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Fixed points for Kannan type contractions in uniform spaces endowed with a graph

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Abstract. In this paper, we give two sufficient conditions for the existence of a fixed point for a Kannan type contraction in a p-complete separated uniform space endowed with a graph and equipped with an A- or an E-distance p. We also discuss the uniqueness of the fixed point and show that Banach and Kannan type contractions are independent.

Keywords: separated uniform space, Kannan $\widetilde{G}$-p-contraction, fixed point.

1 Introduction and preliminaries

In [9, 10], Kannan investigated the existence and uniqueness of fixed points for mappings T defined on a (not necessarily complete) metric space (X, d) satisfying

$$d(Tx, Ty) \leqslant \alpha d(x, Tx) + \beta d(y, Ty) \tag{1}$$

for all $x, y \in X$, where $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$. In fact, in [9], Kannan showed that in a complete metric space (X, d), a mapping $T : X \to X$ satisfying (1) is a Picard operator, i.e. T has a unique fixed point $x^* \in X$ and $T^n x \to x^*$ for all $x \in X$, but in [10], he omitted the completeness of (X, d) and added some new hypotheses such as continuity of T at a single point of X and then proved the existence and uniqueness of a fixed point for T. Since the contractive condition (1) is independent of

$$d(Tx, Ty) \leqslant \alpha d(x, y) \quad (x, y \in X),$$

where $\alpha \in [0, 1)$, used in the well-known Banach contraction principle, it follows that Kannan's results are neither a generalization nor a special case of the Banach contraction principle.

Recently in [8], Jachymski investigated the Banach contraction principle in metric spaces with a graph and generalized the same results in metric and partially ordered metric

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spaces simultaneously. This idea was followed by Bojor for Kannan contractions (see [6]) and by the authors in uniform and modular spaces (see [2, 3, 4, 5]). In [1], the concepts of A- and E-distances were introduced in uniform spaces as a generalization of a metric and a w-distance for rewriting different types of nonlinear contractions in uniform spaces, especially those which cannot be presented via entourages.

The main purpose of the present work is studying Kannan contractions in uniform spaces endowed with a graph using A- and E-distances and investigating the existence and uniqueness of fixed points for them. Our main result generalizes Kannan's fixed point theorem in metric spaces and it is a new version of Theorem 3 in [6] in uniform spaces endowed with a graph and equipped with an A- or an E-distance.

We start with some basic notions about uniformities and uniform spaces which are used in this paper. For more details, the reader is referred to [11].

By a uniform space, we mean a pair $(X, \mathscr{U})$, briefly denoted by X, where X is a nonempty set and $\mathscr{U}$ is a uniformity on X (see [11, Def. 35.2]). The members of $\mathscr{U}$ are called the entourages of X.

It is well known that a uniformity $\mathscr{U}$ on a nonempty set X is separating if the intersection of all entourages of X coincides with the diagonal $\Delta(X) = \{(x, x)\colon x \in X\}$. Moreover, if $\mathscr{U}$ is a separating uniformity on a nonempty set X, then the uniform space X is called separated.

In particular, if (X, d) is a metric space, then the family $\mathscr{U}_d$ consisting of all supersets of the sets

$$\{(x, y) \in X \times X\colon d(x, y) < \varepsilon\} \quad (\varepsilon > 0)$$

is a uniformity on X called the uniformity induced by the metric d. It is clear that the uniform space $(X, \mathscr{U}_d)$ is separated.

We next recall the definition of an A- and an E-distance on a uniform space as well as the new notions of convergence, Cauchyness and completeness using A-distances.

Definition 1. (See [1].) If X is a uniform space, then a function $p : X \times X \to [0, +\infty)$ is called an A-distance on X whenever for all entourages $U \in \mathscr{U}$, there exists a $\delta > 0$ such that $p(z, x) \leqslant \delta$ and $p(z, y) \leqslant \delta$ imply $(x, y) \in U$ for all $x, y, z \in X$. If, further, p satisfies the triangular inequality, then p is called an E-distance on X.

An A-distance p on a uniform space X is called symmetric if $p(x, y) = p(y, x)$ for all $x, y \in X$.

Consider the set $X = [0, +\infty)$ with the uniformity induced by the usual metric. Then it is not difficult to verify that all the three functions $p_1, p_2, p_3 : X \times X \to [0, +\infty)$ defined by the rules

$$p_1(x, y) = y, \quad p_2(x, y) = \max\{x, y\} \quad \text{and} \quad p_3(x, y) = ax + by \quad (x, y \in X),$$

where $a \geqslant 0$ and $b > 0$, are E-distances on X. As seen, none of these three functions fulfills $p(x, x) = 0$ for all $x \in X$, and just p_2 is symmetric.

Remark 1. Note that even a symmetric E-distance on a uniform space is not a metric or even a pseudometric in general, i.e. a symmetric E-distance may not be vanished on the diagonal. For instance, if (X, d) is a metric space and $c > 0$, then the function $p : X \times X \to [0, +\infty)$ defined by the rule $p(x, y) = d(x, y) + c$ for all $x, y \in X$ is a symmetric E-distance on the uniform space $(X, \mathscr{U}_d)$. Indeed, the symmetry of p and the triangular inequality for p follow immediately from the symmetry of d and the triangular inequality for d, respectively. Furthermore, given any $\varepsilon > 0$, it suffices to put $\delta = c/2$ in the definition of an A-distance. But p vanishes nowhere on the diagonal of X because $p(x, x) = c > 0$ for all $x \in X$.

A sequence $\{x_n\}$ in X is called p-convergent to an $x \in X$, denoted by $x_n \overset{p}{\to} x$, whenever $p(x_n, x) \to 0$ as $n \to \infty$, and p-Cauchy whenever $p(x_m, x_n) \to 0$ as $m, n \to \infty$. Finally, the uniform space X is said to be p-complete if each p-Cauchy sequence in X is p-convergent to a point of X.

In the next lemma, a useful property of A-distances is given in separated uniform spaces.

Lemma 1. (See [1].) *Let p be an A-distance on a separated uniform space X and $\{x_n\}$ be a sequence in X. If $x_n \overset{p}{\to} x \in X$ and $x_n \overset{p}{\to} y \in X$, then $x = y$. In particular, if $x, y, z \in X$ and $p(z, x) = p(z, y) = 0$, then $x = y$.*

We are now going to recall some notions in graph theory as well as the way that a uniform space is endowed with a directed graph. For more details, the reader is referred to [7, 8].

Let X be a uniform space and G be a directed graph such that the set $V(G)$ of the vertices of G coincides with X, i.e. $V(G) = X$, and the set $E(G)$ of the edges of G contains all loops, i.e. $E(G) \supseteq \varDelta(X)$, but no parallel edges. In this case, the graph G can be simply denoted by the pair $G = (V(G), E(G)) = (X, E(G))$. If G is such a graph, then it is said that the uniform space X is endowed with the graph G.

By the notation $\widetilde{G}$, it is meant the undirected graph obtained from G by ignoring the directions of the edges of G, i.e.

$$V(\widetilde{G}) = X \quad \text{and} \quad E(\widetilde{G}) = \{(x, y) \in X \times X\colon (x, y) \in E(G) \vee (y, x) \in E(G)\}.$$

A graph $G = (V(G), E(G))$ is said to be symmetric if $(x, y) \in E(G)$ implies $(y, x) \in E(G)$ for all $x, y \in V(G)$. It is clear that the undirected graph $\widetilde{G}$ is symmetric.

A graph $H = (V(H), E(H))$ is said to be a subgraph of $G = (V(G), E(G))$ whenever $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and if $(x, y) \in E(H)$, then we have $x, y \in V(H)$ for all $x, y \in V(G)$.

Finally, if x and y are two vertices of a graph $G = (V(G), E(G))$, then a finite sequence $(x_j)_{j=0}^N$ of $N + 1$ vertices of G is called a path in G from x to y of length N whenever $x_0 = x$, $x_N = y$ and $(x_{j-1}, x_j) \in E(G)$ for $j = 1, \dots, N$. The graph G is called weakly connected if there exists a path in $\widetilde{G}$ between each two vertices of G.

2 Main results

Suppose that X is a uniform space endowed with a graph G and $T : X \to X$ is any arbitrary mapping. Throughout this section, the set of all fixed points of T is denoted by

$\mathrm{Fix}(T)$, and by C_T, it is meant the set of all points $x \in X$ such that $(T^m x, T^n x) \in E(\widetilde{G})$ for all integers $m, n \geqslant 0$. According to these notations, it is clear that $\mathrm{Fix}(T) \subseteq C_T$.

We first introduce the concept of Kannan G-p-contractions in uniform spaces endowed with a graph using an A-distance.

Definition 2. Let p be an A-distance on a uniform space X endowed with a graph G. We say that a mapping $T : X \to X$ is a Kannan G-p-contraction if

(K1) T preserves the egdes of G, i.e. $(x, y) \in E(G)$ implies $(Tx, Ty) \in E(G)$ for all $x, y \in X$;

(K2) there exist $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$ such that

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty)$$

for all $x, y \in X$ with $(x, y) \in E(G)$.

If $T : X \to X$ is a Kannan G-p-contraction, then we call α and β in (K2) the contractive constants of T.

Remark 2. Let p be a symmetric A-distance on a uniform space X endowed with a symmetric graph G and $T : X \to X$ be a Kannan G-p-contraction. If $x, y \in X$ are such that $(x, y) \in E(G)$, then by the symmetry of G, we have $(y, x) \in E(G)$, and so from (K2), we get

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty)$$

and

$$p(Ty, Tx) \leqslant \alpha p(y, Ty) + \beta p(x, Tx),$$

where $\alpha, \beta \geqslant 0$ are the contractive constants of T. Because p is symmetric, we have

$$p(Tx, Ty) \leqslant \frac{\alpha + \beta}{2} \bigl[p(x, Tx) + p(y, Ty)\bigr].$$

Since T preserves the edges of G, it follows that T is a Kannan G-p-contraction whose contractive constants are equal. Note that since $\alpha + \beta < 1$, it follows that the new contractive constant $(\alpha + \beta)/2$ belongs to $[0, 1/2)$.

Hence, whenever the A-distance p and the graph G are both symmetric, one can assume without loss of generality that every Kannan G-p-contraction has equal contractive constants.

We now give some examples of Kannan G-p-contractions in metric and uniform spaces endowed with a graph.

Example 1. Let p be an A-distance on an arbitrary uniform space X endowed with a graph G and x_0 be a point in X such that $p(x_0, x_0) = 0$. Since $(x_0, x_0) \in E(G)$, it follows that the constant mapping $x \overset{T}{\mapsto} x_0$ preserves the edges of G, and since $p(x_0, x_0) = 0$, it follows that T satisfies (K2) for any arbitrary $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$. Hence T is a Kannan G-p-contraction whose contractive constants are any $\alpha, \beta \geqslant 0$ satisfying $\alpha + \beta < 1$. In fact, each constant mapping on X is a Kannan G-p-contraction if and only if $p(x, x) = 0$ for all $x \in X$, i.e. p vanishes on the diagonal of X.

Example 2. Let (X, d) be a metric space and $T : X \to X$ be a mapping satisfying

$$d(Tx, Ty) \leqslant \alpha d(x, Tx) + \beta d(y, Ty) \quad (x, y \in X),$$

where $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$. If we consider the E-distance d on the uniform space $(X, \mathscr{U}_d)$, then T is a Kannan G_0-d-contraction, where G_0 is the complete graph with $V(G_0) = X$, i.e. $E(G_0) = X \times X$. Moreover, since the E-distance d and the complete graph G_0 are both symmetric, it follows by Remark 2 that one can assume without loss of generality that $\alpha = \beta$, i.e. T has equal contractive constants. Hence Kannan G-p-contractions are a generalization of Kannan contractions from metric to uniform spaces endowed with a graph. The existence and uniqueness of fixed points for Kannan contractions were investigated by Kannan in the 1960s (see [9, 10]).

Example 3. Let p be an A-distance on a partially ordered uniform space X, i.e. a uniform space X equipped with a partial order $\preccurlyeq$, and define the poset graphs G_1 and G_2 by

$$V(G_1) = X \quad \text{and} \quad E(G_1) = \bigl\{(x, y) \in X \times X\colon x \preccurlyeq y\bigr\},$$

and $G_2 = \widetilde{G}_1$. Then Kannan G_1-p-contractions are precisely the ordered Kannan p-contractions, i.e. the nondecreasing mappings $T : X \to X$ for which there exist $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$ such that

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty)$$

for all $x, y \in X$ with $x \preccurlyeq y$. Also, Kannan G_2-p-contractions are precisely the mappings $T : X \to X$ which are order-preserving (i.e. the mappings which map comparable elements of X onto comparable elements), and there exist $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$ such that

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty)$$

for all comparable elements $x, y \in X$.

Remark 3. Let X be a uniform space and $T : X \to X$ be any arbitrary mapping. If X is endowed with the complete graph G_0, then $C_T = X$. Also, if X is endowed with either G_1 or G_2 induced by a partial order $\preccurlyeq$, then the set C_T consists of all elements $x \in X$ such that $T^m x$ and $T^n x$ are comparable for all integers $m, n \geqslant 0$. In particular, whenever T is either monotone nondecreasing or monotone nonincreasing, then an $x \in X$ belongs to C_T if and only if it satisfies either $x \preccurlyeq Tx$ or $Tx \preccurlyeq x$.

The concept of Banach G-p-contractions can be defined in uniform spaces endowed with a graph using an A-distance as a counterpart of Banach G-contractions in metric spaces endowed with a graph introduced by Jachymski (see [8, Def. 2.1]). Indeed, if p is an A-distance on a uniform space X endowed with a graph G, then an edge-preserving mapping $T : X \to X$ is called a Banach G-p-contraction if there exists an $\alpha \in [0, 1)$ such that $p(Tx, Ty) \leqslant \alpha p(x, y)$ for all $x, y \in X$ with $(x, y) \in E(G)$.

In the next example, we see that the concepts of Banach and Kannan G-p-contractions are independent, i.e. there exists a Banach G-p-contraction which fails to be a Kannan G-p-contraction and vice versa.

Example 4. Let $(X, \|\cdot\|)$ be a nontrivial real normed space and uniformize X with the trivial uniformity, i.e. $\mathscr{U} = \{X \times X\}$. It is clear that the function $p : X \times X \to [0, +\infty)$ defined by

$$p(x, y) = \|x - y\|^2 \quad (x, y \in X)$$

is an A-distance on X (in fact, each $p : X \times X \to [0, +\infty)$ defines an A-distance on the trivial uniform space X). Now, pick a nonzero vector $x_0 \in X$ and consider the mapping $T : X \to X$ with $Tx = (1/2)x_0$ if $x \neq x_0$, and $Tx_0 = (1/10)x_0$. Then T is a Kannan G_0-p-contraction with the contractive constants $\alpha = 64/81$ and $\beta = 16/81$ (note that $\alpha + \beta = 80/81 < 1$). In fact, given any $x, y \in X$, we have the following three cases:

Case 1. If either $x = y = x_0$ or $x, y \neq x_0$, then $p(Tx, Ty) = 0$, and so there remains nothing to prove;

Case 2. If $x = x_0$ and $y \neq x_0$, then we have

$$\begin{aligned} p(Tx, Ty) &= \frac{4}{25}\|x_0\|^2 \leqslant \frac{16}{25}\|x_0\|^2 + \frac{16}{81}\left\|y - \frac{1}{2}x_0\right\|^2 \\ &= \frac{64}{81}p(x, Tx) + \frac{16}{81}p(y, Ty); \end{aligned}$$

Case 3. Finally, if $x \neq x_0$ and $y = x_0$, then we have

$$\begin{aligned} p(Tx, Ty) &= \frac{4}{25}\|x_0\|^2 \leqslant \frac{64}{81}\left\|x - \frac{1}{2}x_0\right\|^2 + \frac{4}{25}\|x_0\|^2 \\ &= \frac{64}{81}p(x, Tx) + \frac{16}{81}p(y, Ty). \end{aligned}$$

It is worth mentioning that since the A-distance p and the complete graph G_0 are both symmetric, it follows from Remark 2 that T is also a Kannan G_0-p-contraction with the contractive constants $\alpha = \beta = 40/81 \in [0, 1/2)$ (note that $40/81 \in [0, 1/2)$). On the other hand, T fails to be a Banach G_0-p-contraction because given any arbitrary $\alpha \in [0, 1)$, we have

$$p\left(Tx_0, T\frac{3}{5}x_0\right) = \frac{4}{25}\|x_0\|^2 > \frac{4\alpha}{25}\|x_0\|^2 = \alpha p\left(x_0, \frac{3}{5}x_0\right).$$

Conversely, it is clear that the mapping $S : X \to X$ defined by the rule $Sx = (2/3)x$ for all $x \in X$ is a Banach G_0-p-contraction with $\alpha = 2/3$. But given any arbitrary $\alpha, \beta \geqslant 0$ with $\alpha + \beta < 1$, we have

$$p(Sx, S0) = \frac{4}{9}\|x\|^2 > \frac{\alpha}{9}\|x\|^2 = \alpha p(x, Sx) + \beta p(0, S0) \quad \bigl(x \in X \setminus \{0\}\bigr),$$

i.e. S is not a Kannan G_0-p-contraction. More generally, there is no graph G such that $(x, 0) \in E(G)$ for some nonzero vector $x \in X$ and the mapping S is a Kannan G-p-contraction.

Remark 4. Note that the mapping T in Example 4 is not p-continuous on X, i.e. if $\{x_n\}$ is a sequence in X p-convergent to an $x \in X$, then we may not have $Tx_n \xrightarrow{p} Tx$. For instance, the sequence $\{(1 + 1/n)x_0\}$ is p-convergent to x_0 because

$$p(x_n, x_0) = \left\| \left(1 + \frac{1}{n}\right)x_0 - x_0 \right\|^2 = \frac{1}{n^2}\|x_0\|^2 \to 0$$

as $n \to \infty$. But $\{T(1 + 1/n)x_0\}$ fails to p-converge to $Tx_0 = (1/10)x_0$ because

$$p(Tx_n, Tx_0) = \left\| \frac{1}{10}x_0 - \frac{1}{2}x_0 \right\|^2 = \frac{4}{25}\|x_0\|^2 > 0$$

for all integers $n \geqslant 1$. Hence despite Banach G_0-p-contractions are all p-continuous on X, a Kannan G_0-p-contraction need not be a p-continuous mapping.

We are now ready to discuss the fixed points of Kannan G-p-contractions. We begin with an interesting and important property which is needed in the sequel.

Proposition 1. *Let p be an A-distance on a uniform space X endowed with a graph G and $T : X \to X$ be a Kannan G-p-contraction. Then $p(x, x) = p(y, y) = p(x, y) = 0$ for all $x, y \in \mathrm{Fix}(T)$ with $(x, y) \in E(G)$.*

Proof. Suppose that $x, y \in \mathrm{Fix}(T)$ are such that $(x, y) \in E(G)$. Note first that since $(x, x) \in E(G)$, it follows by (K2) that

$$p(x, x) = p(Tx, Tx) \leqslant \alpha p(x, Tx) + \beta p(x, Tx) = (\alpha + \beta)p(x, x),$$

where $\alpha, \beta \geqslant 0$ are the contractive constants of T. Since $\alpha + \beta < 1$, it follows that $p(x, x) = 0$. Similarly, one can show that $p(y, y) = 0$.

Now, using (K2) once more and the observation above, we have

$$p(x, y) = p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty) = 0.$$

Hence $p(x, y) = 0$. □

Remark 5. According to Proposition 1, if X is a uniform space endowed with a graph G, p is an A-distance on X, and there exists a Kannan G-p-contraction $T : X \to X$ such that $x \in \mathrm{Fix}(T)$, then $p(x, x) = 0$. Moreover, if X is separated, then Proposition 1 ensures that there is not Kannan G-p-contraction with two distinct fixed points $x, y \in X$ such that $(x, y) \in E(G)$. Roughly speaking, no Kannan G-p-contraction can keep the vertices of any edge of G fixed. In particular, in partially ordered separated uniform spaces equipped with an A-distance p, there is neither an ordered Kannan p-contraction nor a Kannan G_2-p-contraction having two distinct comparable fixed points.

To prove the existence of a fixed point for a Kannan G-p-contraction, the following lemma is used.

Lemma 2. *Let p be an A-distance on a uniform space X endowed with a graph G and $T : X \to X$ be a Kannan G-p-contraction. Then $p(T^n x, T^{n+1} x) \to 0$ as $n \to \infty$ for all $x \in X$ such that $(x, Tx) \in E(G)$.*

Proof. Suppose that $x \in X$ is given such that $(x, Tx) \in E(G)$. Since T preserves the edges of G, it follows by induction that $(T^n x, T^{n+1} x) \in E(G)$ for all integers $n \geqslant 0$, and so by (K2), we have

$$p\big(T^n x, T^{n+1} x\big) \leqslant \alpha p\big(T^{n-1} x, T^n x\big) + \beta p\big(T^n x, T^{n+1} x\big)$$

for all integers $n \geqslant 1$, where $\alpha, \beta \geqslant 0$ are the contractive constants of T. Because $\alpha + \beta < 1$, we have $\alpha/(1-\beta) < 1$, and hence by induction, we get

$$\begin{aligned} p\big(T^n x, T^{n+1} x\big) &\leqslant \frac{\alpha}{1-\beta} p\big(T^{n-1} x, T^n x\big) \leqslant \cdots \\ &\leqslant \left(\frac{\alpha}{1-\beta}\right)^n p(x, Tx) \to 0 \end{aligned}$$

as $n \to \infty$. □

Definition 3. Let p be an A-distance on a uniform space X endowed with a graph G and $T : X \to X$ be a mapping. We say that:

(i) T is orbitally p-continuous on X if for all $x, y \in X$ and all sequences $\{a_n\}$ of positive integers, $T^{a_n} x \xrightarrow{p} y$ as $n \to \infty$ implies $T(T^{a_n} x) \xrightarrow{p} Ty$ as $n \to \infty$.

(ii) T is orbitally p-G-continuous on X if for all $x, y \in X$ and all sequences $\{a_n\}$ of positive integers satisfying $(T^{a_n} x, T^{a_{n+1}} x) \in E(G)$ for $n = 1, 2, \dots$, $T^{a_n} x \xrightarrow{p} y$ as $n \to \infty$ implies $T(T^{a_n} x) \xrightarrow{p} Ty$ as $n \to \infty$.

(iii) T is a p-Picard operator if T has a unique fixed point $x^* \in X$ and $T^n x \xrightarrow{p} x^*$ for all $x \in X$.

(iv) T is a weakly p-Picard operator if $\{T^n x\}$ is p-convergent to a fixed point of T for all $x \in X$.

It is clear that each p-Picard operator is weakly p-Picard but the converse is true if and only if the fixed point is unique.

Our first main theorem guarantees the existence of a fixed point for a Kannan $\widetilde{G}$-p-contraction T defined on a p-complete separated uniform space X endowed with a graph G and equipped with an A- or an E-distance p whenever T is orbitally p-$\widetilde{G}$-continuous on X or the triple (X, p, G) has a suitable property.

Theorem 1. *Let p be an A-distance on a separated uniform space X endowed with a graph G such that X is p-complete and $T : X \to X$ be a Kannan $\widetilde{G}$-p-contraction. Then $T|_{C_T}$ is a weakly p-Picard operator if one of the following assertions holds*:

(i) *T is orbitally p-$\widetilde{G}$-continuous on X*;

(ii) *p is a symmetric E-distance on X and the triple (X, p, G) satisfies the following property*:

($\star$) *If $\{x_n\}$ is a sequence in X, p-convergent to an $x \in X$ such that $(x_n, x_{n+1}) \in E(\widetilde{G})$ for all integers $n \geqslant 1$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(\widetilde{G})$ for all integers $k \geqslant 1$.*

In particular, whenever (i) *or* (ii) *holds, then* $\mathrm{Fix}(T) \neq \emptyset$ *if and only if* $C_T \neq \emptyset$.

Proof. If $C_T = \emptyset$, then there remains nothing to prove. Otherwise, note first that since T preserves the edges of $\widetilde{G}$, it follows immediately that C_T is T-invariant, i.e. T maps C_T into itself.

Now, suppose that $x \in C_T$ is given. We are first going to show that the sequence $\{T^n x\}$ is p-Cauchy. To this end, note that by (K2), we have

$$p\big(T^m x, T^n x\big) \leqslant \alpha p\big(T^{m-1}x, T^m x\big) + \beta p\big(T^{n-1}x, T^n x\big)$$

for all integers $m, n \geqslant 1$, where $\alpha, \beta \geqslant 0$ are the contractive constants of T. Since $(x, Tx) \in E(\widetilde{G})$ and T is a Kannan $\widetilde{G}$-p- contraction, it follows immediately by Lemma 2 that $p(T^{m-1}x, T^m x) \to 0$ as $m \to \infty$ and $p(T^{n-1}x, T^n x) \to 0$ as $n \to \infty$. Hence $p(T^m x, T^n x) \to \infty$ as $m, n \to \infty$, i.e. $\{T^n x\}$ is p-Cauchy. Since X is p-complete, there exists an $x^* \in X$ (depends on x) such that $T^n x \xrightarrow{p} x^*$ as $n \to \infty$.

We shall show that x^* is a fixed point for T. So, suppose first that T is orbitally p-$\widetilde{G}$-continuous on X. Because $x \in C_T$, we have $(T^n x, T^{n+1}x) \in E(\widetilde{G})$ for all integers $n \geqslant 0$, and so $T^{n+1}x \xrightarrow{p} Tx^*$ as $n \to \infty$. Therefore, because X is separated, Lemma 1 implies that $Tx^* = x^*$.

On the other hand, if p is a symmetric E-distance on X and the triple (X, p, G) satisfies Property ($\star$), then there exists a strictly increasing sequence $\{n_k\}$ of positive integers such that $(T^{n_k}x, x^*) \in E(\widetilde{G})$ for all integers $k \geqslant 1$. Therefore, from (K2), we get

$$\begin{aligned}
p\big(T^{n_k+1}x, Tx^*\big) &\leqslant \alpha p\big(T^{n_k}x, T^{n_k+1}x\big) + \beta p(x^*, Tx^*)\\
&\leqslant \alpha p\big(T^{n_k}x, T^{n_k+1}x\big) + \beta\big(p\big(x^*, T^{n_k+1}x\big) + p\big(T^{n_k+1}x, Tx^*\big)\big)\\
&= \alpha p\big(T^{n_k}x, T^{n_k+1}x\big) + \beta\big(p\big(T^{n_k+1}x, x^*\big) + p\big(T^{n_k+1}x, Tx^*\big)\big)
\end{aligned}$$

for all integers $k \geqslant 1$. Hence

$$p\big(T^{n_k+1}x, Tx^*\big) \leqslant \frac{\alpha}{1-\beta} p\big(T^{n_k}x, T^{n_k+1}x\big) + \frac{\beta}{1-\beta} p\big(T^{n_k+1}x, x^*\big) \quad (k \geqslant 1).$$

Because $(x, Tx) \in E(\widetilde{G})$, letting $k \to \infty$, we find from Lemma 2 that $p(T^{n_k}x, T^{n_k+1}x) \to 0$. So $T^{n_k+1}x \xrightarrow{p} Tx^*$. Since X is separated, Lemma 1 implies again that $Tx^* = x^*$.

Finally, it is clear that $x^* \in \mathrm{Fix}(T) \subseteq C_T$, and consequently, $T|_{C_T}$ is a weakly p-Picard operator. $\square$

Setting $G = G_0$ in Theorem 1, we obtain the following generalization of Kannan's fixed point theorem [9] from metric uniform spaces equipped with an A- or an E-distance.

Corollary 1. *Let p be an A-distance on a separated uniform space X such that X is p-complete and a mapping $T : X \to X$ satisfies*

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty) \tag{2}$$

for all $x, y \in X$, where $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$. Then T is a p-Picard operator if either T is orbitally p-continuous (in particular, p-continuous) or p is a symmetric E-distance on X.

Proof. The set C_T is nonempty because $C_T = X$. Therefore, by Theorem 1, the mapping $T = T|_{C_T}$ is a weakly p-Picard operator. In particular, T has a fixed point in X. To show that T is a p-Picard operator, we prove that the fixed point of T is unique. To this end, suppose that $x^*, y^* \in X$ are two fixed points for T. Then by (2) and Proposition 1, we have $p(x^*, x^*) = p(x^*, y^*) = 0$. Since X is separated, it follows by Lemma 1 that $x^* = y^*$. $\square$

Since $\widetilde{G}_1 = \widetilde{G}_2 = G_2$, setting $G = G_1$ or $G = G_2$ in Theorem 1, we obtain the ordered version of Kannan's fixed point theorem in partially ordered uniform spaces equipped with an A- or an E-distance as follows:

Corollary 2. *Let p be an A-distance on a partially ordered separated uniform space X such that X is p-complete and a mapping $T : X \to X$ satisfies*

$$p(Tx, Ty) \leqslant \alpha p(x, Tx) + \beta p(y, Ty)$$

for all comparable elements $x, y \in X$, where $\alpha, \beta \geqslant 0$ and $\alpha + \beta < 1$. Then the restriction of T to the set of all elements $x \in X$ such that $T^m x$ and $T^n x$ are comparable for all integers $m, n \geqslant 0$ is a weakly p-Picard operator if one of the following assertions holds:

(i) *T is orbitally p-G_2-continuous on X*;

(ii) *p is a symmetric E-distance on X and the partially ordered uniform space X satisfies the following property*:

If $\{x_n\}$ is a sequence in X with successive comparable terms, p-convergent to an $x \in X$, then there exists a subsequence of $\{x_n\}$ whose terms are comparable to x.

In particular, whenever (i) *or* (ii) *holds, then* $\mathrm{Fix}(T) \neq \emptyset$ *if and only if there exists an $x \in X$ such that $T^m x$ and $T^n x$ are comparable for all integers $m, n \geqslant 0$.*

Now, we give two sufficient conditions for the uniqueness of the fixed point for a Kannan $\widetilde{G}$-p-contraction.

Theorem 2. *Let p be an A-distance on a separated uniform space X endowed with a graph G and $T : X \to X$ be a Kannan $\widetilde{G}$-p-contraction. If either for all $x, y \in X$ there exists a $z \in X$ such that $(z, Tz), (z, x), (z, y) \in E(\widetilde{G})$, or the subgraph of G with the vertices* $\mathrm{Fix}(T)$ *is weakly connected, then T has at most one fixed point in X.*

Proof. Suppose that $x^*, y^* \in X$ are two fixed points for T. If there exists a $z \in X$ such that $(z, Tz), (z, x^*), (z, y^*) \in E(\widetilde{G})$, then by (K2), Proposition 1 and Lemma 2, we have

$$\begin{aligned} p\bigl(T^n z, x^*\bigr) &= p\bigl(T^n z, T^n x^*\bigr) \leqslant \alpha p\bigl(T^{n-1} z, T^n z\bigr) + \beta p\bigl(T^{n-1} x^*, T^n x^*\bigr) \\ &= \alpha p\bigl(T^{n-1} z, T^n z\bigr) \to 0 \end{aligned}$$

as $n \to \infty$, where $\alpha, \beta \geqslant 0$ are the contractive constants of T. Therefore, $T^n z \xrightarrow{p} x^*$. Similarly, one can show that $T^n z \xrightarrow{p} y^*$, and because X is separated, Lemma 1 ensures that $x^* = y^*$.

On the other hand, if the subgraph of G with the vertices $\mathrm{Fix}(T)$ is weakly connected, then there exists a path $(x_j)_{j=0}^N$ in $\widetilde{G}$ from x^* to y^* such that $x_1, \ldots, x_{N-1} \in \mathrm{Fix}(T)$, i.e. $x_0 = x^*$, $x_N = y^*$ and $(x_{j-1}, x_j) \in E(\widetilde{G})$ for $j = 1, \ldots, N$. Since $E(\widetilde{G})$ contains all loops, one may assume without loss of generality that the length of $(x_j)_{j=0}^N$, i.e. the integer N is even. Moreover, by Proposition 1, we have

$$p(x_{j-1}, x_j) = p(x_j, x_{j-1}) = 0 \quad \text{for } j = 1, \ldots, N.$$

Because N is even, using Lemma 1 finitely many times, we get $x^* = x_0 = x_2 = \cdots = x_N = y^*$. Consequently, T has at most one fixed point in X. $\square$

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