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Fixed points of multivalued nonlinear F-contractions on complete metric spaces

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Abstract. We introduce a new concept for multivalued maps, also called multivalued nonlinear F-contraction, and give a fixed point result. Our result is a proper generalization of some recent fixed point theorems including the famous theorem of Klim and Wardowski [D. Klim, D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, *J. Math. Anal. Appl.*, 334(1):132–139, 2007].

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1 Introduction and preliminaries

Let (X,d) be a metric space. $P(X)$ denotes the family of all nonempty subsets of X, $C(X)$ denotes the family of all nonempty, closed subsets of X, $CB(X)$ denotes the family of all nonempty, closed, and bounded subsets of X, and $K(X)$ denotes the family of all nonempty compact subsets of X. It is clear that, $K(X)\subseteq CB(X)\subseteq C(X)\subseteq P(X)$. For $A,B\in C(X)$, let

$$H(A,B)=\max\Bigl\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\Bigr\},$$

where $d(x,B)=\inf\{d(x,y)\colon y\in B\}$. Then H is called generalized Pompeiu–Hausdorff distance on $C(X)$. It is well known that H is a metric on $CB(X)$, which is called Pompeiu–Hausdorff metric induced by d. We can find detailed information about the Pompeiu–Hausdorff metric in [1, 5, 9]. Let $T: X\to CB(X)$ be a map, then T is called multivalued contraction (see [14]) if for all $x,y\in X$, there exists $L\in[0,1)$ such that

$$H(Tx,Ty)\leqslant Ld(x,y).$$

In 1969, Nadler [14] proved that every multivalued contraction on complete metric space has a fixed point.

Nadler's fixed point theorem has been extended in many directions [4, 6, 7, 10, 13, 16, 17]. The following generalization of it is given by Feng and Liu [8].

Theorem 1. (*See* [8].) *Let (X,d) be a complete metric space and $T: X \to C(X)$. Assume that the following conditions hold*:

(i) *the map $x \to d(x,Tx)$ is lower semi-continuous*;
(ii) *there exist $b,c \in (0,1)$ with $b < c$ such that for any $x \in X$, there is $y \in I_b^x$ satisfying*

$$d(y,Ty) \leqslant cd(x,y),$$

where

$$I_b^x = \bigl\{y \in Tx\colon bd(x,y) \leqslant d(x,Tx)\bigr\}.$$

Then T has a fixed point.

Recently, another interesting result have been obtained by Klim and Wardowski [11]. They proved the following theorem.

Theorem 2. (*See* [11].) *Let (X,d) be a complete metric space and $T: X \to C(X)$. Assume that the following conditions hold*:

(i) *the map $x \to d(x,Tx)$ is lower semi-continuous*;
(ii) *there exists $b \in (0,1)$ and a function $\varphi : [0,\infty) \to [0,b)$ satisfying*

$$\limsup_{t\to s^+} \varphi(t) < b \quad \text{for } s \geqslant 0$$

and for any $x \in X$, there is $y \in I_b^x$ satisfying

$$d(y,Ty) \leqslant \varphi\bigl(d(x,y)\bigr)d(x,y).$$

Then T has a fixed point.

In this paper, we introduce a new class of multivalued maps and give a fixed point result, which extend and generalize many fixed point theorems including Theorems 1 and 2. Our results are based on F-contraction which is a new approach to contraction mapping. The concept of F-contraction for single valued maps on complete metric space was introduced by Wardowski [18]. First, we recall this new concept and some related results.

Let $F : (0,\infty) \to \mathbb{R}$ be a function. For the sake of completeness, we will consider the following conditions:

(F1) F is strictly increasing, i.e., for all $\alpha,\beta \in (0,\infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.
(F2) For each sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{if and only if} \quad \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F3) There exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
(F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

We represent the set of all functions F satisfying (F1)–(F3) and (F1)–(F4) by $\mathcal{F}$ and $\mathcal{F}_*$, respectively. It is clear that $\mathcal{F}_* \subset \mathcal{F}$ and some examples of the functions belonging to $\mathcal{F}_*$ are $F_1(\alpha) = \ln\alpha$, $F_2(\alpha) = \alpha + \ln\alpha$, $F_3(\alpha) = -1/\sqrt{\alpha}$ and $F_4(\alpha) = \ln(\alpha^2 + \alpha)$. If we define $F_5(\alpha) = \ln\alpha$ for $\alpha \leqslant 1$ and $F_5(\alpha) = 2\alpha$ for $\alpha > 1$, then $F_5 \in \mathcal{F} \setminus \mathcal{F}_*$.

Remark 1. If F satisfies (F1), then it satisfies (F4) if and only if it is right continuous.

Definition 1. (See [18].) Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is an F-contraction if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that for all $x, y \in X$,

$$d(Tx,Ty) > 0 \quad \Longrightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \leqslant F\bigl(d(x,y)\bigr). \tag{1}$$

If we take $F(\alpha) = \ln\alpha$ in Definition 1, inequality (1) turns into

$$d(Tx,Ty) \leqslant \mathrm{e}^{-\tau} d(x,y) \quad \text{for all } x,y \in X,\ Tx \neq Ty. \tag{2}$$

It is clear that for $x, y \in X$ such that $Tx = Ty$, the inequality $d(Tx,Ty) \leqslant \mathrm{e}^{-\tau} d(x,y)$ also holds. Thus, T is an ordinary contraction with contractive constant $\mathrm{e}^{-\tau}$. Therefore, every ordinary contraction is also an F-contraction with $F(\alpha) = \ln\alpha$, but the converse may not be true as shown in Example 2.5 of [18]. If we choose $F(\alpha) = \alpha + \ln\alpha$, inequality (1) turns into

$$\frac{d(Tx,Ty)}{d(x,y)} \mathrm{e}^{d(Tx,Ty)-d(x,y)} \leqslant \mathrm{e}^{-\tau} \quad \text{for all } x,y \in X,\ Tx \neq Ty. \tag{3}$$

In addition, Wardowski showed that every F-contraction T is a contractive mapping, i.e.,

$$d(Tx,Ty) < d(x,y) \quad \text{for all } x,y \in X,\ Tx \neq Ty.$$

Thus, every F-contraction is a continuous map. Also, Wardowski concluded that if $F_1, F_2 \in \mathcal{F}$ with $F_1(\alpha) \leqslant F_2(\alpha)$ for all $\alpha > 0$ and $G = F_2 - F_1$ is nondecreasing, then every F_1-contraction T is an F_2-contraction. He noted that for the mappings $F_1(\alpha) = \ln\alpha$ and $F_2(\alpha) = \alpha + \ln\alpha$, $F_1 < F_2$ and the mapping $F_2 - F_1$ is strictly increasing. Hence, every Banach contraction satisfies the contractive condition (3). On the other hand, Example 2.5 in [18] shows that the mapping T is not F_1-contraction (Banach contraction), but still is an F_2-contraction. Thus, the following theorem is a proper generalization of Banach contraction principle.

Theorem 3. (*See* [18].) *Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point in X.*

By combining the ideas of Wardowski's and Nadler's, Altun et al. [3] introduced the concept of multivalued F-contractions and obtained some fixed point results for these type mappings on complete metric space.

Definition 2. (See [3].) Let (X,d) be a metric space and $T:X\to CB(X)$ be a mapping. Then T is a multivalued F-contraction if $F\in\mathcal{F}$ and there exists $\tau>0$ such that for all $x,y\in X$,

$$H(Tx,Ty)>0 \quad\Longrightarrow\quad \tau+F\big(H(Tx,Ty)\big)\leqslant F\big(d(x,y)\big).$$

By considering $F(\alpha)=\ln\alpha$, every multivalued contraction in the sense of Nadler is also a multivalued F-contraction.

Theorem 4. (*See* [3].) *Let (X,d) be a complete metric space and $T:X\to K(X)$ be a multivalued F-contraction, then T has a fixed point in X.*

At this point, one can ask if $CB(X)$ can be used instead of $K(X)$ in Theorem 4. As shown in Example 1 of [2], the answer is negative. But, by adding condition (F4) on F, we can we take $CB(X)$ instead of $K(X)$.

Theorem 5. (*See* [3].) *Let (X,d) be a complete metric space and $T:X\to CB(X)$ be a multivalued F-contraction. Suppose $F\in\mathcal{F}_*$, then T has a fixed point in X.*

On the other hand, Olgun et al. [15] proved the following theorems. Theorem 7 is a generalization of famous Mizoguchi–Takahashi's fixed point theorem for multivalued contraction maps. These results are nonlinear cases of Theorems 4 and 5, respectively.

Theorem 6. (*See* [15].) *Let (X,d) be a complete metric space and $T:X\to K(X)$. If there exists $F\in\mathcal{F}$ and $\tau:(0,\infty)\to(0,\infty)$ such that*

$$\liminf_{t\to s^+}\tau(t)>0 \quad \text{for all } s\geqslant 0$$

and for all $x,y\in X$,

$$H(Tx,Ty)>0 \quad\Longrightarrow\quad \tau\big(d(x,y)\big)+F\big(H(Tx,Ty)\big)\leqslant F\big(d(x,y)\big),$$

then T has a fixed point in X.

Theorem 7. (*See* [15].) *Let (X,d) be a complete metric space and $T:X\to CB(X)$. If there exists $F\in\mathcal{F}_*$ and $\tau:(0,\infty)\to(0,\infty)$ such that*

$$\liminf_{t\to s^+}\tau(t)>0 \quad \text{for all } s\geqslant 0$$

and for all $x,y\in X$,

$$H(Tx,Ty)>0 \quad\Longrightarrow\quad \tau\big(d(x,y)\big)+F\big(H(Tx,Ty)\big)\leqslant F\big(d(x,y)\big),$$

then T has a fixed point in X.

2 Main results

Let $T:X\to P(X)$ be a multivalued map, $F\in\mathcal{F}$ and $\sigma\geqslant 0$. For $x\in X$ with $d(x,Tx)>0$, define a set $F_\sigma^x\subseteq X$ as

$$F_\sigma^x=\big\{y\in Tx\colon\ F\big(d(x,y)\big)\leqslant F\big(d(x,Tx)\big)+\sigma\big\}.$$

We need to consider the following cases.

Case 1. If $T: X \to K(X)$, then for all $\sigma \geqslant 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, since Tx is compact, for every $x \in X$, we have $y \in Tx$ such that $d(x, y) = d(x, Tx)$. Therefore, for every $x \in X$ with $d(x, Tx) > 0$, we have $F(d(x, y)) = F(d(x, Tx))$. Thus, $y \in F_\sigma^x$ for all $\sigma \geqslant 0$.

Case 2. If $T: X \to C(X)$, then F_σ^x may be empty for some $x \in X$ and $\sigma \geqslant 0$. For example, let $F(\alpha) = \ln \alpha$ for $\alpha \leqslant 1$ and $F(\alpha) = 2\alpha$ for $\alpha > 1$ and let $X = \{0\} \cup (1, 2)$ with the usual metric. Define $T: X \to C(X)$ by $T0 = (1, 2)$ and $Tx = \{0\}$ for $x \in (1, 2)$. Then for $x = 0$, we have (note that $d(0, T0) = 1 > 0$)

$$\begin{aligned} F_1^0 &= \bigl\{y \in T0\colon F\bigl(d(0, y)\bigr) \leqslant F\bigl(d(0, T0)\bigr) + 1\bigr\} \\ &= \bigl\{y \in (1, 2)\colon F(y) \leqslant F(1) + 1\bigr\} = \bigl\{y \in (1, 2)\colon 2y \leqslant 1\bigr\} = \emptyset. \end{aligned}$$

Case 3. If $T: X \to C(X)$ (even if $T: X \to P(X)$) and $F \in \mathcal{F}_*$, then for all $\sigma > 0$ and $x \in X$ with $d(x, Tx) > 0$, we have $F_\sigma^x \neq \emptyset$. Indeed, by (F4), we have

$$\begin{aligned} F_\sigma^x &= \bigl\{y \in Tx\colon F\bigl(d(x, y)\bigr) \leqslant F\bigl(d(x, Tx)\bigr) + \sigma\bigr\} \\ &= \bigl\{y \in Tx\colon F\bigl(d(x, y)\bigr) \leqslant F\bigl(\inf\bigl\{d(x, y)\colon y \in Tx\bigr\}\bigr) + \sigma\bigr\} \\ &= \bigl\{y \in Tx\colon F\bigl(d(x, y)\bigr) \leqslant \inf\bigl\{F\bigl(d(x, y)\bigr)\colon y \in Tx\bigr\} + \sigma\bigr\} \neq \emptyset. \end{aligned}$$

Mınak et al. [12] proved the following fixed point theorems. Note that Theorem 1 is a special case of Theorem 9.

Theorem 8. *Let (X, d) be a complete metric space, $T: X \to K(X)$ and $F \in \mathcal{F}$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F\bigl(d(y, Ty)\bigr) \leqslant F\bigl(d(x, y)\bigr),$$

where

$$F_\sigma^x = \bigl\{y \in Tx\colon F\bigl(d(x, y)\bigr) \leqslant F\bigl(d(x, Tx)\bigr) + \sigma\bigr\},$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

Theorem 9. *Let (X, d) be a complete metric space, $T: X \to C(X)$ and $F \in \mathcal{F}_*$. If there exists $\tau > 0$ such that for any $x \in X$ with $d(x, Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying*

$$\tau + F\bigl(d(y, Ty)\bigr) \leqslant F\bigl(d(x, y)\bigr),$$

then T has a fixed point in X provided $\sigma < \tau$ and $x \to d(x, Tx)$ is lower semi-continuous.

By considering the above facts, we give the following theorems, which are nonlinear form of Theorems 8 and 9. Note that Theorem 10 is a proper generalization of Theorem 2.

Theorem 10. *Let (X,d) be a complete metric space, $T: X \to C(X)$ and $F \in \mathcal{F}_*$. Assume that the following conditions hold*:

(i) *the map $x \to d(x,Tx)$ is lower semi-continuous*;
(ii) *there exist $\sigma > 0$ and a function $\tau : (0,\infty) \to (\sigma,\infty)$ such that*

$$\liminf_{t\to s^+} \tau(t) > \sigma \quad \textit{for all } s \geqslant 0$$

and for any $x \in X$ with $d(x,Tx) > 0$, there exists $y \in F_\sigma^x$ satisfying

$$\tau\bigl(d(x,y)\bigr) + F\bigl(d(y,Ty)\bigr) \leqslant F\bigl(d(x,y)\bigr).$$

Then T has a fixed point.

Proof. Suppose that T has no fixed point. Then for all $x \in X$, we have $d(x,Tx) > 0$. Since $Tx \in C(X)$ for every $x \in X$, the set F_σ^x is nonempty for any $\sigma > 0$. Let $x_0 \in X$ be any initial point, then there exists $x_1 \in F_\sigma^{x_0}$ such that

$$\tau\bigl(d(x_0,x_1)\bigr) + F\bigl(d(x_1,Tx_1)\bigr) \leqslant F\bigl(d(x_0,x_1)\bigr)$$

and for $x_1 \in X$, there exists $x_2 \in F_\sigma^{x_1}$ satisfying

$$\tau\bigl(d(x_1,x_2)\bigr) + F\bigl(d(x_2,Tx_2)\bigr) \leqslant F\bigl(d(x_1,x_2)\bigr).$$

Continuing this process, we get an iterative sequence $\{x_n\}$, where $x_{n+1} \in F_\sigma^{x_n}$ and

$$\tau\bigl(d(x_n,x_{n+1})\bigr) + F\bigl(d(x_{n+1},Tx_{n+1})\bigr) \leqslant F\bigl(d(x_n,x_{n+1})\bigr). \tag{4}$$

We will verify that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in F_\sigma^{x_n}$, we have

$$F\bigl(d(x_n,x_{n+1})\bigr) \leqslant F\bigl(d(x_n,Tx_n)\bigr) + \sigma. \tag{5}$$

From (4) and (5) we have

$$F\bigl(d(x_{n+1},Tx_{n+1})\bigr) \leqslant F\bigl(d(x_n,Tx_n)\bigr) + \sigma - \tau\bigl(d(x_n,x_{n+1})\bigr) \tag{6}$$

and

$$F\bigl(d(x_{n+1},x_{n+2})\bigr) \leqslant F\bigl(d(x_n,x_{n+1})\bigr) + \sigma - \tau\bigl(d(x_n,x_{n+1})\bigr). \tag{7}$$

Let $a_n = d(x_n,x_{n+1})$ for $n \in \mathbb{N}$, then $a_n > 0$ and from (7) $\{a_n\}$ is decreasing. Therefore, there exists $\delta \geqslant 0$ such that $\lim_{n\to\infty} a_n = \delta$. Now let $\delta > 0$. Using (7), the following holds:

$$\begin{aligned} F(a_{n+1}) &\leqslant F(a_n) + \sigma - \tau(a_n) \\ &\leqslant F(a_{n-1}) + 2\sigma - \tau(a_n) - \tau(a_{n-1}) \\ &\;\vdots \\ &\leqslant F(a_0) + n\sigma - \tau(a_n) - \tau(a_{n-1}) - \cdots - \tau(a_0). \end{aligned} \tag{8}$$

Let $\tau(a_{p_n}) = \min\{\tau(a_0), \tau(a_1), \dots, \tau(a_n)\}$ for all $n \in \mathbb{N}$. From (8) we get

$$F(a_n) \leqslant F(a_0) + n\big(\sigma - \tau(a_{p_n})\big). \tag{9}$$

In a similar way, from (6) we can obtain

$$F\big(d(x_{n+1}, Tx_{n+1})\big) \leqslant F\big(d(x_0, Tx_0)\big) + n\big(\sigma - \tau(a_{p_n})\big). \tag{10}$$

Now consider the sequence $\{\tau(a_{p_n})\}$. We distinguish two cases.

Case 1. For each $n \in \mathbb{N}$, there is $m > n$ such that $\tau(a_{p_n}) > \tau(a_{p_m})$. Then we obtain a subsequence $\{a_{p_{n_k}}\}$ of $\{a_{p_n}\}$ with $\tau(a_{p_{n_k}}) > \tau(a_{p_{n_{k+1}}})$ for all k. Since $a_{p_{n_k}} \to \delta^+$, we deduce that

$$\liminf_{k\to\infty} \tau(a_{p_{n_k}}) > \sigma.$$

Hence, $F(a_{n_k}) \leqslant F(a_0) + n_k(\sigma - \tau(a_{p_{n_k}}))$ for all k. Consequently, $\lim_{k\to\infty} F(a_{n_k}) = -\infty$, and by (F2), $\lim_{k\to\infty} a_{p_{n_k}} = 0$, which contradicts that $\lim_{n\to\infty} a_n > 0$.

Case 2. There is $n_0 \in \mathbb{N}$ such that $\tau(a_{p_{n_0}}) = \tau(a_{p_m})$ for all $m > n_0$. Then $F(a_m) \leqslant F(a_0) + m(\sigma - \tau(a_{p_{n_0}}))$ for all $m > n_0$. Hence, $\lim_{m\to\infty} F(a_m) = -\infty$, so $\lim_{m\to\infty} a_m = 0$, which contradicts that $\lim_{m\to\infty} a_m > 0$. Thus, $\lim_{n\to\infty} a_n = 0$. From (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} a_n^k F(a_n) = 0.$$

By (9), the following holds for all $n \in \mathbb{N}$:

$$a_n^k F(a_n) - a_n^k F(a_0) \leqslant a_n^k n\big(\sigma - \tau(a_{p_n})\big) \leqslant 0. \tag{11}$$

Letting $n \to \infty$ in (11), we obtain that

$$\lim_{n\to\infty} n a_n^k = 0. \tag{12}$$

From (12) there exits $n_0 \in \mathbb{N}$ such that $n a_n^k \leqslant 1$ for all $n \geqslant n_0$. So, for all $n \geqslant n_0$, we have

$$a_n \leqslant \frac{1}{n^{1/k}}. \tag{13}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. Using the triangular inequality for the metric and from (13) we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty}(i^{-1/k})$, passing to limit $n, m \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$. On the other hand, from (10) and (F2) we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Since $x \to d(x, Tx)$ is lower semi-continuous, then

$$0 \leqslant d(z, Tz) \leqslant \liminf_{n\to\infty} d(x_n, Tx_n) = 0.$$

This is a contradiction. Hence, T has a fixed point. □

In the following example, we show that there are some multivalued maps such that our result can be applied, but Theorem 2 can not.

Example 1. Let $X = \{x_n = n(n+1)/2,\ n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define a mapping $T : X \to C(X)$ as

$$Tx = \begin{cases} \{x_1\}, & x = x_1, \\ \{x_1, x_{n-1}\}, & x = x_n. \end{cases}$$

Then, since τ_d is discrete topology, the map $x \to d(x, Tx)$ is continuous. Now we claim that condition (ii) of Theorem 2.1 of [11] is not satisfied. Indeed, let $x = x_n$ for $n > 1$, then $Tx = \{x_1, x_{n-1}\}$. In this case, for all $b \in (0,1)$, there exists $n_0(b) \in \mathbb{N}$ such that for all $n \geqslant n_0(b)$, $I_b^{x_n} = \{x_{n-1}\}$. Thus, for $n \geqslant n_0(b)$, we have

$$d(y, Ty) = n - 1, \qquad d(x,y) = n.$$

Therefore, since $d(y,Ty)/d(x,y) = (n-1)/n$, we can not find a function $\varphi : [0,\infty) \to [0,b)$ satisfying

$$d(y, Ty) \leqslant \varphi\bigl(d(x,y)\bigr) d(x,y).$$

Now we show that condition (ii) of Theorem 10 is satisfied with $F(\alpha) = \alpha + \ln\alpha$, $\sigma = 1/2$ and $\tau(t) = 1/t + 1/2$. Note that if $d(x,Tx) > 0$, then $x = x_n$ for $n > 1$. In this case, $d(x_n, Tx_n) = n$. Therefore, for $y = x_{n-1} \in Tx_n$, we have $y \in F_{1/2}^{x_n}$ and

$$\begin{aligned} \tau\bigl(d(x,y)\bigr) + F\bigl(d(y,Ty)\bigr) &= \tau(n) + F(n-1) \\ &= \frac{1}{n} + \frac{1}{2} + n - 1 + \ln(n-1) \\ &\leqslant n + \ln n = F(n) = F\bigl(d(x,Tx)\bigr). \end{aligned}$$

Remark 2. If we take $K(X)$ instead of $C(X)$ in Theorem 10, we can remove condition (F4) on F. Further, by taking into account Case 1, we can take $\sigma \geqslant 0$. Therefore, the proof of the following theorem is obvious.

Theorem 11. *Let (X,d) be a complete metric space and $T : X \to K(X)$. Assume that the following conditions hold*:

(i) *the map* $x \to d(x, Tx)$ *is lower semi-continuous*;
(ii) *there exists* $\sigma \geqslant 0$, $F \in \mathcal{F}$ *and a function* $\tau : (0, \infty) \to (\sigma, \infty)$ *such that*

$$\liminf_{t \to s^+} \tau(t) > \sigma \quad \text{for all } s \geqslant 0$$

and for any $x \in X$ *with* $d(x, Tx) > 0$, *there exists* $y \in F_\sigma^x$ *satisfying*

$$\tau\big(d(x, y)\big) + F\big(d(y, Ty)\big) \leqslant F\big(d(x, y)\big).$$

Then T *has a fixed point.*

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