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Control and synchronization of Julia sets of the complex dissipative standard system*

Weihua Sun, Yongping Zhang

School of Mathematics and Statistics, Shandong University, Weihai,
Weihai 264209, China
whsun_maths@163.com; ypzhangsd@126.com

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Abstract. The fractal behaviors of the complex dissipative standard system are discussed in this paper. By using the boundedness of the forward and backward orbits, Julia set of the system is introduced and visualization of Julia set is also given. Then a controller is designed to achieve Julia set shrinking or expanding with the changing of the control parameter. And synchronization of two different Julia sets is discussed by adding a coupling item, which makes one Julia set change to be the other. The simulations illustrate the efficacy of these methods.

Keywords: Julia set, complex dissipative standard system, control, synchronization.

1 Introduction

The study of fractal is one of the important contents in nonlinear science and many complicated phenomena can be explained better by using the characters of fractals. Julia set is an important set in fractals. At present, not only the properties, graph of Julia set is studied, but also the applications of Julia set is discussed. For example, based on the particle dynamics characteristics, the physical meaning of the generalized M–J set is discussed in [15] and the fractal structure characteristics of the generalized M–J set could visually reflect the change rule of the particle's velocity. It is also pointed out that for the system $z_{n+1} = z_n^q + c$, where c is a complex number and $q > 1$, the generalized Julia set gives the closure of all the possible unstable orbit of the particle within the velocity space. When $c \to 0$, the absolute velocity of the particle is $v = 1$. It rotates unstably at the top of the potential barrier. In [14], the authors reduce the Langevin equation to a class complex mapping system by selecting an appropriate magnetic intensity and time interval. And if the choices of the magnetic intensity and time interval is changed, the generalized Julia sets may emerge the interior-filling structure feature, i.e. "explosion" phenomena appear in the closure of the unstable periodic orbits of the particle in the velocity space.

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In the study of dissipative dynamical systems, the sine circle map is used as a model, which has the following form:

$$x_{n+1} = x_n + \alpha + \beta \sin(x_n), \tag{1}$$

where α and β are real parameters. System (1) investigated in [3, 4] exhibits a mode-locking phase diagram, which contains infinitely many Arnold tongues in which the dynamics is locked to periodic orbits separated by gaps in which the trajectories are quasiperiodic. The sine circle map is a special case for the dissipative standard map discussed in [19] taking the form

$$\begin{aligned} p_{n+1} &= bp_n + (1-b)\Omega + \frac{V}{2\pi}\sin(2\pi x_n), \\ x_{n+1} &= x_n + p_{n+1}, \end{aligned}$$

where $0 \leqslant b \leqslant 1$.

For the study of dynamical systems, periodic orbits provide a useful and systematic framework. It was shown that periodic orbits can be used to obtain many interesting properties of the strange attractor, such as the Liapunov exponent, fractal dimension and topology entropy. In [19], the authors use a numerical technique for the calculation of unstable and stable periodic orbits in dissipative maps. By applying this technique to the dissipative standard map, a classification of the periodic orbits and a numerical procedure for the calculation of any particular one are obtained. Adopting the periodicity orbit search and comparison technique in [10], the structure topological inflexibility and the discontinuity evolution law of the generalized M–J sets generated from the complex mapping $z \rightarrow z^\alpha + c$ $(\alpha \in R)$ are researched, and the structure and distributing of periodicity petal and topological law of periodicity orbits of the generalized M sets are explored. And the generalized M set contains abundant information of structure of the generalized J sets by founding the whole portray of the generalized J sets based on the generalized M set qualitatively.

When systems are considered in the complex plane, people's attentions are focused on the fractal properties, such as Julia set. For example, the authors presented the definition and constructing arithmetic of the generalized Mandelbrot–Julia sets in bicomplex numbers system and studied the connectedness of the generalized M–J sets, the feature of the generalized Tetrabrot, and the relationship between the generalized M sets and its corresponding generalized J sets for bicomplex numbers in theory in [16]. Fractal nature exists not only in complex plane but also in higher dimensional space. The general Mandelbrot sets and Julia sets on the mapping $z \leftarrow z^\alpha + c$ $(\alpha \in N)$ are discussed and the 3-D projections are constructed using escape time algorithm and ray-tracing method in [17] and the connectness of the general quaternionic M sets is proved and the boundary of the stability region of the fixed point is calculated. And the hyperdimensional generalized Mandelbrot–Julia sets in hypercomplex number system are listed out in [13]. In [12], using the escape-time method, Julia set of quasi-sine Fibonacci function is constructed and the authors discover that the Julia set is fractal and is on the x-axis symmetry and the Mandelbrot set is also on the x-axis symmetry. Adopting the experimental mathematics method combining complex variable function theory with computer aided drawing, the structural characteristic and the fission-evolution law of additive and stochastic perturbed generalized Mandelbrot–Julia sets are researched in [11] and [18] respectively. And the corresponding relationship between point coordinates in generalized M set and the general structure of generalized J sets has been founded qualitatively and the physical meaning of the generalized M–J sets has been expounded in [11]. The influence of stochastic perturbed parameters of the structure of generalized M-sets is analyzed in [18]. And the complex dynamical properties and the existence of hairs in the Julia set of the map (1) in the complex plane have been discussed in [1].

These results about Julia sets generated from one complex iterative equation are related to the research on the properties, graphs of a single Julia set. However, many phenomena depicted by Julia set are related to multiple complex iterative equations, such as the complex dissipative standard map. Sometimes the behavior described by Julia set need to show different forms and one behavior need to change to be another behavior according to the practical applications. Therefore, control and synchronization of Julia sets are necessary to consider. Recently, some control methods are applied to control the Julia sets of complex systems in [8, 25, 26]. And in [24], feedback control is taken to achieve the control and synchronization of Julia sets of the system (1).

The dissipative standard map discussed in [19] is defined in the real domain and properties about the periodic orbits are studied. In this paper, we discuss the fractal dynamics of the dissipative standard system, which is in the complex plane and Julia set of the complex dissipative standard system is introduced by use of the boundedness of the orbits. Rewrite the system in the complex plane as

$$\begin{aligned} p_{n+1} &= bp_n + (1-b)\alpha + \beta\sin(q_n), \\ q_{n+1} &= q_n + p_{n+1}, \end{aligned} \tag{2}$$

where α and β are complex numbers, b is real number and $0 \leqslant b \leqslant 1$. By using the boundedness of the forward and backward orbits, Julia set of system (2) is defined, which is generated from two complex iterative equations. And visualization of Julia set of system (2) in the complex plane $\mathbb{C}^2$ is also given in Section 2. Many control methods are associated with the fixed points or the periodic orbits, which appear in the controller. In Section 3, a controller is designed to control the Julia set of system (2) and the value of the fixed points do not need to know in advance. In Section 4, relations about two different Julia sets of the system (2) with different parameters is discussed and a coupling item is also designed to achieve the synchronization of them, which makes one Julia set change to be the other. The simulations illustrate the efficacy of these methods.

2 Preliminaries

Let $F: \mathbb{C}^2 \to \mathbb{C}^2$ be the complex dissipative standard map (2), that is, $F(p,q) = (bp + (1-b)\alpha + \beta\sin q,\ q + (bp + (1-b)\alpha + \beta\sin q))$. Let

$$K^+ = \{(p,q) \in \mathbb{C}^2 \mid \{F^n(p,q)\}_{n\geqslant 1} \text{ is bounded in } \mathbb{C}\},$$

and

$$K^- = \{(p,q) \in \mathbb{C}^2 \mid \{F^n(p,q)\}_{n\leqslant 1} \text{ is bounded in } \mathbb{C}\}$$

denote the sets of points whose forward and backward orbits are bounded respectively. And let

$$J = \partial K^+ \cap \partial K^-,$$

which is called the Julia set of the complex dissipative standard map, where ∂K^+, ∂K^- denote the boundary of K^+, K^-, respectively.

Since p, q are complex numbers, we suppose $(p,q) = (x,y,s,t)$, where x,y,s,t are real numbers. We know Julia set of F is a four dimensional set from the definition. In order to depict Julia set of F with simulation, one of the simplest method is to fix one of the complex coordinates in J such as q, that is, we can consider the set with the form $J_{q_0} = J \cap \{q = q_0\}$. Then we can depict Julia set in the $p = (x,y)$ plane. The other method is to fix one of the four real coordinates and we can depict Julia set by three dimensional simulations. For example, we can fix $t = t_0$ and consider the set of points with the form (x,y,s,t_0) and depict them in three dimension (x,y,s). In this paper, we take the second method to depict Julia set of the complex dissipative standard map. Changing the values of t, we can get the four dimensional description of J. Specially, in this paper, we take $t = 0$.

3 Control of Julia sets of the dissipative standard map

In the investigation of means of controlling trajectory of systems, many methods have been introduced in [2, 6, 7, 21, 22, 24], such as the OGY method, the feedback control method and so on. Control of Julia sets is an important topic to study in nonlinear complex dynamical system. Julia set of the dissipative standard map (2) is closely related to the trajectory of the system from the definition of Julia set of system (2). In this section, control of Julia set of system (2) is achieved by changing the trajectory of the system.

Julia set of the complex dissipative standard map (2) is determinate once the system parameters α, β and b are given. However, sometimes we hope the behavior depicted by Julia set could change according to our needs. Add the controller $u_n(k) = [u_n^1(k),\, u_n^2(k)]^{\mathrm{T}}$ to (2), where k is the control parameter, then we get the controlled system

$$\begin{aligned} p_{n+1} &= bp_n + (1-b)\alpha + \beta\sin(q_n) + u_n^1(k), \\ q_{n+1} &= q_n + p_{n+1} + u_n^2(k). \end{aligned} \tag{3}$$

Julia sets of system (3) are different when k are taken different values. Thus, we can get the desired Julia set by means of taking the appropriate value of k.

From the definition of Julia set of the complex dissipative standard map, it is easy to see that the boundedness of the iteration of the system is important to the structure of its Julia set. Therefore, Julia set control can be achieved by controlling the trajectories of the dissipative standard map. Specially, we can consider the stability of the fixed point of the complex dissipative standard map. Thus, we can achieve the control of the attractive domain of the fixed point and then control of Julia set will be achieved.

Take the controller to be

$$u_n = \begin{bmatrix} u_n^1(k) \\ u_n^2(k) \end{bmatrix} = \begin{bmatrix} -k[(b-1)p_n + (1-b)\alpha + \beta\sin(q_n)] \\ -kp_{n+1} \end{bmatrix}, \tag{4}$$

where k is the complex control parameter. Then we get the controlled system

$$\begin{aligned} p_{n+1} &= bp_n + (1-b)\alpha + \beta\sin(q_n) - k\big[(b-1)p_n + (1-b)\alpha + \beta\sin(q_n)\big], \\ q_{n+1} &= q_n + p_{n+1} - kp_{n+1}. \end{aligned} \tag{5}$$

It is obvious that we do not need to compute the fixed points in advance since it does not appear in the controller (4). However, for convenience to discuss, let (p^*, q^*) be the fixed point of the complex dissipative standard map (2) and it is also the fixed point of the controlled system. Then $p^* = 0$, $\sin q^* = (b-1)\alpha/\beta$ and the Jacobian matrix at the fixed point (p^*, q^*) of system (5) is

$$M = \begin{bmatrix} k + b(1-k) & (1-k)\sqrt{\beta^2 - (b-1)^2\alpha^2} \\ (1-k)[k + b(1-k)] & 1 + (1-k)^2\sqrt{\beta^2 - (b-1)^2\alpha^2} \end{bmatrix}.$$

So the eigenvalues of M are obtained

$$\begin{aligned} \lambda_{1,2} = {}& \frac{[k + b(1-k) + 1 + (1-k)^2\sqrt{\beta^2 - (b-1)^2\alpha^2}]}{2} \\ & \pm \frac{\sqrt{[k + b(1-k) + 1 + (1-k)^2]^2 - 4(k + b(1-k))}}{2}. \end{aligned}$$

It is well known that $|\lambda_i| < 1$ $(i = 1, 2)$ are one of the conditions to guarantee the stability of the fixed point.

We have carried out the numerical calculations and simulations for the control of Julia set of complex dissipative standard map (2). The parameters used in present work are $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$. Figure 1 is the Julia set without control. In order to observe Julia set clearly, we select four cross sections in simulation.

Then the fixed point $(p^*, q^*) = (0, -1.5708 + 1.2801\mathrm{i})$ is obtained. Consider the stability of the fixed point of the controlled system (5). The range of the control parameter k can be obtained from

$$\begin{aligned} \big| & \big[0.5 + 0.5k + 1.3276(1-k)^2\big] \\ & \pm \sqrt{[0.5 + 0.5k + 1.3276(1-k)^2]^2 - (2 + 2k)}\big| < 2 \end{aligned}$$

since $|\lambda_i| < 1$ $(i = 1, 2)$. In order to observe the progress of changing of Julia set of the controlled system, we select some special values of the control parameter k, see Fig. 2.

From Fig. 2, we can see that Julia sets of the controlled system are shrinking with decreasing of the control parameter k. Obviously, we can take the appropriate values of k to get the desired Julia sets.

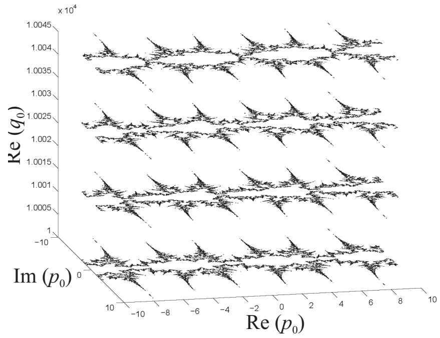
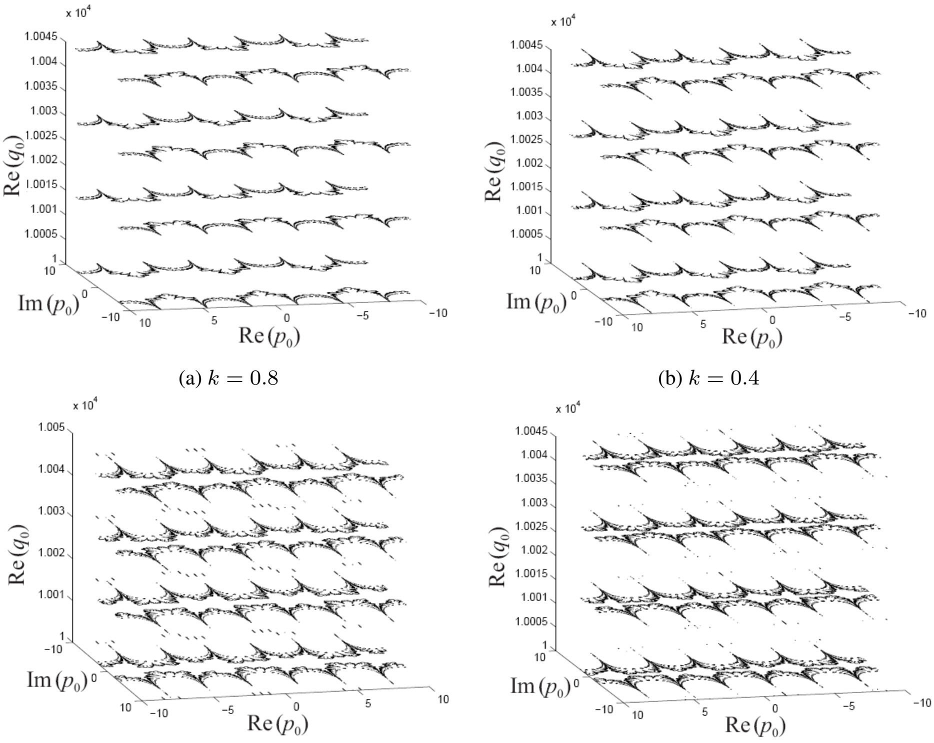


Figure 1. Julia sets of the dissipative standard map (2) when $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$.



(a) $k = 0.8$ (b) $k = 0.4$

(c) $k = -0.2$ (d) $k = -0.6$

Figure 2. The changing of Julia sets of the controlled system (10) when $a = 0.07$, $b = 0.02$, $r = 0.01$, $c = 0.2$, $d = 1.5$.

4 Synchronization of Julia sets of the dissipative standard map

Synchronization is a widespread phenomenon occurring in different fields. Many applications of the synchronization in mechanics, communication and so on have shown that synchronization is extremely important in nonlinear science discussed in [5, 9, 20]. It is well known that once the system parameters are given, the corresponding Julia sets are determinate. Though much work has been done on Julia set in the past few years, the work mainly deals with the structure, properties, graph of a single Julia set. However, we also need to consider the relations and change of two different Julia sets. In recent years, synchronization of Julia sets are discussed [23, 26]. Consider two multi-dimensional systems

$$z_{n+1} = f(z_n), \tag{6}$$

$$w_{n+1} = g(w_n). \tag{7}$$

Their Julia sets are determinate when the parameters of the systems are given and denote them by J^* and J respectively. In order to make the Julia sets of (6) and (7) associated, we couple systems (6) and (7). Add a coupling item in system (7)

$$w_{n+1} = g(w_n) + h(z_n, w_n; l), \tag{8}$$

where $h(z_n, w_n; l)$ is the coupling item about z_n and w_n, l is the coupling strength. It is obvious for every l, there exists a Julia set denoted by J_l for (8). Julia sets synchronization of (6) and (8) occurs if

$$\lim_{l \to l_0} (J_l \cup J^* - J_l \cap J^*) = \emptyset \tag{9}$$

for some l_0 (l_0 may be ∞, l and l_0 are multi-dimensional vectors), which means that Julia sets J_l will change to be arbitrarily approximate to Julia set J^* as $l \to l_0$.

From the definition of Julia set, the synchronization of Julia sets is achieved by taking the limitation of the coupling strength l. In simulation, we can not depict Julia sets synchronization as chaos synchronization because the Julia set is determinate once the parameters are given. However we can exhibit the progress of synchronization by taking different corresponding values of l, which can be seen in the succeeding examples. And l illustrates the synchronization status of the corresponding Julia sets J_l, where we take the same iterative original values in systems (6) and (8).

Consider a system which has the same structure to the system (2) but with different system parameters

$$\begin{aligned} s_{n+1} &= b s_n + (1-b)\alpha + \beta' \sin(t_n), \\ t_{n+1} &= t_n + s_{n+1}, \end{aligned} \tag{10}$$

where $\beta \neq \beta'$. We call the system (10) the response system and the system (2) the driving system. Julia sets of systems (2) and (10) are determinate once the parameters are given. In order to associate the Julia sets of systems (2) and (10), we add the coupled items to the response system (10). Then we have

$$\begin{aligned} s_{n+1} &= b s_n + (1-b)\alpha + \beta' \sin(t_n) \\ &\quad + l\big[\big(b p_n + \beta \sin(q_n)\big) - \big(b s_n + \beta' \sin(t_n)\big)\big], \\ t_{n+1} &= t_n + s_{n+1} + l\big[(q_n + p_{n+1}) - (t_n + s_{n+1})\big], \end{aligned} \tag{11}$$

where l is the coupling strength. Therefore,

$$s_{n+1} - p_{n+1} = (1-l)\big[b(s_n - p_n) + \big(\beta' \sin(t_n) - \beta \sin(q_n)\big)\big].$$

From the definition of Julia set of the complex system, we can suppose that the sets $\{t_n\}_{n=1}^{\infty}$ and $\{q_n\}_{n=1}^{\infty}$ are bounded. So there is a $M > 0$ satisfying $|\beta' \sin(t_n) - \beta \sin(q_n)| < M$. Thus,

$$\begin{aligned}
&|s_{n+1} - p_{n+1}| \\
&\quad \leqslant |1-l||b||s_n - p_n| + |1-l|M \\
&\quad \leqslant |1-l||b|\big(|1-l||b||s_{n-1} - p_{n-1}| + |1-l|M\big) + |1-l|M \\
&\quad = |1-l|^2|b|^2|s_{n-1} - p_{n-1}| + M\big(|1-l|^2|b| + |1-l|\big) \leqslant \cdots \\
&\quad \leqslant |1-l|^n|b|^n|s_1 - p_1| + M\big(|1-l|^n|b|^{n-1} + \cdots + |1-l|^2|b| + |1-l|\big) \\
&\quad = |1-l|^n|b|^n|s_1 - p_1| + M\frac{|1-l|(1-|1-l|^n|b|^n)}{1-|1-l||b|}.
\end{aligned}$$

Since $0 \leqslant b \leqslant 1$, we have $|s_{n+1} - p_{n+1}| \to 0$ as $n \to \infty$ and $l \to 1$. Therefore, there is a $T > 0$ such that $|s_{n+1} - p_{n+1}| < T$ for n being large enough and l being close to 1. Then

$$|t_{n+1} - q_{n+1}| \leqslant |1-l||t_n - q_n| + |1-l|T.$$

With the similar previous analysis, we have $|t_{n+1} - q_{n+1}| \to 0$ as $n \to \infty$ and $l \to 1$. So the synchronization of the trajectories of systems (2) and (11) is achieved and the Julia sets of system (11) are changing to be the Julia set of the driving system (2). Therefore, the synchronization of Julia sets of the driving system and the response system is achieved.

For example, we take $\alpha = 3.1$, $\beta = 0.1$, $b = 0.5$ in the driving system and Julia set is shown in Fig. 3. And we take $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$ in the responsive system and Julia set is shown in Fig. 1.

Figure 4 is the changing of Julia sets of the system (11) with the parameter l. It is obvious that Julia sets are changing to be the Julia set of the driving system shown in Fig. 3 as $l \to 1$.

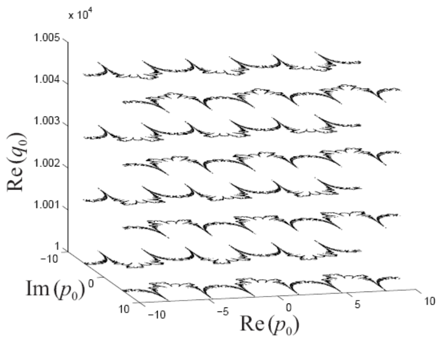
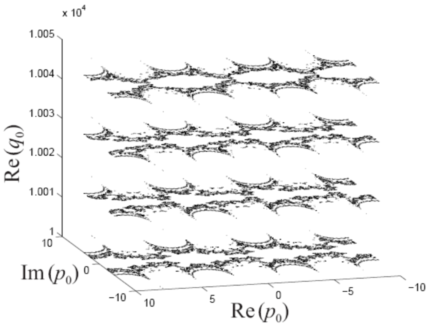
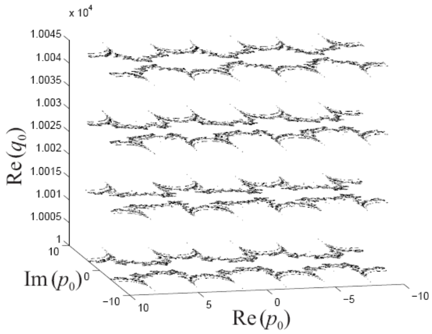


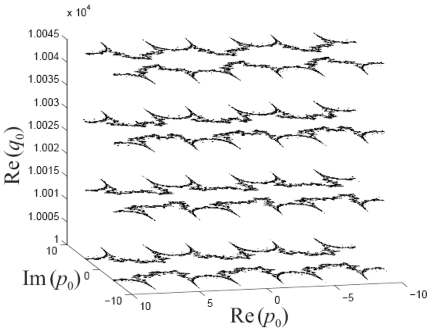
Figure 3. Julia sets of the dissipative standard map (2) when $\alpha = 3.1$, $\beta = 0.1$, $b = 0.5$.



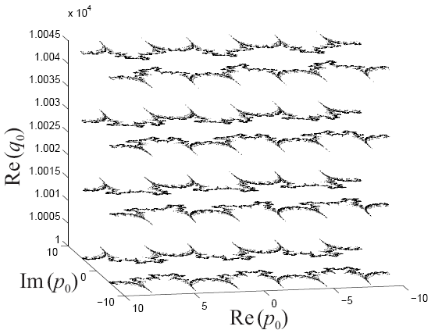
(a) $l = 0.1$



(b) $l = 0.3$

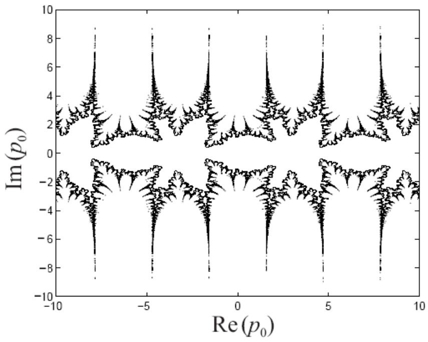


(c) $l = 0.5$

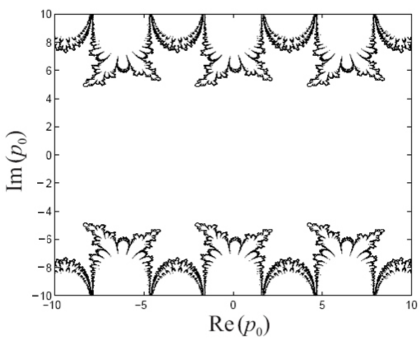


(d) $l = 0.8$

Figure 4. The changing of Julia sets of the controlled system (11) when $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$.



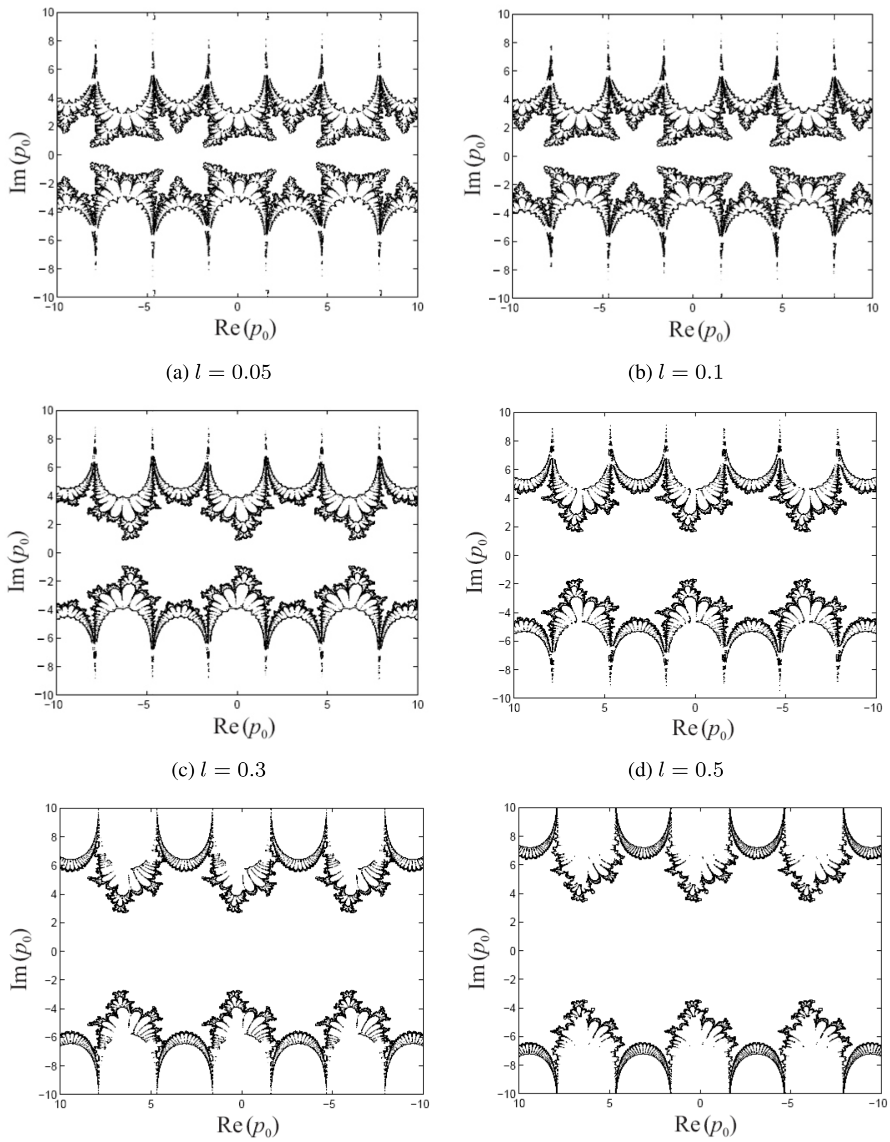
(a)



(b)

Figure 5. Julia sets of the dissipative standard map (2) when $\alpha = 3.1$, $\beta = 0.1$, $b = 0.5$.

To observe the synchronization of Julia sets more clearly, we take a cross section of Julia set in the three dimensional simulation since they are unclear in the three dimensional simulation.



(a) $l = 0.05$ (b) $l = 0.1$

(c) $l = 0.3$ (d) $l = 0.5$

(e) $l = 0.7$ (f) $l = 0.8$

Figure 6. The changing of Julia sets of the controlled system (11) when $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$.

Figures 5a and 5b are the cross sections of the original Julia sets when $\alpha = 3.1$, $\beta = 0.8$, $b = 0.5$ and $\alpha = 3.1$, $\beta = 0.01$, $b = 0.5$, respectively. Figure 6 is the changing of the cross sections of Julia sets of the system (11) with the parameter l. It is obvious that Julia sets are changing to be the Julia set of the driving system shown in Fig. 5b as $l \to 1$.

5 Conclusions

Julia set of the dissipative standard map (2) is introduced in this paper. According to the practical circumstance, the behavior depicted by Julia set need to be controlled. A controller is designed to make Julia set of the complex dissipative system shrink or expand with changing of the control parameter. And we do not need to know the value of the fixed point in advance for this control method. In order to discuss the relations of two Julia sets, coupling items are designed to realize one Julia set change to be the other. The efficacy of these methods is illustrated by simulations.

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