_\alpha(\xi)}{\cos_\alpha(\xi)}, \qquad \cot_\alpha(\xi) = \frac{\cos_\alpha(\xi)}{\sin_\alpha(\xi)},$$

$$\tanh_\alpha(\xi) = \frac{\sinh_\alpha(\xi)}{\cosh_\alpha(\xi)}, \qquad \coth_\alpha(\xi) = \frac{\cosh_\alpha(\xi)}{\sinh_\alpha(\xi)},$$

here $E_\alpha(\xi) = \sum_{k=0}^\infty \xi^k/\Gamma(1+k\alpha)$ ($\alpha > 0$) represents the Mittag–Leffler function in one parameter.

Once again, consider the homogeneous balance principle, we get

$$u(\xi) = a_0 + a_1\psi. \tag{22}$$

Putting (20), (21) with (22) into (19), one can get algebraic equations about l, a_0, a_1 with the coefficients of $(\phi)^i$. Solving them, one can derive:

Case 1.

$$B = B, \quad D = D, \quad a = a, \quad b = b, \quad c = c, \quad d = d, \quad \alpha = \alpha,$$
$$e = e, \quad k = k, \quad l = l, \quad \sigma = -\frac{D^2}{B^2}, \quad a_0 = a_0, \quad a_1 = a_1.$$

Case 2.

$$B = B, \quad D = D, \quad a = a, \quad b = b, \quad c = aa_0a_1 - \frac{1}{2}ba_1 + da_0,$$
$$d = d, \quad e = e, \quad l = \left(-\frac{1}{3}a\sigma a_1^2 + a{a_0}^2 - \frac{2}{3}d\sigma a_1 + \frac{2}{3}e\sigma a_1 - ba_0\right)^{1/\alpha},$$
$$k = -\frac{1}{6}aa_1^2 - \frac{1}{3}da_1 - \frac{1}{6}ea_1, \quad \alpha = \alpha, \quad \sigma = \sigma, \quad a_0 = a_0, \quad a_1 = a_1.$$

In Case 1, one can obtain new solutions of Eq. (2) as follows:

$$u = a_0 + a_1 \frac{-\sigma B - D\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi, \alpha)}{D - B\sqrt{-\sigma}\tanh(\sqrt{-\sigma}\xi, \alpha)},$$

where $\sigma < 0$, $\xi = x + lt$;

$$u = a_0 + a_1 \frac{-\sigma B - D\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi, \alpha)}{D - B\sqrt{-\sigma}\coth(\sqrt{-\sigma}\xi, \alpha)},$$

where $\sigma < 0$, $\xi = x + lt$;

$$u = a_0 + a_1 \frac{-\sigma B + D\sqrt{\sigma}\tan(\sqrt{\sigma}\xi, \alpha)}{D + B\sqrt{\sigma}\tan(\sqrt{\sigma}\xi, \alpha)},$$

where $\sigma > 0$, $\xi = x + lt$;

$$u = a_0 + a_1 \frac{-\sigma B - D\sqrt{\sigma}\cot(\sqrt{\sigma}\xi, \alpha)}{D - B\sqrt{\sigma}\cot(\sqrt{\sigma}\xi, \alpha)},$$

where $\sigma > 0$, $\xi = x + lt$;

$$u = a_0 + a_1 \frac{D\Gamma(1+\alpha)}{-D(\xi^\alpha + \omega) + B\Gamma(1+\alpha)},$$

where $\sigma = 0$, $\xi = x + lt$.

Remark 4. In Case 2, one can also get other exact solutions of (2). We do not list all of them here.

6 Concluding remarks

In the present paper, by using the Lie symmetry groups, we studied the symmetry properties, similarity reduction forms and explicit solutions of nonlinear perturbed Burgers equation and the time fractional nonlinear perturbed Burgers equation. We note that there is the essential difference between the fractional nonlinear perturbed Burgers equation and the nonlinear perturbed Burgers equation. Furthermore, it should be also stressed that the obtained point transformation groups of the fractional equation (2) are relatively fewer than the evolution equation (1). At last, some exact and explicit solutions of the equations are presented. The obtained results are helpful to better understand the intricate nonlinear physical real world.

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