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Solvability of boundary value problem for second order impulsive differential equations with one-dimensional p -Laplacian on whole line*

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Abstract. This paper is concerned with a class of boundary value problems of the impulsive differential equations with one-dimensional p -Laplacian on whole line with a non-Carathéodory nonlinearity. Sufficient conditions to guarantee the existence of solutions are established. Some examples are given to illustrate the main results.

Keywords: second order impulsive differential equation on whole line, one-dimensional p -Laplacian, boundary value problem, solution, fixed point theorem.

1 Introduction

The motivation for the present work stems from both practical and theoretical aspects. In fact, boundary value problems on the whole line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations modelling various physical phenomena such as unsteady flow of gas through a whole, porous media, and the theory of drain flows.

The asymptotic theory of ordinary differential equations is an area in which there is great activity among a large number of investigators. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole line (half line). The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole line (half line).

In recent years, the existence of solutions of boundary value problems of the differential equations governed by nonlinear differential operator $[\Phi(u')] = [|u'|^{p-2}u']'$ has been studied by many authors, see [5, 7, 8, 9, 10, 11, 12, 13, 15, 17].

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Impulsive differential equation is one of the main tools to study the dynamics of processes in which sudden changes occur. The theory of impulsive differential equation has recently received considerable attention. However, the study on existence of positive solutions of nonlocal boundary value problems for impulsive differential equations on whole real line has not been sufficiently developed [1, 2, 3, 4, 5, 6, 16].

In all above mentioned papers, the boundary conditions are subjected to the two end points 0 and $+\infty$ (or $-\infty$ and $+\infty$) and the obtained solutions are defined on $[0, +\infty)$ (or \mathbb{R}). An interesting question occurs: when one subjects the boundary conditions on two intermediate points ξ, η , how can we get solutions defined on \mathbb{R} of a boundary value problem of differential equations on whole line? On the other hand, in known papers [7, 8, 9, 10, 11, 12, 13, 14], concerning the differential equations $[\Phi(\rho(t)x'(t))] + p(t)f(t, x(t), \rho(t)x'(t)) = 0$, it is supposed that $(t, u, v) \rightarrow p(t)f(t, u, v)$ is a Carathéodory function. To the best of our knowledge, there has been no paper concerning the solvability of $[\Phi(\rho(t)x'(t))] + p(t)f(t, x(t), \rho(t)x'(t)) = 0$ with $(t, u, v) \rightarrow p(t)f(t, u, v)$ being a non-Carathéodory function.

Motivated by mentioned papers, to fill this gap, we consider the following boundary value problem for the impulsive singular differential equation on the whole line:

$$\begin{aligned} & [\Phi(\rho(t)x'(t))] + p(t)f(t, x(t), \rho(t)x'(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, \\ & x(\xi) = \int_{-\infty}^{+\infty} m(s)\phi(s, x(s), \rho(s)x'(s)) ds, \\ & x(\eta) = \int_{-\infty}^{+\infty} n(s)\psi(s, x(s), \rho(s)x'(s)) ds, \\ & \Delta x(t_i) = I(t_i, x(t_i), \rho(t_i)x'(t_i)), \quad i \in \mathbb{Z}, \\ & \Delta \Phi(\rho(t_i)x'(t_i)) = J(t_i, x(t_i), \rho(t_i)x'(t_i)), \quad i \in \mathbb{Z}, \end{aligned} \tag{1}$$

where $\Phi(x) = |x|^{k-2}x$ with $k > 1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $1/k + 1/l = 1$, $\xi < \eta$ are constants, p is nonnegative and satisfies $p \in L^1_{\text{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t) dt = \int_0^{+\infty} p(s) ds = +\infty$, $\rho : \mathbb{R} \rightarrow [0, \infty)$ with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot) \in L^1_{\text{loc}}(\mathbb{R})$, $\tau(t) = 1 + |\int_{\eta}^t p(s) ds|$, and the following cases will be discussed:

- Case 1. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u) du = \int_0^{+\infty} \Phi^{-1}(\tau(u))/\rho(u) du = +\infty$;
- Case 2. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u) du < +\infty$, $\int_0^{+\infty} \Phi^{-1}(\tau(u))/\rho(u) du < +\infty$;
- Case 3. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u) du = +\infty$, $\int_0^{+\infty} \Phi^{-1}(\tau(u))/\rho(u) du < +\infty$;
- Case 4. $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u) du < +\infty$, $\int_0^{+\infty} \Phi^{-1}(\tau(u))/\rho(u) du = +\infty$;

$f, \phi, \psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ are strong Carathéodory functions, $m, n \in L^1(\mathbb{R})$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $\{t_i : i \in \mathbb{Z}\}$ is a increasing sequence with $\lim_{i \rightarrow -\infty} t_i = -\infty$ and $\lim_{i \rightarrow +\infty} t_i = +\infty$, $\Delta x(t_i) = \lim_{t \rightarrow t_i^+} x(t) - x(t_i)$, $\Delta \Phi(\rho(t_i)x'(t_i)) = \lim_{t \rightarrow t_i^+} \Phi(\rho(t)x'(t)) - \Phi(\rho(t_i)x'(t_i))$, $I, J : \{t_i : i \in \mathbb{Z}\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are discrete Carathéodory functions.

The homogeneous boundary conditions $x(\xi) = 0, x(\eta) = 0$ of special case in (1) come from the four-point boundary conditions $a \lim_{t \rightarrow -\infty} x(t) - bx(\xi) = c \lim_{t \rightarrow +\infty} (t) - dx(\eta) = 0$ (if $a = c = 0$ and $b = d = 1$), which arise in the study of heat flow problems involving a bar of unit length with two controllers at $t = -\infty$ and $t = +\infty$ adding or removing heat according to the temperatures detected by two sensors at $t = \xi$ and $t = \eta$. It is well known that

$$\begin{aligned} & [\Phi(\rho(t)x'(t))] + p(t)f(t, x(t), \rho(t)x'(t)) = 0, \quad \text{a.e. } t \in [\xi, \eta], \\ x(\xi) &= \int_{\xi}^{\eta} m(s)\phi(s, x(s), \rho(s)x'(s)) \, ds, \\ x(\eta) &= \int_{\xi}^{\eta} n(s)\psi(s, x(s), \rho(s)x'(s)) \, ds \end{aligned}$$

is called Dirichlet boundary value problem with integral boundary conditions whose solutions are defined on $[\xi, \eta]$. In this sense, BVP (1) is a generalization of Dirichlet boundary value problem.

Consider the problem $(tx'(t))' = 1, \text{ a.e. } t \in \mathbb{R}, x(-1) = x(1) = 1$. It is easy to know from $(tx'(t))' = 1, \text{ a.e. } t \in \mathbb{R}$, that $x(t) = c_1 - c_2 \ln |t| + t$. Thus this problem has no continuous solution. Consider the problem $(\sqrt{|t|x'(t)})' = 1, \text{ a.e. } t \in \mathbb{R}, x(-1) = x(1) = 1$. One can get from $(\sqrt{|t|x'(t)})' = 1, \text{ a.e. } t \in \mathbb{R}$, that $x(t) = (2/3)t^{3/2} + 2c_1\sqrt{|t|} + c_2$ for $t > 0$ and $x(t) = -(2/3)|t|^{3/2} - 2c_1\sqrt{|t|} + c_2$ for $t < 0$. Thus the mentioned problem has infinitely many continuous solutions

$$x(t) = \begin{cases} \frac{2}{3}t^{3/2} + 2c_1\sqrt{t} + c_2, & t \geq 0, \\ -\frac{2}{3}|t|^{3/2} - 2c_3\sqrt{|t|} + c_2, & t < 0. \end{cases}$$

Here $c_1, c_2, c_3 \in \mathbb{R}$. So this kind of problem is interesting.

Our purpose is to establish sufficient conditions for the existence of solutions of BVP (1) in Cases 1–4, respectively. The remainder of this paper is organized as follows: the first result are given in Section 2 in Case 1, the existence result of solutions of BVP (1) in Cases 2 is given in Section 3, and similarly, we can establish existence results in Cases 3 and 4, respectively, we omit the details. Finally, in Section 4, two examples are given to illustrate the main results.

2 Solvability of (1) in Case 1

In this section, we present existence result of BVP (1) in Case 1. Denote

$$\tau(t) = 1 + \left| \int_{\eta}^t p(s) \, ds \right|, \quad \sigma(t) = 1 + \left| \int_{\xi}^t \frac{\Phi^{-1}(\tau(u))}{\rho(u)} \, du \right|. \tag{2}$$

It is easy to show that σ, τ are continuous on \mathbb{R} and $\lim_{t \rightarrow \pm\infty} \sigma(t) = \lim_{t \rightarrow \pm\infty} \tau(t) = +\infty$.

Definition 1. $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a strong Carathéodory function if

- (i) $t \rightarrow F(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ is measurable on \mathbb{R} for any $u, v \in \mathbb{R}$;
- (ii) $(u, v) \rightarrow F(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ is continuous on \mathbb{R}^2 for a.e. $t \in \mathbb{R}$;
- (iii) for each $r > 0$, there exists nonnegative function $M_r \geq 0$ such that $|u|, |v| \leq r$ implies $|F(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| \leq M_r$, a.e. $t \in \mathbb{R}$.

Definition 2. $H : \{t_i : i \in \mathbb{Z}\} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a discrete Carathéodory function if

- (i) $(x, y) \rightarrow H(t_s, \sigma(t_s)x, \Phi^{-1}(\tau(t_s))y)$ is continuous on \mathbb{R}^2 for all $s \in \mathbb{Z}$;
- (ii) for each $r > 0$, there exists nonnegative constants $M_{ir} \geq 0$ ($i \in \mathbb{Z}$) such that $|x|, |y| \leq r$ implies $|H(t_s, \sigma(t_s)x, \Phi^{-1}(\tau(t_s))y)| \leq M_{sr}$, $s \in \mathbb{Z}$, $\sum_{s=-\infty}^{+\infty} M_{sr} < +\infty$.

Definition 3. Let X be a real Banach space. An operator $T : X \rightarrow X$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Choose

$$X = \left\{ x : \mathbb{R} \rightarrow \mathbb{R} : x|_{(t_s, t_{s+1}]}, \rho x'|_{(t_s, t_{s+1}]} \text{ is continuous, } s \in \mathbb{Z}; \right.$$

the following limits exist and are finite: $\lim_{t \rightarrow t_s^+} x(t), \lim_{t \rightarrow t_s^+} \rho(t)x'(t), s \in \mathbb{Z}$,

$$\left. \lim_{t \rightarrow -\infty} \frac{x(t)}{\sigma(t)}, \lim_{t \rightarrow +\infty} \frac{x(t)}{\sigma(t)}, \lim_{t \rightarrow -\infty} \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}, \lim_{t \rightarrow +\infty} \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \right\}.$$

For $x \in X$, define $\|x\| = \max\{\sup_{t \in \mathbb{R}} |x(t)|/\sigma(t), \sup_{t \in \mathbb{R}} \rho(t)|x'(t)|/\Phi^{-1}(\tau(t))\}$.

Lemma 1. X is a Banach space with $\|\cdot\|$ defined.

Proof. It is easy to see that X is a normed linear space. Let $\{x_u\}$ be a Cauchy sequence in X . Then $\|x_u - x_v\| \rightarrow 0, u, v \rightarrow +\infty$. We will prove that there $x_0 \in X$ such that $x_u \rightarrow x_0$ as $u \rightarrow +\infty$. Since $x_u \in X$, we have

$$\sup_{t \in \mathbb{R}} \frac{|x_u(t) - x_v(t)|}{\sigma(t)} \rightarrow 0, \quad \sup_{t \in \mathbb{R}} \frac{\rho(t)|x'_u(t) - x'_v(t)|}{\Phi^{-1}(\tau(t))} \rightarrow 0, \quad u, v \rightarrow +\infty.$$

So

$$\sup_{t \in (t_s, t_{s+1}]} \frac{|x_u(t) - x_v(t)|}{\sigma(t)} \rightarrow 0, \quad u, v \rightarrow +\infty,$$

$$\sup_{t \in (t_s, t_{s+1}]} \frac{\rho(t)|x'_u(t) - x'_v(t)|}{\Phi^{-1}(\tau(t))} \rightarrow 0, \quad u, v \rightarrow +\infty,$$

$$\lim_{t \rightarrow t_s^+} \frac{x_u(t)}{\sigma(t)}, \lim_{t \rightarrow t_s^+} \frac{\rho(t)x'_u(t)}{\Phi^{-1}(\tau(t))} \text{ exists, } s \in \mathbb{Z}.$$

Then there exists functions $x_{s,0}, y_{s,0} \in C^0[t_s, t_{s+1}]$ such that $\lim_{u \rightarrow +\infty} x_u(t)/\sigma(t) = x_{s,0}(t)$ and $\lim_{u \rightarrow +\infty} \rho(t)x'_u(t)/\Phi^{-1}(\tau(t)) = y_{s,0}(t)$ uniformly on $[t_s, t_{s+1}]$.

Define $x_0(t) = x_{s,0}(t), y_0(t) = y_{s,0}(t)$ for all $t \in (t_s, t_{s+1}] (s \in \mathbb{Z})$. Then $x_0, y_0 : \mathbb{R} \rightarrow \mathbb{R}$ is well defined on \mathbb{R} , and

$$\lim_{u \rightarrow +\infty} \frac{x_u(t)}{\sigma(t)} = x_0(t), \quad \lim_{u \rightarrow +\infty} \frac{\rho(t)x'_u(t)}{\Phi^{-1}(\tau(t))} = y_0(t), \quad t \in \mathbb{R}.$$

It follows that

$$\sup_{t \in (t_s, t_{s+1}]} \left| \frac{x_u(t)}{\sigma(t)} - x_0(t) \right| \rightarrow 0, \quad \sup_{t \in (t_s, t_{s+1}]} \left| \frac{\rho(t)x_u(t)}{\Phi^{-1}(\tau(t))} - y_0(t) \right|, \quad u \rightarrow +\infty.$$

Now we do the following three steps:

- Step 1. Prove that $\sigma(\cdot)x_0(\cdot), \Phi^{-1}(\tau(\cdot))y_0(\cdot) \in C^0(t_s, t_{s+1}]$ and $\lim_{t \rightarrow t_s^+} \sigma(t)x_0(t)$ and $\lim_{t \rightarrow t_s^+} \Phi^{-1}(\tau(t))y_0(t)$ exist.
- Step 2. Prove that the limits $\lim_{t \rightarrow -\infty} x_0(t), \lim_{t \rightarrow +\infty} x_0(t), \lim_{t \rightarrow -\infty} y_0(t), \lim_{t \rightarrow +\infty} y_0(t)$ exist.
- Step 3. Prove that $y_0(t) = \rho(t)[\sigma(t)x_0(t)]'/\Phi^{-1}(\tau(t))$.

The details are omitted. It follows that $x_u \rightarrow x_0$ as $u \rightarrow +\infty$. So X is a Banach space. □

Lemma 2. *Let M be a subset of X . Then M is relatively compact if and only if the following conditions are satisfied:*

- (i) both $\{t \rightarrow x(t)/\sigma(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are uniformly bounded;
- (ii) both $\{t \rightarrow x(t)/\sigma(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are equicontinuous in $(t_s, t_{s+1}] (s \in \mathbb{N})$;
- (iii) both $\{t \rightarrow x(t)/\sigma(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are equi-convergent as $t \rightarrow \pm\infty$.

Proof. (\Leftarrow) From Lemma 1 we know X is a Banach space. In order to prove that the subset M is relatively compact in X , we only need to show M is totally bounded in X , that is, for all $\epsilon > 0, M$ has a finite ϵ -net.

For any given $\epsilon > 0$, by (i)–(iii) there exist constants $M > 0, \delta > 0, t_{s_0} > 0$ and $t_{-s_0} < 0$ such that

$$\sup_{t \in \mathbb{R}} \frac{|x(t)|}{\sigma(t)}, \sup_{t \in \mathbb{R}} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))} \leq M, \quad x \in M,$$

$$\left| \frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)} \right| \leq \frac{\epsilon}{3}, \quad \left| \frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))} \right| < \frac{\epsilon}{3}, \quad x \in M,$$

$$w_1, w_2 \in (t_s, t_{s+1}], |w_1 - w_2| < \delta, s = -s_0, -s_0 + 1, \dots, s_0 - 1,$$

$$\left| \frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)} \right| \leq \frac{\epsilon}{3}, \quad \left| \frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))} \right| < \frac{\epsilon}{3}, \quad x \in M,$$

$$w_1, w_2 \leq t_{-s_0} \text{ or } w_1, w_2 \geq t_{s_0}.$$

Define

$$X|_{(-t_{s_0}, t_{s_0})} = \left\{ x: x, \rho(t)x' \in C(t_s, t_{s+1}], s = -s_0, -s_0 + 1, \dots, s_0 - 1, \right.$$

$$\lim_{t \rightarrow t_s^+} \frac{x(t)}{\sigma(t)} \text{ exist, } s = -s_0, -s_0 + 1, \dots, s_0 - 1,$$

$$\left. \lim_{t \rightarrow t_s^+} \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \text{ exist, } s = -s_0, -s_0 + 1, \dots, s_0 - 1 \right\}.$$

For $x \in X|_{(-t_{s_0}, t_{s_0})}$, define

$$\|x\|_{s_0} = \max \left\{ \max_{t \in (-t_{s_0}, t_{s_0})} \frac{|x(t)|}{\sigma(t)}, \max_{t \in (-t_{s_0}, t_{s_0})} \frac{\rho(t)|x'(t)|}{\Phi^{-1}(\tau(t))} \right\}.$$

Similarly to Lemma 1, we can prove that $X|_{(-t_{s_0}, t_{s_0})}$ is a Banach space.

Let $M|_{(-t_{s_0}, t_{s_0})} = \{t \rightarrow x(t), t \in (-t_{s_0}, t_{s_0}): x \in M\}$. Then $M|_{(-t_{s_0}, t_{s_0})}$ is a subset of $X|_{(-t_{s_0}, t_{s_0})}$. By (i), (ii), and Ascoli–Arzela theorem, we can know that $M|_{(-t_{s_0}, t_{s_0})}$ is relatively compact. Thus there exist $x_1, x_2, \dots, x_k \in M$ such that, for any $x \in M$, we have that there exists some $i = 1, 2, \dots, k$ such that

$$\|x - x_i\|_{s_0} = \max \left\{ \sup_{t \in (-t_{s_0}, t_{s_0})} \frac{|x(t) - x_i(t)|}{\sigma(t)}, \sup_{t \in (-t_{s_0}, t_{s_0})} \frac{\rho(t)|x'(t) - x_i'(t)|}{\Phi^{-1}(\tau(t))} \right\} < \frac{\epsilon}{3}.$$

Therefore, for $x \in M$, we can get $\|x - x_i\|_X \leq \epsilon$. So, for any $\epsilon > 0$, M has a finite ϵ -net $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$, that is, M is totally bounded in X . Hence M is relatively compact in X .

(\Rightarrow) Assume that M is relatively compact, then for any $\epsilon > 0$, there exists a finite ϵ -net of M . Let the finite ϵ -net be $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$ with $x_i \in M$. Then for any $x \in M$, there exists U_{x_i} such that $x \in U_{x_i}$ and $\|x\| \leq \|x - x_i\| + \|x_i\| \leq \epsilon + \max\{\|x_i\|: i = 1, 2, \dots, k\}$. It follows that both $\{t \rightarrow x(t)/\sigma(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are uniformly bounded. Then (i) holds.

Furthermore, there exists $t_{-s_0} < 0$ and $t_{s_0} > 0$ such that $|x_i(w_1) - x_i(w_2)| < \epsilon$ for all $w_1, w_2 \geq t_{s_0}$ and all $w_1, w_2 \leq t_{-s_0}$ and $i = 1, 2, \dots, k$. Then we have for $w_1, w_2 \geq t_{s_0}$, all $w_1, w_2 \leq t_{-s_0}$ and $x \in M$ that

$$\left| \frac{x(w_1)}{\sigma(w_1)} - \frac{x(w_2)}{\sigma(w_2)} \right| \leq \left| \frac{x(w_1)}{\sigma(w_1)} - \frac{x_i(w_1)}{\sigma(w_1)} \right| + \left| \frac{x_i(w_1)}{\sigma(w_1)} - \frac{x_i(w_2)}{\sigma(w_2)} \right|$$

$$+ \left| \frac{x_i(w_2)}{\sigma(w_2)} - \frac{x(w_2)}{\sigma(w_2)} \right| < 3\epsilon.$$

Similarly, for $w_1, w_2 \geq t_{s_0}$, all $w_1, w_2 \leq t_{-s_0}$ and $x \in M$, we have that

$$\left| \frac{\rho(w_1)x'(w_1)}{\Phi^{-1}(\tau(w_1))} - \frac{\rho(w_2)x'(w_2)}{\Phi^{-1}(\tau(w_2))} \right| < 3\epsilon.$$

Thus (iii) is valid. Similarly, we can prove that (ii) holds. Consequently, the lemma is proved. \square

For ease expression, denote $G_x(t) = G(t, x(t), \rho(t)x'(t))$ for a function $G : \mathbb{R}^3 \rightarrow \mathbb{R}$ and $x \in X$.

Lemma 3. *Suppose that $x \in X$. Then there exists a unique $A_x \in \mathbb{R}$ such that*

$$\int_{\xi}^{\eta} \frac{\Phi^{-1}(A_x + \sum_{\xi \leq t_s < u} J_x(t_s) - \int_{\xi}^u p(w)f_x(w) \, dw)}{\rho(u)} \, du + \int_{-\infty}^{+\infty} m(w)\phi_x(w) \, dw - \int_{-\infty}^{+\infty} n(w)\psi_x(w) \, dw + \sum_{\xi \leq t_s < \eta} I_x(t_s) = 0, \quad (3)$$

and A_x satisfies

$$|A_x| \leq \sum_{\xi \leq t_s < \eta} |J_x(t_s)| + \int_{\xi}^{\eta} p(w)|f_x(w)| \, dw + \Phi \left(\frac{\int_{-\infty}^{+\infty} |m(w)||\phi_x(w)| \, dw + \int_{-\infty}^{+\infty} |n(w)||\psi_x(w)| \, dw + \sum_{\xi \leq t_s < \eta} |I_x(t_s)|}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right). \quad (4)$$

Proof. Denote

$$\begin{aligned} \sigma_1 &= \sum_{\xi \leq t_s < \eta} |J_x(t_s)| + \int_{\xi}^{\eta} p(w)|f_x(w)| \, dw \\ &\quad + \Phi \left(\frac{\int_{-\infty}^{+\infty} n(w)\psi_x(w) \, dw - \int_{-\infty}^{+\infty} m(w)\phi_x(w) \, dw - \sum_{\xi \leq t_s < \eta} I_x(t_s)}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right), \\ \sigma_2 &= - \sum_{\xi \leq t_s < \eta} |J_x(t_s)| - \int_{\xi}^{\eta} p(w)|f_x(w)| \, dw \\ &\quad + \Phi \left(\frac{\int_{-\infty}^{+\infty} n(w)\psi_x(w) \, dw - \int_{-\infty}^{+\infty} m(w)\phi_x(w) \, dw - \sum_{\xi \leq t_s < \eta} I_x(t_s)}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right). \end{aligned}$$

Let

$$G(c) = \int_{\xi}^{\eta} \frac{1}{\rho(u)} \Phi^{-1} \left(c + \sum_{\xi \leq t_s < u} J_x(t_s) - \int_{\xi}^u p(w) f_x(w) \, dw \right) \, du$$

$$+ \int_{-\infty}^{+\infty} m(w) \phi_x(w) \, dw - \int_{-\infty}^{+\infty} n(w) \psi_x(w) \, dw + \sum_{\xi \leq t_s < \eta} I_x(t_s).$$

Then we find that G is increasing on \mathbb{R} , $G(\sigma_1) \geq 0$ and $G(\sigma_2) \leq 0$. So there exists a unique $A_x \in [\sigma_2, \sigma_1]$ such that $G(A_x) = 0$. This is (3). Furthermore, we get from the definitions of σ_1, σ_2 that (4) holds. The proof is completed. \square

We define $\sum_{a \leq t_s < b} k_s := -\sum_{b \leq t_s < a} k_s$ for $a > b$. For $x \in X$, define $(Tx)(t)$ by

$$(Tx)(t) = \int_{-\infty}^{+\infty} m(s) \phi_x(s) \, ds + \sum_{\xi \leq t_s < t} I_x(t_s)$$

$$+ \int_{\xi}^t \frac{1}{\rho(u)} \Phi^{-1} \left(A_x + \sum_{\eta \leq t_s < u} J_x(t_s) - \int_{\eta}^u p(w) f_x(w) \, dw \right) \, du, \quad t \in \mathbb{R},$$

where A_x is defined by (3).

Lemma 4. *Suppose that f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$. Then*

- (i) $T : X \rightarrow X$ is well defined;
- (ii) $x \in X$ is a solution of (1) if and only if $x \in X$ is a fixed point of T in X ;
- (iii) T is completely continuous.

Proof. (i) From Lemma 1, X is a Banach space. For $x \in X$, we have $\|x\| \leq r$ for some $r \geq 0$. Then there exists constants $M_{r,f} \geq 0, M_{r,J,s} \geq 0$ and $M_{r,I,s} \geq 0$ such that

$$|f_x(t)| = \left| f \left(t, \sigma(t) \frac{x(t)}{\sigma(t)}, \Phi^{-1}(\tau(t)) \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \right) \right| \leq M_{r,f}, \quad \text{a.e. } t \in \mathbb{R},$$

$$|\phi_x(t)| = |\phi(t, x(t), \rho(t)x'(t))| \leq M_{r,\phi}, \quad \text{a.e. } t \in \mathbb{R},$$

$$|\psi_x(t)| = |\psi(t, x(t), \rho(t)x'(t))| \leq M_{r,\psi}, \quad \text{a.e. } t \in \mathbb{R},$$

$$|I_x(t_s)| = |I(t_s, x(t_s), \rho(t_s)x'(t_s))| \leq M_{r,I,s}, \quad s \in \mathbb{Z}, \quad \sum_{s=-\infty}^{+\infty} M_{r,I,s} < +\infty, \tag{5}$$

$$|J_x(t_s)| = |J(t_s, x(t_s), \rho(t_s)x'(t_s))| \leq M_{r,J,s}, \quad s \in \mathbb{Z}, \quad \sum_{s=-\infty}^{+\infty} M_{r,J,s} < +\infty.$$

One finds from the definition of Tx that $(Tx)(\cdot)|_{(t_s, t_{s+1}]}, \rho(\cdot)(Tx)'(\cdot)|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}]$, and the limits $\lim_{t \rightarrow t_s^+} (Tx)(t) (s \in \mathbb{Z})$ and $\lim_{t \rightarrow t_s^+} \rho(t)(Tx)'(t) (s \in \mathbb{Z})$ exist. One can show easily from $\lim_{t \rightarrow \pm\infty} \sigma(t) = +\infty, \lim_{t \rightarrow \pm\infty} \tau(t) = +\infty$ and (5) that

$$\begin{aligned} \lim_{t \rightarrow \pm\infty} \frac{\int_{-\infty}^{+\infty} m(s)\phi_x(s) ds + \sum_{\xi \leq t_s < t} I_x(t_s)}{\sigma(t)} &= 0, \\ \lim_{t \rightarrow \pm\infty} \frac{A_x + \sum_{\eta \leq t_s < t} J_x(t_s)}{\tau(t)} &= 0. \end{aligned}$$

On the other hand, for $t > \xi$, we have by using l'Hôpital's rule that

$$\begin{aligned} \lim_{t \rightarrow +\infty} \frac{\int_{\xi}^t \frac{1}{\rho(u)} \Phi^{-1}(A_x + \sum_{\eta \leq t_s < u} J_x(t_s) - \int_{\eta}^u p(w)f_x(w) dw) du}{\sigma(t)} \\ = - \lim_{t \rightarrow +\infty} \Phi^{-1} \left(f \left(t, \sigma(t) \frac{x(t)}{\sigma(t)}, \Phi^{-1}(\tau(t)) \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \right) \right). \end{aligned}$$

Since for each $r > 0, f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$, then

$$\lim_{t \rightarrow +\infty} \frac{\int_{\xi}^t \frac{1}{\rho(u)} \Phi^{-1}(A_x + \sum_{\eta \leq t_s < u} J_x(t_s) - \int_{\eta}^u p(w)f_x(w) dw) du}{\sigma(t)} \text{ exists.}$$

So $\lim_{t \rightarrow +\infty} (Tx)(t)/\sigma(t)$ exists. Similarly we can show that $\lim_{t \rightarrow -\infty} (Tx)(t)/\sigma(t)$ and $\lim_{t \rightarrow \pm\infty} \rho(t)(Tx)'(t)/\Phi^{-1}(\tau(t))$ exist. It follows that $Tx \in X$. Hence $T : X \rightarrow X$ is well defined.

(ii) By direct computation, we can get

$$\begin{aligned} [\Phi(\rho(t)(Tx)'(t))] + p(t)f(t, x(t), \rho(t)x'(t)) &= 0, \quad \text{a.e. } t \in \mathbb{R}, \\ (Tx)(\xi) &= \int_{-\infty}^{+\infty} m(s)\phi(s, x(s), \rho(s)x'(s)) ds, \\ (Tx)(\eta) &= \int_{-\infty}^{+\infty} n(s)\psi(s, x(s), \rho(s)x'(s)) ds, \\ \Delta(Tx)(t_i) &= I(t_i, x(t_i), \rho(t_i)x'(t_i)), \quad i \in \mathbb{Z}, \\ \Delta\Phi(\rho(t_i)(Tx)'(t_i)) &= J(t_i, x(t_i), \rho(t_i)x'(t_i)), \quad i \in \mathbb{Z}. \end{aligned}$$

Thus it follows that $x \in X$ is a solution of (1) if and only if $x \in X$ is a fixed point of T in X .

(iii) Now we prove that T is completely continuous. The following five steps are needed (Steps 1–2 imply that $T : X \rightarrow X$ is continuous, and Steps 3–5 imply that T maps bounded sets into relatively compact sets). We omit the details of the proofs.

- Step 1. We prove that the function $A_x : X \rightarrow \mathbb{R}$ is continuous in x .
- Step 2. We show that T is continuous on X . Since A_x is continuous, f, ϕ, ψ are strong Carathéodory functions, I, J are discrete Carathéodory functions, then the result follows.
- Step 3. We show that T maps bounded subsets into bounded sets.
- Step 4. We prove that both $\{t \rightarrow x(t)/\sigma(t) : x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)) : x \in M\}$ are equicontinuous in $(t_s, t_{s+1}] (s \in \mathbb{Z})$.
- Step 5. We show that both $\{t \rightarrow x(t)/\sigma(t) : x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)) : x \in M\}$ are equi-convergent as $t \rightarrow \pm\infty$.

From Steps 3–5 and Lemma 2 we see that T maps bounded sets into relatively compact sets.

Therefore, the operator $T : X \rightarrow X$ is completely continuous. The proof of (iii) is complete. The proof is complete. \square

Now, we address the first result of this paper. We need the following assumption.

Assumption A. There exist nonnegative constants $A_j, a_{ij}, b_{ij} \geq 0$ ($i = 1, 2, j = 0, 1, 2, \dots, m$), $\phi_s \geq 0, \psi_s \geq 0$ ($s \in \mathbb{Z}$) and $k_j, l_j \geq 0$ ($j = 1, 2, \dots, m$) with $k_j + l_j > 0$ and $\sum_{s=-\infty}^{+\infty} \phi_s < +\infty, \sum_{s=-\infty}^{+\infty} \psi_s < +\infty$ and

$$\begin{aligned}
 |f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| &\leq \Phi \left(A_0 + \sum_{j=1}^m A_j |u|^{k_j} |v|^{l_j} \right), \quad u, v \in \mathbb{R}, \text{ a.e. } t \in \mathbb{R}, \\
 |I(t_s, \sigma(t_s)u, \Phi^{-1}(\tau(t_s))v)| &\leq \phi_s \left[b_{10} + \sum_{j=1}^m b_{1j} |u|^{k_j} |v|^{l_j} \right], \quad u, v \in \mathbb{R}, s \in \mathbb{Z}, \\
 |J(t_s, \sigma(t_s)u, \Phi^{-1}(\tau(t_s))v)| &\leq \psi_s \Phi \left(b_{20} + \sum_{j=1}^m b_{2j} |u|^{k_j} |v|^{l_j} \right), \quad u, v \in \mathbb{R}, s \in \mathbb{Z}, \\
 |\phi(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| &\leq a_{10} + \sum_{j=1}^m a_{1j} |u|^{k_j} |v|^{l_j}, \quad u, v \in \mathbb{R}, \text{ a.e. } t \in \mathbb{R}, \\
 |\psi(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)| &\leq a_{20} + \sum_{j=1}^m a_{2j} |u|^{k_j} |v|^{l_j}, \quad u, v \in \mathbb{R}, \text{ a.e. } t \in \mathbb{R}.
 \end{aligned}$$

We denote $\sigma = \max\{k_j + l_j : j = 1, 2, \dots, m\}$ and

$$\begin{aligned}
 \bar{A}_1 &= \left(\frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) \|m\|_1 a_{10} + \sum_{s=-\infty}^{+\infty} \phi_s \left(1 + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right) b_{10} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{20} \\
 &\quad + 2\Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) c_l b_{2,0} + \left(1 + \Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) \right) c_l A_0,
 \end{aligned}$$

$$\begin{aligned} \bar{B}_{1,j} = & \left(\frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) \|m\|_1 a_{1j} + \sum_{s=-\infty}^{+\infty} \phi_s \left(\frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) b_{1j} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{2j} \\ & + 2\Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) c_l b_{2,j} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_j, \end{aligned}$$

$$\begin{aligned} \bar{A}_2 = & \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|m\|_1 a_{10} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{20} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{10} \\ & + 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{20} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_0, \end{aligned}$$

$$\begin{aligned} \bar{B}_{2,j} = & \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|m\|_1 a_{1j} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{2j} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{1j} \\ & + 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{2j} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_j, \end{aligned}$$

and

$$A = \max\{\bar{A}_1, \bar{A}_2\}, \quad B = \max\left\{ \sum_{j=1}^m \bar{B}_{1,j}, \sum_{j=1}^m \bar{B}_{2,j} \right\}.$$

Theorem 1. *Suppose that Assumption A holds, f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$. Then BVP (1) has at least one solution if*

- (i) $\sigma \in (0, 1)$ or
- (ii) $\sigma = 1$ and $B < 1$ or
- (iii) $\sigma > 1$ and $B(A + B)^{\sigma-1} \leq (\sigma - 1)^{\sigma-1} / \sigma^{\sigma}$.

Proof. Let X and T be defined above. From Lemma 3, $T : X \rightarrow X$ is well defined and is a completely continuous operator. We prove that T has a fixed point in X to get a solution of BVP (1). For $x \in X$, we have $\|x\| \leq r < +\infty$. Then Assumption A implies that

$$\begin{aligned} |f(t, x(t), \rho(t)x'(t))| &= \left| f\left(t, \sigma(t) \frac{x(t)}{\sigma(t)}, \Phi^{-1}(\tau(t)) \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))}\right) \right| \\ &\leq \Phi \left(A_0 + \sum_{j=1}^m A_j \left| \frac{x(t)}{\sigma(t)} \right|^{k_j} \left| \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \right|^{l_j} \right) \\ &\leq \Phi \left(A_0 + \sum_{j=1}^m A_j r^{k_j+l_j} \right), \quad \text{a.e. } t \in \mathbb{R}, \end{aligned}$$

$$\begin{aligned}
|I(t_s, x(t_s), \rho(t_s)x'(t_s))| &\leq \phi_s \left(b_{10} + \sum_{j=1}^m b_{1j} r^{k_j+l_j} \right), \quad s \in \mathbb{Z}, \\
|J(t_s, x(t_s), \rho(t_s)x'(t_s))| &\leq \psi_s \Phi \left(b_{20} + \sum_{j=1}^m b_{2j} r^{k_j+l_j} \right), \quad s \in \mathbb{Z}, \\
|\phi(t, x(t), \rho(t)x'(t))| &\leq a_{10} + \sum_{j=1}^m a_{1j} r^{k_j+l_j}, \quad \text{a.e. } t \in \mathbb{R}, \\
|\psi(t, x(t), \rho(t)x'(t))| &\leq a_{20} + \sum_{j=1}^m a_{2j} r^{k_j+l_j}, \quad \text{a.e. } t \in \mathbb{R}.
\end{aligned}$$

By the definition of T , we get by using (4) that

$$\begin{aligned}
|A_x| &\leq \sum_{\xi \leq t_s < \eta} \psi_s \Phi \left(b_{20} + \sum_{j=1}^m b_{2j} r^{k_j+l_j} \right) + \int_{\xi}^{\eta} p(w) \, dw \, \Phi \left(A_0 + \sum_{j=1}^m A_j r^{k_j+l_j} \right) \\
&\quad + \Phi \left(\frac{\|m\|_1 (a_{10} + \sum_{j=1}^m a_{1j} r^{k_j+l_j}) + \|n\|_1 (a_{20} + \sum_{j=1}^m a_{2j} r^{k_j+l_j})}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right. \\
&\quad \left. + \frac{\sum_{\xi \leq t_s < \eta} \phi_s (b_{10} + \sum_{j=1}^m b_{1j} r^{k_j+l_j})}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{|(Tx)(t)|}{\sigma(t)} &\leq \|m\|_1 \left[a_{10} + \sum_{j=1}^m a_{1j} r^{k_j+l_j} \right] + \sum_{s=-\infty}^{+\infty} \phi_s \left[b_{1,0} + \sum_{j=1}^m b_{1,j} r^{k_j+l_j} \right] \\
&\quad + \Phi^{-1} \left(\sum_{\xi \leq t_s < \eta} \psi_s \Phi \left(b_{20} + \sum_{j=1}^m b_{2j} r^{k_j+l_j} \right) \right) \\
&\quad + \int_{\xi}^{\eta} p(w) \, dw \, \Phi \left(A_0 + \sum_{j=1}^m A_j r^{k_j+l_j} \right) \\
&\quad + \Phi \left(\frac{\|m\|_1 (a_{10} + \sum_{j=1}^m a_{1j} r^{k_j+l_j}) + \|n\|_1 (a_{20} + \sum_{j=1}^m a_{2j} r^{k_j+l_j})}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right. \\
&\quad \left. + \frac{\sum_{\xi \leq t_s < \eta} \phi_s (b_{10} + \sum_{j=1}^m b_{1j} r^{k_j+l_j})}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right) \\
&\quad + \sum_{s=-\infty}^{+\infty} \psi_s \Phi \left(b_{2,0} + \sum_{j=1}^m b_{2,j} r^{k_j+l_j} \right) + \Phi \left(A_0 + \sum_{j=1}^m A_j r^{k_j+l_j} \right).
\end{aligned}$$

One knows that

$$\Phi^{-1}(u + v) \leq c_l [\Phi^{-1}(u) + \Phi^{-1}(v)], \quad u, v \geq 0 \text{ with } c_l = \begin{cases} 1, & 1 < l < 2, \\ 2^{l-1}, & l \geq 2. \end{cases}$$

It follows that

$$\begin{aligned} \sup_{t \in \mathbb{R}} \frac{|(Tx)(t)|}{\sigma(t)} &\leq \left(\frac{c_l \|m\|_1}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + \|m\|_1 \right) a_{10} + \left(\sum_{s=-\infty}^{+\infty} \phi_s + \frac{c_l \sum_{\xi \leq t_s < \eta} \phi_s}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right) b_{10} \\ &+ \frac{c_l \|n\|_1}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} a_{20} + \left(c_l \Phi^{-1} \left(\sum_{\xi \leq t_s < \eta} \psi_s \right) + c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) \right) b_{2,0} \\ &+ \left(c_l + c_l \Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) \right) A_0 \\ &+ \sum_{j=1}^m \left[a_{1j} \left(\frac{c_l \|m\|_1}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + \|m\|_1 \right) + b_{1j} \left(\frac{c_l \sum_{\xi \leq t_s < \eta} \phi_s}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + \sum_{s=-\infty}^{+\infty} \phi_s \right) \right. \\ &+ a_{2j} \frac{c_l \|n\|_1}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + b_{2,j} \left(c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) + c_l \Phi^{-1} \left(\sum_{\xi \leq t_s < \eta} \psi_s \right) \right) \\ &\left. + A_j \left(c_l \Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + c_l \right) \right] r^{k_j+l_j}. \end{aligned}$$

Then

$$\sup_{t \in \mathbb{R}} \frac{|(Tx)(t)|}{\sigma(t)} \leq \bar{A}_1 + \sum_{j=1}^m \bar{B}_{1,j} r^{k_j+l_j}. \tag{6}$$

On the other hand, we have

$$\begin{aligned} \frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))} &\leq \frac{\|m\|_1 c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} a_{10} + \frac{\|n\|_1 c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} a_{20} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{10} \\ &+ 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{20} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_0 \\ &+ \sum_{j=1}^m \left[c_l \Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) A_j + \frac{\|m\|_1 c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} a_{1j} + \frac{\|n\|_1 c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} a_{2j} \right. \\ &\left. + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{1j} + 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{2j} + c_l A_j \right] r^{k_j+l_j}. \end{aligned}$$

It follows that

$$\sup_{t \in \mathbb{R}} \frac{\rho(t)|(Tx)'(t)|}{\Phi^{-1}(\tau(t))} \leq \bar{A}_2 + \sum_{j=1}^m \bar{B}_{2,j} \|x\|^{k_j+l_j}. \quad (7)$$

It follows from (6) and (7) that $\|Tx\| \leq A + B \max\{\|(x, y)\|^\sigma, 1\} \leq A + B + B\|x\|^\sigma$.

(i) $\sigma \in (0, 1)$. Since $\sigma \in (0, 1)$, choose $r_0 > 0$ such that $A + B + Br_0^\sigma \leq r_0$. Let $\Omega_0 = \{x \in X: \|x\| \leq r_0\}$. Then we get $\|Tx\| \leq A + B + Br_0^\sigma \leq r_0$. So $T\bar{\Omega}_0 \subset \bar{\Omega}_0$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\bar{\Omega}_0$. So BVP (1) has at least one solution.

(ii) $\sigma = 1$ and $B < 1$. Let $r_0 = (A + B)/(1 - B)$ such that $A + B + Br_0 = r_0$. Let $\Omega_0 = \{x \in X: \|x\| \leq r_0\}$. Then we get $\|Tx\| \leq A + B + Br_0 \leq r_0$. So $T\bar{\Omega}_0 \subset \bar{\Omega}_0$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\bar{\Omega}_0$. So BVP (1) has at least one solution.

(iii) $\sigma > 1$ and $B(A + B)^{\sigma-1} \leq (\sigma - 1)^{\sigma-1}/\sigma^\sigma$. Let $r_0 = ((A + B)/(B(\sigma - 1))^{1/\sigma})$. It is easy to show from $(A + B)^{\sigma-1}\sigma^\sigma/(\sigma - 1)^{\sigma-1} \leq 1/B$ that $A + B + Br_0^\sigma \leq r_0$. Let $\Omega_0 = \{x \in X: \|x\| \leq r_0\}$. Then we get $\|Tx\| \leq A + B + Br_0^\sigma \leq r_0$. So $T\bar{\Omega}_0 \subset \bar{\Omega}_0$. Thus Schauder's fixed point theorem implies that the operator T has at least one fixed point in $\bar{\Omega}_0$. So BVP(1) has at least one solution.

The proof of Theorem 1 is completed. \square

3 Solvability of (1) in Case 2

In this section, we present existence result of BVP (1) in Case 2. Denote $\tau(t) = 1 + |\int_\eta^t p(s) ds|$. It is easy to show that τ are continuous on \mathbb{R} and $\lim_{t \rightarrow \pm\infty} \tau(t) = +\infty$.

Definition 4. $F : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a strong Carathéodry function if

- (i) $t \rightarrow F(t, u, \Phi^{-1}(\tau(t))v)$ is measurable on \mathbb{R} for any $u, v \in \mathbb{R}$;
- (ii) $(u, v) \rightarrow F(t, u, \Phi^{-1}(\tau(t))v)$ is continuous on \mathbb{R}^2 for a.e. $t \in \mathbb{R}$;
- (iii) for each $r > 0$, there exists nonnegative function $M_r \geq 0$ such that $|u|, |v| \leq r$ implies $|F(t, u, \Phi^{-1}(\tau(t))v)| \leq M_r$, a.e. $t \in \mathbb{R}$.

Definition 5. $H : \{t_i: i \in \mathbb{Z}\} \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is called a discrete Carathéodry function if

- (i) $(x, y) \rightarrow H(t_s, x, \Phi^{-1}(\tau(t_s))y)$ is continuous on \mathbb{R}^2 for all $s \in \mathbb{Z}$;
- (ii) for each $r > 0$, there exists nonnegative constants $M_{ir} \geq 0$ ($i \in \mathbb{Z}$) such that $|x|, |y| \leq r$ implies $|H(t_s, x, \Phi^{-1}(\tau(t_s))y)| \leq M_{sr}$, $s \in \mathbb{Z}$, $\sum_{s=-\infty}^{+\infty} M_{sr} < +\infty$.

Choose

$$X = \left\{ x : \mathbb{R} \rightarrow \mathbb{R}: x|_{(t_s, t_{s+1}]}, \rho x'|_{(t_s, t_{s+1}]} \text{ is continuous, } s \in \mathbb{Z}; \right.$$

the following limits exist and are finite:

$$\left. \lim_{t \rightarrow t_s^+} x(t), \lim_{t \rightarrow t_s^+} \rho(t)x'(t), s \in \mathbb{Z}, \lim_{t \rightarrow \pm\infty} x(t), \lim_{t \rightarrow \pm\infty} \frac{\rho(t)x'(t)}{\Phi^{-1}(\tau(t))} \right\}.$$

For $x \in X$, define $\|x\| = \max\{\sup_{t \in \mathbb{R}} |x(t)|, \sup_{t \in \mathbb{R}} \rho(t)|x'(t)|/\Phi^{-1}(\tau(t))\}$.

Lemma 5. X is a Banach space with $\|\cdot\|$ defined.

Proof. It is similar to the proof of Lemma 1 and is omitted. □

Lemma 6. Let M be a subset of X . Then M is relatively compact if and only if the following conditions are satisfied:

- (i) both $\{t \rightarrow x(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are uniformly bounded;
- (ii) both $\{t \rightarrow x(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are equicontinuous in $(t_s, t_{s+1}]$ ($s \in \mathbb{N}$);
- (iii) both $\{t \rightarrow x(t): x \in M\}$ and $\{t \rightarrow \rho(t)x'(t)/\Phi^{-1}(\tau(t)): x \in M\}$ are equiconvergent as $t \rightarrow \pm\infty$.

Proof. It is similar to the proof of Lemma 2 and is omitted. □

For $x \in X$, let $(Tx)(t)$ be defined by

$$(Tx)(t) = \int_{-\infty}^{+\infty} m(s)\phi_x(s) ds + \sum_{\xi \leq t_s < t} I_x(t_s) + \int_{\xi}^t \frac{1}{\rho(u)} \Phi^{-1} \left(A_x + \sum_{\eta \leq t_s < u} J_x(t_s) - \int_{\eta}^u p(w)f_x(w) dw \right) du, \quad t \in \mathbb{R},$$

where A_x is defined by (3).

Lemma 7. Suppose that f is a strong Carathéodory function and I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$. Then $T : X \rightarrow X$ is well defined and is completely continuous, $x \in X$ is a solution of (1) if and only if $x \in X$ is a fixed point of T in X .

Proof. It is similar to the proof of Lemma 4 and is omitted. □

To state and prove Theorem 2, we need the following assumption.

Assumption B. There exist nonnegative constants $A_j, a_{ij}, b_{ij} \geq 0$ ($i = 1, 2, j = 0, 1, 2, \dots, m$), $\phi_s \geq 0, \psi_s \geq 0$ ($s \in \mathbb{Z}$) and $k_j, l_j \geq 0$ ($j = 1, 2, \dots, m$) with $k_j + l_j > 0$ and $\sum_{s=-\infty}^{+\infty} \phi_s < +\infty, \sum_{s=-\infty}^{+\infty} \psi_s < +\infty$ and

$$|f(t, u, \Phi^{-1}(\tau(t))v)| \leq \Phi \left(A_0 + \sum_{j=1}^m A_j |u|^{k_j} |v|^{l_j} \right), \quad u, v \in \mathbb{R}, \text{ a.e. } t \in \mathbb{R},$$

$$|I(t_s, u, \Phi^{-1}(\tau(t_s))v)| \leq \phi_s \left[b_{10} + \sum_{j=1}^m b_{1j} |u|^{k_j} |v|^{l_j} \right], \quad u, v \in \mathbb{R}, s \in \mathbb{Z},$$

$$|J(t_s, u, \Phi^{-1}(\tau(t_s))v)| \leq \psi_s \Phi(b_{20} + \sum_{j=1}^m b_{2j} |u|^{k_j} |v|^{l_j}), \quad u, v \in \mathbb{R}, \quad s \in \mathbb{Z},$$

$$|\phi(t, u, \Phi^{-1}(\tau(t))v)| \leq a_{10} + \sum_{j=1}^m a_{1j} |u|^{k_j} |v|^{l_j}, \quad u, v \in \mathbb{R}, \quad \text{a.e. } t \in \mathbb{R},$$

$$|\psi(t, u, \Phi^{-1}(\tau(t))v)| \leq a_{20} + \sum_{j=1}^m a_{2j} |u|^{k_j} |v|^{l_j}, \quad u, v \in \mathbb{R}, \quad \text{a.e. } t \in \mathbb{R}.$$

We denote $\sigma = \max\{k_j + l_j: j = 1, 2, \dots, m\}$ and

$$\begin{aligned} \bar{A}_1 &= \left(\frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) \|m\|_1 a_{10} + \sum_{s=-\infty}^{+\infty} \phi_s \left(1 + \frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \right) b_{10} \\ &\quad + \frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{20} + 2\Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du c_l b_{2,0} \\ &\quad + \left(1 + \Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) \right) \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du c_l A_0, \end{aligned}$$

$$\begin{aligned} \bar{B}_{1,j} &= \left(\frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) \|m\|_1 a_{1j} + \sum_{s=-\infty}^{+\infty} \phi_s \left(\frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} + 1 \right) b_{1j} \\ &\quad + \frac{c_l \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{2j} + 2\Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du c_l b_{2,j} \\ &\quad + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du c_l A_j, \end{aligned}$$

$$\begin{aligned} \bar{A}_2 &= \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|m\|_1 a_{10} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{20} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{10} \\ &\quad + 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{20} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_0, \end{aligned}$$

$$\begin{aligned} \bar{B}_{2,j} &= \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|m\|_1 a_{1j} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \|n\|_1 a_{2j} + \frac{c_l}{\int_{\xi}^{\eta} \frac{du}{\rho(u)}} \sum_{s=-\infty}^{+\infty} \phi_s b_{1j} \\ &\quad + 2c_l \Phi^{-1} \left(\sum_{s=-\infty}^{+\infty} \psi_s \right) b_{2j} + \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) c_l A_j, \end{aligned}$$

and

$$A = \max\{\bar{A}_1, \bar{A}_2\}, \quad B = \max\left\{\sum_{j=1}^m \bar{B}_{1,j}, \sum_{j=1}^m \bar{B}_{2,j}\right\}.$$

Theorem 2. *Suppose that Assumption B holds, f is a strong Carathéodory function, I, J are discrete Carathéodory functions, and for each $r > 0$, $f(t, u, \Phi^{-1}(\tau(t))v)$ converges uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$. Then BVP (1) has at least one solution if*

- (i) $\sigma \in (0, 1)$ or
- (ii) $\sigma = 1$ and $B < 1$ or
- (iii) $\sigma > 1$ and $B(A + B)^{\sigma-1} \leq (\sigma - 1)^{\sigma-1} / \sigma^\sigma$.

Proof. Let X and T be defined above. From Lemma 7, $T : X \rightarrow X$ is well defined and is a completely continuous operator. We prove that T has a fixed point in X to get a solution of BVP (1). The proof is similar to that of Theorem 1 and is omitted. \square

Remark 1. By constructing suitable Banach spaces similar to X in Sections 2 and 3, one can establish existence results on solvability of BVP (1) for Cases 3 and 4, respectively.

4 Examples

In this section, we present examples to illustrate the main result.

Example 1. We consider the following BVP:

$$\begin{aligned} & [|t - 1.5| |x'(t)| x'(t)]' + \frac{1}{2\sqrt{|t - 1.5|}} \frac{1}{1 + |t|} g(t, x(t), x'(t)) = 0, \\ & \text{a.e. } t \in \mathbb{R}, \\ & x(-1.5) = \sqrt{\pi}, \quad x(1.5) = 2\sqrt{\pi}, \\ & \Delta x(s) = 2^{-|s|}, \quad \Delta [|s - 1.5| |x'(s)| x'(s)] = 2^{-|s|}, \quad s \in \mathbb{Z}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} g(t, u, v) &= \left(A_0 + \sum_{j=1}^m A_j \left| \frac{u}{\sigma_0(t)} \right|^{k_j} \left| \frac{v}{\sqrt{1 + |t - 1.5|}} \right|^{l_j} \right)^2, \\ \sigma_0(t) &= \begin{cases} -\frac{5}{3} + \frac{4}{3}(1 + \sqrt{t - 1.5})^{3/2} + \frac{4}{3}(1 + \sqrt{3})^{3/2}, & t \geq 1.5, \\ 1 + \frac{4}{3}(1 + \sqrt{3})^{3/2} - \frac{4}{3}(1 + \sqrt{1.5 - t})^{3/2}, & t \in [-1.5, 1.5), \\ 1 + \frac{4}{3}(1 + \sqrt{1.5 - t})^{3/2} - \frac{4}{3}(1 + \sqrt{3})^{3/2}, & t \leq -1.5, \end{cases} \end{aligned}$$

$$A_j \geq 0, k_j, l_j \geq 0 \quad (j = 0, 1, 2, \dots, m).$$

Corresponding to BVP (1), we have

$$\begin{aligned}\Phi(x) &= |s|s, & \Phi^{-1}(x) &= |s|^{-1/2}s, \\ \rho(t) &= \sqrt{|t-1.5|}, & p(t) &= \frac{1}{2\sqrt{|t-1.5|}}, & f(t, u, v) &= \frac{1}{1+|t|}g(t, u, v), \\ \xi &= -1.5, & \eta &= 1.5, & t_s &= s, \quad s \in \mathbb{Z}, \\ \phi(t, u, v) &= 1, & \psi(t, u, v) &= 2, & m(t) &= n(t) = e^{-t^2}, \\ I(s, u, v) &= J(s, u, v) = 2^{-|s|}, & s &\in \mathbb{Z}.\end{aligned}$$

One sees that

- (i) $\Phi(x) = |x|^{k-2}x$ with $k = 3 > 1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $l = 3/2$, then $c_l = 1$.
- (ii) p is nonnegative and satisfies $p \in L^1_{\text{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t) dt = \int_0^{+\infty} p(s) ds = +\infty$.
- (iii) $\rho : \mathbb{R} \rightarrow [0, +\infty)$ and

$$\begin{aligned}\tau(t) &= 1 + \left| \int_{\eta}^t p(s) ds \right| = 1 + \left| \int_{1.5}^t p(s) ds \right| = 1 + \sqrt{|t-1.5|}, \\ \sigma(t) &= \begin{cases} -\frac{5}{3} + \frac{4}{3}(1 + \sqrt{t-1.5})^{3/2} + \frac{4}{3}(1 + \sqrt{3})^{3/2}, & t \geq 1.5, \\ 1 + \frac{4}{3}(1 + \sqrt{3})^{3/2} - \frac{4}{3}(1 + \sqrt{1.5-t})^{3/2}, & t \in [-1.5, 1.5), \\ 1 + \frac{4}{3}(1 + \sqrt{1.5-t})^{3/2} - \frac{4}{3}(1 + \sqrt{3})^{3/2}, & t \leq -1.5, \end{cases} \\ &= \sigma_0(t)\end{aligned}$$

with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 \Phi^{-1}(\tau(u))/\rho(u) du = \int_0^{+\infty} \Phi^{-1}(\tau(u))/\rho(u) du = +\infty$, and f is a strong Carathéodory function and I, J are discrete Carathéodory functions and for each $r > 0$,

$$f(t, \sigma(t)u, \Phi^{-1}(\tau(t))v) = \frac{1}{1+|t|} \left(A_0 + \sum_{j=1}^m A_j |u|^{k_j} |v|^{l_j} \right)^2 \rightarrow 0$$

uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$.

- (iv) $f, \phi, \psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ are strong Carathéodory functions, $m, n \in L^1(\mathbb{R})$.
- (v) $\{t_s : s \in \mathbb{Z}\}$ is a increasing sequence with $\lim_{s \rightarrow -\infty} t_s = -\infty$ and $\lim_{s \rightarrow +\infty} t_s = +\infty$.
- (vi) $I, J : \{t_s : s \in \mathbb{Z}\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are discrete Carathéodory functions.

Choose A_j ($j = 0, 1, 2, \dots, m$) and

$$\begin{aligned}b_{1,0} &= b_{2,0} = 1, & b_{1,j} &= b_{2,j} = 0 = 0, & j &= 1, 2, \dots, m, & \phi_s &= 2^{-|s|}, & s &\in \mathbb{Z}, \\ a_{1,0} &= 1, & a_{2,0} &= 2, & a_{1,j} &= a_{2,j} = 0, & j &= 1, 2, \dots, m, & \psi_s &= 2^{-|s|}, & s &\in \mathbb{Z}.\end{aligned}$$

Then Assumption A holds and $\sum_{s=-\infty}^{+\infty} \phi_s = \sum_{s=-\infty}^{+\infty} \psi_s = 3$. Denote $\sigma = \max\{k_j + l_j : j = 1, 2, \dots, m\}$ and by direct computation, we get

$$\int_{\xi}^{\eta} \rho(w) \, dw = \int_{-1.5}^{1.5} \sqrt{|s - 1.5|} \, ds = 2\sqrt{3},$$

$$\int_{\xi}^{\eta} p(w) \, dw = \int_{-1.5}^{1.5} \frac{1}{2\sqrt{|s - 1.5|}} \, ds = \sqrt{3},$$

$$\bar{A}_1 = \frac{2 + \sqrt{3}}{2} \sqrt{\pi} + \frac{6 + 5\sqrt{3}}{2} + (1 + \sqrt[4]{3})A_0, \quad \bar{B}_{1,j} = (1 + \sqrt[4]{3})A_j,$$

$$\bar{A}_2 = \frac{\sqrt{3}}{2} \sqrt{\pi} + \frac{5\sqrt{3}}{2} + (1 + \sqrt[4]{3})A_0, \quad \bar{B}_{2,j} = (1 + \sqrt[4]{3})A_j,$$

and

$$A = \max\{\bar{A}_1, \bar{A}_2\} = \frac{2 + \sqrt{3}}{2} \sqrt{\pi} + \frac{6 + 5\sqrt{3}}{2} + (1 + \sqrt[4]{3})A_0,$$

$$B = \max\left\{ \sum_{j=1}^m \bar{B}_{1,j}, \sum_{j=1}^m \bar{B}_{2,j} \right\} = (1 + \sqrt[4]{3}) \sum_{j=1}^m A_j.$$

By Theorem 1, BVP (8) has at least one solution if (i) $\sigma \in (0, 1)$ or (ii) $\sigma = 1$ and $(\sqrt[4]{3} + 1) \sum_{j=1}^m A_j < 1$ or (iii) $\sigma > 1$ and

$$\left(\frac{2 + \sqrt{3}}{2} \sqrt{\pi} + \frac{6 + 5\sqrt{3}}{2} + (1 + \sqrt[4]{3})A_0 + (\sqrt[4]{3} + 1) \sum_{j=1}^m A_j \right)^{\sigma-1} \sum_{j=1}^m A_j \leq \frac{(\sigma-1)^{\sigma-1}}{(\sqrt[4]{3} + 1)\sigma^{\sigma}}.$$

Example 2. We consider the following BVP:

$$\begin{aligned} & [(\rho(t)x'(t))^3]' + \frac{1}{1 + \sqrt{|t|}} g(t, x(t), \sqrt{|t|x'(t)}) = 0, \quad \text{a.e. } t \in \mathbb{R}, \\ & x(-1) = \sqrt{\pi}, \quad x(1) = 2\sqrt{\pi}, \\ & \Delta x(s) = \lim_{t \rightarrow s^+} x(t) - x(s) = 2^{-|s|}, \quad s \in \mathbb{Z}, \\ & \Delta(\sqrt{|s|x'(s)})^3 = \lim_{t \rightarrow s^+} (\sqrt{|t|x'(t)})^3 - (\sqrt{|s|x'(s)})^3 = 2^{-|s|}, \quad s \in \mathbb{Z}, \end{aligned} \tag{9}$$

where

$$\rho(t) = \begin{cases} \sqrt{|t-1|}, & t \in [0, 2], \\ (t-1)^4, & |t-1| > 1, \end{cases}$$

$$g(t, u, v) = \left(A_0 + \sum_{j=1}^m A_j |u|^{k_j} \left| \frac{v}{\sqrt[3]{1 + |t-1|}} \right|^{l_j} \right)^3,$$

and $A_j \geq 0, k_j, l_j \geq 0 (j = 0, 1, 2, \dots, m)$.

Corresponding to BVP (1), we have

$$\begin{aligned}\Phi(x) &= x^3, & \Phi^{-1}(x) &= x^{1/3}, & \xi &= -1, & \eta &= 1, & t_s &= s, & s &\in \mathbb{Z}, \\ p(t) &= 1, & f(t, u, v) &= \frac{1}{1 + \sqrt{|t|}} g(t, u, v), \\ \phi(t, u, v) &= 1, & \psi(t, u, v) &= 2, & m(t) &= n(t) &= e^{-t^2}, \\ I(s, u, v) &= J(s, u, v) &= 2^{-|s|}, & s &\in \mathbb{Z}.\end{aligned}$$

One sees that

- (i) $\Phi(x) = |x|^{k-2}x$ with $k = 4 > 1$, the inverse of Φ is denoted by Φ^{-1} and $\Phi^{-1}(x) = |x|^{l-2}x$ with $l = 4/3$, then $c_l = 1$.
(ii) $p : \mathbb{R} \rightarrow [0, +\infty)$ with $p \in L^1_{\text{loc}}(\mathbb{R})$ and $\int_{-\infty}^0 p(t) dt = \int_0^{+\infty} p(s) ds = +\infty$.
(iii) $\rho : \mathbb{R} \rightarrow [0, +\infty)$ with $\Phi^{-1}(\tau(\cdot))/\rho(\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ and

$$\begin{aligned}\tau(t) &= 1 + \left| \int_{\eta}^t p(s) ds \right| = 1 + |t - 1|, \\ \int_0^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du &= \int_0^2 \frac{\sqrt[3]{1+|u-1|}}{\sqrt{|u-1|}} du + \int_2^{+\infty} \frac{\sqrt[3]{1+|u-1|}}{(u-1)^4} du \\ &< \int_0^1 \frac{\sqrt[3]{2}}{\sqrt{1-u}} du + \int_1^2 \frac{\sqrt[3]{2}}{\sqrt{u-1}} du + \int_1^{+\infty} \frac{\sqrt[3]{2v}}{v^4} dv \\ &= \frac{35\sqrt[3]{2}}{8} < +\infty, \\ \int_{-\infty}^0 \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du &= \int_{-\infty}^0 \frac{\sqrt[3]{1+|u-1|}}{(u-1)^4} du < \int_{-\infty}^0 \frac{1 + \sqrt[3]{1-u}}{(u-1)^4} du \\ &= \frac{17}{24} < +\infty.\end{aligned}$$

- (iv) $f, \phi, \psi : \mathbb{R}^3 \rightarrow \mathbb{R}$ are strong Carathéodory functions, $m, n \in L^1(\mathbb{R})$, and for each $r > 0$,

$$f(t, u, \Phi^{-1}(\tau(t))v) = \frac{1}{1 + \sqrt{|t|}} \left(A_0 + \sum_{j=1}^m A_j |u|^{k_j} |v|^{l_j} \right)^3 \rightarrow 0$$

uniformly as $t \rightarrow \pm\infty$ on $[-r, r] \times [-r, r]$.

- (v) $\{t_s : s \in \mathbb{Z}\}$ is a increasing sequence with $\lim_{s \rightarrow -\infty} t_s = -\infty$ and $\lim_{s \rightarrow +\infty} t_s = +\infty$.
(vi) $I, J : \{t_s : s \in \mathbb{Z}\} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are discrete Carathéodory functions.

Choose A_j ($j = 0, 1, 2, \dots, m$) and

$$\begin{aligned} b_{1,0} = b_{2,0} = 1, \quad b_{1,j} = b_{2,j} = 0, \quad j = 1, 2, \dots, m, \quad \phi_s = 2^{-|s|}, \quad s \in \mathbb{Z}, \\ a_{1,0} = 1, \quad a_{2,0} = 2, \quad a_{1,j} = a_{2,j} = 0, \quad j = 1, 2, \dots, m, \quad \psi_s = 2^{-|s|}, \quad s \in \mathbb{Z}. \end{aligned}$$

Then Assumption B holds and $\sum_{s=-\infty}^{+\infty} \phi_s = \sum_{s=-\infty}^{+\infty} \psi_s = 3$. Denote $\sigma = \max\{k_j + l_j: j = 1, 2, \dots, m\}$ and by direct computation and (iii), we get

$$\begin{aligned} \int_{\xi}^{\eta} p(s) ds = 2, \quad \int_{\xi}^{\eta} \rho(s) ds = \frac{103}{15}, \\ \bar{A}_1 < \frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24} \\ + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}A_0, \\ \bar{A}_2 = \frac{45 + 45\pi}{103} + 2\sqrt[3]{3} + (\sqrt[3]{2} + 1)A_0, \\ \bar{B}_{1,j} = \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) \int_{-\infty}^{+\infty} \frac{\Phi^{-1}(\tau(u))}{\rho(u)} du A_j < \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24} A_j, \\ \bar{B}_{2,j} = \left(\Phi^{-1} \left(\int_{\xi}^{\eta} p(w) dw \right) + 1 \right) A_j = (\sqrt[3]{2} + 1) A_j, \end{aligned}$$

and the definition of A, B imply that

$$\begin{aligned} A < \frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24} \\ + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24}A_0, \\ B < \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24} \sum_{j=1}^m A_j. \end{aligned}$$

By Theorem 2, BVP (9) has at least one solution if (i) $\sigma \in (0, 1)$ or (ii) $\sigma = 1$ and $((105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)/24) \sum_{j=1}^m A_j < 1$ or (iii) $\sigma > 1$ and

$$\begin{aligned} \left(\frac{105\sqrt[3]{2} + 12377}{12360}\pi + \frac{315\sqrt[3]{2} + 12401}{4120} + \frac{2\sqrt[3]{3}(105\sqrt[3]{2} + 17)}{24} \right. \\ \left. + \frac{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)}{24} \sum_{j=0}^m A_j \right)^{\sigma-1} \sum_{j=1}^m A_j \\ \leq \frac{24}{(105\sqrt[3]{2} + 17)(\sqrt[3]{2} + 1)} \frac{(\sigma - 1)^{\sigma-1}}{\sigma^{\sigma}}. \end{aligned}$$

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