Nonlinear Analysis: Modelling and Control, Vol. 21, No. 4, 547–563 ISSN 1392-5113
http://dx.doi.org/10.15388/NA.2016.4.8

Nabla derivatives associated with nonlinear control systems on homogeneous time scales

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Received: March 11, 2015 / **Revised:** January 13, 2016 / **Published online:** April 7, 2016

Abstract. The backward shift and nabla derivative operators, defined by the control system on homogeneous time scale, in vector spaces of one-forms and vector fields are introduced, and some of their properties are proven. In particular, the formulas for components of the backward shift and nabla derivative of an arbitrary vector field are presented.

Keywords: nonlinear system, time scale, vector field, one-form, nabla and delta derivatives.

1 Introduction

In papers [2, 3], an algebraic formalism for nonlinear control systems, defined on homogeneous time scales, has been developed. It has already found applications in the solution of several control problems like system reduction [17], realization of the external system description in the state space form [9]. Note that the properties of both the continuous- and discrete-time control systems can be studied, characterized and checked by the same mathematical techniques, theorems and algorithms. The formalism is based on differential one-forms and vector fields. In our set-up,the vector fields may have infinite

[1]The author supported by the Bialystok University of Technology grant No. S/WI/1/2016.
[2]Supported by the project financed under the Human Capital Operational Programme no KSI POKL.04.0.01-00-143/09 with funds from the European Social Fund and the Polish State budget.
[3]Supported by the Estonian Research Council grant PUT 481.
[4]Supported by the Bialystok University of Technology grant No. S/WM/1/2016.

number of terms. Though in the classical control theory the vector fields are defined in the finite-dimensional spaces, the infinite dimensional versions are not completely new. For instance, in [12], the infinite vector fields are considered. They have been also used in problems related to dynamic state feedback like in the studies of flatness [18]. An inversive σ_f-differential field $\mathcal{K}^*$ of meromorphic functions in system variables is constructed and equipped with two operators, delta derivative Δ_f and forward shift σ_f, defined by system equations. Then two vector spaces over $\mathcal{K}^*$, of one-forms and of vector fields, respectively, were introduced, and the operators Δ_f and σ_f were extended to these vector spaces.

The goal of this paper is to introduce the nabla derivative operator in these vector spaces and prove a number of its properties. Note that the literature on control systems on time scales is, up to now, mostly limited by application of delta derivatives, see, for example, [4, 8, 9]. However, some recent applications in economics, neural networks, control and other topics [1, 11, 14, 16] have suggested that in some situations nabla derivative is more natural. In particular, backward differences are preferable in numerical and computational methods due to practical implementation (information availability) reasons. Moreover, in control and neural networks, nabla derivative is a natural tool that allows to handle delays in system [13, 16]. In differential geometric approach to control problems, in majority of cases (though not always), nabla derivative of a vector field is much easier to compute than the delta derivative. Though the computation of both derivatives requires to apply the backward shift operator, defined by the inverse map of the extended system equations, in the case of delta derivative, the shift is applied to a one-form whereas in the case of nabla derivative, it is applied to a function. Since in the continuous-time case both delta and nabla derivatives coincide, they yield the same concept and whenever one assumes the finite-dimensional space, both result in classical Lie derivative along the vector field, defined by system equations (see Corollary 1 below). Therefore, it is natural to assume that several distributions, important for control theory, expressed in terms of Lie derivatives of vector fields (such as accessibility distribution) can be extended into the time-scale domain using either nabla or delta derivatives. Only further studies will reveal which of those derivatives are better suited in addressing different problems.

Some extensions of the Lie derivative of the vector field into the discrete-time domain were made in [19], but in this paper, the discrete-time system (i) was described in terms of the shift operator and not via difference operator as in this paper, (ii) the vectors were assumed to belong into the finite-dimensional spaces and (iii) though the extension was in fact nabla derivative, it was not recognized as such. In [19], two possible definitions of Lie derivative are recalled, both yielding in the same result in the continuous-time case, but when extended to discrete-time case, one of them will yield nabla derivative and the other delta derivative.

2 Time scale calculus

For a general introduction to the calculus on time scales, see [6, 7]. Here we give only those notions and facts that we need in our paper and most of them was taken from [6, 7].

The main task is to introduce the concept of derivatives for real functions defined on a time scale.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but also $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k \mid k \in \mathbb{Z}\} \cup \{0\}$ is a time scale. However, the set of rational numbers and the open interval (a, b), $a < b$, are not the examples of time scales. For $t \in \mathbb{T}$, the *forward* and *backward jump operators* $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} \mid s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} \mid s < t\}$, respectively. In addition, we set $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$. Since $\mathbb{T}$ is a closed subset of $\mathbb{R}$, both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. Finally, the *graininess* functions $\mu, \nu : \mathbb{T} \to [0, \infty)$ are defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$ for all $t \in \mathbb{T}$, respectively. A time scale is called *homogeneous* if μ and ν are constant functions.

Let $\mathbb{T}^\kappa$ denote truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point. The reason to omit a maximal left-scattered point is to guarantee uniqueness of f^Δ, defined below.

Definition 1. The *delta derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}^\kappa$ is the real number $f^\Delta(t)$ (provided it exists) such that for each $\varepsilon > 0$, there exists a neighborhood $U(\varepsilon)$ of t, $U(\varepsilon) \subset \mathbb{T}$, such that for all $\tau \in U(\varepsilon)$, $|(f(\sigma(t)) - f(\tau)) - f^\Delta(t)(\sigma(t) - \tau)| \leqslant \varepsilon|\sigma(t) - \tau|$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Let $\mathbb{T}_\kappa$ denote truncated set consisting of $\mathbb{T}$ except for a possible right-scattered minimal point. The reason to omit a maximal right-scattered point is to guarantee uniqueness of f^∇, defined below.

Definition 2. The *nabla derivative* of a function $f : \mathbb{T} \to \mathbb{R}$ at $t \in \mathbb{T}_\kappa$ is the real number $f^\nabla(t)$ (provided it exists) such that for each $\varepsilon > 0$, there exists a neighborhood $U(\varepsilon)$ of t, $U(\varepsilon) \subset \mathbb{T}$ such that for all $\tau \in U(\varepsilon)$, $|(f(\rho(t)) - f(\tau)) - f^\nabla(t)(\rho(t) - \tau)| \leqslant \varepsilon|\rho(t) - \tau|$. Moreover, we say that f is *nabla differentiable* on $\mathbb{T}_\kappa$ provided $f^\nabla(t)$ exists for all $t \in \mathbb{T}_\kappa$.

The delta and nabla derivatives of higher order are defined inductively. The kth-order delta derivative of function f is denoted by $f^{[k]}$, and the kth-order nabla derivative of function f is denoted by $f^{\{k\}}$.

For $f : \mathbb{T} \to \mathbb{R}$, we define $f^\sigma := f \circ \sigma : \mathbb{T} \to \mathbb{R}$ and $f^\rho := f \circ \rho : \mathbb{T} \to \mathbb{R}$ and call them respectively the *forward* and *backward time shift* of f. Denote $f^{\Delta\sigma} := (f^\Delta)^\sigma$, $f^{\sigma\Delta} := (f^\sigma)^\Delta$, $f^{\nabla\rho} := (f^\nabla)^\rho$, $f^{\rho\nabla} := (f^\rho)^\nabla$.

If f and f^Δ are delta differentiable functions, then for a homogeneous time scale, one has $f^{\sigma\Delta} = f^{\Delta\sigma}$. Similarly, if f and f^∇ are nabla differentiable functions, then for a homogeneous time scale, one has $f^{\rho\nabla} = f^{\nabla\rho}$.

Theorem 1. (*See* [6].) *Let* $f : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ *be two delta* (*nabla*) *differentiable functions defined on* $\mathbb{T}$*, and let* $t \in \mathbb{T}$*. Then*:

(i) f *is continuous at* t;
(ii) $f^\sigma = f + \mu f^\Delta$ *and* $f^\rho = f - \nu f^\nabla$;
(iii) $(\alpha f + \beta g)^\Delta = \alpha f^\Delta + \beta g^\Delta$ *and* $(\alpha f + \beta g)S^\nabla = \alpha f^\nabla + \beta g^\nabla$ *for constants* α *and* β;

(iv) $(fg)^{\Delta} = f^{\sigma}g^{\Delta} + f^{\Delta}g = fg^{\Delta} + f^{\Delta}g^{\sigma}$ *and* $(fg)^{\nabla} = f^{\rho}g^{\nabla} + f^{\nabla}g = fg^{\nabla} + f^{\nabla}g^{\rho}$;

(v) *if* $gg^{\sigma} \neq 0$, *then* $(f/g)^{\Delta} = (f^{\Delta}g - fg^{\Delta})/(gg^{\sigma})$, *and if* $gg^{\rho} \neq 0$, *then* $(f/g)^{\nabla} = (f^{\nabla}g - fg^{\nabla})/(gg^{\rho})$.

Proposition 1. (*See* [6, 7].)

(i) *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^{\kappa}$. *Then* f *is nabla differentiable at* t *and*

$$f^{\nabla}(t) = f^{\Delta}\big(\rho(t)\big) \tag{1}$$

for $t \in \mathbb{T}_{\kappa}$ *such that* $\sigma(\rho(t)) = t$. *If, in addition,* f^{Δ} *is continuous on* $\mathbb{T}^{\kappa}$, *then* f *is nabla differentiable at* t *and* (1) *holds for any* $t \in \mathbb{T}_{\kappa}$.

(ii) *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is nabla differentiable on* $\mathbb{T}_{\kappa}$. *Then* f *is delta differentiable at* t *and*

$$f^{\Delta}(t) = f^{\nabla}\big(\sigma(t)\big) \tag{2}$$

for $t \in \mathbb{T}^{\kappa}$ *such that* $\rho(\sigma(t)) = t$. *If, in addition,* f^{∇} *is continuous on* $\mathbb{T}_{\kappa}$, *then* f *is delta differentiable at* t *and* (2) *holds for any* $t \in \mathbb{T}^{\kappa}$.

3 Differential fields

Let us recall the facts for a σ_f-differential field given in [2] and introduce the concept of a ρ_f-differential field.

From now on we assume that $\mathbb{T}$ is homogeneous. Note that for homogeneous time scales, $\nu = \mu = \mathrm{const}$. Consider now the control system, defined on $\mathbb{T}$,

$$x^{\Delta}(t) = f\big(x(t), u(t)\big), \tag{3}$$

where $(x(t), u(t)) \in U$, U is an open subset of $\mathbb{R}^n \times \mathbb{R}^m$, $m \leqslant n$, x is a state, u is a control (input) of the system and function $f : U \to \mathbb{R}^n$ is analytic. Let us define $\tilde{f}(x, u) := x + \mu f(x, u)$ and assume that there exists a map $\varphi : U \to \mathbb{R}^m$ such that $\Phi = (\tilde{f}, \varphi)^{\mathrm{T}}$ is an analytic diffeomorphism[5] from the set U onto U. This means that from $(\bar{x}, z) = (\tilde{f}(x, u), \varphi(x, u)) = \Phi(x, u)$ we can uniquely compute (x, u) as an analytic function of $(\bar{x}, z)$. For $\mu = 0$, this condition is always satisfied with $\varphi(x, u) = u$. In the case $\mu > 0$, system (3) can be rewritten in the following equivalent form:

$$x^{\sigma}(t) = \tilde{f}\big(x(t), u(t)\big). \tag{4}$$

For notational convenience, $(x_1, \ldots, x_n)$ will simply be written as x, and for $k \geqslant 0$, $(u_1^{[k]}, \ldots, u_m^{[k]})$ will be written as $u^{[k]}$. For $i \leqslant k$, let $u^{[i \ldots k]} := (u^{[i]}, \ldots, u^{[k]})$. We assume that the control (input) applied to system (3) is infinitely many times delta differentiable,

[5]This assumption guarantees that the system $x^{\sigma} = \tilde{f}(x, u)$ is submersive, i.e. generically $\mathrm{rank}(\partial \tilde{f}(x, u)/\partial(x, u)) = n$.

i.e. $u^{[0\dots k]}$ exists for all $k \geqslant 0$. Consider the infinite set of real (independent) indeterminates

$$\mathcal{C} = \{x_i,\ i = 1,\dots,n,\ u_j^{[k]},\ j = 1,\dots,m,\ k \geqslant 0\}, \tag{5}$$

and let $\mathcal{K}$ be the (commutative) field of meromorphic functions in a finite number of the variables from the set $\mathcal{C}$. Let $\sigma_f : \mathcal{K} \to \mathcal{K}$ be an operator defined by

$$\sigma_f(F)\big(x, u^{[0\dots k+1]}\big) := F\big(x + \mu f(x,u),\ u^{[0\dots k]} + \mu u^{[1\dots k+1]}\big), \tag{6}$$

where $F \in \mathcal{K}$ depends on x and $u^{[0\dots k]}$. For particular cases when F, F_k, $k \geqslant 0$, are coordinate functions, i.e. $F(x) = x$, $F_k(u^{[k]}) = u^{[k]}$, $k \geqslant 0$, we get $\sigma_f(x) = x + \mu f(x,u) = \tilde{f}(x,u)$, $\sigma_f(u^{[k]}) = u^{k]} + \mu u^{[k+1]}$, $k \geqslant 0$. We assume that $(x,u) \in U$ and the other variables are restricted in such a way that σ_f is well defined. Under the assumption about the existence of φ such that $\Phi = (\tilde{f}, \varphi)$ is an analytic diffeomorphism, σ_f is injective endomorphism.

The field $\mathcal{K}$ can be equipped with a delta derivative operator $\Delta_f : \mathcal{K} \to \mathcal{K}$ defined by

$$\Delta_f(F)\big(x, u^{[0\dots k+1]}\big) = \begin{cases} \frac{1}{\mu}[F(x + \mu f(x,u),\ u^{[0\dots k]} + \mu u^{[1\dots k+1]}) - F(x, u^{[0\dots k]})] & \text{if } \mu \neq 0, \\ \frac{\partial F}{\partial x}(x, u^{[0\dots k]}) f(x,u) + \sum_{k\geqslant 0} \frac{\partial F}{\partial u^{[0\dots k]}}(x, u^{[0\dots k]}) u^{[1\dots k+1]} & \text{if } \mu = 0, \end{cases} \tag{7}$$

where $F \in \mathcal{K}$ depends on x and $u^{[0\dots k]}$.

The more compact notations F^{σ_f} and F^{Δ_f} will be sometimes used instead of $\sigma_f(F)$ and $\Delta_f(F)$. Note that when x is considered as the function defined on the homogeneous time scale $\mathbb{T}$, then the notations x^σ and x^Δ will be used to denote the functions $x^\sigma : \mathbb{T} \to \mathbb{R}^n$ and $x^\Delta : \mathbb{T} \to \mathbb{R}^n$ such that $x^\Delta(\cdot) = f(x(\cdot), u(\cdot))$ and $x^\sigma(\cdot) = x(\cdot) + \mu f(x(\cdot), u(\cdot))$. Whereas the notations x^{σ_f} and x^{Δ_f} are used for the image of coordinate function $F(x) = x$ with respect to operators σ_f and Δ_f. Similarly, in the case $u^{[k]}$, $k \geqslant 0$.

The delta derivative Δ_f satisfies, for all $F, G \in \mathcal{K}$, the conditions:

(i) $\Delta_f(F + G) = \Delta_f(F) + \Delta_f(G)$,
(ii) $\Delta_f(FG) = \Delta_f(F)G + \sigma_f(F)\Delta_f(G)$ (generalized Leibniz rule).

An operator satisfying the generalized Leibniz rule is called a *"σ_f-derivation"*, and a commutative field endowed with a σ_f-derivation is called a *σ_f-differential field* [10]. Therefore, under the assumption about the existence of φ such that $\Phi = (\tilde{f}, \varphi)$ is an analytic diffeomorphism, $\mathcal{K}$ endowed with the delta derivative Δ_f is a σ_f-differential field. For $\mu = 0$, $\sigma_f = \sigma_f^{-1} = \mathrm{id}$ and $\mathcal{K}$ is inversive. Though $\mathcal{K}$ is not inversive in general, it is always possible to embed $\mathcal{K}$ into an inversive σ_f-differential overfield $\mathcal{K}^*$ called the *inversive closure* of $\mathcal{K}$ [10]. Since σ_f is an injective endomorphism, it can be extended to $\mathcal{K}^*$, so that $\sigma_f : \mathcal{K}^* \to \mathcal{K}^*$ is an automorphism.

Let $\rho_f : \mathcal{K}^* \to \mathcal{K}^*$ be an operator defined by

$$\rho_f := \sigma_f^{-1}. \tag{8}$$

Hence one gets $\rho_f(F) = F \circ \rho_f$ for $F \in \mathcal{K}^*$. It was shown in [3] that, for $\mu \neq 0$, the inversive closure of $\mathcal{K}$ may be constructed as the field of meromorphic functions in a finite number of the independent variables $\mathcal{C}^* = \mathcal{C} \cup \{z_s^{\langle -\ell \rangle},\ s = 1, \dots, m,\ \ell \geqslant 1\}$, where the new variables are related by σ_f as follows: $z_i^{\langle -k \rangle} = \sigma_f(z_i^{\langle -k-1 \rangle})$ and $z_i = \varphi_i(x, u) = \sigma_f(z_i^{\langle -1 \rangle})$.

Let $z := (z_1, \dots, z_m)$. Then $(\rho_f(x), \rho_f(u)) = \Psi(x, z^{\langle -1 \rangle})$, where Ψ is a certain vector valued function, determined by f in (3), and the extension $z = \varphi(x, u)$. Although the choice of variables z is not unique, all possible choices yield isomorphic field extensions. We extend the operator Δ_f to new variables by

$$\Delta_f(z^{\langle -\ell \rangle}) := \frac{z^{\langle -\ell+1 \rangle} - z^{\langle -\ell \rangle}}{\mu}, \quad l \geqslant 1.$$

The extension of operator Δ_f to $\mathcal{K}^*$ can be made in analogy to (7). Such operator Δ_f is now a σ_f-derivation of $\mathcal{K}^*$. A practical procedure for construction of $\mathcal{K}^*$ (for $\mu \neq 0$) is given in [3].

Additionally, the field $\mathcal{K}^*$ can be equipped with a nabla derivative operator $\nabla_f : \mathcal{K}^* \to \mathcal{K}^*$ defined by

$$\nabla_f(F) := \begin{cases} \frac{1}{\mu}[F - \rho_f(F)] & \text{if } \mu \neq 0, \\ \Delta_f(F) & \text{if } \mu = 0, \end{cases} \tag{9}$$

where $F \in \mathcal{K}^*$.

Note that for $\mu > 0$, we get

$$\begin{aligned} (\rho_f \circ \Delta_f)(F) &= \rho_f\left(\frac{1}{\mu}\big[\sigma_f(F) - F\big]\right) = \frac{1}{\mu}\big[F - \rho_f(F)\big] = \nabla_f(F), \\ (\Delta_f \circ \rho_f)(F) &= \Delta_f\big(\rho_f(F)\big) = \frac{1}{\mu}\big[\sigma_f\big(\rho_f(F)\big) - \rho_f(F)\big] \\ &= \frac{1}{\mu}\big[F - \rho_f(F)\big] = \nabla_f(F) \end{aligned}$$

and for $\mu = 0$, $\rho_f = \mathrm{id}$, so on homogeneous time scales, we get the following relation between the operators Δ_f and ∇_f:

$$\nabla_f = \rho_f \circ \Delta_f = \Delta_f \circ \rho_f. \tag{10}$$

Moreover, applying the operator σ_f to (10), one gets

$$\sigma_f \circ \nabla_f = \nabla_f \circ \sigma_f = \Delta_f. \tag{11}$$

The nabla derivative ∇_f satisfies, for all $F, G \in \mathcal{K}$, the conditions:

(i) $\nabla_f(F + G) = \nabla_f(F) + \nabla_f(G)$,
(ii) $\nabla_f(FG) = \nabla_f(F)G + \rho_f(F)\nabla_f(G)$ (generalized Leibniz rule).

Therefore, the operator ∇_f is a "ρ_f-derivation", and the commutative field $\mathcal{K}^*$ endowed with a ρ_f-derivation is called a ρ_f-*differential field* [10].

Similarly as in the case of the operators σ_f and Δ_f, the more compact notations F^{ρ_f} and F^{∇_f} will be sometimes used instead of $\rho_f(F)$ and $\nabla_f(F)$.

From now on

$$\mathcal{C}^* = \begin{cases} \mathcal{C} & \text{if } \mu = 0, \\ \mathcal{C} \cup \{z^{\langle -\ell\rangle} \mid \ell \geqslant 1\} & \text{if } \mu \neq 0. \end{cases}$$

Consider the infinite set of differentials of indeterminates $\mathrm{d}\mathcal{C}^* = \{\mathrm{d}\zeta_i,\, \zeta_i \in \mathcal{C}^*\}$ and define $\mathcal{E} := \operatorname{span}_{\mathcal{K}^*} \mathrm{d}\mathcal{C}^*$. Any element of $\mathcal{E}$ is a vector of the form

$$\omega = \sum_{\ell \geqslant 1} \sum_{s=1}^{m} C_{s\ell}\, \mathrm{d}z_s^{\langle -\ell\rangle} + \sum_{i=1}^{n} A_i\, \mathrm{d}x_i + \sum_{k \geqslant 0} \sum_{j=1}^{m} B_{jk}\, \mathrm{d}u_j^{[k]},$$

where only a finite number of coefficients B_{jk} and $C_{s\ell}$ are nonzero elements of $\mathcal{K}^*$.

The elements of $\mathcal{E}$ are called differential *one-forms*. Let $\mathrm{d} : \mathcal{K}^* \to \mathcal{E}$ be defined in the standard manner:

$$\mathrm{d}F := \sum_{\ell \geqslant 1} \sum_{s=1}^{m} \frac{\partial F}{\partial z_s^{\langle -\ell\rangle}}\, \mathrm{d}z_s^{\langle -\ell\rangle} + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, \mathrm{d}x_i + \sum_{k \geqslant 0} \sum_{j=1}^{m} \frac{\partial F}{\partial u_j^{[k]}}\, \mathrm{d}u_j^{[k]}. \tag{12}$$

One says that $\omega \in \mathcal{E}$ is an *exact one-form* if $\omega = \mathrm{d}F$ for some $F \in \mathcal{K}^*$. We will refer to $\mathrm{d}F$ as to the *total differential* (or simply the *differential*) of F.

If $\omega = \sum_i A_i \mathrm{d}\zeta_i$ is a one-form, where $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}^*$, one can define the operators $\Delta_f : \mathcal{E} \to \mathcal{E}$ and $\sigma_f : \mathcal{E} \to \mathcal{E}$ by

$$\Delta_f(\omega) := \sum_i \bigl\{ \Delta_f(A_i)\, \mathrm{d}\zeta_i + \sigma_f(A_i)\, \mathrm{d}\bigl[\Delta_f(\zeta_i)\bigr] \bigr\} \tag{13}$$

and

$$\sigma_f(\omega) := \sum_i \sigma_f(A_i)\, \mathrm{d}\bigl[\sigma_f(\zeta_i)\bigr]. \tag{14}$$

By (13) and (14) we get

$$\sigma_f(\omega) = \omega + \mu \Delta_f(\omega).$$

Additionally, if

$$\omega = \sum_{\ell \geqslant 1} \sum_{s=1}^{m} \omega_{z_s^{\langle -\ell\rangle}}\, \mathrm{d}z_s^{\langle -\ell\rangle} + \sum_{i=1}^{n} \omega_{x_i}\, \mathrm{d}x_i + \sum_{k \geqslant 0} \sum_{j=1}^{m} \omega_{u_j^{[k]}}\, \mathrm{d}u_j^{[k]},$$

then we have

$$\bigl(\omega^{\Delta_f}\bigr)_{z_s^{\langle -\ell\rangle}} = \bigl(\omega_{z_s^{\langle -\ell\rangle}}\bigr)^{\Delta_f}, \tag{15a}$$

$$\bigl(\omega^{\Delta_f}\bigr)_{x_i} = \frac{1}{\mu} \sum_{s=1}^{m} \bigl(\omega_{z_s^{\langle -1\rangle}}\bigr)^{\sigma_f} \frac{\partial \varphi_s}{\partial x_i} + (\omega_{x_i})^{\Delta_f} + \sum_{k=1}^{n} (\omega_{x_k})^{\sigma_f} \frac{\partial f_k}{\partial x_i}, \tag{15b}$$

$$\left(\omega^{\Delta_f}\right)_{u_j} = \frac{1}{\mu}\sum_{s=1}^{m}(\omega_{z_s^{\langle -1\rangle}})^{\sigma_f}\frac{\partial \varphi_s}{\partial u_j} + (\omega_{u_j})^{\Delta_f} + \sum_{k=1}^{n}(\omega_{x_k})^{\sigma_f}\frac{\partial f_k}{\partial u_j}, \tag{15c}$$

$$\left(\omega^{\Delta_f}\right)_{u_j^{[k]}} = (\omega_{u_j^{[k]}})^{\Delta_f} + (\omega_{u_j^{[k-1]}})^{\sigma_f}, \quad k \geqslant 1, \tag{15d}$$

where $\Delta_f(\omega) = \sum_i (\omega^{\Delta_f})_{\zeta_i}\,\mathrm{d}\zeta_i$, $\zeta_i \in \mathcal{C}^*$, $(\varphi_1, \dots, \varphi_m)^{\mathrm{T}} = \varphi$ and $(f_1, \dots, f_n)^{\mathrm{T}} = f$, and

$$\left(\omega^{\sigma_f}\right)_{z_s^{\langle -\ell\rangle}} = (\omega_{z_s^{\langle -\ell\rangle}})^{\sigma_f}, \tag{16a}$$

$$\left(\omega^{\sigma_f}\right)_{x_i} = \sum_{s=1}^{m}(\omega_{z_s^{\langle -1\rangle}})^{\sigma_f}\frac{\partial \varphi_s}{\partial x_i} + (\omega_{x_i})^{\sigma_f} + \mu\sum_{k=1}^{n}(\omega_{x_k})^{\sigma_f}\frac{\partial f_k}{\partial x_i}, \tag{16b}$$

$$\left(\omega^{\sigma_f}\right)_{u_j} = \sum_{s=1}^{m}(\omega_{z_s^{\langle -1\rangle}})^{\sigma_f}\frac{\partial \varphi_s}{\partial u_j} + (\omega_{u_j})^{\sigma_f} + \mu\sum_{k=1}^{n}(\omega_{x_k})^{\sigma_f}\frac{\partial f_k}{\partial u_j}, \tag{16c}$$

$$\left(\omega^{\sigma_f}\right)_{u_j^{[k]}} = (\omega_{u_j^{[k]}})^{\sigma_f} + \mu(\omega_{u_j^{[k-1]}})^{\sigma_f}, \quad k \geqslant 1, \tag{16d}$$

where $\sigma_f(\omega) = \sum_i (\omega^{\sigma_f})_{\zeta_i}\,\mathrm{d}\zeta_i$, $\zeta_i \in \mathcal{C}^*$.

The operator $\sigma_f : \mathcal{E} \to \mathcal{E}$ is invertible, and the inverse operator $\rho_f := \sigma_f^{-1} : \mathcal{E} \to \mathcal{E}$ is defined by

$$\rho_f\left(\sum_i A_i\,\mathrm{d}\zeta_i\right) := \sum_i \rho_f(A_i)\,\mathrm{d}\big[\rho_f(\zeta_i)\big] \tag{17}$$

for $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}^*$. Moreover, one can define $\nabla_f : \mathcal{E} \to \mathcal{E}$ by

$$\nabla_f(\omega) := \sum_i \big\{\nabla_f(A_i)\,\mathrm{d}\zeta_i + \rho_f(A_i)\,\mathrm{d}\big[\nabla_f(\zeta_i)\big]\big\}. \tag{18}$$

By (17) and (18) we get

$$\rho_f(\omega) = \omega - \mu\nabla_f(\omega). \tag{19}$$

Since (19) holds, then

$$\begin{aligned}
\nabla_f(\omega) &= \sum_i \big\{\nabla_f(A_i)\,\mathrm{d}\zeta_i + \big(A_i - \mu\nabla_f(A_i)\big)\,\mathrm{d}\big[\nabla_f(\zeta_i)\big]\big\}, \\
&= \sum_i \big\{\nabla_f(A_i)\,\mathrm{d}\big[\zeta_i - \mu\nabla_f(\zeta_i)\big] + A_i\,\mathrm{d}\big[\nabla_f(\zeta_i)\big]\big\}, \\
&= \sum_i \big\{\nabla_f(A_i)\,\mathrm{d}\big[\rho_f(\zeta_i)\big] + A_i\,\mathrm{d}\big[\nabla_f(\zeta_i)\big]\big\}.
\end{aligned}$$

Note that, in order to find the components of ω^{ρ_f} and ω^{∇_f}, we have to know the formula for inverse of the map Φ. Then $(\rho_f(x), \rho_f(u)) = \Phi^{-1}(x, z^{\langle -1\rangle})$ and for $k \geqslant 2$, we have $\rho_f(u^{[k]}) = (u^{[k-1]} - \rho_f(u^{[k-1]}))/\mu = \nabla_f(u^{[k-1]})$, so, in the general case, one cannot give the explicit formulas for components as it was given for components of ω^{Δ_f} and ω^{σ_f} in (15) and (16).

Proposition 2. *Let $\mathbb{T}$ be the homogeneous time scale and $F \in \mathcal{K}^*$. Then*

$$\Delta_f(\mathrm{d}F) = \mathrm{d}\big[\Delta_f(F)\big], \tag{20a}$$

$$\sigma_f(\mathrm{d}F) = \mathrm{d}\big[\sigma_f(F)\big], \tag{20b}$$

$$\nabla_f(\mathrm{d}F) = \mathrm{d}\big[\nabla_f(F)\big], \tag{20c}$$

$$\rho_f(\mathrm{d}F) = \mathrm{d}\big[\rho_f(F)\big]. \tag{20d}$$

Proof. Formulas (20a) and (20b) have been shown in [3]. Then by (10) and (20b) we get

$$\mathrm{d}\big[\rho_f(F)\big] = (\rho_f \circ \sigma_f)\,\mathrm{d}\big[\rho_f(F)\big] = \rho_f\big(\mathrm{d}\big[(\sigma_f \circ \rho_f)(F)\big]\big) = \rho_f(\mathrm{d}F),$$

so (20d) holds. Next, by (10) and (20a) we have

$$\nabla_f(\mathrm{d}F) = (\rho_f \circ \Delta_f)(\mathrm{d}F) = \rho_f\big(\mathrm{d}\big[\Delta_f(F)\big]\big) = \mathrm{d}\big[(\rho_f \circ \Delta_f)(F)\big] = \mathrm{d}\big[\nabla_f(F)\big],$$

so (20c) also holds. □

For one-forms, similarly as for functions, the more compact notations ω^{Δ_f} and ω^{σ_f}, ω^{∇_f} and ω^{ρ_f} will be used instead of $\Delta_f(\omega)$ and $\sigma_f(\omega)$, $\nabla_f(\omega)$ and $\rho_f(\omega)$, respectively. The relations between Δ_f and ∇_f operators given for functions from $\mathcal{K}^*$ in (11) and (10) hold also for one-forms.

Proposition 3. *Let $\mathbb{T}$ be the homogeneous time scale and $\omega \in \mathcal{E}$. Then*

$$\big(\omega^{\nabla_f}\big)^{\sigma_f} = \omega^{\Delta_f}, \tag{21a}$$

$$\big(\omega^{\sigma_f}\big)^{\nabla_f} = \omega^{\Delta_f}, \tag{21b}$$

$$\big(\omega^{\Delta_f}\big)^{\rho_f} = \omega^{\nabla_f}, \tag{21c}$$

$$\big(\omega^{\rho_f}\big)^{\Delta_f} = \omega^{\nabla_f}. \tag{21d}$$

Proof. Note that if $\omega = \sum_i A_i\,\mathrm{d}\zeta_i$, $A_i \in \mathcal{K}^*$ and $\zeta_i \in \mathcal{C}^*$, then by the previous properties we have

$$\begin{aligned}
\big(\omega^{\nabla_f}\big)^{\sigma_f} &= \sigma_f\biggl(\sum_i \big\{\nabla_f(A_i)\,\mathrm{d}\zeta_i + \rho_f(A_i)\,\mathrm{d}\big[\nabla_f(\zeta_i)\big]\big\}\biggr)\\
&= \biggl(\sum_i \big\{(\sigma_f \circ \nabla_f)(A_i)\,\mathrm{d}\big[\sigma_f(\zeta_i)\big] + (\sigma_f \circ \rho_f)(A_i)\,\mathrm{d}\big[(\sigma_f \circ \nabla_f)(\zeta_i)\big]\big\}\biggr)\\
&= \biggl(\sum_i \big\{\Delta_f(A_i)\,\mathrm{d}\big(\zeta_i + \mu\Delta_f(\zeta_i)\big) + A_i\,\mathrm{d}\big[\Delta_f(\zeta_i)\big]\big\}\biggr)\\
&= \biggl(\sum_i \big\{\Delta_f(A_i)\,\mathrm{d}\zeta_i + \sigma_f(A_i)\,\mathrm{d}\big[\Delta_f(\zeta_i)\big]\big\}\biggr) = \omega^{\Delta_f}.
\end{aligned} \tag{22}$$

Additionally, similarly as in (22), one can prove that (21b), (21c) and (21d) hold. □

4 The space of vector fields

Let $\mathcal{E}'$ be the dual vector space of $\mathcal{E}$, i.e. the space of linear mappings from $\mathcal{E}$ to $\mathcal{K}^*$. The elements of $\mathcal{E}'$ are called the vector fields, and they are of the form

$$X = \sum_{\ell \geqslant 1} \sum_{s=1}^{m} X_{z_s^{\langle -\ell \rangle}} \frac{\partial}{\partial z_s^{\langle -\ell \rangle}} + \sum_{i=1}^{n} X_{x_i} \frac{\partial}{\partial x_i} + \sum_{k \geqslant 0} \sum_{j=1}^{m} X_{u_j^{[k]}} \frac{\partial}{\partial u_j^{[k]}}, \tag{23}$$

where $X_{z_s^{\langle -\ell \rangle}}, X_{x_i}, X_{u_j^{[k]}} \in \mathcal{K}^*$. Taking $\omega = \sum_{\ell=1}^{q} \sum_{s=1}^{m} C_{s\ell} \, \mathrm{d}z_s^{\langle -\ell \rangle} + \sum_{i=1}^{n} A_i \, \mathrm{d}x_i + \sum_{k=0}^{p} \sum_{j=1}^{m} B_{jk} \, \mathrm{d}u_j^{[k]} \in \mathcal{E}$ and the vector field $X \in \mathcal{E}'$ of the form (23), we get

$$\langle \omega, X \rangle := \sum_{\ell=1}^{q} \sum_{s=1}^{m} X_{z_s^{\langle -\ell \rangle}} C_{s\ell} + \sum_{i=1}^{n} X_{x_i} A_i + \sum_{k=0}^{p} \sum_{j=1}^{m} X_{u_j^{[k]}} B_{jk}. \tag{24}$$

Note that though the (formal) sum (23) is infinite, $\langle \omega, X \rangle$ is always a sum with only finitely many nonzero terms in (24).

The delta-derivative X^{Δ_f} and forward-shift X^{σ_f} of $X \in \mathcal{E}'$ may be defined uniquely by the equations

$$\langle \omega, X^{\Delta_f} \rangle = \langle \rho_f(\omega), X \rangle^{\Delta_f} - \langle [\rho_f(\omega)]^{\Delta_f}, X \rangle \tag{25}$$

and

$$\langle \omega, X^{\sigma_f} \rangle = \langle \rho_f(\omega), X \rangle^{\sigma_f}, \tag{26}$$

respectively, where ω is an arbitrary one-form. Note that $\langle \rho_f(\omega), X \rangle \in \mathcal{K}^*$, so $\langle \rho_f(\omega), X \rangle^{\sigma_f}$ and $\langle \rho_f(\omega), X \rangle^{\Delta_f}$ are well defined.

Remark 1. Note that for $\mu = 0$, we have $\sigma_f = \rho_f = \mathrm{id}$ and

$$X = \sum_{i=1}^{n} X_{x_i} \frac{\partial}{\partial x_i} + \sum_{k \geqslant 0} \sum_{j=1}^{m} X_{u_j^{[k]}} \frac{\partial}{\partial u_j^{[k]}},$$

where $X_{x_i}, X_{u_j^{[k]}} \in \mathcal{K}^*$, so in (23), $X_{z_s^{\langle -\ell \rangle}} = 0$ for $s = 1, \dots, m$ and $\ell \geqslant 1$.

In [2], we show that evaluating (25) and (26) with the elements of canonical basis (i.e. with the elements from the set $\mathrm{d}\mathcal{C}^*$), we obtain two sets of explicit formulas that define the coefficients of X^{Δ_f} and X^{σ_f}, respectively. Note that, in order to find the components of the vector field X^{Δ_f} and X^{σ_f}, we have to know the formula for Φ^{-1}.

Proposition 4. (*See* [2].) *Let $X \in \mathcal{E}'$. Then for arbitrary $\omega \in \mathcal{E}$,*

$$X^{\sigma_f} = X + \mu X^{\Delta_f}, \tag{27}$$

$$\langle \omega, X \rangle^{\Delta_f} = \langle \omega, X^{\Delta_f} \rangle + \langle \omega^{\Delta_f}, X^{\sigma_f} \rangle. \tag{28}$$

Note that for $\mu = 0$, formula (27) gives $X^{\sigma_f} = X$, so $\sigma_f = \mathrm{id}$.

Now let us define the backward-shift and nabla-derivative of the vector field X. The backward-shift X^{ρ_f} and nabla-derivative X^{∇_f} of $X \in \mathcal{E}'$ may be defined uniquely by the equations

$$\langle \omega, X^{\rho_f} \rangle = \langle \omega^{\sigma_f}, X \rangle^{\rho_f} \tag{29}$$

and

$$\langle \omega, X^{\nabla_f} \rangle = \langle \omega^{\sigma_f}, X \rangle^{\nabla_f} - \langle \omega^{\Delta_f}, X \rangle, \tag{30}$$

respectively, where ω is an arbitrary one-form.

Proposition 5. *Let $X \in \mathcal{E}'$. Then $X^{\rho_f}, X^{\nabla_f} \in \mathcal{E}'$ and*

$$X^{\rho_f} = X - \mu X^{\nabla_f}, \tag{31}$$

$$\langle \omega, X \rangle^{\nabla_f} = \langle \omega^{\rho_f}, X^{\nabla_f} \rangle + \langle \omega^{\nabla_f}, X \rangle, \tag{32}$$

$$\langle \omega, X \rangle^{\nabla_f} = \langle \omega, X^{\nabla_f} \rangle + \langle \omega^{\nabla_f}, X^{\rho_f} \rangle. \tag{33}$$

Proof. Let $X \in \mathcal{E}'$. Note that $\langle \omega, X \rangle \in \mathcal{K}^*$ for arbitrary $\omega \in \mathcal{E}$ and

$$\langle \omega, X \rangle^{\rho_f} = \langle \omega, X \rangle - \mu \langle \omega, X \rangle^{\nabla_f}.$$

Then by (29) and (30), for arbitrary $\omega \in \mathcal{E}$, we get

$$\begin{aligned} \langle \omega, X^{\rho_f} \rangle &= \langle \omega^{\sigma_f}, X \rangle^{\rho_f} = \langle \omega^{\sigma_f}, X \rangle - \mu \big[\langle \omega, X^{\nabla_f} \rangle + \langle \omega^{\Delta_f}, X \rangle \big] \\ &= \langle \omega^{\sigma_f} - \mu \omega^{\Delta_f}, X \rangle - \mu \langle \omega, X^{\nabla_f} \rangle = \langle \omega, X \rangle - \mu \langle \omega, X^{\nabla_f} \rangle \\ &= \langle \omega, X - \mu X^{\nabla_f} \rangle. \end{aligned} \tag{34}$$

Hence by (34) the relation (31) holds.

Moreover, from (30) we get

$$\langle \omega, X \rangle^{\nabla_f} = \langle \omega^{\rho_f}, X^{\nabla_f} \rangle + \langle (\omega^{\rho_f})^{\Delta_f}, X \rangle = \langle \omega^{\rho_f}, X^{\nabla_f} \rangle + \langle \omega^{\nabla_f}, X \rangle$$

for arbitrary $\omega \in \mathcal{E}$. Therefore, (32) holds. By (19) we get

$$\langle \omega, X \rangle^{\nabla_f} = \langle \omega, X^{\nabla_f} \rangle + \langle \omega^{\nabla_f}, X \rangle - \mu \langle \omega^{\nabla_f}, X^{\nabla_f} \rangle$$

and, consequently, $\langle \omega, X \rangle^{\nabla_f} = \langle \omega, X^{\nabla_f} \rangle + \langle \omega^{\nabla_f}, X - \mu X^{\nabla_f} \rangle$, so from (31) formula (33) holds. □

Proposition 6. *Let $X \in \mathcal{E}'$. Then*

$$(X^{\sigma_f})^{\rho_f} = (X^{\rho_f})^{\sigma_f} = X, \tag{35}$$

$$(X^{\nabla_f})^{\sigma_f} = (X^{\sigma_f})^{\nabla_f} = X^{\Delta_f}, \tag{36}$$

$$(X^{\Delta_f})^{\rho_f} = (X^{\rho_f})^{\Delta_f} = X^{\nabla_f}. \tag{37}$$

Proof. Let $X \in \mathcal{E}'$. Then for arbitrary $\omega \in \mathcal{E}$, we get

$$\left\langle \omega, \left(X^{\sigma_f}\right)^{\rho_f} \right\rangle = \left\langle \omega^{\sigma_f}, X^{\sigma_f} \right\rangle^{\rho_f} = (\rho_f \circ \sigma_f)\langle \omega, X\rangle = \langle \omega, X\rangle$$

and

$$\left\langle \omega, \left(X^{\rho_f}\right)^{\sigma_f} \right\rangle = \left\langle \omega^{\rho_f}, X^{\rho_f} \right\rangle^{\sigma_f} = (\sigma_f \circ \rho_f)\langle \omega, X\rangle = \langle \omega, X\rangle.$$

Therefore, (35) holds.

Additionally, by (26), (28), (30) and (11) we have

$$\begin{aligned}\left\langle \omega, \left(X^{\nabla_f}\right)^{\sigma_f} \right\rangle &= \left\langle \omega^{\rho_f}, X^{\nabla_f} \right\rangle^{\sigma_f} = \left[\langle \omega, X\rangle^{\nabla_f} - \left\langle \left(\omega^{\rho_f}\right)^{\Delta_f}, X\right\rangle\right]^{\sigma_f} \\ &= \langle \omega, X\rangle^{\Delta_f} - \left\langle \omega^{\nabla_f}, X\right\rangle^{\sigma_f} = \langle \omega, X\rangle^{\Delta_f} - \left\langle \omega^{\Delta_f}, X^{\sigma_f}\right\rangle \\ &= \left\langle \omega, X^{\Delta_f}\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle \omega, \left(X^{\sigma_f}\right)^{\nabla_f} \right\rangle &= \left\langle \omega^{\sigma_f}, X^{\sigma_f} \right\rangle^{\nabla_f} - \left\langle \omega^{\Delta_f}, X^{\sigma_f}\right\rangle \\ &= \langle \omega, X\rangle^{\Delta_f} - \left\langle \omega^{\Delta_f}, X^{\sigma_f}\right\rangle = \left\langle \omega, X^{\Delta_f}\right\rangle\end{aligned}$$

for arbitrary $\omega \in \mathcal{E}$, so (36) holds. Similarly, by (25), (29), (33) and (10) we have

$$\begin{aligned}\left\langle \omega, \left(X^{\Delta_f}\right)^{\rho_f} \right\rangle &= \left\langle \omega^{\sigma_f}, X^{\Delta_f} \right\rangle^{\rho_f} = \left[\langle \omega, X\rangle^{\Delta_f} - \left\langle \omega^{\Delta_f}, X\right\rangle\right]^{\rho_f} \\ &= \langle \omega, X\rangle^{\nabla_f} - \left\langle \omega^{\Delta_f}, X\right\rangle^{\rho_f} = \langle \omega, X\rangle^{\nabla_f} - \langle \omega^{\nabla_f}, X^{\rho_f}\rangle \\ &= \left\langle \omega, X^{\nabla_f}\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle \omega, \left(X^{\rho_f}\right)^{\Delta_f} \right\rangle &= \left\langle \omega^{\rho_f}, X^{\rho_f} \right\rangle^{\Delta_f} - \left\langle \omega^{\nabla_f}, X^{\rho_f}\right\rangle = \langle \omega, X\rangle^{\nabla_f} - \left\langle \omega^{\nabla_f}, X^{\rho_f}\right\rangle \\ &= \left\langle \omega, X^{\nabla_f}\right\rangle\end{aligned}$$

for arbitrary $\omega \in \mathcal{E}$. Therefore, (37) holds. □

Now let us present the formulas for components of the backward shift X^{ρ_f} and nabla derivative X^{∇_f} of an arbitrary vector field X.

Proposition 7. *For $\mu > 0$, we have*

$$\begin{aligned}X &= \sum_{\ell\geqslant 1}\sum_{s=1}^{m} X_{z_s^{\langle -\ell\rangle}} \frac{\partial}{\partial z_s^{\langle -\ell\rangle}} + \sum_{i=1}^{n} X_{x_i}\frac{\partial}{\partial x_i} + \sum_{k\geqslant 0}\sum_{j=1}^{m} X_{u_j^{[k]}}\frac{\partial}{\partial u_j^{[k]}}, \\ X^{\rho_f} &= \sum_{\ell\geqslant 1}\sum_{s=1}^{m} \left(X^{\rho_f}\right)_{z_s^{\langle -\ell\rangle}} \frac{\partial}{\partial z_s^{\langle -\ell\rangle}} + \sum_{i=1}^{n} \left(X^{\rho_f}\right)_{x_i}\frac{\partial}{\partial x_i} + \sum_{k\geqslant 0}\sum_{j=1}^{m} \left(X^{\rho_f}\right)_{u_j^{[k]}}\frac{\partial}{\partial u_j^{[k]}}, \\ X^{\nabla_f} &= \sum_{\ell\geqslant 1}\sum_{s=1}^{m} \left(X^{\nabla_f}\right)_{z_s^{\langle -\ell\rangle}} \frac{\partial}{\partial z_s^{\langle -\ell\rangle}} + \sum_{i=1}^{n} \left(X^{\nabla_f}\right)_{x_i}\frac{\partial}{\partial x_i} + \sum_{k\geqslant 0}\sum_{j=1}^{m} \left(X^{\nabla_f}\right)_{u_j^{[k]}}\frac{\partial}{\partial u_j^{[k]}},\end{aligned}$$

where

$$\big(X^{\rho_f}\big)_{z_s^{\langle -\ell\rangle}} = \rho_f\big(X_{z_s^{\langle -\ell+1\rangle}}\big) \quad \textit{for } \ell \geqslant 2, \tag{38a}$$

$$\big(X^{\rho_f}\big)_{z_s^{\langle -1\rangle}} = \sum_{i=1}^{n} \rho_f\left(\frac{\partial \varphi_s}{\partial x_i} X_{x_i}\right) + \sum_{k=1}^{m} \rho_f\left(\frac{\partial \varphi_s}{\partial u_k} X_{u_k}\right), \tag{38b}$$

$$\big(X^{\rho_f}\big)_{x_i} = \rho_f(X_{x_i}) + \mu \sum_{j=1}^{n} \rho_f\left(\frac{\partial f_i}{\partial x_j} X_{x_j}\right) + \mu \sum_{k=1}^{m} \rho_f\left(\frac{\partial f_i}{\partial u_k} X_{u_k}\right), \tag{38c}$$

$$\big(X^{\rho_f}\big)_{u_j^{[k]}} = \rho_f\big(X_{u_j^{[k]}}\big) + \mu \rho_f\big(X_{u_j^{[k+1]}}\big) \quad \textit{for } k \geqslant 0 \tag{38d}$$

and

$$\big(X^{\nabla_f}\big)_{z_s^{\langle -\ell\rangle}} = \frac{1}{\mu}\big(X_{z_s^{\langle -\ell\rangle}} - \rho_f\big(X_{z_s^{\langle -\ell+1\rangle}}\big)\big) \quad \textit{for } \ell \geqslant 2, \tag{39a}$$

$$\big(X^{\nabla_f}\big)_{z_s^{\langle -1\rangle}} = \frac{1}{\mu}\left(X_{z_s^{\langle -1\rangle}} - \sum_{i=1}^{n} \rho_f\left(\frac{\partial \varphi_s}{\partial x_i} X_{x_i}\right) - \sum_{k=1}^{m} \rho_f\left(\frac{\partial \varphi_s}{\partial u_k} X_{u_k}\right)\right), \tag{39b}$$

$$\big(X^{\nabla_f}\big)_{x_i} = \nabla_f(X_{x_i}) - \sum_{j=1}^{n} \rho_f\left(\frac{\partial f_i}{\partial x_j} X_{x_j}\right) - \sum_{k=1}^{m} \rho_f\left(\frac{\partial f_i}{\partial u_k} X_{u_k}\right), \tag{39c}$$

$$\big(X^{\nabla_f}\big)_{u_j^{[k]}} = \nabla_f\big(X_{u_j^{[k]}}\big) - \rho_f\big(X_{u_j^{[k+1]}}\big) \quad \textit{for } k \geqslant 0. \tag{39d}$$

For $\mu = 0$, one gets

$$X = \sum_{i=1}^{n} X_{x_i} \frac{\partial}{\partial x_i} + \sum_{k\geqslant 0} \sum_{j=1}^{m} X_{u_j^{[k]}} \frac{\partial}{\partial u_j^{[k]}}, \quad X^{\rho_f} = X,$$
$$X^{\nabla_f} = \sum_{i=1}^{n} \big(X^{\nabla_f}\big)_{x_i} \frac{\partial}{\partial x_i} + \sum_{k\geqslant 0} \sum_{j=1}^{m} \big(X^{\nabla_f}\big)_{u_j^{[k]}} \frac{\partial}{\partial u_j^{[k]}},$$

where

$$\big(X^{\nabla_f}\big)_{x_i} = \nabla_f(X_{x_i}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} X_{x_j} - \sum_{k=1}^{m} \frac{\partial f_i}{\partial u_k} X_{u_k}, \tag{40a}$$

$$\big(X^{\nabla_f}\big)_{u_j^{[k]}} = \nabla_f\big(X_{u_j^{[k]}}\big) - X_{u_j^{[k+1]}} \quad \textit{for } k \geqslant 0. \tag{40b}$$

Proof. Let $\mu > 0$ and $s = 1, \dots, m$. Taking $\omega = \mathrm{d}z_s^{\langle -\ell\rangle}$, $\ell \geqslant 2$, from (29) one gets

$$\begin{aligned}\big(X^{\rho_f}\big)_{z_s^{\langle -\ell\rangle}} &= \big\langle \mathrm{d}z_s^{\langle -\ell\rangle}, X^{\rho_f}\big\rangle = \big\langle \mathrm{d}z_s^{\langle -\ell+1\rangle}, X\big\rangle^{\rho_f} \\ &= \big(X_{z_s^{\langle -\ell+1\rangle}}\big)^{\rho_f} = \rho_f\big(X_{z_s^{\langle -\ell+1\rangle}}\big),\end{aligned}$$

and for $\omega = \mathrm{d}z_s^{\langle -1\rangle}$, we have

$$\begin{aligned}\left(X^{\rho_f}\right)_{z_s^{\langle -1\rangle}} &= \langle \mathrm{d}z_s, X\rangle^{\rho_f} = \langle \mathrm{d}\varphi_s, X\rangle^{\rho_f} \\ &= \rho_f\left(\sum_{i=1}^{n}\frac{\partial\varphi_s}{\partial x_i}X_{x_i} + \sum_{k=1}^{m}\frac{\partial\varphi_s}{\partial u_k}X_{u_k}\right) \\ &= \sum_{i=1}^{n}\rho_f\left(\frac{\partial\varphi_s}{\partial x_i}X_{x_i}\right) + \sum_{k=1}^{m}\rho_f\left(\frac{\partial\varphi_s}{\partial u_k}X_{u_k}\right).\end{aligned}$$

Similarly, for $\omega = \mathrm{d}x_i$, $i = 1,\dots,n$, from (29) one gets

$$\begin{aligned}\left(X^{\rho_f}\right)_{x_i} &= \left\langle \mathrm{d}\sigma_f(x_i), X\right\rangle^{\rho_f} = \left\langle \mathrm{d}\big(x_i + \mu f_i(x,u)\big), X\right\rangle^{\rho_f} \\ &= \rho_f(X_{x_i}) + \mu\sum_{j=1}^{n}\rho_f\left(\frac{\partial f_i}{\partial x_j}X_{x_j}\right) + \mu\sum_{k=1}^{m}\rho_f\left(\frac{\partial f_i}{\partial u_k}X_{u_k}\right),\end{aligned}$$

and for $\omega = \mathrm{d}u_j^{[k]}$, $j = 1,\dots,m$, $k \geqslant 0$, we have

$$\begin{aligned}\left(X^{\rho_f}\right)_{u_j^{[k]}} &= \left\langle \mathrm{d}\sigma_f\big(u_j^{[k]}\big), X\right\rangle^{\rho_f} = \left\langle \mathrm{d}\big(u_j^{[k]} + \mu u_j^{[k+1]}\big),\ X\right\rangle^{\rho_f} \\ &= \rho_f(X_{u_j^{[k]}}) + \mu\rho_f(X_{u_j^{[k+1]}}).\end{aligned}$$

Hence the components of X^{ρ_f} satisfy (38). Using (31) given in Proposition 5, we get $X^{\nabla_f} = (X - X^{\rho_f})/\mu$ for $\mu > 0$. Hence we obtain the formulas for components of X^{∇_f}, i.e. (39).

Let now $\mu = 0$. For $\omega = \mathrm{d}x_i$, $i = 1,\dots,n$, from (30) one gets

$$\begin{aligned}\left(X^{\nabla_f}\right)_{x_i} &= \langle \mathrm{d}x_i, X\rangle^{\nabla_f} - \left\langle \mathrm{d}x_i^{\Delta}, X\right\rangle \\ &= \nabla_f(X_{x_i}) - \sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}X_{x_j} - \sum_{k=1}^{m}\frac{\partial f_i}{\partial u_k}X_{u_k},\end{aligned}$$

and for $\omega = \mathrm{d}u_j^{[k]}$, $j = 1,\dots,m$, $k \geqslant 0$, we have

$$\left(X^{\nabla_f}\right)_{u_j^{[k]}} = \left\langle \mathrm{d}u_j^{[k]}, X\right\rangle^{\nabla_f} - \left\langle \mathrm{d}u_j^{[k+1]}, X\right\rangle = \nabla_f(X_{u_j^{[k]}}) - X_{u_j^{[k+1]}}.$$

Hence, for $\mu = 0$, the components of X^{∇_f} satisfy (40). □

Corollary 1. *Let $\mu = 0$. Then the nabla derivative of the vector field X equals the delta derivative of X, i.e. $X^{\nabla_f} = X^{\Delta_f}$. Moreover, if $x^{\Delta} = f(x)$ and $f = (f_1,\dots,f_n)^{\mathrm{T}}$, then $X^{\nabla_f} = X^{\Delta_f} = L_f X$, where*

$$L_f X = [f, X] = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial X_{x_i}}{\partial x_j}f_j - \sum_{j=1}^{n}\frac{\partial f_i}{\partial x_j}X_j\right)\frac{\partial}{\partial x_i}.$$

Proof. Note that for $\mu = 0$, $\sigma_f = \rho_f = \mathrm{id}$. Let $X \in \mathcal{E}'$. From (36) or (37) it follows that in this case, $X^{\nabla_f} = X^{\Delta_f}$. Moreover, when additionally $x^\Delta = f(x)$, by (40a) we get

$$\left(X^{\nabla_f}\right)_{x_i} = \nabla_f(X_{x_i}) - \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j} X_{x_j}.$$

Then by (7) and (9) we get $\nabla_f(X_{x_i}) = \sum_{j=1}^n (\partial X_{x_i}/\partial x_j) f_j$ and, consequently, $X^{\nabla_f} = \sum_{i=1}^n (\sum_{j=1}^n (\partial X_{x_i}/\partial x_j) f_j - \sum_{j=1}^n (\partial f_i/\partial x_j) X_j)(\partial/\partial x_i) = L_f X$. □

We show below on the simple example how to compute X^{ρ_f}, X^{∇_f} and $X^{\nabla_f^2}$.

Example. Consider the system described by

$$x_1^\Delta = x_2^2, \qquad x_2^\Delta = u.$$

For $\mu > 0$, the system can be rewritten in the form (4), i.e.

$$\begin{aligned} x_1^\sigma &= x_1 + \mu x_2^2, \\ x_2^\sigma &= x_2 + \mu u. \end{aligned} \tag{41}$$

We shall need x_2 and x_2^σ are different than 0, so we define the subset U of $\mathbb{R}^3$ that satisfies the following inequalities:

$$x_2 \neq 0, \qquad x_2 + \mu u \neq 0.$$

Then the inversive closure of $\mathcal{K}$ can be chosen as the field of meromorphic functions in a finite number of variables x_1, x_2, $u^{[k]}$, $z^{\langle -\ell \rangle}$, $k \geqslant 0$, $\ell \geqslant 1$, where $z^{\langle -1 \rangle} = \sigma_f^{-1}(z)$ and $z^{\langle -\ell \rangle} = \sigma_f^{-1}(z^{\langle -\ell+1 \rangle})$. We construct the field extension in two different ways (choosing z as u or x_1) and compute X^{ρ_f}, X^{∇_f}, $X^{\nabla_f^2}$ using the formulas given in Proposition 7.

Let $X = \partial/\partial u$ be an element of $\mathcal{E}'$. Then $X_{z^{\langle -\ell \rangle}} = X_{x_1} = X_{x_2} = X_{u^{[k]}} = 0$, $\ell \geqslant 1$, $k \geqslant 1$, and $X_u = 1$.

Case 1: $z = u$. By (38) one gets

$$X^{\rho_f} = \begin{cases} \frac{\partial}{\partial z^{\langle -1 \rangle}} + \mu \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u} & \text{for } \mu > 0, \\ \frac{\partial}{\partial u} & \text{for } \mu = 0 \end{cases}$$

and by (39)

$$X^{\nabla_f} = \begin{cases} -\frac{1}{\mu} \frac{\partial}{\partial z^{\langle -1 \rangle}} - \frac{\partial}{\partial x_2} & \text{for } \mu > 0, \\ -\frac{\partial}{\partial x_2} & \text{for} \mu = 0. \end{cases}$$

Moreover, by (39) we have

$$X^{\nabla_f^2} = \left(X^{\nabla_f}\right)^{\nabla_f} = \begin{cases} \frac{1}{\mu^2} \frac{\partial}{\partial z^{\langle -2 \rangle}} - \frac{1}{\mu^2} \frac{\partial}{\partial z^{\langle -1 \rangle}} + 2(x_2 - \mu z^{\langle -1 \rangle}) \frac{\partial}{\partial x_1} & \text{for } \mu > 0, \\ 2x_2 \frac{\partial}{\partial x_1} & \text{for } \mu = 0. \end{cases}$$

Case 2: $z = x_1$. By (38) one gets

$$X^{\rho_f} = \mu \frac{\partial}{\partial x_2} + \frac{\partial}{\partial u}$$

and by (39)

$$X^{\nabla_f} = -\frac{\partial}{\partial x_2}.$$

Moreover, by (39) we have

$$X^{\nabla_f^2} = \left(X^{\nabla_f}\right)^{\nabla_f} = \begin{cases} 2\rho_f(x_2)\frac{\partial}{\partial x_1} & \text{for } \mu > 0, \\ 2x_2\frac{\partial}{\partial x_1} & \text{for } \mu = 0, \end{cases}$$

where $\rho_f(x_2)$ satisfies the following relation: $\mu(\rho_f(x_2))^2 = x_1 - z^{\langle -1 \rangle}$, and the inequality $\mu(\rho_f(x_2))^2 = x_1 - z^{\langle -1 \rangle} \geqslant 0$ follows from the first equation of the considered system.

5 Conclusions

The paper may be understood as continuation of papers [2, 3, 4], focused on the development of algebraic tools that allow to study the properties of nonlinear control systems, defined on time scales. In this paper, we have extended the backward shift and nabla derivative operators, defined by the control system on homogeneous time scale, for one-forms and vector fields, and proved a number of their properties. The operator of nabla derivative (applied to the vector fields) is useful tool. In future studies, the nabla derivative operator can be applied, for instance, in computation of the accessibility distribution and checking accessibility property of the system, defined on time scales, extending the results for nonlinear continuous-time systems from [15]. In the continuous-time case, the nabla derivative is equal to the delta derivative, whereas in the discrete-time case the two derivatives are different. The unification aspect of time scale approach entails considerable simplification in the software development on NLControl website [5].

References

1. F.M. Atici, F. Usynal, A producion-inventory model of HMMS model on time scales, *Appl. Math. Lett.*, **21**(3):236–243, 2008.
2. Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz, M. Tõnso, M. Wyrwas, Algebraic formalism of differential p-forms and vector fields for nonlinear control systems on homogeneous time scales, *Proc. Est. Acad. Sci.*, **62**(4):215–226, 2013.
3. Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz, M. Wyrwas, Algebraic formalism of differential one-forms for nonlinear control systems on time scales, *Proc. Est. Acad. Sci., Phys. Math.*, **56**:264–282, 2007.
4. Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz, M. Wyrwas, Control systems on regular time scales and their differential rings, *Math. Control Signals Syst.*, **22**:185–201, 2011.

5. J. Belikov, V. Kaparin, Ü. Kotta, M. Tõnso, NLControl website, 2014, available from: `www.nlcontrol.ioc.ee`.
6. M. Bohner, A. Peterson, *Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2001.
7. M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
8. M. Bohner, N. Wintz, The Kalman for linear systems on time scales, *J. Math. Anal. Appl.*, **406**(2):419–436, 2013.
9. D. Casagrande, Ü. Kotta, M. Tõnso, M. Wyrwas, Transfer equivalence and realization of nonlinear input-output delta-differential equations on homogeneous time scales, *IEEE Trans. Autom. Control*, **55**:2601–2606, 2010.
10. R.M. Cohn, *Difference Algebra*, Wiley, New York, 1965.
11. B. Do, Y. Liu, H.A. Batarfi, F.E. Alsaadi, Almost periodic solution for a neutral-type neural networks with distributed leakage delays on time scales, *Neurocomputing*, **173**:921–929, 2016.
12. M. Fliess, J. Levine, P. Martin, F. Olivier, P. Rouchon, A remark on nonlinear accessibility conditions and infinite prolongations, *Syst. Control Lett.*, **31**:77–83, 1997.
13. J. Gao, Q.-R. Wang, L.-W. Zhang, Existence and stability of almost-periodic solutions for cellular neural networks with time-varying delays in leakage terms on time scales, *Appl. Math. Comput.*, **237**:639–649, 2015.
14. R. Hilscher, V. Zeidan, Nabla time scale symplectic systems and related quadratic functionals, *Differ. Equ. Dyn. Syst.*, **18**(1):163–198, 2010.
15. A. Isidori, *Nonlinear Control Systems*, Springer, London, 1995.
16. B.J. Jackson, Adaptive control in the nabla setting, *Neural Parallel Sci. Comput.*, **16**:253–272, 2008.
17. Ü. Kotta, Z. Bartosiewicz, E. Pawłuszewicz, M. Wyrwas, Irreducibility, reduction and transfer equivalence of nonlinear input-output equations on homogeneous time scales, *Syst. Control Lett.*, **58**(9):646–651, 2009.
18. J. Levine, *Analysis and Control of Nonlinear Systems*, Springer-Verlag, Berlin, Heidelberg, 2009.
19. T. Mullari, Ü. Kotta, Z. Bartosiewicz, E. Pawluszewicz, Discrete-time Lie derivative with respect to system dynamics, in S. Bittanti (Ed.), *Proceedings of the 18th IFAC World Congress, Milano, Italy, August 28–September 2, 2011*, International Federation of Automatic Control, 2011, pp. 11000–11005.