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Multivalued generalizations of fixed point results in fuzzy metric spaces*

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Abstract. This paper attempts to prove fixed and coincidence point results in fuzzy metric space using multivalued mappings. Altering distance function and multivalued strong $\{b_n\}$ -fuzzy contraction are used in order to do that. Presented theorems are generalization of some well known single valued results. Two examples are given to support the theoretical results.

Keywords: fixed point, coincidence point, fuzzy metric space, multivalued mappings, altering distance.

1 Introduction

Banach contraction principle [1] was motivation for many fixed point studies in various spaces [2,3,4,5,6,7,10,12,13,14,15,16,17,18,22,24,25,26,30]. In particular, multivalued generalization of this principle in metric space (X, d) is done by Nadler [27] on the following way: there exist $k \in (0, 1)$ so that, for every $x, y \in X$,

$$H(fx, fy) \leq kd(x, y), \quad (1)$$

where H is Hausdorff–Pompeiu metric and f is multivalued mapping from X to the family of its non-empty, closed and bounded subsets. Later on, the probabilistic versions

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of condition (1) are given in [12, 13, 15, 16], where notions of weakly demicompact mapping, f -strongly demicompact and weakly commuting mapping are introduced. Further, Hausdorff distance between sets in fuzzy metric spaces is introduced [22] and used in [7] for study of existence of coincidence point using two multivalued and one single valued mappings.

Also, Banach contraction principle in metric spaces is improved by Khan, Swaleh and Sessa [19], where control function, called altering distance function, is introduced. This type of function is used in [30] in fuzzy metric space (X, M, T) with the following condition:

$$\varphi(M(fx, fy, t)) \leq k(t) \cdot \varphi(M(x, y, t)), \quad x, y \in X, t > 0, 0 < k(t) < 1, \quad (2)$$

where φ is altering distance function. Note that condition (2) is improved in [5]. Moreover, many functions of this type are used in the study of fixed point problems [3, 4, 26].

Another classes of contraction, so called (strong) $\{b_n\}$ -probabilistic contraction, are introduced in [6, 24] and used in the study of fixed point problems in multivalued case in probabilistic spaces [25].

Our aim in present paper is to study the multivalued generalization in fuzzy metric spaces of results given in [6, 19, 30]. First, we use altering distance function in the style of condition (2) to obtain coincidence point results. That is realized through two theorems using strong fuzzy metric space with t -norm of H -type in the first and f -strongly demicompact mappings in the second one. On the other side, result given in [6] is transferred to multivalued case by introducing multivalued strong $\{b_n\}$ -fuzzy contraction.

2 Preliminaries

In order to make paper more readable, first, we list the definitions of basic notions important to further work. Using the results of Menger and Zadeh [23, 31], Kramosil and Michalek [21] introduced the notion of fuzzy metric space. Later, George and Veermani [8, 9] modified their definition in way to associate each fuzzy metric to a Hausdorff topology.

Definition 1. (See [29].) A mapping $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (t -norm) if the following conditions are satisfied:

- (T1) $T(a, 1) = a, a \in [0, 1]$,
- (T2) $T(a, b) = T(b, a), a, b \in [0, 1]$,
- (T3) $a \geq b, c \geq d \Rightarrow T(a, c) \geq T(b, d), a, b, c, d \in [0, 1]$,
- (T4) $T(a, T(b, c)) = T(T(a, b), c), a, b, c \in [0, 1]$.

Definition 2. (See [21].) The 3-tuple (X, M, T) is said to be a KM fuzzy metric space in the sense of Kramosil and Michalek if X is an arbitrary set, T is a t -norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions:

- (KM1) $M(x, y, 0) = 0, x, y \in X$,
- (KM2) $M(x, y, t) = 1, t > 0 \Leftrightarrow x = y$,

- (KM3) $M(x, y, t) = M(y, x, t)$, $x, y \in X$, $t > 0$,
 (KM4) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$, $x, y, z \in X$, $t, s > 0$,
 (KM5) $M(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]$ is left-continuous for every $x, y \in X$.

Definition 3. (See [8, 9].) The 3-tuple (X, M, T) is said to be a fuzzy metric space in the sense of George and Veeramani if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions:

- (GV1) $M(x, y, t) > 0$, $x, y \in X$, $t > 0$,
 (GV2) $M(x, y, t) = 1$, $t > 0 \Leftrightarrow x = y$,
 (GV3) $M(x, y, t) = M(y, x, t)$, $x, y \in X$, $t > 0$,
 (GV4) $T(M(x, y, t), M(y, z, s)) \leq M(x, z, t + s)$, $x, y, z \in X$, $t, s > 0$,
 (GV5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0, 1]$ is continuous for every $x, y \in X$.

If (GV4) is replaced by condition

- (GV4') $T(M(x, y, t), M(y, z, t)) \leq M(x, z, t)$, $x, y, z \in X$, $t > 0$,

then (X, M, T) is called a strong fuzzy metric space [11].

Moreover, if (X, M, T) is a fuzzy metric space, then M is a continuous function on $X \times X \times (0, \infty)$ [28] and $M(x, y, \cdot)$ is non-decreasing for all $x, y \in X$ [10].

If (X, M, T) is a fuzzy metric space, then M generates the Hausdorff topology on X (see [8, 9]) with base of open sets $\{U(x, r, t) : x \in X, r \in (0, 1), t > 0\}$, where $U(x, r, t) = \{y : y \in X, M(x, y, t) > 1 - r\}$.

A function $\varphi : [0, 1] \rightarrow [0, 1]$ is called an altering distance function [26, 30] if it satisfies the following properties:

- (AD1) φ is strictly decreasing and left continuous;
 (AD2) $\varphi(\lambda) = 0$ if and only if $\lambda = 1$.

It is obvious that $\lim_{\lambda \rightarrow 1^-} \varphi(\lambda) = \varphi(1) = 0$.

Definition 4. (See [8, 9]) Let (X, M, T) be a fuzzy metric space.

- (a) A sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in (X, M, T) if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, $n, m \geq n_0$, $t > 0$.
 (b) A sequence $\{x_n\}_{n \in \mathbb{N}}$ converges to x in (X, M, T) if, for every $\varepsilon \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - \varepsilon$, $n \geq n_0$, $t > 0$. Then we say that $\{x_n\}_{n \in \mathbb{N}}$ is convergent. Every convergent sequence is a Cauchy sequence.
 (c) A fuzzy metric space (X, M, T) is complete if every Cauchy sequence in (X, M, T) is convergent.

Definition 5. (See [14].) Let T be a t-norm and $T_n : [0, 1] \rightarrow [0, 1]$, $n \in \mathbb{N}$, be defined in the following way:

$$T_1(x) = T(x, x), \quad T_{n+1}(x) = T(T_n(x), x), \quad n \in \mathbb{N}, x \in [0, 1].$$

We say that t-norm T is of H -type if the family $\{T_n(x)\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$.

Each t-norm T can be extended (see [20]) (by associativity) in a unique way to an n -ary operation taking for $(x_1, \dots, x_n) \in [0, 1]^n$ the values

$$\mathbf{T}_{i=1}^0 x_i = 1, \quad \mathbf{T}_{i=1}^n x_i = T(\mathbf{T}_{i=1}^{n-1} x_i, x_n).$$

A t-norm T can be extended to a countable infinite operation taking for any sequence $(x_n)_{n \in \mathbb{N}}$ from $[0, 1]$ the value

$$\mathbf{T}_{i=1}^\infty x_i = \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^n x_i.$$

The sequence $(\mathbf{T}_{i=1}^n x_i)_{n \in \mathbb{N}}$ is non-increasing and bounded from below. Hence, the limit $\mathbf{T}_{i=1}^\infty x_i$ exists.

In the fixed point theory (see [15, 17]), it is of interest to investigate the classes of t-norms T and sequences (x_n) from the interval $[0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$ and

$$\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^\infty x_i = \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^\infty x_{n+i} = 1. \quad (3)$$

In [15], the following proposition is obtained.

Proposition 1. *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of numbers from $[0, 1]$ such that $\lim_{n \rightarrow \infty} x_n = 1$ and t-norm T is of H -type. Then $\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^\infty x_i = \lim_{n \rightarrow \infty} \mathbf{T}_{i=1}^\infty x_{n+i} = 1$.*

Definition 6. (See [12, 15].) Let (X, M, T) be a fuzzy metric space, A a non-empty subset of X and $f : A \rightarrow 2^X \setminus \{\emptyset\}$. The mapping f is weakly demicompact if, for every sequence $\{x_n\}_{n \in \mathbb{N}}$ from A such that $x_{n+1} \in f x_n$, $n \in \mathbb{N}$, and $\lim_{n \rightarrow \infty} M(x_{n+1}, x_n, t) = 1$, $t > 0$, there exists a convergent subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$.

Throughout the paper by $C(X)$ is denoted a family of all non-empty and closed subsets of X .

Definition 7. (See [22].) Let (X, M, T) be a fuzzy metric space, A a non-empty subset of X , $f : A \rightarrow A$ and $F : A \rightarrow C(A)$. The mapping F is a f -strongly demicompact if, for every sequence $\{x_n\}_{n \in \mathbb{N}}$ from A such that $\lim_{n \rightarrow \infty} M(f x_n, y_n, t) = 1$, $t > 0$, for some sequence $\{y_n\}_{n \in \mathbb{N}}$, $y_n \in F x_n$, $n \in \mathbb{N}$, there exists a convergent subsequence $\{f x_{n_k}\}_{k \in \mathbb{N}}$.

Definition 8. (See [13, 15].) A mapping $F : X \rightarrow C(X)$ is weakly commuting with $f : X \rightarrow X$ if, for all $x \in X$, it holds $f(Fx) \subseteq F(fx)$.

3 Main results

3.1 Multivalued mappings using altering distance

Main result of this section is an extension of results given in [30] to the case of multivalued mappings.

Let A and B be two nonempty subsets of X , define the Hausdorff–Pompeiu fuzzy metric as

$$\widetilde{M}(A, B, t) = \min \left\{ \inf_{x \in A} E(x, B, t), \inf_{y \in B} E(y, A, t) \right\}, \quad t > 0,$$

where $E(x, B, t) = \sup_{y \in B} M(x, y, t)$.

Theorem 1. Let (X, M, T) be a complete strong fuzzy metric space and T is t -norm of H -type. Let $f : X \rightarrow X$ be a continuous mapping and $F, G : X \rightarrow C(X)$ are weakly commuting with f . If there exist $k : (0, \infty) \rightarrow (0, 1)$ and altering distance function φ such that the following condition is satisfied:

$$\varphi(\widetilde{M}(Fx, Gy, t)) \leq k(t) \cdot \varphi(M(fx, fy, t)), \quad x, y \in X, x \neq y, t > 0, \quad (4)$$

then there exists $x \in X$ such that $fx \in Fx \cap Gx$.

Proof. Let $x_0 \in X$. Since Fx_0 is a non-empty subset of X , there exist $x_1 \in X$ such that $fx_1 \in Fx_0$. Let $t_0 > 0$ be arbitrary. Continuity of M and the fact that $k(t) < 1, t > 0$, implies that, for $\varepsilon_1 > 0$, the following inequality holds:

$$k(t_0) \cdot \varphi(M(fx_0, fx_1, t_0)) < \varphi(M(fx_0, fx_1, t_0) + \varepsilon_1). \quad (5)$$

By definition of Hausdorff fuzzy metric, for $\varepsilon_1 > 0$ given in (5), there exist $x_2 \in X, fx_2 \in Gx_1$ and $l_1 \in \mathbb{N} \setminus \{0\}$ such that

$$\widetilde{M}(Fx_0, Gx_1, t_0) \leq M(fx_1, fx_2, t_0) + \frac{\varepsilon_1}{2^{l_1}}. \quad (6)$$

Now, by (4), (5) and (6), using that φ is strictly decreasing, we conclude that

$$M(fx_0, fx_1, t_0) < M(fx_1, fx_2, t_0). \quad (7)$$

Similarly, we can find $x_3 \in X, fx_3 \in Fx_2$, and $l_2 \in \mathbb{N}, l_2 > l_1$ such that

$$k(t) \cdot \varphi(M(fx_1, fx_2, t_0)) < \varphi(M(fx_1, fx_2, t_0) + \varepsilon_1) \quad (8)$$

and

$$\widetilde{M}(Gx_1, Fx_2, t_0) \leq M(fx_2, fx_3, t_0) + \frac{\varepsilon_1}{2^{l_2}}. \quad (9)$$

By (8) and (9) we have

$$M(fx_1, fx_2, t_0) < M(fx_2, fx_3, t_0). \quad (10)$$

Repeating the procedure presented above, we define a sequence $\{x_n\}_{n \in \mathbb{N}}$ from X and strictly increasing sequence $\{l_n\}_{n \in \mathbb{N}}$ from \mathbb{N} such that the following conditions are satisfied:

- (i) $fx_{2n+1} \in Fx_{2n}, fx_{2n+2} \in Gx_{2n+1}, n \in \mathbb{N}$,
- (ii) $M(fx_{n-1}, fx_n, t) < M(fx_n, fx_{n+1}, t), t > 0, n \in \mathbb{N}$,

where

$$\widetilde{M}(Fx_{2n}, Gx_{2n+1}, t) \leq M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^n}, \quad t > 0, n \in \mathbb{N}. \quad (11)$$

Hence, the sequence $\{M(fx_n, fx_{n+1}, t)\}_{n \in \mathbb{N}}, t > 0$, is non-decreasing and bounded, so there exist $a : (0, \infty) \rightarrow [0, 1]$ such that

$$\lim_{n \rightarrow \infty} M(fx_n, fx_{n+1}, t) = a(t), \quad t > 0. \quad (12)$$

By (4), (11) and (12), for $n \in \mathbb{N}, t > 0$, we have

$$\begin{aligned} & \varphi\left(M(fx_{2n+1}, fx_{2n+2}, t) + \frac{\varepsilon_1}{2^n}\right) \\ & < \varphi(\widetilde{M}(Fx_{2n}, Gx_{2n+1}, t)) < k(t) \cdot \varphi(M(fx_{2n}, fx_{2n+1}, t)). \end{aligned} \quad (13)$$

Letting $n \rightarrow \infty$ in (13), we get

$$\varphi(a(t)) \leq k(t) \cdot \varphi(a(t)), \quad t > 0, \quad (14)$$

and we conclude that $\varphi(a(t)) = 0$ for all $t > 0$ so that $a \equiv 1$.

Further, we will prove that $\{fx_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ and $s \in \mathbb{N}$. Since t-norm T is of H -type, using (12) and Proposition 1, we have that there exist $n_0 \in \mathbb{N}$ such that

$$\mathbf{T}_{i=n}^{\infty} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0, n \geq n_0. \quad (15)$$

Since (X, M, T) is strong fuzzy metric space and $\{\mathbf{T}_{i=1}^n M(fx_i, fx_{i+1}, t)\}_{n \in \mathbb{N}}$ is non-increasing sequence, by (15), we have that

$$M(fx_{n+s+1}, fx_n, t) \geq \mathbf{T}_{i=n}^{n+s} M(fx_i, fx_{i+1}, t) > 1 - \varepsilon, \quad t > 0, n \geq n_0. \quad (16)$$

So, $\{fx_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and, since the space (X, M, T) is complete, there exist $x \in X$ such that

$$x = \lim_{n \rightarrow \infty} fx_n. \quad (17)$$

It remains to prove that $fx \in Fx \cap Gx$. As $Fx \cap Gx = \overline{Fx} \cap \overline{Gx}$, we need to show that, for every $t > 0$ and $\lambda \in (0, 1)$, there exists $r_1 = r_1(t, \lambda) \in Fx$ and $r_2 = r_2(t, \lambda) \in Gx$ such that $r_1, r_2 \in U(fx, t, \lambda)$, i.e. $M(fx, r_1, t) > 1 - \lambda$ and $M(fx, r_2, t) > 1 - \lambda$.

Let $t_0 > 0$ and $\lambda \in (0, 1)$. Since t-norm T is continuous, it follows that there exist $\delta = \delta(\lambda) \in (0, 1)$ such that

$$T(1 - \delta, T(1 - \delta, 1 - \delta)) > 1 - \lambda. \quad (18)$$

By the continuity of f and (17) there exist $n_1 \in \mathbb{N}$ such that

$$M\left(fx, ffx_{2n}, \frac{t_0}{3}\right) > 1 - \delta, \quad n \geq n_1. \quad (19)$$

By (12) there exists $n_2 \in \mathbb{N}$ such that

$$M\left(ffx_{2n}, ffx_{2n+1}, \frac{t_0}{3}\right) > 1 - \delta, \quad n \geq n_2.$$

Since f is weakly commuting with F , we have

$$ffx_{2n+1} \in f(Fx_{2n}) \subseteq F(fx_{2n}). \quad (20)$$

Also, there exist $\varepsilon^* \in (0, 1)$ such that

$$k\left(\frac{t_0}{3}\right) \cdot \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right)\right) < \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) + \varepsilon^*\right) \quad (21)$$

for arbitrary $n_0 \geq \max\{n_1, n_2\}$. By (20) and definition of Hausdorff fuzzy metric there exist $r_2 \in Gx$ such that, for $\varepsilon^* > 0$ (defined in (21)), the following is satisfied:

$$\widetilde{M}\left(Gx, F(fx_{2n_0}), \frac{t_0}{3}\right) \leq M\left(r_2, ffx_{2n_0+1}, \frac{t_0}{3}\right) + \varepsilon^*. \quad (22)$$

By (4), (20) and (21) we have:

$$\begin{aligned} & \varphi\left(M\left(r_2, ffx_{2n_0+1}, \frac{t_0}{3}\right) + \varepsilon^*\right) \\ & \leq \varphi\left(\widetilde{M}\left(Gx, F(fx_{2n_0}), \frac{t_0}{3}\right)\right) \leq k\frac{t_0}{3} \cdot \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right)\right) \\ & < \varphi\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) + \varepsilon^*\right). \end{aligned}$$

Now, by (19) follows that

$$M\left(r_2, ffx_{2n_0}, \frac{t_0}{3}\right) > M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right) > 1 - \delta.$$

Finally, using (18), we get

$$\begin{aligned} M(fx, r_2, t_0) & \geq T\left(M\left(fx, ffx_{2n_0}, \frac{t_0}{3}\right), \right. \\ & \quad \left. T\left(M\left(ffx_{2n_0}, ffx_{2n_0+1}, \frac{t_0}{3}\right), M\left(ffx_{2n_0+1}, r_2, \frac{t_0}{3}\right)\right)\right) \\ & \geq T(1 - \delta, T(1 - \delta, 1 - \delta)) > 1 - \lambda. \end{aligned}$$

So, $r_2 \in U(fx, t_0, \lambda)$ for arbitrary $t_0 > 0$ and $\lambda \in (0, 1)$, i.e. $fx \in Gx$. Similarly, it can be shown that $r_1 \in U(fx, t, \lambda)$, $t > 0$, $\lambda \in (0, 1)$, which implies that $fx \in Fx$, too. \square

Theorem 2. Let (X, M, T) be a complete fuzzy metric space and $f : X \rightarrow X$ be a continuous mapping. Let $F, G : X \rightarrow C(X)$ are weakly commuting with f and F or G is f -strongly demicontact. If, for some $k : (0, \infty) \rightarrow (0, 1)$ and altering distance function φ , the following condition is satisfied:

$$\varphi(\widetilde{M}(Fx, Gy, t)) \leq k(t) \cdot \varphi(M(fx, fy, t)), \quad x, y \in X, x \neq y, t > 0, \quad (23)$$

then there exists $x \in X$ such that $fx \in Fx \cap Gx$.

Proof. The proof is similar with that of the Theorem 1, except in the part related to Cauchy sequence. Namely, since F or G is f -strongly demicontact, $fx_{2n+1} \in Fx_{2n}$ or $fx_{2n+2} \in Gx_{2n+1}$ and $\lim_{n \rightarrow \infty} M(fx_{2n}, fx_{2n+1}, t) = 1, t > 0$, we conclude that there exist convergent subsequence $\{fx_{2n_p}\}_{p \in \mathbb{N}}$ or $\{fx_{2n_p+1}\}_{p \in \mathbb{N}}$, respectively, such that

$$\lim_{p \rightarrow \infty} fx_{2n_p} = x. \quad (24)$$

The last part of the proof is analogous as in Theorem 1, where instead of sequence $\{fx_n\}_{n \in \mathbb{N}}$, we deal with subsequences $\{fx_{2n_p}\}_{p \in \mathbb{N}}$ and $\{fx_{2n_p+1}\}_{p \in \mathbb{N}}$. \square

If in Theorems 1 and 2, we take that $F = G$ and that f is the identity mapping, we get the following corollary.

Corollary 1. Let (X, M, T) be a complete fuzzy metric space, $F : X \rightarrow C(X)$, and one of the following conditions is satisfied:

(a) F is weakly demicontact mapping,

or

(b) (X, M, T) is strong fuzzy metric space and T is t -norm of H -type.

If there exist $k : (0, \infty) \rightarrow (0, 1)$ and altering distance function φ such that:

$$\varphi(\widetilde{M}(Fx, Fy, t)) \leq k(t) \cdot \varphi(M(x, y, t)), \quad x, y \in X, t > 0, \quad (25)$$

then there exists $x \in X$ such that $x \in Fx$.

Moreover, if the mapping F in Corollary 1 is single-valued we got the result in [30].

Example 1.

- (a) Let $X = [0, 2], T = T_P, M(x, y, t) = t/(t + d(x, y))$, where d is Euclidian metric. Then (X, M, T) is a fuzzy metric space. Let $F(x) = \{1, 2\}, x \in X$. Since F is weakly demicontact and condition (25) is satisfied, by Corollary 1(a) follows that there exists $x \in X$ such that $x \in Fx$.
- (b) Let $X = [0, 2], T = T_M, M^*(x, y, t) = t/(t + d^*(x, y))$, where d^* is ultrametric. Ultrametric space is metric space, where instead of triangle inequality condition, the following is satisfied: $d^*(x, z) \leq \max\{d^*(x, y), d^*(y, z)\}$. Then (X, M^*, T) is a strong fuzzy metric space [11]. For $F(x) = \{1, 2\}, x \in X$, condition (25) is satisfied and by Corollary 1(b) follows that there exists $x \in X$ such that $x \in Fx$.

3.2 Multivalued strong $\{b_n\}$ -fuzzy contraction

In this part, we present multivalued extension of results given in [6] using multivalued strong $\{b_n\}$ -fuzzy contraction.

Definition 9. Let (X, M, T) be a fuzzy metric space and $\{b_n\}_{n \in \mathbb{N}}$ a sequence from $(0, 1)$ such that $\lim_{n \rightarrow \infty} b_n = 1$. The mapping $F : X \rightarrow C(X)$ is a multivalued strong $\{b_n\}$ -fuzzy contraction if there exist $q \in (0, 1)$ such that

$$M(x, y, t) > b_n \implies \widetilde{M}(Fx, Fy, qt) > b_{n+1}, \quad x, y \in X, t > 0, n \in \mathbb{N}. \quad (26)$$

Theorem 3. Let (X, M, T) be a complete KM fuzzy metric space such that $\lim_{t \rightarrow \infty} M(x, y, t) = 1$, $x, y \in X$, $\sup_{a < 1} T(a, a) = 1$. Let $\{b_n\} \subset (0, 1)$ be a sequence such that $\lim_{n \rightarrow \infty} b_n = 1$ and $F : X \rightarrow C(X)$ be a multivalued strong $\{b_n\}$ -fuzzy contraction. If t -norm T satisfies the following condition:

$$\lim_{n \rightarrow \infty} \mathbf{T}_{i=n}^\infty b_i = 1, \quad (27)$$

then there exists $x \in X$ such that $x \in Fx$.

Proof. Let $x_0, x_1 \in X$, where $x_1 \in Fx_0$. By (27), for arbitrary $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that

$$\mathbf{T}_{i=n_0}^\infty b_i > 1 - \varepsilon \quad \text{and} \quad M(x_0, x_1, t_0) > b_{n_0}. \quad (28)$$

Then by condition (26), for some $q \in (0, 1)$ and $\varepsilon_0 > 0$, we have

$$\widetilde{M}(Fx_0, Fx_1, qt_0) > b_{n_0+1} + \varepsilon_0. \quad (29)$$

Keeping the same ε_0 and using definition of Hausdorff metric, we can find $x_2 \in Fx_1$ such that

$$\widetilde{M}(Fx_0, Fx_1, qt_0) \leq M(x_1, x_2, qt_0) + \varepsilon_0. \quad (30)$$

By (26), (29) and (30) we obtain

$$M(x_1, x_2, qt_0) > b_{n_0+1} \implies \widetilde{M}(Fx_1, Fx_2, q^2t_0) > b_{n_0+2}.$$

Repeating the same procedure, we get

$$M(x_k, x_{k+1}, q^k t_0) > b_{n_0+k}, \quad k \in \mathbb{N}. \quad (31)$$

Let $\varepsilon > 0$ and $t > 0$. If we choose $k_0 \in \mathbb{N}$, $k_0 > n_0$, such that $\sum_{k=k_0}^\infty q^k < t/t_0$, then, for every $l, r \in \mathbb{N}$, $r > 1$, we have

$$\begin{aligned} & M(x_{k_0+l}, x_{k_0+l+r}, t) \\ & \geq M\left(x_{k_0+l}, x_{k_0+l+r}, t_0 \sum_{k=k_0}^\infty q^k\right) \geq M\left(x_{k_0+l}, x_{k_0+l+r}, t_0 \sum_{k=k_0+l}^{k_0+l+r-1} q^k\right) \\ & \geq \underbrace{T(T \dots T)}_{(r-1)\text{-times}} \left(M(x_{k_0+l}, x_{k_0+l+1}, t_0 q^{k_0+l}), \dots\right), \\ & \qquad M(x_{k_0+l+r-1}, x_{k_0+l+r}, t_0 q^{k_0+l+r-1}) \\ & \geq \mathbf{T}_{i=n_0}^\infty b_i > 1 - \varepsilon, \end{aligned}$$

where is used (28) and (31). So, $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence and, since (X, M, T) is complete, there exist $x \in X$ so that

$$\lim_{n \rightarrow \infty} x_n = x. \quad (32)$$

It is remain to prove that $x \in Fx$. As $Fx = \overline{Fx}$, it is enough to show that, for every $\lambda \in (0, 1)$ and $t > 0$, there exists $r = r(t, \lambda) \in Fx$ such that $M(x, r, t) > 1 - \lambda$.

Let $t_0 > 0$ and $\lambda \in (0, 1)$. Since $\sup_{a < 1} T(a, a) = 1$, there exist $\delta = \delta(\lambda) \in (0, 1)$ such that

$$T(T(1 - \delta, 1 - \delta), 1 - \delta) > 1 - \lambda. \quad (33)$$

From $\lim_{n \rightarrow \infty} b_n = 1$, for δ defined in (33), there exist $p_0 \in \mathbb{N}$ such that

$$b_p > 1 - \delta, \quad p \geq p_0. \quad (34)$$

By (32), for p_0 given above, it is possible to find $n_0 \in \mathbb{N}$ such that

$$M\left(x_n, x, \frac{t_0}{3}\right) > b_{p_0} > 1 - \delta, \quad n \geq n_0, \quad (35)$$

and

$$M\left(x_n, x_{n+1}, \frac{t_0}{3}\right) > b_{p_0} > 1 - \delta, \quad n \geq n_0. \quad (36)$$

Now, by (26) there exist $\varepsilon^* > 0$ such that

$$\widetilde{M}\left(Fx_n, Fx, q \frac{t_0}{3}\right) > b_{p_0+1} + \varepsilon^*, \quad n \geq n_0.$$

For the same ε^* there exist $r \in Fx$ such that

$$M\left(x_{n+1}, r, q \frac{t_0}{3}\right) + \varepsilon^* \geq \widetilde{M}\left(Fx_n, Fx, q \frac{t_0}{3}\right) > b_{p_0+1} + \varepsilon^*,$$

i.e.

$$M\left(x_{n+1}, r, \frac{t_0}{3}\right) > M\left(x_{n+1}, r, q \frac{t_0}{3}\right) > b_{p_0+1} > 1 - \delta, \quad n \geq n_0. \quad (37)$$

Finally, by (33), (35), (36) and (37) we get

$$\begin{aligned} M(x, r, t_0) &\geq T\left(T\left(M\left(x, x_n, \frac{t_0}{3}\right), M\left(x_n, x_{n+1}, \frac{t_0}{3}\right)\right), M\left(x_{n+1}, r, \frac{t_0}{3}\right)\right) \\ &> 1 - \lambda, \end{aligned}$$

which means $x \in Fx$. □

4 Conclusion

In this paper we prove several fixed point and coincidence point results, which presented fuzzy generalization of Nadler fixed point result using altering distance function, as well as a multivalued generalizations of strong fuzzy $\{b_n\}$ -contractions.

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