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Positive solutions for a class of fractional boundary value problems*

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Abstract. In this work, by virtue of the Krasnoselskii–Zabreiko fixed point theorem, we investigate the existence of positive solutions for a class of fractional boundary value problems under some appropriate conditions concerning the first eigenvalue of the relevant linear operator. Moreover, we utilize the method of lower and upper solutions to discuss the unique positive solution when the nonlinear term grows sublinearly.

Keywords: fractional boundary value problem, Krasnoselskii–Zabreiko fixed point theorem, positive solution, uniqueness.

1 Introduction

In this paper we consider the existence of positive solutions for the boundary value problem of fractional order involving Riemann–Liouville's derivative

$$\begin{aligned} &D_{0+}^{\alpha} D_{0+}^{\alpha} u = f\bigl(t, u, u', -D_{0+}^{\alpha} u\bigr), \quad t \in [0,1], \\ &u(0) = u'(0) = u'(1) = D_{0+}^{\alpha} u(0) = D_{0+}^{\alpha+1} u(0) = D_{0+}^{\alpha+1} u(1) = 0, \end{aligned} \tag{1}$$

where $\alpha \in (2,3]$ is a real number, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α and $f \in C([0,1] \times \mathbb{R}_+^3, \mathbb{R}_+)$ ($\mathbb{R}_+ := [0,+\infty)$).

Recently, the fractional differential calculus and fractional differential equation have drawn more and more attention due to the applications of such constructions in various

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sciences such as physics, mechanics, chemistry, engineering, etc. Many books on fractional calculus, fractional differential equations have appeared, for instance, see [7,10,11]. This may explain the reason that the last two decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published. We refer the interested reader to [1,2,4,5,6,12,13,14,15] and the references therein.

In [4,6], by using the fixed point index theory and Krein–Rutman theorem, Jiang et al. obtained the existence of positive solutions for the multi-point boundary value problems of fractional differential equations

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1,\ 1 < \alpha \leqslant 2, \\ &u(0) = 0, D_{0+}^{\beta}u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^{\beta} u(\xi_i), \end{aligned} \tag{2}$$

and

$$\begin{aligned} &D^{\alpha}u - Mu = \lambda f\big(t, u(t)\big), \quad t \in [0,1],\ 0 < \alpha < 1, \\ &u(0) = \sum_{i=1}^{n} \beta_i u(\xi_i). \end{aligned} \tag{3}$$

In this paper, we first construct an integral operator for the corresponding linear boundary value problem and find out its first eigenvalue and eigenfunction. Then we establish a special cone associated with the Green's function of (1). Finally, by employing the Krasnoselskii–Zabreiko fixed point theorem, combined with a priori estimates of positive solutions, we obtain the existence of positive solutions for (1). Note that our nonlinear term f involves the fractional derivatives of the dependent variable–this is seldom studied in the literature and most research articles on boundary value problems consider nonlinear terms that involve the unknown function u only, for example, [1,2,4,5,6,12,13,15]. Moreover, we adopt the method of lower and upper solutions to discuss the uniqueness of positive solutions for (1), and prove that the unique positive solution can be uniformly approximated by an iterative sequence beginning with any function which is continuous, nonnegative and not identically vanishing on $[0,1]$ This, together with the fact that our nonlinearity may be of distinct growth, means that our methodology and main results here are entirely different from those in the above papers.

2 Preliminaries

For convenience, we give some background materials from fractional calculus theory to facilitate analysis of problem (1). These materials can be found in the recent books, see [7,10,11].

Definition 1. (See [7,10], [11, pp. 36–37].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_0^t (t-s)^{n-\alpha-1} f(s)\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2. (See [11, Def. 2.1].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to (-\infty, +\infty)$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann–Liouville derivative, we can obtain the following statement.

Lemma 1. (*See* [1].) *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation $D_{0+}^{\alpha} u(t) = 0$ has a unique solution*

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R},\ i = 1, 2, \dots, N,$$

where N is the smallest integer greater than or equal to α.

Lemma 2. (*See* [1].) *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R},\ i = 1, 2, \dots, N,$$

where N is the smallest integer greater than or equal to α.

In what follows, we shall discuss some properties of the Green's function for fractional boundary value problem (1). Let

$$G_1(t,s) := \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{4}$$

Then we can easily obtain that

$$\begin{aligned} G_2(t,s) &:= \frac{\partial}{\partial t} G_1(t,s) \\ &= \frac{\alpha-1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-2}(1-s)^{\alpha-2} - (t-s)^{\alpha-2}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-2}(1-s)^{\alpha-2}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \end{aligned} \tag{5}$$

Lemma 3. (*See* [2, *Lemma* 2.7].) *Let f be as in* (1) *and $-D_{0+}^{\alpha} u := v$. Then* (1) *is equivalent to*

$$v(t) = \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau) v(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau) v(\tau)\,\mathrm{d}\tau, v(s)\right) \mathrm{d}s. \tag{6}$$

Lemma 4. (*See* [2, *Lemma* 2.8] *and* [5, *Thms.* 1.1, 1.2].) *The functions* $G_i(t,s) \in C([0,1]\times[0,1],\mathbb{R}_+)$ $(i=1,2)$*, moreover, the following two inequalities hold*:

$$\begin{aligned} &t^{\alpha-1}s(1-s)^{\alpha-2} \\ &\quad \leqslant \Gamma(\alpha)G_1(t,s) \leqslant s(1-s)^{\alpha-2} \quad \forall t,s\in[0,1]. \end{aligned} \tag{7}$$

$$\begin{aligned} &(\alpha-1)(\alpha-2)t^{\alpha-2}(1-t)s(1-s)^{\alpha-2} \\ &\quad \leqslant \Gamma(\alpha)G_2(t,s) \leqslant (\alpha-1)t^{\alpha-3}s(1-s)^{\alpha-2} \quad \forall t,s\in[0,1]. \end{aligned} \tag{8}$$

In what follows, we shall define two extra functions by G_1, G_2. Let

$$\begin{aligned} G_3(t,s) &:= \int_0^1 G_1(t,\tau)G_1(\tau,s)\,\mathrm{d}\tau \quad \forall t,s\in[0,1], \\ G_4(t,s) &:= \int_0^1 G_1(t,\tau)G_2(\tau,s)\,\mathrm{d}\tau \quad \forall t,s\in[0,1]. \end{aligned} \tag{9}$$

Then $G_i(t,s) \in C([0,1]\times[0,1],\mathbb{R}_+)$ $(i=3,4)$. Moreover, by Lemma 4, we easily have

$$\begin{aligned} &\frac{\alpha}{(\alpha-1)\Gamma(2\alpha)}t^{\alpha-1}s(1-s)^{\alpha-2} \\ &\quad = \int_0^1 \frac{t^{\alpha-1}\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)}\cdot\frac{\tau^{\alpha-1}s(1-s)^{\alpha-2}}{\Gamma(\alpha)}\,\mathrm{d}\tau \leqslant G_3(t,s) \\ &\quad \leqslant \int_0^1 \frac{s(1-s)^{\alpha-2}\tau(1-\tau)^{\alpha-2}}{\Gamma^2(\alpha)}\,\mathrm{d}\tau = \frac{s(1-s)^{\alpha-2}}{\alpha(\alpha-1)\Gamma^2(\alpha)}. \end{aligned} \tag{10}$$

Similarly,

$$\begin{aligned} &\frac{(\alpha-1)(\alpha-2)}{\Gamma(2\alpha)}t^{\alpha-1}s(1-s)^{\alpha-2} \\ &\quad = \int_0^1 \frac{t^{\alpha-1}\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)}\cdot\frac{(\alpha-1)(\alpha-2)\tau^{\alpha-2}(1-\tau)s(1-s)^{\alpha-2}}{\Gamma(\alpha)}\,\mathrm{d}\tau \\ &\quad \leqslant G_4(t,s) \leqslant \int_0^1 \frac{(\alpha-1)\tau^{\alpha-3}s(1-s)^{\alpha-2}\tau(1-\tau)^{\alpha-2}}{\Gamma^2(\alpha)}\,\mathrm{d}\tau \\ &\quad = \frac{s(1-s)^{\alpha-2}}{(\alpha-1)\Gamma(2\alpha-2)}. \end{aligned} \tag{11}$$

Let

$$E := C[0,1], \qquad \|v\| := \max_{t\in[0,1]}\bigl|v(t)\bigr|, \qquad P := \bigl\{v\in E\colon\ v(t)\geqslant 0\ \forall t\in[0,1]\bigr\}.$$

Then $(E, \|\cdot\|)$ becomes a real Banach space and P is a cone on E. Define $B_\rho := \{v \in E: \|v\| < \rho\}$ for $\rho > 0$ in the sequel.

Let

$$(Av)(t) := \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)v(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)v(\tau)\,\mathrm{d}\tau, v(s)\right) \mathrm{d}s \quad (12)$$

for all $v \in E$. The continuity of G_1, G_2 and f implies that $A : E \to E$ is a completely continuous nonlinear operator. As mentioned in Lemma 3, $-D_{0+}^{\alpha} u = v$, together with the boundary conditions $u(0) = u'(0) = u'(1) = 0$, we have

$$u(t) = \int_0^1 G_1(t,s)v(s)\,\mathrm{d}s, \quad (13)$$

where G_1 is determined by (4). Therefore, we find the existence of solutions of (1) is equivalent to that of fixed points of A.

For $a, b, c \geqslant 0$ with $a^2 + b^2 + c^2 \neq 0$, let

$$G_{a,b,c}(t,s) := aG_3(t,s) + bG_4(t,s) + cG_1(t,s) \quad \forall t, s \in [0,1],$$

and define a linear operator $L_{a,b,c}$ as follows:

$$(L_{a,b,c}v)(t) = \int_0^1 G_{a,b,c}(t,s)v(s)\,\mathrm{d}s \quad \forall v \in E. \quad (14)$$

Obviously, $L_{a,b,c}$ is positive, i.e., $L_{a,b,c}(P) \subset P$. The continuity of G_1, G_3, G_4 implies that $L_{a,b,c}$ is a completely continuous operator. From now on, we utilize $r(L_{a,b,c})$ to denote the spectral radius of $L_{a,b,c}$. Furthermore, Gelfand's theorem enables us to obtain the following result.

Lemma 5. *Let*

$$\xi_{a,b,c} := \frac{a\alpha}{(\alpha-1)\Gamma(2\alpha)} + \frac{b(\alpha-1)(\alpha-2)}{\Gamma(2\alpha)} + \frac{c}{\Gamma(\alpha)},$$
$$\eta_{a,b,c} := \frac{a}{\alpha(\alpha-1)\Gamma^2(\alpha)} + \frac{b}{(\alpha-1)\Gamma(2\alpha-2)} + \frac{c}{\Gamma(\alpha)}.$$

Then

$$\frac{\xi_{a,b,c}\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)} \leqslant r(L_{a,b,c}) \leqslant \frac{\eta_{a,b,c}}{\alpha(\alpha-1)}.$$

Proof. By (7), (10), and (11), we obtain

$$\|L_{a,b,c}\| = \max_{t\in[0,1]} \int_0^1 G_{a,b,c}(t,s)\,\mathrm{d}s \leqslant \eta_{a,b,c} \int_0^1 s(1-s)^{\alpha-2}\,\mathrm{d}s = \frac{\eta_{a,b,c}}{\alpha(\alpha-1)}.$$

Similarly, we find, for all $n \in \mathbb{N}_+$,

$$\begin{aligned}\left\|L_{a,b,c}^n\right\| &= \max_{t\in[0,1]} \underbrace{\int_0^1 \cdots \int_0^1}_{n} G_{a,b,c}(t,s_{n-2})\cdots G_{a,b,c}(s_2,s_1)G_{a,b,c}(s_1,s)\\ &\qquad \times G_{a,b,c}(s,\tau)\,\mathrm{d}s_{n-2}\cdots\,\mathrm{d}s_1\,\mathrm{d}s\,\mathrm{d}\tau\\ &\leqslant \left[\frac{\eta_{a,b,c}}{\alpha(\alpha-1)}\right]^n.\end{aligned}$$

Gelfand's theorem implies that

$$r(L_{a,b,c}) = \lim_{n\to\infty}\sqrt[n]{\left\|L_{a,b,c}^n\right\|} \leqslant \frac{\eta_{a,b,c}}{\alpha(\alpha-1)}.$$

On the other hand,

$$\begin{aligned}\|L_{a,b,c}\| &= \max_{t\in[0,1]}\int_0^1 G_{a,b,c}(t,s)\,\mathrm{d}s \geqslant \max_{t\in[0,1]}\int_0^1 \xi_{a,b,c}t^{\alpha-1}s(1-s)^{\alpha-2}\,\mathrm{d}s\\ &= \frac{\xi_{a,b,c}}{\alpha(\alpha-1)}.\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}\left\|L_{a,b,c}^2\right\| &= \max_{t\in[0,1]}\int_0^1\int_0^1 G_{a,b,c}(t,s)G_{a,b,c}(s,\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\ &\geqslant \max_{t\in[0,1]}\int_0^1\int_0^1 \xi_{a,b,c}^2 t^{\alpha-1}s(1-s)^{\alpha-2}s^{\alpha-1}\tau(1-\tau)^{\alpha-2}\,\mathrm{d}\tau\,\mathrm{d}s\\ &= \xi_{a,b,c}^2\int_0^1 s^\alpha(1-s)^{\alpha-2}\,\mathrm{d}s\int_0^1\tau(1-\tau)^{\alpha-2}\,\mathrm{d}\tau\end{aligned}$$

and

$$\left\|L_{a,b,c}^3\right\| \geqslant \xi_{a,b,c}^3\left(\int_0^1 s^\alpha(1-s)^{\alpha-2}\,\mathrm{d}s\right)^2\int_0^1\tau(1-\tau)^{\alpha-2}\,\mathrm{d}\tau.$$

Therefore, for all $n \in \mathbb{N}_+$,

$$\left\|L_{a,b,c}^n\right\| \geqslant \xi_{a,b,c}^n\left(\int_0^1 s^\alpha(1-s)^{\alpha-2}\,\mathrm{d}s\right)^{n-1}\int_0^1\tau(1-\tau)^{\alpha-2}\,\mathrm{d}\tau.$$

By Gelfand's theorem, we see

$$\begin{aligned} r(L_{a,b,c}) &= \lim_{n\to\infty} \sqrt[n]{\|L_{a,b,c}^n\|} \geqslant \xi_{a,b,c} \int_0^1 s^\alpha (1-s)^{\alpha-2}\,\mathrm{d}s \\ &= \frac{\xi_{a,b,c}\Gamma(\alpha+1)\Gamma(\alpha-1)}{\Gamma(2\alpha)}. \end{aligned}$$

This completes the proof. □

By Lemma 5, we see $r(L_{a,b,c}) > 0$, and thus the Krein–Rutman theorem [9] asserts that there are $\varphi_{a,b,c} \in P \setminus \{0\}$ and $\psi_{a,b,c} \in P \setminus \{0\}$ such that

$$\begin{aligned} &\int_0^1 G_{a,b,c}(t,s)\varphi_{a,b,c}(s)\,\mathrm{d}s = r(L_{a,b,c})\varphi_{a,b,c}(t), \\ &\int_0^1 G_{a,b,c}(t,s)\psi_{a,b,c}(t)\,\mathrm{d}t = r(L_{a,b,c})\psi_{a,b,c}(s). \end{aligned} \tag{15}$$

Note that we can normalize $\psi_{a,b,c}$ such that

$$\int_0^1 \psi_{a,b,c}(t)\,\mathrm{d}t = 1. \tag{16}$$

Let $\omega_{a,b,c} = \xi_{a,b,c}\eta_{a,b,c}^{-1}\int_0^1 t^{\alpha-1}\psi_{a,b,c}(t)\,\mathrm{d}t$ and define

$$P_0 := \left\{ v \in P\colon \int_0^1 v(t)\psi_{a,b,c}(t)\,\mathrm{d}t \geqslant \omega_{a,b,c}\|v\| \right\}.$$

Clearly, P_0 is also a cone of E.

Lemma 6. $L_{a,b,c}(P) \subset P_0$.

Proof. We easily have the following inequality:

$$G_{a,b,c}(t,s) \geqslant \xi_{a,b,c}\eta_{a,b,c}^{-1} t^{\alpha-1} G_{a,b,c}(\tau,s) \quad \forall t,s,\tau \in [0,1].$$

For $v(t) \geqslant 0$, $t \in [0,1]$, we have

$$\int_0^1 (L_{a,b,c}v)(t)\psi_{a,b,c}(t)\,\mathrm{d}t = \int_0^1 \int_0^1 G_{a,b,c}(t,s)v(s)\psi_{a,b,c}(t)\,\mathrm{d}s\,\mathrm{d}t$$

$$\geqslant \int_0^1 \int_0^1 \xi_{a,b,c}\eta_{a,b,c}^{-1} t^{\alpha-1} G_{a,b,c}(\tau,s)v(s)\psi_{a,b,c}(t)\,\mathrm{d}s\,\mathrm{d}t$$

$$= \xi_{a,b,c}\eta_{a,b,c}^{-1} \int_0^1 t^{\alpha-1}\psi_{a,b,c}(t)\,\mathrm{d}t \int_0^1 G_{a,b,c}(\tau,s)v(s)\,\mathrm{d}s \quad \forall \tau \in [0,1].$$

Consequently, we see

$$\int_0^1 (L_{a,b,c}v)(t)\psi_{a,b,c}(t)\,\mathrm{d}t \geqslant \omega_{a,b,c}\|L_{a,b,c}v\|.$$

This completes the proof. □

Lemma 7. (*See* [8].) *Let E be a real Banach space and W a cone of E. Suppose that $A : (\overline{B}_R \setminus B_r) \cap W \to W$ is a completely continuous operator with $0 < r < R$. If either*

(i) *$Au \nleqslant u$ for each $\partial B_r \cap W$ and $Au \ngeqslant u$ for each $\partial B_R \cap W$ or*
(ii) *$Au \ngeqslant u$ for each $\partial B_r \cap W$ and $Au \nleqslant u$ for each $\partial B_R \cap W$,*

then A has at least one fixed point on $(\overline{B}_R \setminus B_r) \cap W$.

Lemma 8. (*See* [3].) *Let E be a partial order Banach space, and $x_0, y_0 \in E$ with $x_0 \leqslant y_0$, $D = [x_0, y_0]$. Suppose that $A : D \to E$ satisfies the following conditions*:

(i) *A is an increasing operator*;
(ii) *$x_0 \leqslant Ax_0$, $y_0 \geqslant Ay_0$, i.e., x_0 and y_0 is a subsolution and a supersolution of A*;
(iii) *A is a completely continuous operator.*

Then A has the smallest fixed point x^* and the largest fixed point y^* in $[x_0, y_0]$, respectively. Moreover, $x^* = \lim_{n\to\infty} A^n x_0$ and $y^* = \lim_{n\to\infty} A^n y_0$.

3 Main results

We first offer twelve fixed numbers $\alpha_i, \beta_i, \gamma_i \geqslant 0$ which are not all zero and let $r^{-1}(L_{\alpha_i,\beta_i,\gamma_i}) = \lambda_{\alpha_i,\beta_i,\gamma_i}$ for $i = 1, 2, 3, 4$. Now, we list our assumptions on f:

(H1) $f \in C([0,1] \times \mathbb{R}_+^3, \mathbb{R}_+)$; (H1)′ $f \in C([0,1] \times \mathbb{R}_+^3, (0,+\infty))$.

(H2)
$$\liminf_{\alpha_1 x_1+\beta_1 x_2+\gamma_1 x_3 \to +\infty} \frac{f(t,x_1,x_2,x_3)}{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3} > \lambda_{\alpha_1,\beta_1,\gamma_1} \tag{17}$$

uniformly for $t \in [0,1]$.

(H3)
$$\limsup_{\alpha_2 x_1+\beta_2 x_2+\gamma_2 x_3\to 0^+} \frac{f(t,x_1,x_2,x_3)}{\alpha_2 x_1+\beta_2 x_2+\gamma_2 x_3} < \lambda_{\alpha_2,\beta_2,\gamma_2} \tag{18}$$
uniformly for $t \in [0,1]$.

(H4)
$$\liminf_{\alpha_3 x_1+\beta_3 x_2+\gamma_3 x_3\to 0^+} \frac{f(t,x_1,x_2,x_3)}{\alpha_3 x_1+\beta_3 x_2+\gamma_3 x_3} > \lambda_{\alpha_3,\beta_3,\gamma_3} \tag{19}$$
uniformly for $t \in [0,1]$.

(H5)
$$\limsup_{\alpha_4 x_1+\beta_4 x_2+\gamma_4 x_3\to +\infty} \frac{f(t,x_1,x_2,x_3)}{\alpha_4 x_1+\beta_4 x_2+\gamma_4 x_3} < \lambda_{\alpha_4,\beta_4,\gamma_4} \tag{20}$$
uniformly for $t \in [0,1]$.

(H6) There exists a positive constant $\mu < 1$ such that
$$\kappa^\mu f(t,x_1,x_2,x_3) \leqslant f(t,\kappa x_1,\kappa x_2,\kappa x_3) \quad \forall \kappa \in (0,1).$$

(H7) $f(t,x_1,x_2,x_3)$ is increasing in x_1,x_2,x_3, that is, the inequality
$$f(t,x_1,x_2,x_3) \leqslant f(t,x_1',x_2',x_3')$$
holds for $x_1 \leqslant x_1', x_2 \leqslant x_2', x_3 \leqslant x_3'$.

3.1 Existence of positive solutions

Theorem 1. *Assume that* (H1)–(H3) *hold. Then* (1) *has at least one positive solution.*

Proof. (H2) implies that there are $\varepsilon > 0$ and $c_1 > 0$ such that
$$f(t,x_1,x_2,x_3) \geqslant (\lambda_{\alpha_1,\beta_1,\gamma_1}+\varepsilon)(\alpha_1 x_1+\beta_1 x_2+\gamma_1 x_3)-c_1 \quad \forall x_i \in \mathbb{R}_+,\ t\in[0,1]. \tag{21}$$
Let $\mathscr{M}_1 := \{v \in P\colon\ v \geqslant Av\}$. We claim that $\mathscr{M}_1$ is bounded in P. Indeed, if $v \in \mathscr{M}_1$, by (12) and (21), we can obtain
$$\begin{aligned}
v(t) &\geqslant \int_0^1 G_1(t,s) f\left(s,\ \int_0^1 G_1(s,\tau)v(\tau)\,\mathrm{d}\tau,\ \int_0^1 G_2(s,\tau)v(\tau)\,\mathrm{d}\tau,\ v(s)\right)\mathrm{d}s \\
&\geqslant (\lambda_{\alpha_1,\beta_1,\gamma_1}+\varepsilon)\left[\int_0^1 \alpha_1 G_3(t,\tau)v(\tau)\,\mathrm{d}\tau + \int_0^1 \beta_1 G_4(t,\tau)v(\tau)\,\mathrm{d}\tau \right.\\
&\quad \left. + \int_0^1 \gamma_1 G_1(t,s)v(s)\,\mathrm{d}s\right] - \frac{c_1}{\alpha(\alpha-1)\Gamma(\alpha)} \\
&= (\lambda_{\alpha_1,\beta_1,\gamma_1}+\varepsilon)\int_0^1 G_{\alpha_1,\beta_1,\gamma_1}(t,s)v(s)\,\mathrm{d}s - \frac{c_1}{\alpha(\alpha-1)\Gamma(\alpha)}.
\end{aligned} \tag{22}$$

Multiply (22) by $\psi_{\alpha_1,\beta_1,\gamma_1}(t)$ on both sides and integrate over $[0,1]$ and use (15), (16) to obtain

$$\int_0^1 v(t)\psi_{\alpha_1,\beta_1,\gamma_1}(t)\,\mathrm{d}t \geqslant \frac{\lambda_{\alpha_1,\beta_1,\gamma_1}+\varepsilon}{\lambda_{\alpha_1,\beta_1,\gamma_1}} \int_0^1 v(t)\psi_{\alpha_1,\beta_1,\gamma_1}(t)\,\mathrm{d}t - \frac{c_1}{\alpha(\alpha-1)\Gamma(\alpha)}. \tag{23}$$

Therefore, we have

$$\int_0^1 v(t)\psi_{\alpha_1,\beta_1,\gamma_1}(t)\,\mathrm{d}t \leqslant \frac{\varepsilon^{-1}\lambda_{\alpha_1,\beta_1,\gamma_1}c_1}{\alpha(\alpha-1)\Gamma(\alpha)}. \tag{24}$$

Consequently, Lemma 6 implies that

$$\omega_{\alpha_1,\beta_1,\gamma_1}\|v\| \leqslant \frac{\varepsilon^{-1}\lambda_{\alpha_1,\beta_1,\gamma_1}c_1}{\alpha(\alpha-1)\Gamma(\alpha)}, \tag{25}$$

and hence,

$$\|v\| \leqslant \frac{\varepsilon^{-1}\omega^{-1}_{\alpha_1,\beta_1,\gamma_1}\lambda_{\alpha_1,\beta_1,\gamma_1}c_1}{\alpha(\alpha-1)\Gamma(\alpha)} \tag{26}$$

for all $v \in \mathscr{M}_1$. Taking $R > \sup\{\|v\|\colon v \in \mathscr{M}_1\}$, we obtain

$$v \ngeqslant Av \quad \forall v \in \partial B_R \cap P. \tag{27}$$

On the other hand, by (H3), there exist $r \in (0,R)$ and $\varepsilon \in (0, \lambda_{\alpha_2,\beta_2,\gamma_2})$ such that

$$f(t,x_1,x_2,x_3) \leqslant (\lambda_{\alpha_2,\beta_2,\gamma_2}-\varepsilon)(\alpha_2 x_1+\beta_2 x_2+\gamma_2 x_3) \tag{28}$$

for all $x_i \in [0,r]$ and $t \in [0,1]$. This implies that

$$\begin{aligned}(Av)(t) &\leqslant (\lambda_{\alpha_2,\beta_2,\gamma_2}-\varepsilon)\int_0^1 G_1(t,s)\\ &\quad\times\left(\alpha_2\int_0^1 G_1(s,\tau)v(\tau)\,\mathrm{d}\tau+\beta_2\int_0^1 G_2(s,\tau)v(\tau)\,\mathrm{d}\tau+\gamma_2 v(s)\right)\mathrm{d}s\\ &= (\lambda_{\alpha_2,\beta_2,\gamma_2}-\varepsilon)\int_0^1 G_{\alpha_2,\beta_2,\gamma_2}(t,s)v(s)\,\mathrm{d}s\end{aligned} \tag{29}$$

for all $v \in \overline{B}_r \cap P$. Let $\mathscr{M}_2 := \{v \in \overline{B}_r \cap P\colon v \leqslant Av\}$. Now, we claim $\mathscr{M}_2 = \{0\}$. Indeed, if $v \in \mathscr{M}_2$, by (29), we have

$$v(t) \leqslant (\lambda_{\alpha_2,\beta_2,\gamma_2}-\varepsilon)\int_0^1 G_{\alpha_2,\beta_2,\gamma_2}(t,s)v(s)\,\mathrm{d}s.$$

Multiply (22) by $\psi_{\alpha_2,\beta_2,\gamma_2}(t)$ on both sides and integrate over $[0,1]$ and use (15), (16) to obtain

$$\int_0^1 v(t)\psi_{\alpha_2,\beta_2,\gamma_2}(t)\,\mathrm{d}t \leqslant (\lambda_{\alpha_2,\beta_2,\gamma_2} - \varepsilon)\lambda^{-1}_{\alpha_2,\beta_2,\gamma_2} \int_0^1 v(t)\psi_{\alpha_2,\beta_2,\gamma_2}(t)\,\mathrm{d}t$$

and thus $\int_0^1 v(t)\psi_{\alpha_2,\beta_2,\gamma_2}(t)\,\mathrm{d}t = 0$. Consequently, we have $v(t) \equiv 0$, i.e., $\mathscr{M}_2 = \{0\}$. Therefore,

$$v \nleqslant Av \quad \forall v \in \partial B_r \cap P. \tag{30}$$

Now Lemma 7 indicates that the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. That is, (1) has at least one positive solution. This completes the proof. $\square$

Theorem 2. *Assume that* (H1), (H4) *and* (H5) *hold. Then* (1) *has at least one positive solution.*

Proof. By (H4), there exist $r > 0$ and $\varepsilon > 0$ such that

$$f(t,x_1,x_2,x_3) \geqslant (\lambda_{\alpha_3,\beta_3,\gamma_3} + \varepsilon)(\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3) \quad \forall x_i \in [0,r],\ t \in [0,1]. \tag{31}$$

This implies

$$(Av)(t) \geqslant (\lambda_{\alpha_3,\beta_3,\gamma_3} + \varepsilon)\int_0^1 G_{\alpha_3,\beta_3,\gamma_3}(t,s)v(s)\,\mathrm{d}s \tag{32}$$

for all $v \in \overline{B}_r \cap P$. Let $\mathscr{M}_3 := \{v \in \overline{B}_r \cap P\colon\ v \geqslant Av\}$. We claim that $\mathscr{M}_3 = \{0\}$. Indeed, if $v \in \mathscr{M}_3$, combining with (32), we find

$$v(t) \geqslant (\lambda_{\alpha_3,\beta_3,\gamma_3} + \varepsilon)\int_0^1 G_{\alpha_3,\beta_3,\gamma_3}(t,s)v(s)\,\mathrm{d}s. \tag{33}$$

Multiply (33) by $\psi_{\alpha_3,\beta_3,\gamma_3}(t)$ on both sides and integrate over $[0,1]$ and use (15), (16) to obtain

$$\int_0^1 v(t)\psi_{\alpha_3,\beta_3,\gamma_3}(t)\,\mathrm{d}t \geqslant (\lambda_{\alpha_3,\beta_3,\gamma_3} + \varepsilon)\lambda^{-1}_{\alpha_3,\beta_3,\gamma_3} \int_0^1 v(t)\psi_{\alpha_3,\beta_3,\gamma_3}(t)\,\mathrm{d}t$$

and thus $\int_0^1 v(t)\psi_{\alpha_3,\beta_3,\gamma_3}(t)\,\mathrm{d}t = 0$. Hence, we see $v(t) \equiv 0$, i.e., $\mathscr{M}_3 = \{0\}$. Consequently,

$$v \ngeqslant Av \quad \forall v \in \partial B_r \cap P. \tag{34}$$

In addition, by (H5), there exist $\varepsilon \in (0, \lambda_{\alpha_4,\beta_4,\gamma_4})$ and $c_2 > 0$ such that

$$f(t,x_1,x_2,x_3) \leqslant (\lambda_{\alpha_4,\beta_4,\gamma_4} - \varepsilon)(\alpha_4 x_1 + \beta_4 x_2 + \gamma_4 x_3) + c_2 \quad \forall x_i \geqslant 0,\ t \in [0,1]. \tag{35}$$

Let $\mathscr{M}_4 := \{v \in P\colon\ v \leqslant Av\}$. We shall prove that $\mathscr{M}_4$ is bounded in P. Indeed, if $v \in \mathscr{M}_4$, then we have

$$v(t) \leqslant (\lambda_{\alpha_4,\beta_4,\gamma_4} - \varepsilon) \int_0^1 G_{\alpha_4,\beta_4,\gamma_4}(t,s)v(s)\,\mathrm{d}s + \frac{c_2}{\alpha(\alpha-1)\Gamma(\alpha)}. \tag{36}$$

Multiply (36) by $\psi_{\alpha_4,\beta_4,\gamma_4}(t)$ on both sides and integrate over $[0,1]$ and use (15), (16) to obtain

$$\int_0^1 v(t)\psi_{\alpha_4,\beta_4,\gamma_4}(t)\,\mathrm{d}t \leqslant (\lambda_{\alpha_4,\beta_4,\gamma_4} - \varepsilon)\lambda^{-1}_{\alpha_4,\beta_4,\gamma_4} \int_0^1 v(t)\psi_{\alpha_4,\beta_4,\gamma_4}(t)\,\mathrm{d}t + \frac{c_2}{\alpha(\alpha-1)\Gamma(\alpha)}$$

and then

$$\int_0^1 v(t)\psi_{\alpha_4,\beta_4,\gamma_4}(t)\,\mathrm{d}t \leqslant \frac{\varepsilon^{-1}\lambda_{\alpha_4,\beta_4,\gamma_4}c_2}{\alpha(\alpha-1)\Gamma(\alpha)}.$$

It follows from Lemma 6 that

$$\|v\| \leqslant \frac{\varepsilon^{-1}\omega^{-1}_{\alpha_4,\beta_4,\gamma_4}\lambda_{\alpha_4,\beta_4,\gamma_4}c_2}{\alpha(\alpha-1)\Gamma(\alpha)} \tag{37}$$

for all $v \in \mathscr{M}_4$. Choosing $R > \sup\{\|v\|\colon\ v \in \mathscr{M}_4\}$ and $R > r$, we have

$$v \nleqslant Av \quad \forall v \in \partial B_R \cap P. \tag{38}$$

Now Lemma 7 implies that A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. Equivalently, (1) has at least one positive solution. This completes the proof. $\square$

3.2 Uniqueness of positive solutions

In order to obtain our main results in this subsection, we first offer some lemmas. From now on, we always assume that (H1)$'$ holds.

Lemma 9. *If $v(t) \in C[0,1]$ is a positive fixed point of A in* (12)*, then there exist two positive constants a_v and b_v such that $a_v\rho(t) \leqslant v(t) \leqslant b_v\rho(t)$, where $\rho(t) = \int_0^1 G_1(t,s)\,\mathrm{d}s$.*

Proof. The continuity of G_1, G_2 and v implies that there exists $M > 0$ such that $|v(t)| \leqslant M$ and $|\int_0^1 G_i(t,s)v(s)\,\mathrm{d}s| \leqslant M$ for all $t \in [0,1]$. Taking

$$a_v = \min_{(t,x_1,x_2,x_3)\in[0,1]\times[0,M]^3} f(t,x_1,x_2,x_3) > 0,$$
$$b_v = \max_{(t,x_1,x_2,x_3)\in[0,1]\times[0,M]^3} f(t,x_1,x_2,x_3) > 0.$$

Then we have

$$\begin{aligned} a_v\rho(t) &\leqslant v(t) = (Av)(t) \\ &= \int_0^1 G_1(t,s) f\left(s,\, \int_0^1 G_1(s,\tau)v(\tau)\,\mathrm{d}\tau,\, \int_0^1 G_2(s,\tau)v(\tau)\,\mathrm{d}\tau,\, v(s)\right)\mathrm{d}s \\ &\leqslant b_v\rho(t). \end{aligned}$$

This completes the proof. □

Lemma 10. *Suppose that* (H1)′, (H4)–(H7) *hold. Then the operator A has exactly one positive fixed point.*

Proof. By Theorem 2, A has at least one positive fixed point. It then remains to prove that A has at most one positive fixed point. Indeed, if v_1 and v_2 are two positive fixed points of A, then

$$v_i(t) = \int_0^1 G_1(t,s) f\left(s,\, \int_0^1 G_1(s,\tau)v_i(\tau)\,\mathrm{d}\tau,\, \int_0^1 G_2(s,\tau)v_i(\tau)\,\mathrm{d}\tau,\, v_i(s)\right)\mathrm{d}s,$$

where $i = 1, 2$. By Lemma 9, there exist four positive numbers a_i, b_i for which $a_i\rho(t) \leqslant v_i(t) \leqslant b_i\rho(t)$ for $t \in [0,1]$ and $i = 1, 2$. Clearly, $v_2 \geqslant (a_2/b_1)v_1$.

Let $\gamma_0 := \sup\{\gamma > 0\colon\ v_2 \geqslant \gamma v_1\}$ $(\neq \emptyset)$. Then $\gamma_0 > 0$ and $v_2 \geqslant \gamma_0 v_1$. We shall claim that $\gamma_0 \geqslant 1$. Suppose the contrary. Then $\gamma_0 < 1$ and

$$\begin{aligned} v_2(t) &\geqslant \int_0^1 G_1(t,s) f\left(s,\, \int_0^1 G_1(s,\tau)\gamma_0 v_1(\tau)\,\mathrm{d}\tau,\, \int_0^1 G_2(s,\tau)\gamma_0 v_1(\tau)\,\mathrm{d}\tau,\, \gamma_0 v_1(s)\right)\mathrm{d}s \\ &= \int_0^1 G_1(t,s) g(s)\,\mathrm{d}s + \gamma_0^\mu v_1(t), \end{aligned}$$

where

$$\begin{aligned} g(s) = f&\left(s,\, \int_0^1 G_1(s,\tau)\gamma_0 v_1(\tau)\,\mathrm{d}\tau,\, \int_0^1 G_2(s,\tau)\gamma_0 v_1(\tau)\,\mathrm{d}\tau,\, \gamma_0 v_1(s)\right) \\ &- \gamma_0^\mu f\left(s,\, \int_0^1 G_1(s,\tau)v_1(\tau)\,\mathrm{d}\tau,\, \int_0^1 G_2(s,\tau)v_1(\tau)\,\mathrm{d}\tau,\, v_1(s)\right). \end{aligned}$$

(H6) implies that $g \in P\setminus\{0\}$ and there is a $a_3 > 0$ such that $\int_0^1 G_1(t,s)g(s)\,\mathrm{d}s \geqslant a_3\rho(t)$ by Lemma 9. Consequently, $v_2(t) \geqslant a_3\rho(t) + \gamma_0^\mu v_1(t) \geqslant (a_3/b_1)v_1(t) + \gamma_0 v_1(t)$, which contradicts the definition of γ_0. As a result, $\gamma_0 \geqslant 1$ and $v_2 \geqslant v_1$. Similarly, $v_1 \geqslant v_2$. Hence, $v_1 = v_2$. This completes the proof. □

Theorem 3. *Let all the conditions in Lemma* 10 *hold and $v^*(t)$ be the unique positive solution of A. Then for any $v_0 \in P \setminus \{0\}$, we have $A^n v_0 \to v^*(n \to \infty)$ uniformly in $t \in [0,1]$.*

Proof. Clearly, $\rho(t) = \int_0^1 G_1(t,s)\,\mathrm{d}s$ is a bounded function on $[0,1]$. Then by Lemma 9, there exist $a_\rho, b_\rho > 0$ such that

$$\begin{aligned} a_\rho \rho(t) &\leqslant \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \rho(s)\right)\mathrm{d}s \\ &:= \eta(t) \leqslant b_\rho \rho(t). \end{aligned}$$

Let $\beta_1(t) = \delta\eta(t)$ with $0 < \delta < \min\{1/b_\rho, a_\rho^{\mu/(1-\mu}\}$. Then we can choose $0 < \varepsilon < \min\{1, a_\rho\}$, and

$$\begin{aligned} &(A\varepsilon\beta_1)(t) \\ &\quad= \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\varepsilon\beta_1(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\varepsilon\beta_1(\tau)\,\mathrm{d}\tau, \varepsilon\beta_1(s)\right)\mathrm{d}s \\ &\quad= \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\frac{\varepsilon\beta_1(\tau)}{\rho(\tau)}\rho(\tau)\,\mathrm{d}\tau,\right. \\ &\qquad\qquad \left.\int_0^1 G_2(s,\tau)\frac{\varepsilon\beta_1(\tau)}{\rho(\tau)}\rho(\tau)\,\mathrm{d}\tau, \frac{\varepsilon\beta_1(s)}{\rho(s)}\rho(s)\right)\mathrm{d}s \\ &\quad\geqslant \varepsilon^\mu(\delta a_\rho)^\mu \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \rho(s)\right)\mathrm{d}s \\ &\quad= \varepsilon^\mu(\delta a_\rho)^\mu \eta(t) \geqslant \varepsilon^\mu \delta\eta(t) \geqslant \varepsilon\delta\eta(t) = \varepsilon\beta_1(t). \end{aligned}$$

Thus we have $A\varepsilon\beta_1 \geqslant \varepsilon\beta_1$. On the other hand, let $\beta_2(t) = \xi\eta(t)$ with $\xi > \max\{1/a_\rho, b_\rho^{\mu/(1-\mu)}\}$. Taking $\overline{\varepsilon} > \max\{1, b_\rho\}$, we find

$$\begin{aligned} &\overline{\varepsilon}\beta_2(t) \geqslant \overline{\varepsilon}^\mu \xi\eta(t) \\ &\quad= \overline{\varepsilon}^\mu \xi \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\rho(\tau)\,\mathrm{d}\tau, \rho(s)\right)\mathrm{d}s \\ &\quad\geqslant \overline{\varepsilon}^\mu \xi \int_0^1 G_1(t,s) f\left(s, \int_0^1 \frac{G_1(s,\tau)\rho(\tau)\overline{\varepsilon}\beta_2(\tau)}{\overline{\varepsilon}\beta_2(\tau)}\,\mathrm{d}\tau,\right. \\ &\qquad\qquad \left.\int_0^1 \frac{G_2(s,\tau)\rho(\tau)\overline{\varepsilon}\beta_2(\tau)}{\overline{\varepsilon}\beta_2(\tau)}\,\mathrm{d}\tau, \frac{\rho(s)\overline{\varepsilon}\beta_2(s)}{\overline{\varepsilon}\beta_2(s)}\right)\mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&\geqslant \bar{\varepsilon}^{\mu}\xi\bar{\varepsilon}^{-\mu}(\xi b_\rho)^{-\mu} \\
&\quad\times \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\bar{\varepsilon}\beta_2(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\bar{\varepsilon}\beta_2(\tau)\,\mathrm{d}\tau, \bar{\varepsilon}\beta_2(s)\right) \mathrm{d}s \\
&\geqslant \int_0^1 G_1(t,s) f\left(s, \int_0^1 G_1(s,\tau)\bar{\varepsilon}\beta_2(\tau)\,\mathrm{d}\tau, \int_0^1 G_2(s,\tau)\bar{\varepsilon}\beta_2(\tau)\,\mathrm{d}\tau, \bar{\varepsilon}\beta_2(s)\right) \mathrm{d}s \\
&= (A\bar{\varepsilon}\beta_2)(t).
\end{aligned}$$

Hence, $A\bar{\varepsilon}\beta_2 \leqslant \bar{\varepsilon}\beta_2$.

(H7) implies that A is an increasing operator. It follows from Lemma 8 that A has the smallest fixed point v_{**} and the largest fixed point v^{**} in $[\varepsilon\beta_1, \bar{\varepsilon}\beta_2]$, respectively. Based on this, we first show $v^* \in [\varepsilon\beta_1, \bar{\varepsilon}\beta_2]$. Indeed, for all $n \in \mathbb{N}_+$, we have

$$\varepsilon\beta_1 \leqslant A^n\varepsilon\beta_1 \leqslant A^n\bar{\varepsilon}\beta_2 \leqslant \bar{\varepsilon}\beta_2. \tag{39}$$

Let $n \to \infty$ in (39), we see $\varepsilon\beta_1 \leqslant v_{**} \leqslant v^* \leqslant v^{**} \leqslant \bar{\varepsilon}\beta_2$. For all $\varepsilon\beta_1 \leqslant v_0 \leqslant \bar{\varepsilon}\beta_2$ and $n \in \mathbb{N}_+$, we have $v_0 \in P \setminus \{0\}$ and

$$A^n\varepsilon\beta_1 \leqslant A^n v_0 = v_n \leqslant A^n\bar{\varepsilon}\beta_2. \tag{40}$$

By Theorem 2 and Lemma 10, we know that A has only a positive fixed point, i.e., $\lim_{n\to\infty} A^n\varepsilon\beta_1 = \lim_{n\to\infty} A^n\bar{\varepsilon}\beta_2 = v^*$, and thus $\lim_{n\to\infty} A^n v_0 \to v^*$. This completes the proof. $\square$

To facilitate computations for the following examples, let $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$, $\alpha_3 = \alpha_4$, $\beta_3 = \beta_4$, $\gamma_3 = \gamma_4$ in (H2)–(H5).

Example 1. Let $\alpha = 2.5$, $\alpha_1 = \Gamma^2(\alpha) = 9\pi/16 \approx 1.77$, $\beta_1 = \Gamma(2\alpha) = 24$, $\gamma_1 = \Gamma(\alpha) = 3\sqrt{\pi}/4 \approx 1.33$. Then by Lemma 5, we get $0.23 \leqslant r(L_{\alpha_1,\beta_1,\gamma_1}) \leqslant 2.47$, and $0.40 \leqslant \lambda_{\alpha_1,\beta_1,\gamma_1} \leqslant 4.35$.

Let

$$f(t,x_1,x_2,x_3) = \frac{1}{4}\big|\sin(\alpha_1x_1+\beta_1x_2+\gamma_1x_3)\big| + (\alpha_1x_1+\beta_1x_2+\gamma_1x_3)^2.$$

Then

$$\begin{aligned}
&\liminf_{\alpha_1x_1+\beta_1x_2+\gamma_1x_3\to+\infty} \frac{\frac{1}{4}|\sin(\alpha_1x_1+\beta_1x_2+\gamma_1x_3)| + (\alpha_1x_1+\beta_1x_2+\gamma_1x_3)^2}{\alpha_1x_1+\beta_1x_2+\gamma_1x_3} \\
&\quad = \infty > \lambda_{\alpha_1,\beta_1,\gamma_1},
\end{aligned}$$

and

$$\begin{aligned}
&\limsup_{\alpha_1x_1+\beta_1x_2+\gamma_1x_3\to0^+} \frac{\frac{1}{4}|\sin(\alpha_1x_1+\beta_1x_2+\gamma_1x_3)| + (\alpha_1x_1+\beta_1x_2+\gamma_1x_3)^2}{\alpha_1x_1+\beta_1x_2+\gamma_1x_3} \\
&\quad = \frac{1}{4} < \lambda_{\alpha_1,\beta_1,\gamma_1}
\end{aligned}$$

uniformly for $t \in [0,1]$. All conditions of Theorem 1 hold. Therefore, (1) has at least one positive solution.

Example 2. Let $\alpha = 2.5$, $\alpha_3 = \Gamma^2(\alpha) = 9\pi/16 \approx 1.77$, $\beta_3 = \Gamma(2\alpha - 2) = 2$, $\gamma_3 = \Gamma(\alpha - 1) = \sqrt{\pi}/2 \approx 0.89$. Then from Lemma 5 we have $0.10 \leqslant r(L_{\alpha_3,\beta_3,\gamma_3}) \leqslant 0.43$, and $2.33 \leqslant \lambda_{\alpha_3,\beta_3,\gamma_3} \leqslant 10$.

Let

$$f(t, x_1, x_2, x_3) = \mathrm{e}^t + \ln\bigl(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3)\bigr).$$

Then

$$\liminf_{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 \to 0^+} \frac{\mathrm{e}^t + \ln(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3))}{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3} = \infty > \lambda_{\alpha_3,\beta_3,\gamma_3}$$

and

$$\limsup_{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3 \to +\infty} \frac{\mathrm{e}^t + \ln(1 + (\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3))}{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3} = 0 < \lambda_{\alpha_3,\beta_3,\gamma_3}$$

uniformly for $t \in [0,1]$. Hence, (H4), (H5) hold, and Theorem 2 implies that (1) has at least one positive solution.

Example 3. Let $\alpha = 2.5$, $\alpha_3 = \Gamma^2(2\alpha - 2) = 4$, $\beta_3 = \gamma_3 = \Gamma(\alpha - 2) = \sqrt{\pi} \approx 1.77$. By Lemma 5, we can obtain $\lambda_{\alpha_3,\beta_3,\gamma_3} \in [1.48, 4.90]$.

Let

$$f(t, x_1, x_2, x_3) = e^t + \sqrt{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3}.$$

Similar with Example 2, we can show (H4) and (H5) hold. On the other hand, for any $\kappa \in (0,1)$, we have $\sqrt{\kappa} \leqslant 1$ and

$$\begin{aligned} &\sqrt{\kappa}\bigl[\mathrm{e}^t + \sqrt{\alpha_3 x_1 + \beta_3 x_2 + \gamma_3 x_3}\,\bigr] \\ &\quad = \sqrt{\kappa}\mathrm{e}^t + \sqrt{\alpha_3 \kappa x_1 + \beta_3 \kappa x_2 + \gamma_3 \kappa x_3} \leqslant \mathrm{e}^t + \sqrt{\alpha_3 \kappa x_1 + \beta_3 \kappa x_2 + \gamma_3 \kappa x_3}. \end{aligned}$$

As a result, (H6) is also satisfied. In addition, (H1)$'$ and (H7) automatically hold. Hence, from Theorem 3, (1) has a unique positive solution.

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