ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2016, Vol. 21, No. 4, 498–514
http://dx.doi.org/10.15388/NA.2016.4.5

Approximate finite-dimensional ODE temperature model for microwave heating*

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Received: October 19, 2014 / **Revised:** May 25, 2015 / **Published online:** March 18, 2016

Abstract. In this paper, a finite-dimensional ordinary differential equation (ODE) model is proposed for predicting the temperature profile with microwave heating to accomplish lower computing complexity. The traditional parabolic partial different equation (PDE) model with integrating Maxwell's equation and heat transport equation is not suitable for designing the on-line controller. Based on the obstruction, using an auxiliary function derives an intermediate model, which is analyzed and discussed for model reduction by employing the parameter separation method and Galerkin's method. The simulation experiments on one-dimensional waveguide and cavity demonstrate that the proposed approximate model is effective.

Keywords: model reduction, temperature profile, microwave heating.

1 Introduction

Microwave applications for thermal purposes have obtained vast application in domestic usage and are attracting much attention in industrial applications over several decades [5]. In some certain industrial areas, the microwave heating has become an established technology, which includes food processing, wood drying, organic chemistry and pharmaceutical industry [2, 4, 11, 17, 21]. Under microwave irradiation, the molecules in dielectric material are realigned about a million times per second [27], which can assist

*Financial support from the National Basic Research Program of China (No. 2013CB328903) is gratefully acknowledged.
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the electromagnetic energy being converted to heat energy in the materials. Because of this principle, microwave heating has incomparable advantages in terms of the economy and non-pollution. However, the major drawback encountered during the process of microwave heating is the inhomogeneous of resultant temperature distribution, leading to the hotspots and thermal runaway [23, 25].

With the rapid development of hardware and software, numerical simulations gradually become an indispensable tool for us to further understand the complicated process of microwave heating, predict the temperature distribution and avoid some evitable experimental expenses. Nonhomogeneous parabolic PDE is a typical expression to describe the basic principles relationship for electromagnetic field (Maxwell's equation or Lambert's law) and temperature field (conduction, radiation or convection) [1, 14], especially for the phenomenon of resonant into the sample. In order to obtain the temperature profile from above equations, traditional analytical methods are hardly to get its closed-form solution, which promotes the development of numerical methods, such as, Finite Element Method (FEM), Finite-Different Time-Domain (FDTD) method, Moment of Method (MoM), Finite-Volume Time-Domain (FVTD) method and Transmission Line Matrix (TLM) method [18, 19, 26]. However, some important issues about these methods need to be addressed, such as the excessive computation time and overload CPU utilization. Moreover, the influence of boundary conditions, initial conditions and external disturbances will inevitably increase the complexity of calculation. From the view of cybernetics, the main characteristics of the microwave heating process are that the outputs, inputs, state variables and parameters are changing with the time domain and spatial domain. That is another insufficient for the traditional PDE model to directly design the close-loop controllers [20]. Therefore, it is necessary to readily predict the distribution of temperature in order to lay a foundation for accurate on-line control.

Mathematically, the main characteristics of conventional microwave heating model is that the solution of parabolic PDE can be decomposed as the multiply of time domain and spatial domain [13]. If we only consider the function of spatial differential operator, its eigenspectrum can be divided into infinite discrete components [8]. Zhong applies above method to transform the waveguide heating model into infinite-dimensional ODEs with the zero boundary condition and zero initial condition [28]. While the nonhomogeneous or mixed boundary condition is another important factor to influence the efficiency of heating and it is difficult to obtain eigenfunctions of spatial differential operator directly. To solve these problems, the research efforts focus on applying an auxiliary function to achieve a more complicated equivalent PDE, including not only the states of microwave irradiation and heat transfer, but also the information of boundary conditions. In this case, the Galerkin's method is applied to partition the infinite-dimensional model into a finite-dimensional slow complement and an infinite-dimensional stable fast one [3, 6, 7, 10, 15], where the slow (finite-dimensional) model can also approximately describe the dynamic performance for temperature rise.

In this work, we focus on developing a novel approximate nonlinear temperature model for one-dimensional microwave heating with mixed or nonhomogeneous boundary conditions. The layout of the remainder of the paper is as follows: Some fundamental theories of electromagnetic and thermodynamic fields for one-dimensional microwave traditional is presented in Section 2, such as, Maxwell's equation, heat transport equation and boundary conditions. Then, the intermediate model with homogeneous boundary condition is obtained by introducing an auxiliary function successfully in order to provide facilities for time-spatial separation and model reduction. Because the finite-dimensional ODE can capture the primary dynamics of PDE model, the novel nonlinear temperature model is deduced by applying Galerkin's method and analyzing the relationship between input and output in Section 3. In order to verify the nonlinear model, the results of traditional model are simulated in some points by Matlab to contrast with the novel model in Section 4, whose results demonstrate that the methodology could provide important guarantee for control algorithm in the next step.

2 One-dimensional traditional microwave heating model

With no loss generality, a microwave heating appliance consists of three major components: the magnetrons, transmission lines and cavities [22]. Resulting from the acceleration of charge, electromagnetic radiation is generated in magnetrons, which mainly depends on the interaction with powerful external magnet and cathode voltage. Transmission lines are usually hollow tubes in which the microwave energy can be propagated. For the process of heating materials, a number of modes exist in microwave cavities. However, most of modes in the propagation of electromagnetic wave are usually neglected as they have little effect on temperature. In order to simplify the analysis process for one-dimensional heating, this paper only considers a slender material exposed the TEM modes in the microwave heating appliances.

2.1 Microwave propagation in heating material

Electromagnetic wave is constituted of the electric and magnetic field orthogonal to each other. And we could first assume that the electromagnetic wave propagates along the z-axis with an $\mathrm{e}^{j\omega t}$ dependence. For general materials, the electric and magnetic field can be rewritten as

$$\vec{E}(x,y,z) = \big[\bar{e}(x,y) + \hat{z}e_z(x,y)\big]\mathrm{e}^{-j\beta z},$$
$$\vec{H}(x,y,z) = \big[\bar{h}(x,y) + \hat{z}h_z(x,y)\big]\mathrm{e}^{-j\beta z},$$

where $\bar{e}(x,y)$ and $\bar{h}(x,y)$ denote the transverse $(\hat{x},\hat{y})$ electric and magnetic field components; e_z and h_z are the longitudinal electric and magnetic field components, respectively.

Assuming that the region and conductive material are source-free, linear, isotropic and homogeneous, Maxwell's equations can be deduced as phasor form

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}, \qquad \nabla \times \vec{H} = j\omega\varepsilon\vec{E},$$

where μ and ε are magnetic permeability and permittivity; ω is the radian frequency. Considering the non-magnetic material, we can obtain the simplified equation

$$\nabla^2\vec{E} + \omega^2\mu\varepsilon\vec{E} = 0. \tag{1}$$

For an insulating and non-magnetic material, we can define that

$$k^2 = \omega^2 \mu_0 \varepsilon_0 (\varepsilon' + \mathrm{i}\varepsilon''),$$

where μ_0 and ε_0 are the free space magnetic permeability and permittivity; ε' is the relative dielectric constant which represents the materials ability to store electrical energy and ε'' is the relative dielectric loss which stands for dielectric loss through energy dissipation [1]. Thus, the complex quantity of propagation constant can be expressed as

$$k = \alpha + \mathrm{i}\beta,$$

where

$$\alpha = \frac{\omega}{c}\sqrt{\frac{\varepsilon'(\sqrt{1+\tan^2\delta}+1)}{2}} \quad \text{and} \quad \beta = \frac{\omega}{c}\sqrt{\frac{\varepsilon'(\sqrt{1+\tan^2\delta}-1)}{2}},$$

where the loss tangent ($\tan\delta = \varepsilon''/\varepsilon'$) indicates the ratio of the dielectric loss to the dielectric constant and c is the velocity of light. Then, for one-dimensional microwave heating model, the Helmholtz equation of (1) reduces to

$$\frac{\partial^2 E_x}{\partial z^2} + k^2 E_x = 0. \tag{2}$$

By using substitution method, the solution for the electric field E_x in (2) is obtained

$$E_x = E^+ \mathrm{e}^{-jkz} + E^- \mathrm{e}^{jkz}, \tag{3}$$

where E^+ and E^- denote the arbitrary amplitude constants. And the first and the second terms in (3) represent for the propagation of microwave in the $+z$ and $-z$ direction, respectively.

2.2 Traditional nonhomogeneous heating transport equation

To model the process of microwave heating, an nonhomogeneous heat equation is deduced to describe the distribution of temperature, which depends on thermo physical properties of the material [9, 12, 16, 23]. In order to simplify the process of analysis, a mathematical model is proposed based on the following six assumptions:

(A1) uniform initial temperature,
(A2) homogeneous and isotropic material,
(A3) no mass transfer,
(A4) temperature independent thermal and dielectric properties,
(A5) convective boundary conditions,
(A6) no volume changes.

The energy balance during microwave heating can be expressed as

$$\rho(T)C_p(T)\frac{\partial T}{\partial t} = \mathrm{div}\big(\kappa(T)\nabla T\big) + P_{\mathrm{abs}}(x, y, z, t), \tag{4}$$

where $\rho(T)$, $C_p(T)$ and $\kappa(T)$ are material density, specific heat capacity and thermal conductivity, respectively; $P_{\mathrm{abs}}(x, y, z, t)$ is the internal heating source term. Assuming that we only consider the one-dimensional microwave heating process, (4) can be simplified as

$$\rho(T)C_p(T)\frac{\partial T}{\partial t} = \kappa(T)\frac{\partial^2 T}{\partial z^2} + P_{\mathrm{abs}}(z, t).$$

The following thermal boundary condition is exposed in the air. And both ends of slender materials are assumed to lose heat by natural convection and radiation:

$$\kappa \nabla T = h(T - T_\infty) + \sigma_h \varepsilon_h \left(T^4 - T_\infty^4\right), \tag{5}$$

where T_∞ is the ambient temperature; h is the heat transfer coefficient; ε_h is the emissivity of the sample and σ_h is the Stefan–Boltzmann constant (5.67×10^{-8} $\mathrm{W/m^2K^4}$). In general, $\sigma_h \varepsilon_h (T^4 - T_\infty^4) \to 0$, (5) can be simplified that

$$\kappa \nabla T = h(T - T_\infty).$$

2.3 Estimating internal heat source

For most of inner cavity wall, the microwave energy can be totally reflected by perfect electrical conductors, due to the characteristics of $\sigma \to \infty$, whose penetrated depth is approximately seen as zero. And the wall will also ensure the security for the usage of microwave further. By using Faraday's law and Gauss' theorem [24], the perfect boundary condition can be given as

$$E_{\mathrm{t}} = 0, \qquad H_{\mathrm{n}} = 0,$$

where subscripts t and n denote the components of tangential and normal direction, respectively. If the energy loss about surfaces between different materials is neglected, the continuity boundary condition can be also denoted as

$$E_{\mathrm{t}} = E'_{\mathrm{t}}, \qquad H_{\mathrm{t}} = H'_{\mathrm{t}}, \qquad P_{\mathrm{t}} = P'_{\mathrm{t}},$$

where the superscript denotes one of the different materials.

With knowledge of the mentioned electric field distribution, the absorbed microwave power for the slender sample is

$$P_{\mathrm{abs}}(z) = \frac{1}{2}\omega\varepsilon_0\varepsilon'' E_x E_x^*, \tag{6}$$

where E_x^* is the complex conjugate of E_x. If $E^+ = |E^+|\mathrm{e}^{\mathrm{i}\varphi_A}\mathrm{e}^{-\beta z}$ and $E^- = |E^-|\mathrm{e}^{\mathrm{i}\varphi_B}\mathrm{e}^{\beta z}$, the relationship between the E^+ and E^- can be denoted as

$$E^- = \Gamma\left|E^+\right|\mathrm{e}^{-2\beta l}\mathrm{e}^{\mathrm{i}\varphi_B}\mathrm{e}^{\beta z}, \tag{7}$$

where Γ in (7) denotes the reflection coefficient in the boundary of sample; φ_A and φ_B represent the phase in the different boundary. With (6) and (7), the expression for power dissipated per unit volume by the electric fields is

$$\begin{aligned}P_{\text{abs}}(z) &= \frac{1}{2}\omega\varepsilon_0\varepsilon''\big[\big|E^+\big|^2 \mathrm{e}^{-2\beta z} + \big|E^-\big|^2 \mathrm{e}^{2\beta z} \\ &\quad + 2\big|E^+\big|\big|E^-\big|\cos(\varphi_A - \varphi_B + 2\alpha z)\big] \\ &= \frac{1}{2}\omega\varepsilon_0\varepsilon''\big[\big|E^+\big|^2 \mathrm{e}^{-2\beta z} + \big|\Gamma E^+ \mathrm{e}^{-2\beta l}\big|^2 \mathrm{e}^{2\beta z} \\ &\quad + 2\Gamma\big|E^+\big|^2 \mathrm{e}^{-2\beta l}\cos(\varphi_A - \varphi_B + 2\alpha z)\big].\end{aligned}$$

3 A finite-dimensional ODE model for microwave heating

Mathematically, the process of microwave heating belongs to the multi-physics field coupled system, which is described by PDEs arising from heating balances and microwave transport. For a completely known model of nonlinear parabolic PDE, with mixed or nonhomogeneous boundary conditions, its results could approximate to a lower dimensional ODE through modal decomposition by applying the Galerkin's method.

3.1 Formulation for intermediate PDE model

In terms of the aforementioned one-dimensional process of microwave heating, the fundamental model can be expressed as follows:

$$\begin{aligned}\rho(T)C_p(T)\frac{\partial T}{\partial t} &= \kappa(T)\frac{\partial^2 T}{\partial z^2} + \frac{\omega\varepsilon_0\varepsilon''}{2}\big[\big|E^+\big|^2 \mathrm{e}^{-2\beta z} + \big|\Gamma E^+ \mathrm{e}^{-2\beta l}\big|^2 \mathrm{e}^{2\beta z} \\ &\quad + 2\Gamma\big|E^+\big|^2 \mathrm{e}^{-2\beta l}\cos(\varphi_A - \varphi_B + 2\alpha z)\big], \quad z \in [0, l], \qquad (8)\end{aligned}$$

subject to the boundary condition of heating transport

$$T_z(0, t) = f_1(T), \qquad T_z(l, t) = f_2(T) \qquad (9)$$

with the initial condition

$$T(z, 0) = \varphi(z, 0), \qquad (10)$$

where $f_1(T)$ and $f_2(T)$ denote the nonhomogenous boundary conditions in different positions, respectively; $\varphi(z, 0)$ is the initial temperature distribution.

In order to further deduce (8)–(10) to a class of infinite dimensional ODE equations, it is essential to derive an intermediate PDE model with the homogeneous boundary conditions, which is stated in the following theorem:

Theorem 1. *Assume the nonhomogeneous or mix boundary in* (9) *can be obtained. Then, there exists the following relationship holds*:

$$T(z, t) = \Theta(z, t) + \left(\frac{f_2(T) - f_1(T)}{2l}z^2 + f_1(T)z\right). \qquad (11)$$

Then, the traditional PDE model (8)–(10) *can be equivalent to the following intermediate model*:

$$\frac{\partial \Theta}{\partial t} = \frac{\kappa(T)}{\rho(T)C_p(T)}\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{f_2(T) - f_1(T)}{l}\right) + \frac{1}{\rho(T)C_p(T)} Q(z)u$$
$$= k_1\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{f_2(T) - f_1(T)}{l}\right) + k_2 Q(z)u, \quad z \in [0, l], \tag{12}$$

subject to the homogeneous boundary condition

$$\Theta_z(0, t) = 0, \qquad \Theta_z(l, t) = 0 \tag{13}$$

with the initial condition

$$\Theta(z, 0) = \varphi(z, 0) - \left(\frac{f_2(T) - f_1(T)}{2l} z^2 + f_1(T)z\right), \tag{14}$$

where $k_1 = \kappa(T)/(\rho(T)C_p(T))$ *and* $k_2 = 1/(\rho(T)C_p(T))$; $u = 1/2\,\omega\varepsilon_0\varepsilon''|E^+|^2$; $Q(z) = [\mathrm{e}^{-2\beta z} + |\Gamma \mathrm{e}^{-2\beta l}|^2 \mathrm{e}^{2\beta z} + 2\Gamma \mathrm{e}^{-2\beta l}\cos(\varphi_A - \varphi_B + 2\alpha z)]$.

Proof. To obtain homogeneous boundary conditions, an auxiliary function $\omega'(z, t)$ and a new unknown function $\theta(z, t)$ are introduced. $T(z, t)$ can be divided as

$$T(z, t) = \omega'(z, t) + \theta(z, t),$$

where $\theta(z, t)$ can be obtained by the following equation with homogeneous boundary condition and initial condition:

$$\frac{\partial \theta}{\partial t} = k_1 \frac{\partial^2 \theta}{\partial z^2} + k_2 Q(z)u, \quad z \in [0, l], \tag{15}$$

subject to the homogeneous boundary condition

$$\theta_z(0, t) = 0, \qquad \theta_z(l, t) = 0 \tag{16}$$

with the initial condition

$$\theta(z, 0) = \varphi(z, 0). \tag{17}$$

And the auxiliary function $\omega'(z, t)$ could be determined by

$$\frac{\partial \omega'}{\partial t} = k_1 \frac{\partial^2 \omega'}{\partial z^2}, \quad z \in [0, l], \tag{18}$$

subject to the nonhomogeneous boundary condition

$$\omega'_z(0, t) = f_1(T), \qquad \omega'_z(l, t) = f_2(T) \tag{19}$$

with the initial condition

$$\omega'(z, 0) = 0. \tag{20}$$

It is obvious that the eigenfunction of spatial differential operator in (15)–(17) could be easily obtained. But the one in (18)–(20) is still different to be derived. To this end, we find another function $u_p(z,t)$, which satisfies the boundary conditions in (19). Then, when $v(z,t) = \omega'(z,t) - u_p(z,t)$, the $v(z,t)$ satisfies the homogeneous Neumann boundary conditions. Due to not existing steady-state solution for $u_p(z,t)$, we specially define the following particular solution:

$$u_p(z,t) = gt + h(z), \tag{21}$$

where g is constant, $h(z)$ is the function of z. (21) denotes that the variation of temperature with constant rate. Substituting (21) into (19), the coefficients of (21) can be rewritten as

$$g = (u_p)_t = k_1(u_p)_{zz} = k_1 h_{zz}(z),$$

then

$$h(z) = \frac{g}{2k_1}z^2 + pz + q. \tag{22}$$

Substituting (19) into (22), the solution of (21) can be rewritten as (for simplify, $q = 0$)

$$u_p(z,t) = \frac{f_2(T) - f_1(T)}{l}k_1 t + \frac{f_2(T) - f_1(T)}{2l}z^2 + f_1(T)z. \tag{23}$$

It is interesting to note that $T(z,t)$ can also be separated the following two terms:

$$\begin{aligned} T(z,t) &= \theta(z,t) + v(z,t) + u_p(z,t) \\ &= \Theta^{(1)}(z,t) + u_p(z,t). \end{aligned} \tag{24}$$

Substituting (24) into (8)–(10), the solution of $\Theta^{(1)}(z,t)$ can be determinated by

$$\frac{\partial \Theta^{(1)}}{\partial t} = k_1 \frac{\partial^2 \Theta^{(1)}}{\partial z^2} + k_2 Q(z)u, \quad z \in [0,l], \tag{25}$$

subject to the homogeneous boundary condition

$$\Theta_z^{(1)}(0,t) = 0, \qquad \Theta_z^{(1)}(l,t) = 0 \tag{26}$$

with the initial condition

$$\Theta^{(1)}(z,0) = \varphi(z,0) - \left(\frac{f_2(T) - f_1(T)}{2l}z^2 + f_1(T)z\right). \tag{27}$$

In order to facilitate the following analysis, the temporal term in (23) is substituted into (25)–(27). The intermediate model (11)–(14) can be obtained. □

Remark 1. For the microwave heating transport model (8) with nonhomogeneous boundary condition (9), it is different to directly derive the eigenfunction of the spatial operator. By introducing the auxiliary function, the intermediate PDE (12) with homogeneous boundary condition (13) can be obtained, whose results can be consistent with the traditional PDE model by the relationship (11).

3.2 Formulation of an infinite-dimensional ODE model

According to the theorem of variables separation, the solution of homogeneous parabolic PDEs can be expressed as the time-space decoupled form:

$$\Theta(z,t) = \phi(z)G(t).$$

Thus, the eigenvalue problem of operator $\partial^2 T/\partial z^2$ in (12) takes the solution as

$$\lambda_0 = 0, \qquad \phi_0(z) = \frac{1}{2},$$
$$\lambda_i = -\left(\frac{i\pi}{l}\right)^2, \quad \phi_i(z) = \cos\frac{i\pi z}{l}, \quad i = 1, 2, \dots, \infty,$$

where λ_i denotes eigenvalues and $\phi_i(z)$ denotes the corresponding eigenfunctions.

Thus, the solution of (12)–(14) can be approximately expressed in an orthogonally decoupled series:

$$\Theta(z,t) = \sum_{i=0}^{\infty} \bar{\Theta}_i(t)\phi_i(z), \tag{28}$$

where $\bar{\Theta}_i(t)$, $i = 0, 1, 2, \dots$, represent expansion coefficients associated with $\phi_i(z)$.

Based on the aforementioned analysis, substituting (28) into (12), the superposition of (12) will be written as

$$\sum_{i=0}^{\infty} \dot{\bar{\Theta}}_i(t)\phi_i(z) = \sum_{i=0}^{\infty} \left(-k_1\bar{\Theta}_i(t)\left(\frac{i\pi}{l}\right)^2 \phi_i(z) + \frac{2}{l}\int_0^l \bigl(k_1 g\phi_i(z) + k_2 Q(z)\phi_i(z)u\bigr)\,\mathrm{d}z\,\phi_i(z)\right). \tag{29}$$

By simplifying (29), the infinite-dimensional ODE model can be expressed as

$$\dot{\bar{\Theta}}_i(t) = k_1\lambda_i\bar{\Theta}_i(t) + k_1\bar{g}_i + k_2\bar{Q}_i u, \tag{30}$$

where

$$\bar{g}_i = \frac{2}{l}\int_0^l g\phi_i(z)\,\mathrm{d}z, \qquad \bar{Q}_i = \frac{2}{l}\int_0^l Q(z)\phi_i(z)\,\mathrm{d}z. \tag{31}$$

And the initial condition can be expressed as

$$\bar{\Theta}_i(0) = \frac{2}{l}\int_0^l \Theta(z,0)\phi_i(z)\,\mathrm{d}z. \tag{32}$$

Therefore, referring to (30)–(32), a general form of infinite-dimensional nonlinear form is rewritten as follows:

$$\dot{\bar{\Theta}}(t) = \mathbf{A}\bar{\Theta}(t) + \mathbf{B}u(t) + F, \qquad \Theta(z,t) = \mathbf{C}\bar{\Theta}(t), \tag{33}$$

where

$$\bar{\Theta}(t) = \left[\bar{\Theta}_0(t), \bar{\Theta}_1(t), \bar{\Theta}_2(t), \ldots, \bar{\Theta}_n(t), \ldots\right]^{\mathrm{T}},$$
$$\mathbf{A} = k_1 \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots),$$
$$\mathbf{B} = k_2 \frac{2}{l} \int_0^l Q(z)\left[\phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_n(z), \ldots\right]^{\mathrm{T}} \mathrm{d}z,$$
$$\mathbf{C} = \left[2\phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_n(z), \ldots\right],$$
$$F = k_1 \frac{2}{l} \int_0^l g\left[\phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_n(z), \ldots\right]^{\mathrm{T}} \mathrm{d}z,$$
$$\bar{\Theta}(0) = \frac{2}{l} \int_0^l \Theta(z, 0)\left[\phi_0(z), \phi_1(z), \phi_2(z), \ldots, \phi_n(z), \ldots\right]^{\mathrm{T}} \mathrm{d}z.$$

3.3 Model reduction

To reduce the infinite-dimensional nonlinear ODE model (33) to finite-dimensional nonlinear ODE model, the following three assumptions [8] are made for the eigenspectrum of the spatial differential operator $\partial^2 T/\partial z^2$ in (12). For simplicity, a class of all eigenvalues can be rewritten as $\sigma(\bar{A}) = \{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$.

Assumption 1. $\mathrm{Re}\{\lambda_0\} \geqslant \mathrm{Re}\{\lambda_1\} \geqslant \mathrm{Re}\{\lambda_2\} \geqslant \cdots \geqslant \mathrm{Re}\{\lambda_n\} \geqslant \cdots$, where $\mathrm{Re}\{\lambda_n\}$ represents the real part of λ_n.

Assumption 2. $\sigma(\bar{A})$ can be divided as $\sigma(\bar{A}) = \sigma_1(\bar{A}) + \sigma_2(\bar{A})$, where $\sigma_1(\bar{A})$ consists of first n (with n finite) eigenvalues, i.e. $\sigma_1(\bar{A}) = \{\lambda_0, \lambda_1, \ldots \lambda_n\}$ and $|\mathrm{Re}\{\lambda_1\}|/|\mathrm{Re}\{\lambda_n\}| = O(l)$.

Assumption 3. $\mathrm{Re}\{\lambda_{n+1}\} < 0$ and $|\mathrm{Re}\{\lambda_n\}|/|\mathrm{Re}\{\lambda_{n+1}\}| = O(\varepsilon)$, where $\varepsilon := |\mathrm{Re}\,\lambda_1|/|\mathrm{Re}\,\lambda_{n+1}| < 1$ is a small positive parameter.

Remark 2. The above assumptions for parabolic PDEs are always satisfied the finite number of unstable eigenvalues, which means the eigenspectrum can be divided into an infinite-dimensional stable fast part and a finite-dimensional slow part. It signifies that finite-dimensional ODE can approximately describe the dynamic behaviors for above parabolic PDE.

Thus, $\bar{\Theta}_{\mathrm{s}} = \operatorname{span}\{\phi_0, \phi_1, \phi_2, \ldots, \phi_m\}$ and $\bar{\Theta}_f = \operatorname{span}\{\phi_{m+1}, \phi_{m+2}, \ldots\}$ are defined as the subsets of $\mathbf{A}$. Correspondingly, orthogonal projection operators P_s and P_f are chosen to decompose the state of $\bar{\Theta}$ as

$$\bar{\Theta} = \bar{\Theta}_s + \bar{\Theta}_f = P_s\bar{\Theta} + P_f\bar{\Theta}. \tag{34}$$

Therefore, by substituting (34) into (33), typical finite-dimensional nonlinear model can be expressed in the following equivalent form:

$$\dot{\bar{\Theta}}_s(t) = \mathbf{A}_s\bar{\Theta}_s(t) + \mathbf{B}_s u(t) + F_s, \qquad \Theta_s(z,t) = \mathbf{C}_s\bar{\Theta}_s(t),$$

where

$$\begin{aligned}
\bar{\Theta}_s(\mathrm{t}) &= \left[\bar{\Theta}_0(t), \bar{\Theta}_1(t), \bar{\Theta}_2(t), \dots, \bar{\Theta}_m(t)\right]^{\mathrm{T}},\\
\mathbf{A}_s &= k_1 \operatorname{diag}(\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_m),\\
\mathbf{B}_s &= k_2 \frac{2}{l}\int_0^l Q(z)\left[\phi_0(z), \phi_1(z), \phi_2(z), \dots, \phi_m(z)\right]^{\mathrm{T}} \mathrm{d}z,\\
\mathbf{C}_s &= \left[2\phi_0(z), \phi_1(z), \phi_2(z), \dots, \phi_m(z)\right],\\
F_s &= k_1 \frac{2}{l}\int_0^l g\left[\phi_0(z), \phi_1(z), \phi_2(z), \dots, \phi_m(z)\right]^{\mathrm{T}} \mathrm{d}z,\\
\bar{\Theta}_s(0) &= \frac{2}{l}\int_0^l \Theta(z,0)\left[\phi_0(z), \phi_1(z), \phi_2(z), \dots, \phi_m(z)\right]^{\mathrm{T}} \mathrm{d}z.
\end{aligned}$$

Therefore, based on aforementioned analysis, the approximate temperature distribution in microwave heating process can be rewritten as

$$T(z,t) = \Theta_s(z,t) + \left(\frac{f_2(T) - f_1(T)}{2l} z^2 + f_1(T) z\right).$$

4 Simulation and validation

In this section, we validate and contrast, through computer simulations, the proposed finite-dimensional nonlinear model in aforementioned section. To this end, we first assume a heated sample exposed in microwave irradiation and its length is equal with the wavelength of microwave, whose schematic diagram is shown in Fig. 1.

If the sample is filled with the resonance cavity, the phase of reflection microwave is $\varphi_{\mathrm{ref}}|_{L=l} = \varphi_{\mathrm{Inc}}|_{L=l}$ and reflection coefficient is $\Gamma \equiv 1$. Thus, the expression of nonhomogeneous term in (8) is $P_0[\mathrm{e}^{-2\beta z} + |\mathrm{e}^{-2\beta l}|^2 \mathrm{e}^{2\beta z} + 2\mathrm{e}^{-2\beta l}\cos(2\alpha z)]$.

Throughout the history of microwave heating, food processing has become one of the most successful and largest applications over the three decades. Deionized water is

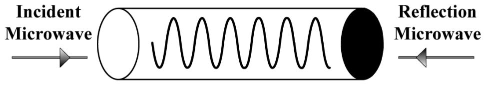

Figure 1. Schematic diagram for the thin water column in resonance cavity.

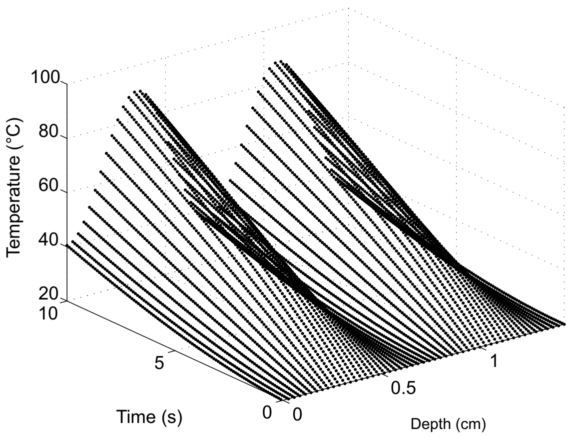

Figure 2. Global temperature estimation for finite-dimensional nonlinear form via resonant cavity heating.

usually chosen to be the experimental and simulation object by many researchers because of the good absorption properties. As an illustration, they can be regarded as constant because of the small variation of the physical characteristics with temperature. Thus, thermodynamic parameters, such as, density ρ, specific heat C_p and heat conductivity κ, can be equaled with $1\ \mathrm{g/cm^3}$, $4.2\ \mathrm{J/g\,^\circ C}$ and $0.0054\ \mathrm{W/cm\,^\circ C}$; At the frequent microwave of 2.45 GHz, the relative dielectric constant ε' and relative dielectric loss ε'' are 72.8 and 6.5, respectively. Based on above priori knowledge and proposed method in Section 3, the global results of simulation for finite-dimensional nonlinear representation with uniform initial temperature $20\,^\circ\mathrm{C}$, nonhomogeneous boundary conditions $f_1(T) = 1\,^\circ\mathrm{C/cm}$ and $f_2(T) = -1\,^\circ\mathrm{C/cm}$, constant absorbed input power density $15\ \mathrm{W/cm^2}$ and finite dimensionality $m = 5$ are shown in Fig. 2.

In order to analyze the validity of above proposed finite-dimensional nonlinear model, we choose three points which are located in 0.37 cm, 0.71 cm and 1.11 cm, respectively, to contrast the trend of temperature rising with the closed-form solutions from traditional numerical model (8)–(10), which is used FEM. And the estimating temperatures by different models in different positions are shown in Fig. 3. For the sake of comparison, it manifests that the estimation temperature used the finite-dimensional nonlinear model can approximately describe the trend of temperature rise.

In microwave heating field, waveguide is also one of the main appliances and its schematic diagram is shown in Fig. 4. In a nutshell, on the condition that the length of sample is greater than the two times of penetration depth, we usually regard both the transmitted microwave and the reflection coefficient Γ as 0. Similarly, the expression of nonhomogeneous term in (8) can be written in $P_0\mathrm{e}^{-2\beta z}$.

Thus, above simulation object is also chosen to validate the proposed finite-dimensional nonlinear model in another microwave heating appliance. For the same initial conditions and input power density but the different finite dimensionality $m = 40$ and homogeneous Neumann boundary conditions, the global temperature estimation in waveguide is shown in Fig. 5. To future verity the finite-dimensional model, the closed-form

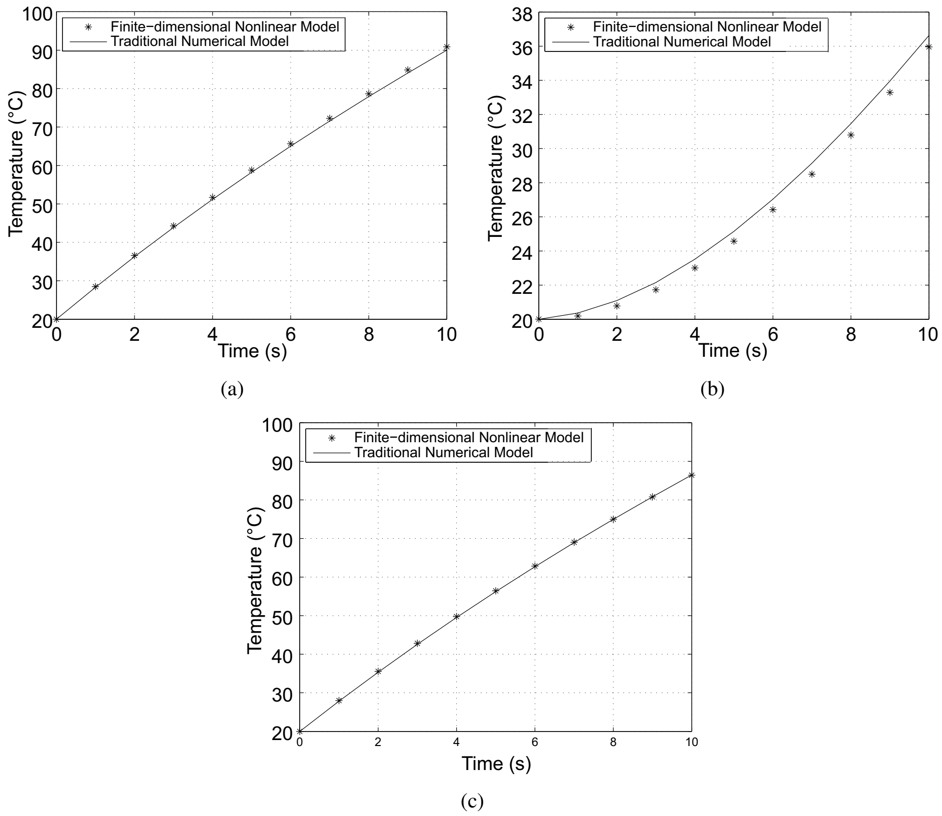

Figure 3. Estimating temperature for the finite-dimensional nonlinear model and traditional numerical model for resonant cavity heating in: (a) 0.37 cm, (b) 0.71 cm, (c) 1.11 cm.

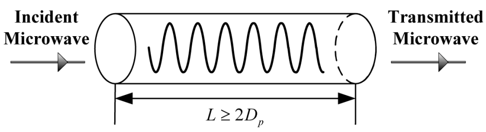

Figure 4. Schematic diagram for the thin water column in waveguide.

solutions for traditional finite-dimensional waveguide model is also chosen to compare the temperature rise curves, which are shown in Fig. 6.

Furthermore, the above two groups of simulation experiments demonstrate that the proposed finite-dimensional nonlinear model could approximately describe the distribution of temperature in the process of microwave for resonant cavity or waveguide heating. And the finite-dimensional nonlinear form also can reduce the time-consuming simulations heavily and obtain the state variable easily, so it expects to be applied in designing and optimizing intelligent controller in the next step.

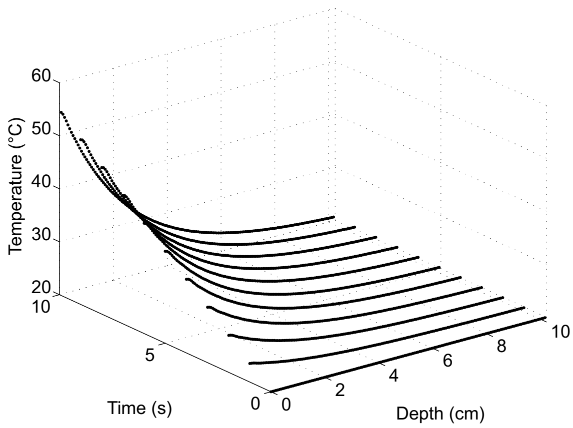

Figure 5. Global temperature estimation for finite-dimensional nonlinear model via waveguide heating.

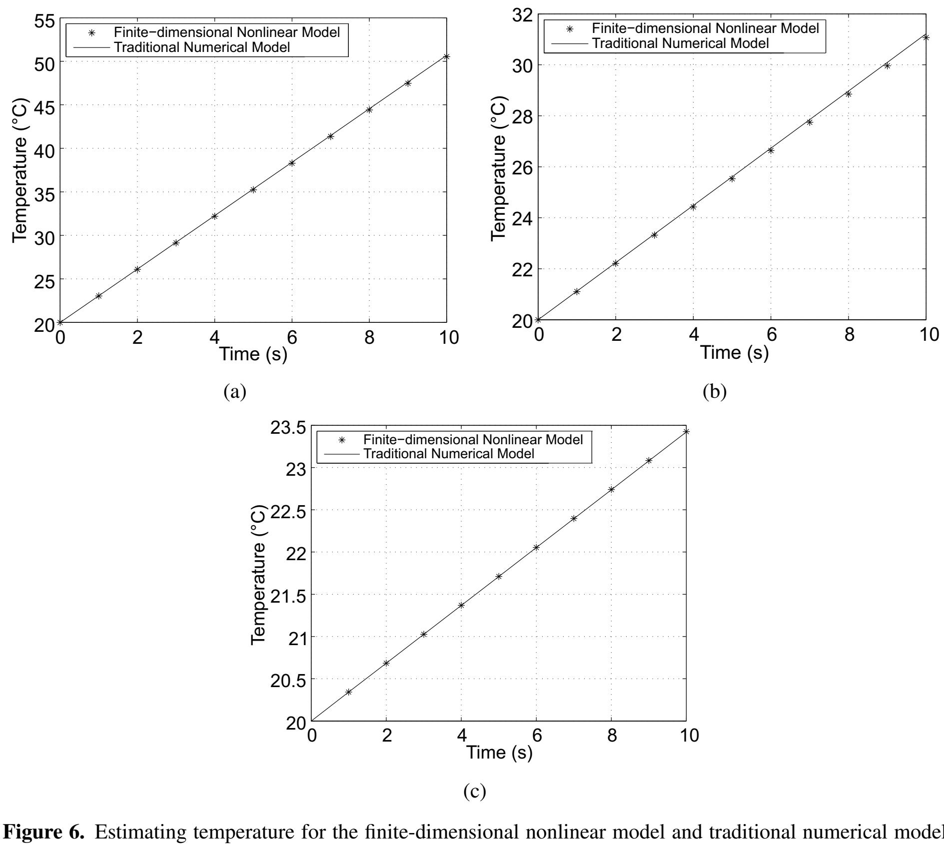

Figure 6. Estimating temperature for the finite-dimensional nonlinear model and traditional numerical model for waveguide heating in: (a) 0.4 cm, (b) 3 cm, (c) 6 cm.

5 Conclusion

In this paper, a novel approximate temperature model for microwave heating process is proposed and validated by model reduction and simulation, which overcomes the insufficient of controller design for parabolic PDEs. The typical characteristic of microwave heating process is strong coupling, especially for the electromagnetic field and thermodynamic field. And the traditional mathematical model concludes a set of relevant equations, which consist of a nonlinear PDE, boundary conditions and initial conditions. But the existence of nonhomogeneous boundary conditions will lead to impossibility for directly deriving the eigenfunctions of spatial operator. Thus, we obtain and prove an intermediate PDE by introducing an auxiliary function, which can be transformed into infinite-dimensional ODE. Therefore, nonlinear temperature model should be acquired as the relationship between input and output solved by the application of Galerkin's method. In order to validate the proposed methodology, we successfully compare the temperature rise curves between the proposed nonlinear model and traditional numerical model. It demonstrates that the finite-dimensional model could also approximately describe distribution of temperature in microwave heat processing. Further researches are underway for the intelligent controller design using the proposed temperature model in industrial microwave ovens.

Acknowledgment. We would also like to thank Professor Qian Chen and Professor Xueke Pu for valuable suggestions and kind help.

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