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Self-approximation of periodic Hurwitz zeta-functions

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Abstract. Let $\zeta(s,\omega;\mathfrak{A})$ be the periodic Hurwitz zeta-function. We look for real numbers α and β for which there exist "many" real numbers τ such that the shifts $\zeta(s+\mathrm{i}\alpha\tau,\omega;\mathfrak{A})$ and $\zeta(s+\mathrm{i}\beta\tau,\omega;\mathfrak{A})$ are "near" each other.

Keywords: self-approximation, periodic Hurwitz zeta-functions.

1 Introduction

Let as usual $s=\sigma+it$ denote a complex variable. Let ω be a fixed real number from the interval $(0,1]$ and denote by $\mathfrak{A}=\{c_m\colon m\in\mathbb{N}_0\}$, $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, a periodic sequence of complex numbers with the smallest period $k\in\mathbb{N}$. For $\sigma>1$, the periodic Hurwitz zeta-function is defined by

$$\zeta(s,\omega;\mathfrak{A})=\sum_{m=0}^{\infty}\frac{c_m}{(m+\omega)^s}.$$

If $\mathfrak{A}=\{1\}$, then $\zeta(s,\omega;\mathfrak{A})$ is the classical Hurwitz zeta-function

$$\zeta(s,\omega)=\sum_{m=0}^{\infty}\frac{1}{(m+\omega)^s},\quad \sigma>1,$$

which has meromorphic continuation to the whole complex plane with a simple pole $s=1$ and residue 1.

If $\omega=1$, then the function $\zeta(s,\omega;\mathfrak{A})$ reduces to the periodic zeta-function

$$\zeta(s;\mathfrak{A})=\sum_{m=1}^{\infty}\frac{c_{m-1}}{m^s},\quad \sigma>1.$$

It is not difficult to see that, for $\sigma > 1$,

$$\zeta(s,\omega;\mathfrak{A}) = \sum_{l=0}^{k-1}\sum_{m=0}^{\infty}\frac{c_l}{(mk+l+\omega)^s} = \frac{1}{k^s}\sum_{l=0}^{k-1}c_l\sum_{m=0}^{\infty}\frac{1}{(m+(l+\omega/k))^s}$$
$$= \frac{1}{k^s}\sum_{l=0}^{k-1}c_l\zeta\left(s,\frac{l+\omega}{k}\right). \tag{1}$$

Therefore, (1) gives the analytic continuation for $\zeta(s,\omega;\mathfrak{A})$ to the whole complex plane, except, perhaps, for a simple pole $s=1$ with residue

$$c = \frac{1}{k}\sum_{l=0}^{k-1}c_l.$$

If $c=0$, then $\zeta(s,\omega;\mathfrak{A})$ is an entire function.

In the case when $\mathfrak{A}=\{1\}$ and $\omega=1$, the function $\zeta(s,\omega;\mathfrak{A})$ becomes the Riemann zeta-function

$$\zeta(s) = \sum_{m=1}^{\infty}\frac{1}{m^s}, \quad \sigma > 1.$$

In 1982, Bagchi [1] proved that the Riemann hypothesis for Dirichlet L-function $L(s,\chi)$ (χ is an arbitrary Dirichlet character) holds if and only if for any compact subset $\mathcal{K}$ of the strip $1/2 < \sigma < 1$ and for any $\varepsilon > 0$:

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]\colon \max_{s\in\mathcal{K}}\big|L(s+\mathrm{i}\tau,\chi)-L(s,\chi)\big| < \varepsilon\Big\} > 0,$$

where meas stands for the Lebesgue measure on $\mathbb{R}$.

Recently, Nakamura in [5] considered joint universality of shifted Dirichlet L-functions, which led to the following generalization of Bagchi's criteria. Assume that $1=d_1$, $d_2,\dots,d_m$ are algebraic real numbers linearly independent over $\mathbb{Q}$ and χ is an arbitrary Dirichlet character. Then, for every $\varepsilon>0$, we have

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\Big\{\tau\in[0,T]\colon \max_{1\leqslant j,k\leqslant m}\max_{s\in\mathcal{K}}\big|L(s+\mathrm{i}d_j\tau,\chi)-L(s+\mathrm{i}d_k\tau,\chi)\big| < \varepsilon\Big\} > 0. \tag{2}$$

For $m=2$, Pańkowski [8] using Six Exponentials Theorem showed that (2) holds as well for every real numbers d_1, d_2 linearly independent over $\mathbb{Q}$. The case where $d_1/d_2\in\mathbb{Q}$ in inequality (2) was considered by Garunkštis [2] and Nakamura [5] independently. It is worth mentioning that the proofs of their results contain gaps. The gaps were filled by Nakamura and Pańkowski in [7], where $d_1=1$ and $d_2=a/b\in\mathbb{Q}$ satisfies $\gcd(a,b)=1$, $|a-b|\neq 1$. It should be mentioned that the general case for $d_1=1$ and for non-zero rational d_2 is still open. Garunkštis and Karikovas [3] investigated the self-approximation property for Hurtwitz zeta-functions with a transcendental parameter ω. Karikovas and Pańkowski [4] deal with Hurwitz zeta-functions with rational ω.

In this paper, we prove two theorems.

Theorem 1. *Let $\mathfrak{A} = \{c_m\colon\ m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$. Let $\omega = a/b$, $\omega \in (0, 1]$, $0 < a < b$, $\gcd(a, b) = 1$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\biggl\{\tau \in [0,T]\colon\ \max_{s\in\mathcal{K}}\biggl|\zeta\biggl(s + \mathrm{i}\alpha\tau, \frac{a}{b}; \mathfrak{A}\biggr) - \zeta\biggl(s + \mathrm{i}\beta\tau, \frac{a}{b}; \mathfrak{A}\biggr)\biggr| < \varepsilon\biggr\} > 0.$$

In the next theorem, we consider the case when the parameter ω is a transcendental number.

Let $d_1, d_2, \ldots, d_k, \omega$ be real numbers and let ω be a transcendental number from the interval $(0, 1]$.

Let

$$A(d_1, d_2, \ldots, d_k; \omega) = \bigl\{d_j \log(n + \omega)\colon\ j = 1, \ldots, k;\ n \in \mathbb{N}_0\bigr\}$$

be a multiset. Note that in a multiset the elements can appear more than once. For example, $\{2, 3\}$ and $\{2, 3, 3\}$ are different multisets, but $\{2, 3\}$ and $\{3, 2\}$ are equal multisets. If a multiset $A(d_1, d_2, \ldots, d_k; \omega)$ is linearly independent over rational numbers, then $A(d_1, d_2, \ldots, d_k; \omega)$ is a set and the numbers $d_1, \ldots, d_k$ are linearly independent over $\mathbb{Q}$.

Denote by $\|x\|$ the minimal distance of $x \in \mathbb{R}$ to an integer.

Theorem 2. *Let $\mathfrak{A} = \{c_m\colon\ m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with the smallest period $k \in \mathbb{N}$. Let ω be a transcendental number from the interval $(0, 1]$. Moreover, suppose that $\alpha, \beta \in \mathbb{R}$ are such that the set $A(\alpha, \beta; \omega)$ is linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\biggl\{\tau \in [0,T]\colon\ \max_{s\in\mathcal{K}}\bigl|\zeta(s + \mathrm{i}\alpha\tau, \omega; \mathfrak{A}) - \zeta(s + \mathrm{i}\beta\tau, \omega; \mathfrak{A})\bigr| < \varepsilon, \\ \biggl\|\frac{(\alpha - \beta)\tau\log k}{2\pi}\biggr\| < \varepsilon\biggr\} > 0.$$

In the next section, we prove Theorem 1. Section 3 is devoted to the proof of Theorem 2.

2 Proof of theorem 1

Recall that, for Lebesgue measurable set $A \subset (0, \infty)$, we define *lower density* of A as

$$\liminf_{T\to\infty}\frac{1}{T}\,\mathrm{meas}\bigl(A \cap (0, T]\bigr).$$

Moreover, if the limit above is positive, then we say that A has a *positive lower density*.

In the proof of Theorem 1, the following statement will be useful.

Lemma 1. *Let $\mathcal{K} \subset D$ be any compact set with connected complement, $\chi_1, \dots, \chi_n$ be pairwise non-equivalent Dirichlet characters and f_j, g_j, $(j = 1, \dots, n)$ be functions which are non-vanishing and continuous on $\mathcal{K}$ and analytic in the interior. Moreover, let α, β be real numbers linearly independent over $\mathbb{Q}$ and B be a finite set of prime numbers.*

Then, for every $\varepsilon > 0$, the set of real numbers τ satisfying

$$\max_{1\leqslant j\leqslant n} \max_{s\in\mathcal{K}} \left|L(s + \mathrm{i}\alpha\tau, \chi_j) - f_j(s)\right| < \varepsilon,$$

$$\max_{1\leqslant j\leqslant n} \max_{s\in\mathcal{K}} \left|L(s + \mathrm{i}\beta\tau, \chi_j) - g_j(s)\right| < \varepsilon,$$

$$\max_{p\in B} \left\|\tau \frac{(\alpha - \beta)\log p}{2\pi}\right\| < \varepsilon$$

has a positive lower density.

Particularly, taking $f_j = g_j$ yields that the set of $\tau \in \mathbb{R}$ satisfying

$$\max_{1\leqslant j\leqslant n} \max_{s\in\mathcal{K}} \left|L(s + \mathrm{i}\alpha\tau, \chi_j) - L(s + \mathrm{i}\beta\tau, \chi_j)\right| < \varepsilon,$$

$$\max_{p\in B} \left\|\tau \frac{(\alpha - \beta)\log p}{2\pi}\right\| < \varepsilon$$

has a positive lower density.

Proof. This is Theorem 4.1 in [4]. □

Theorem 1 will be derived from the following proposition.

Proposition 1. *Let $k, n \in \mathbb{N}$ and $a_1/b_1, \dots, a_n/b_n$ be rational numbers satisfying $0 < a_j < b_j$ and $\gcd(a_j, b_j) = 1$ for $j = 1, 2, \dots, n$. Moreover, suppose that α, β are real numbers linearly independent over $\mathbb{Q}$ and $\mathcal{K}$ is any compact subset of the strip $1/2 < \sigma < 1$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \max_{s\in\mathcal{K}} \max_{1\leqslant j\leqslant n} \left|\zeta\left(s + \mathrm{i}\alpha\tau, \frac{a_j}{b_j}\right) - \zeta\left(s + \mathrm{i}\beta\tau, \frac{a_j}{b_j}\right)\right| < \varepsilon, \right.$$
$$\left. \max_{p|k} \left\|\frac{1}{2\pi}\tau \log p - 1\right\| < \varepsilon\right\} > 0.$$

Let the notation $A \ll B$ means that there exists $c > 0$ such that $|A| \leqslant cB$. Note that the inequality

$$\max_{p|k} \left\|\frac{1}{2\pi}\tau \log p - 1\right\| < \varepsilon$$

implies that

$$\max_{s\in\mathcal{K}} \left|k^{s+\mathrm{i}\tau} - k^s\right| \ll \varepsilon.$$

Proof. Let us consider the set of the functions $\{\zeta(s, a_1/b_1), \zeta(s, a_2/b_2), \dots, \zeta(s, a_n/b_n)\}$.
Since $(a_j, b_j) = 1$ $(j = 1, \dots n)$, we have:

$$\zeta\left(s, \frac{a_j}{b_j}\right) = \frac{b_j^s}{\varphi(b_j)} \sum_{\chi^{(j)} \bmod b_j} \overline{\chi^{(j)}(a_j)} L\bigl(s, \chi^{(j)}\bigr) = \frac{b_j^s}{\varphi(b_j)} \sum_{k=1}^{\varphi(b_j)} \overline{\chi_k^{(j)}(a_j)} L\bigl(s, \chi_k^{(j)}\bigr).$$

Thus

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \frac{b_l^s}{\varphi(b_l)} \sum_{\chi^{(l)} \bmod b_l} \overline{\chi^{(l)}(a_l)} L\bigl(s, \chi^{(l)}\bigr).$$

Two characters, $\chi_1 \bmod k_1$, $\chi_2 \bmod k_2$, are equivalent if they are induced by the same primitive character $\chi^* \bmod k$ with $k|k_1$ and $k|k_2$. Then, for $j = 1, 2$, we have

$$L(s, \chi_j) = L(s, \chi^*) \prod_{p|k_j} \left(1 - \frac{\chi^*(p)}{p^s}\right).$$

Now let us assume that $\chi_k^{(j)}$ is induced by a primitive character $\chi_k^{(j)*}$. Let us observe that every two elements from the set

$$\bigl\{\chi_1^{(1)*}, \chi_2^{(1)*}, \dots, \chi_{\varphi(b_1)}^{(1)*}, \dots, \chi_1^{(n)*}, \chi_2^{(n)*}, \dots, \chi_{\varphi(b_n)}^{(n)*}\bigr\}$$

are non-equivalent either equal.

Let $\chi_1, \dots \chi_N$ denote all distinct characters in the set

$$\bigl\{\chi_1^{(1)*}, \chi_2^{(1)*}, \dots, \chi_{\varphi(b_1)}^{(1)*}, \dots, \chi_1^{(n)*}, \chi_2^{(n)*}, \dots, \chi_{\varphi(b_n)}^{(n)*}\bigr\}.$$

Moreover, put

$$P\bigl(s, \chi^{(j)}\bigr) = \begin{cases} 1 & \text{if } \chi^{(j)} \text{ is primitive,} \\ \prod_{p|q}\bigl(1 - \frac{\chi^{(j)*}(p)}{p^s}\bigr) & \text{if } \chi^{(j)} \text{ is imprimitive character } \bmod q. \end{cases}$$

Let us observe that, for any imprimitive character $\chi^{(j)} \bmod q$, we have

$$\bigl|P\bigl(s + \mathrm{i}\tau, \chi^{(j)}\bigr) - P\bigl(s, \chi^{(j)}\bigr)\bigr| \ll \varepsilon,$$

provided

$$\max_{p|q} \left\| \frac{1}{2\pi} \tau \log p \right\| \ll \varepsilon.$$

Therefore,

$$\zeta\left(s, \frac{a_j}{b_j}\right) = \frac{b_j^s}{\varphi(b_j)} \sum_{k=1}^{\varphi(b_j)} \overline{\chi_k^{(j)}(a_j)} P\bigl(s, \chi_k^{(j)}\bigr) L\bigl(s, \chi_k^{(j)}\bigr).$$

We see that, for any $\varepsilon > 0$, there are $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that

$$\left|\zeta\left(s + \mathrm{i}\tau, \frac{a_j}{b_j}\right) - \zeta\left(s, \frac{a_j}{b_j}\right)\right| < \varepsilon$$

for all $j = 1, \dots, n$ if

$$\begin{aligned} &\left|L(s + \mathrm{i}\tau, \chi_r) - L(s, \chi_r)\right| < \varepsilon_1 \quad \text{for all } r = 1, \dots, N, \\ &\left|P\big(s + \mathrm{i}\tau, \chi_r^{(j)}\big) - P\big(s, \chi_r^{(j)}\big)\right| < \varepsilon_2 \quad \text{for all } j = 1, \dots, n,\ r = 1, \dots, \varphi(b_j). \end{aligned} \tag{3}$$

The above inequalities (3) are implied by Lemma 1. This proves Proposition 1. □

Proof of Theorem 1. From equality (1) for $\omega = a/b \in \mathbb{Q}$ we obtain

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{lb + a}{bk}\right).$$

Obviously, for all l with $0 \leqslant l \leqslant k - 1$, we can find a_l, b_l such that $(a_l, b_l) = 1$ and $(lb + a)/(bk) = a_l/b_l$. Hence

$$\zeta\left(s, \frac{a}{b}, \mathfrak{A}\right) = \frac{1}{k^s} \sum_{l=0}^{k-1} c_l \zeta\left(s, \frac{a_l}{b_l}\right).$$

Now we have that

$$\begin{aligned} &\max_{s \in \mathcal{K}} \left|\zeta(s + \mathrm{i}\alpha\tau, \omega; \mathfrak{A}) - \zeta(s + \mathrm{i}\beta\tau, \omega; \mathfrak{A})\right| \\ &\quad = \max_{s \in \mathcal{K}} \left|\frac{1}{k^{s+\mathrm{i}\alpha\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s + \mathrm{i}\alpha\tau, \frac{a_l}{b_l}\right) - \frac{1}{k^{s+\mathrm{i}\beta\tau}} \sum_{l=0}^{k-1} c_l \zeta\left(s + \mathrm{i}\beta\tau, \frac{a_l}{b_l}\right)\right| \\ &\quad \leqslant \max_{s \in \mathcal{K}} \max_{0 \leqslant l \leqslant k-1} |kc_l| \left|\frac{1}{k^{s+\mathrm{i}\alpha\tau}} \zeta\left(s + \mathrm{i}\alpha\tau, \frac{a_l}{b_l}\right) - \frac{1}{k^{s+i\beta\tau}} \zeta\left(s + \mathrm{i}\beta\tau, \frac{a_l}{b_l}\right)\right|. \end{aligned} \tag{4}$$

Note that $|kc_l| \ll 1$.

In view of (4), it is easy to see that Theorem 1 follows from the Proposition 1. □

3 Proof of the theorem 2

In the proof of Theorem 2 the following lemmas will be useful.

Lemma 2. *Let $l \leqslant m$ be positive integers and let ω be a transcendental number from the interval $(0, 1]$. Let $d_1, \dots, d_l \in \mathbb{R}$ be such that $A(d_1, d_2, \dots, d_l; \omega)$ is linearly independent over $\mathbb{Q}$. For $m > l$, let $d_{l+1}, \dots, d_m \in \mathbb{R}$ be such that each d_k, $k = l + 1, \dots, m$, is a linear combination of $d_1, \dots, d_l$ over $\mathbb{Q}$. Then, for any $\varepsilon > 0$,*

$$\liminf_{T \to \infty} \frac{1}{T} \operatorname{meas}\left\{\tau \in [0, T]\colon \max_{1 \leqslant j,k \leqslant m} \max_{s \in \mathcal{K}} \left|\zeta(s + \mathrm{i}d_j\tau, \omega) - \zeta(s + \mathrm{i}d_k\tau, \omega)\right| < \varepsilon\right\} > 0.$$

Proof. This is Theorem 1 in [3]. □

Note that for any transcendental number ω, $0 < \omega \leqslant 1$, and for any real number d_1, the set $A(d_1;\omega)$ is linearly independent over $\mathbb{Q}$. The following lemma shows that for any positive integer l, "most" collections of real numbers $d_1, d_2, \ldots, d_l, \omega$, where $0 < \omega \leqslant 1$, are such that $A(d_1, d_2, \ldots, d_l; \omega)$ is linearly independent over $\mathbb{Q}$.

Lemma 3. *Let ω be a transcendental number and $l \geqslant 2$. If $A(d_1, d_2, \ldots, d_{l-1}; \omega)$ is linearly independent over $\mathbb{Q}$, then the set*

$$D = \bigl\{d_l \in \mathbb{R}\colon\ A(d_1, d_2, \ldots, d_l; \omega) \textit{ is linearly dependent over } \mathbb{Q}\bigr\}$$

is countable.

Proof. This is Proposition 2 in [3]. □

Next we will prove Theorem 2.

Proof of Theorem 2. Let α be a real number. By Lemma 3, we can find a real number β such that $A(\alpha, \beta; \omega)$ is linearly independent over $\mathbb{Q}$.

We have that

$$\begin{aligned}
&\max_{s\in\mathcal{K}} \bigl|\zeta(s + \mathrm{i}\alpha\tau, \omega; \mathfrak{A}) - \zeta(s + \mathrm{i}\beta\tau, \omega; \mathfrak{A})\bigr| \\
&\quad = \max_{s\in\mathcal{K}} \left|\frac{1}{k^{s+\mathrm{i}\alpha\tau}} \sum_{l=0}^{k-1} c_l\zeta(s + \mathrm{i}\alpha\tau, \omega_l) - \frac{1}{k^{s+\mathrm{i}\beta\tau}} \sum_{l=0}^{k-1} c_l\zeta(s + \mathrm{i}\beta\tau, \omega_l)\right| \\
&\quad \leqslant \max_{s\in\mathcal{K}} \max_{0\leqslant l\leqslant k-1} |kc_l| \left|\frac{1}{k^{s+\mathrm{i}\alpha\tau}}\zeta(s + \mathrm{i}\alpha\tau, \omega_l) - \frac{1}{k^{s+\mathrm{i}\beta\tau}}\zeta(s + \mathrm{i}\beta\tau, \omega_l)\right|.
\end{aligned}$$

Note that $|kc_l| \ll 1$.

Inequality

$$\left\|\tau\frac{(\alpha - \beta)\log k}{2\pi}\right\| < \varepsilon$$

implies that

$$\bigl|k^{s+\mathrm{i}\alpha\tau} - k^{s+\mathrm{i}\beta\tau}\bigr| = \bigl|k^{\sigma}\bigr|\bigl|k^{i(\alpha-\beta)\tau} - 1\bigr| \ll \bigl|k^{\mathrm{i}(\alpha-\beta)\tau} - 1\bigr| \ll \varepsilon.$$

This means that $1/k^{s+\mathrm{i}\alpha\tau}$ is near $1/k^{s+\mathrm{i}\beta\tau}$.

Now we consider linear independence of numbers $\log(n + \omega_l)$ $(n \in \mathbb{N}_0)$ and $\log k$ over $\mathbb{Q}$, where $\omega_l = (l + \omega)/k$ and $l = 0, \ldots, k - 1$.

Assume that there exists a finite sequence of rational numbers

$$d_{ln}, \quad l = 0, \ldots, k - 1,\ n = 0, 1, 2, \ldots, N, \quad \text{and} \quad d$$

such that not all of these numbers are equal to 0 and

$$\sum_{l=0}^{k-1}\sum_{n=0}^{N} d_{ln}\log(n+\omega_l)+d\log k = \sum_{l=0}^{k-1}\sum_{n=0}^{N} d_{ln}\bigl(\log(nk+l+\omega)-\log k\bigr)+d\log k = 0.$$

Then

$$\sum_{l=0}^{k-1}\sum_{n=0}^{N} d_{ln}\log(nk+l+\omega)=\log k^{\gamma},$$

where

$$\gamma=\sum_{l=0}^{k-1}\sum_{n=0}^{N} d_{ln}-d$$

and

$$\prod_{l=0}^{k-1}\prod_{n=0}^{N}(nk+l+\omega)^{d_{ln}}=k^{\gamma}. \tag{5}$$

Numbers d_{ln}, d and γ are rationals. Therefore, it is not difficult to see that we can write (5) in the form $P(\omega)=0$, where $P(\omega)$ is a polynomial. Then ω is a root of this polynomial. But ω is a transcendental number, and we obtain a contradiction. This gives that numbers $\log(n+\omega_l)$ and $\log k$ are linearly independent over $\mathbb{Q}$.

By the linear independence of numbers $\log(n+\omega_l)$ and $\log k$ over $\mathbb{Q}$, and by Lemma 2 (for $m=2$) we obtain

$$\max_{s\in\mathcal{K}}\max_{0\leqslant l\leqslant k-1}\bigl|\zeta(s+\mathrm{i}\alpha\tau,\omega_l)-\zeta(s+\beta\tau,\omega_l)\bigr|\ll\varepsilon,$$

and Theorem 2 follows. □

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