Approximate controllability of a second-order neutral stochastic differential equation with state-dependent delay[*]

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Abstract. In this paper, the existence and uniqueness of mild solution is initially obtained by use of measure of noncompactness and simple growth conditions. Then the conditions for approximate controllability are investigated for the distributed second-order neutral stochastic differential system with respect to the approximate controllability of the corresponding linear system in a Hilbert space. We construct controllability operators by using simple and fundamental assumptions on the system components. We use Lemma 3, which implies the approximate controllability of the associated linear system. Lemma 3 is also described as a geometrical relation between the range of the operator B and the subspaces $\mathcal{N}_i^\perp$, $i = 1, 2, 3$, associated with sine and cosine operators in $L_2([0, a], X)$ and $L_2([0, a], L_Q)$. Eventually, we show that the reachable set of the stochastic control system lies in the reachable set of its associated linear control system. An example is provided to illustrate the presented theory.

Keywords: approximate controllability, cosine family, state-dependent delay, neutral stochastic differential equation, measure of noncompactness.

1 Introduction

Random noise causes fluctuations in deterministic models. So, necessarily, we move from deterministic problems to stochastic ones. Stochastic evolution equations are natural generalizations of ordinary differential equations incorporating the randomness into the equations. Thereby, making the system more realistic, [9, 21] and the references therein explore the qualitative properties of solutions for stochastic differential equations. Considering the environmental disturbances, Kolmanovskii and Myshkis [22] introduced a class of neutral stochastic functional differential equations, which are applicable in several fields, such as chemical engineering, aero-elasticity and so on. In recent years,

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controllability of stochastic infinite-dimensional systems has been extensively studied for various applications. Several papers studied the approximate controllability of semilinear stochastic control systems, see, for instance, [5, 6, 8, 11, 12, 23, 24] and references therein. Controllability results are available in overwhelming majority for abstract stochastic differential delay systems; rather than for neutral second-order stochastic differential with state-dependent delay.

Mahmudov [24] investigated conditions on the system operators so that the semilinear control system is approximately controllable provided the corresponding linear system is approximately controllable. The main drawback of the papers [11, 23, 24] is the need to check the invertibility of the controllability Gramian operator and a associated limit condition, which are practically difficult to verify and apply.

Neutral differential equations appear in several areas of applied mathematics and thus studied in several papers and monographs, see, for instance, [16, 17, 22] and references therein. Differential equations with delay reflect physical phenomena more realistically than those without delay.

Recently, much attention is paid to partial functional differential equation with state-dependent delay. For details, see [1, 3, 18, 19, 20]. As a matter of fact, in these papers, their authors assume severe conditions on the operator family generated by A, which imply that the underlying space X has finite dimension. Thus, the equations treated in these works are really ordinary and not partial equations.

This is an extension of our previous work [10] on approximate controllability of neutral differential equation with state-dependent delay. We also remove the need to assume the invertibility of a controllability operator used by authors in [4, 5, 7, 25], which fails to exist in infinite dimensional spaces if the associated semigroup is compact. Our approach also removes the drawback of the method applied in [11, 23, 24].

Hence, motivated by this fact in this paper, we study the existence and uniqueness of mild solution and approximate controllability of the partial neutral stochastic differential equation of second order with state delay. Specifically, we study the second-order equations modeled in the form

$$
\begin{aligned}
\mathrm{d}\big(x'(t) - g(t, x_t)\big) = &\big[Ax(t) + f(t, x_{\rho(t, x_t)}) + Bu(t)\big]\,\mathrm{d}t \\
&+ G(t, x_t)\,\mathrm{d}W(t), \quad \text{a.e. } t \in J = [0, a], \qquad (1)
\end{aligned}
$$
$$
x_0 = \phi \in \mathfrak{B}, \quad x'(0) = \psi \in X,
$$

where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t), t \in \mathbb{R}\}$ of bounded linear operators on a Hilbert space X. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space together with a normal filtration $\mathcal{F}_t$, $t \geqslant 0$. The state space $x(t) \in X$, and the control $u(t) \in L_2^{\mathcal{F}}(J, U)$, where X and U are separable Hilbert spaces and d is the stochastic differentiation. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; g, f are appropriate functions. B is a bounded linear operator on a Hilbert space U. Let K be a separable Hilbert space, and $\{W(t)\}_{t \geqslant 0}$ is a given K-valued Brownian motion or Wiener process with finite trace nuclear covariance operator $Q > 0$. The functions $f, g : J \times \mathfrak{B} \to X$

are measurable mappings in X norm, and $G : J \times \mathfrak{B} \to L_Q(K, X)$ is a measurable mapping in $L_Q(J, X)$ norm. $L_Q(J, X)$ is the space of all Q-Hilbert–Schmidt operators from K into X. B is a bounded linear operator from U into X. $\phi(t)$ is $\mathfrak{B}$-valued random variable independent of Brownian motion $W(t)$ with finite second moment. Also, $\psi(t)$ is a X-valued $\mathcal{F}_t$-measurable function.

2 Preliminaries

In this section, some definitions, notations and lemmas that are used throughout this paper are stated. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space endowed with complete family of right-continuous increasing sub σ-algebras $\{\mathcal{F}_t, \ t \in J\}$ such that $\mathcal{F}_t \subset \mathcal{F}$. A X-valued random variable is a $\mathcal{F}$-measurable process. A stochastic process is a collection of random variables $S = \{x(t, w) : \Omega \to X, \ t \in J\}$. We usually suppress w and write $x(t)$ instead of $x(t, w)$.

Now suppose $\beta_n(t)$, $n = 1, 2, \ldots$, be a sequence of real-valued one dimensional standard Brownian motions mutually independent over $(\Omega, \mathcal{F}, \mathbf{P})$. Let ς_n be a complete orthonormal basis in K. $Q \in L(K, K)$ be an operator defined by $Q\varsigma_n = \lambda_n \varsigma_n$ with finite trace $\mathrm{Tr}(Q) = \sum_{n=1}^{\infty} \lambda_n \leqslant \infty$. Let us define

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \beta_n(t) \varsigma_n(t), \quad t \geqslant 0,$$

which is a K-valued stochastic process and is called a Q-Wiener process. Let $\mathcal{F}_t = \sigma(W(s), 0 \leqslant s \leqslant t)$ be the σ-algebra generated by W and $\mathcal{F}_a = \mathcal{F}$. Let $\phi \in L(K, X)$, and if

$$\|\phi\|_Q^2 = \mathrm{Tr}(\phi Q \phi^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \phi \varsigma_n\|^2 \leqslant \infty,$$

then ϕ is called a Q-Hilbert–Schmidt operator. The completion $L_Q(K, X)$ of $L(K, X)$, with respect to the topology induced by norm $\|\phi\|_Q^2 = \langle \phi, \phi \rangle$, is a Hilbert space.

The family $\{C(t), t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);
(b) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$;
(c) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

$\{S(t), \ t \in \mathbb{R}\}$ is the strongly continuous sine family associated to the strongly continuous cosine family $\{C(t), \ t \in \mathbb{R}\}$. It is defined as $S(t)x = \int_0^t C(s)x \, \mathrm{d}s$, $x \in X$, $t \in \mathbb{R}$.

The operator A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $C(t))_{t \in R}$, and $S(t)$ is the associated sine function. Let N, $\widetilde{N}$ be certain constants such that $\|C(t)\|^2 \leqslant N$ and $\|S(t)\|^2 \leqslant \widetilde{N}$ for every $t \in J = [0, a]$. For more details, see the book by Fattorini [13]. In this work, we use the axiomatic definition of phase space $\mathfrak{B}$ introduced by Hale and Kato [14].

Definition 1. (See [14].) Let $\mathfrak{B}$ be a linear space of functions mapping $(-\infty, 0]$ into X endowed with the seminorm $\|\cdot\|_{\mathfrak{B}}$ and satisfies the following conditions:

(A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, such that $x_t \in \mathfrak{B}$ and $x|_{[\sigma, \sigma+b]} \in C([\sigma, \sigma + b]; X)$, then for every $t \in [\sigma, \sigma + b)$, the following conditions hold:

    (i) $x_t \in \mathfrak{B}$;

    (ii) $\|x(t)\| \leqslant H\|x_t\|_{\mathfrak{B}}$;

    (iii) $\|x_t\|_{\mathfrak{B}} \leqslant K(t - \sigma) \sup\{\|x(s)\|, \sigma \leqslant s \leqslant t\} + M(t + \sigma)\|x_\sigma\|_{\mathfrak{B}}$, where $H > 0$ is a constant, $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(B) The space $\mathfrak{B}$ is complete.

Lemma 1. (*See* [1].) *If $y : (-\infty, a] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$, then*

$$\|y_{\rho(s, y_s)}\|_{\mathfrak{B}} \leqslant \big(M_a + \widetilde{J^\phi}\big)\|\phi\|_{\mathfrak{B}} + K_a \sup\big\{\|y(\theta)\|, \theta \in [0, \max\{0, s\}]\big\},$$
$$s \in \mathbb{R}(\rho^-) \cup [0, a],$$

where $\widetilde{J^\phi} = \sup_{t \in \mathbb{R}(\rho^-)} J^\phi(t)$, $M_a = \sup_{t \in J} M(t)$ and $K_a = \max_{t \in J} K(t)$.

Let us denote $\mathbf{E}$ as the expectation defined by $\mathbf{E}(h) = \int_\Omega h(w)\,\mathrm{d}\mathbf{P}$. Let $L_2(\Omega, \mathcal{F}, \mathbf{P}; X) \equiv L_2(\Omega; X)$ be the Banach space of all strongly measurable, square integrable, X-valued random variables equipped with the norm $\|x(\cdot)\|_{L_2}^2 = \sup \mathbf{E}\|x(\cdot; w)\|_X^2$. $C(J_1, L_2(\Omega; X))$ denotes the Banach space of all continuous maps from $J_1 = (-\infty, a]$ into $L_2(\Omega; X)$, which satisfy $\sup_{t \in J_1} \mathbf{E}\|x(t)\|^2 < \infty$. $L_2^0(\Omega, X) = \{f \in L_2(\Omega, X) : f$ is $\mathcal{F}_0$-measurable$\}$ denotes an important subspace.

We denote by $\mathcal{C}$ the closed subspace of all continuously differentiable process $x \in C^1(J, L_2(\Omega; X))$ consisting of $\mathcal{F}_t$-adapted measurable processes such that $\phi, \psi \in L_2^0(\Omega; \mathfrak{B})$ and seminorm $\|\cdot\|_{\mathcal{C}}$ defined by $\|x\|_{\mathcal{C}} = (\sup_{t \in J} \|x_t\|_{\mathfrak{B}}^2)^{1/2}$, where

$$\|x_t\|_{\mathfrak{B}} \leqslant M_a E\|\phi\|_{\mathfrak{B}} + K_a \sup\big\{\mathbf{E}\|x(s)\|, 0 \leqslant s \leqslant a\big\},$$
$$\|y_{\rho(s, y_s)}\|_{\mathfrak{B}} \leqslant \big(M_a + \widetilde{J^\phi}\big)\mathbf{E}\|\phi\|_{\mathfrak{B}} + K_a \sup\big\{\mathbf{E}\|y(\theta)\|, \theta \in [0, \max\{0, s\}]\big\},$$
$$s \in \mathbb{R}(\rho^-) \cup [0, a].$$

Here $\widetilde{J^\phi} = \sup_{t \in \mathbb{R}(\rho^-)} J^\phi(t)$, $K_a = \sup_{t \in J} K(t)$ and $M_a = \sup_{t \in J} M(t)$. It can be easily seen that $\mathcal{C}$ endowed with norm topology is a Banach space.

Definition 2. (See [2].) The Hausdorff's measure of noncompactness χ_Y for a bounded set B in any Banach space Y is defined by

$$\chi_Y(B) = \inf\{r > 0, \ B \text{ can be covered by finite number of balls with radii } r\}.$$

Lemma 2. (*See* [2].) *Let Y be a Banach space and $B, C \subset Y$ be bounded, then the following properties hold*:

    (i) *B is pre-compact if and only if $\chi_Y(B) = 0$;*

(ii) $\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(\mathrm{conv}B)$, *where $\overline{B}$ and $\mathrm{conv}B$ are closure and convex hull of B, respectively*;

(iii) $\chi_Y(B) \leqslant \chi_Y(C)$ *when $B \subset C$*;

(iv) $\chi_Y(B + C) \leqslant \chi_Y(B) + \chi_Y(C)$, *where $B + C = \{x + y,\ x \in B,\ y \in C\}$*;

(v) $\chi_Y(B \cup C) = \max\{\chi_Y(B), \chi_Y(C)\}$;

(vi) $\chi_Y(\lambda B) = \|\lambda\|\chi_Y(B)$ *for any $\lambda \in \mathbb{R}$*;

(vii) *If the map $Q : D(Q) \subset Y \to Z$ is Lipschitz continuous with constant k, then $\chi_Z(QB) \leqslant k\chi_Y(B)$ for any bounded subset $B \subset D(Q)$, where Z is a Banach space*;

(viii) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $\lim_{n \to \infty} \chi_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty}$ is nonempty and compact in Y.*

Definition 3. Let X and Y be Banach spaces, and Φ, Ψ be the Measure of Noncompactness (MNC) in X and Y, respectively. If for any continuous function $f : D(f) \subset X \to Y$ and any $O \subset D(f)$, $\Psi[f(O)] \geqslant \Phi(O)$ implies that O is relatively compact, then f is called (Φ, Ψ)-condensing map.

Theorem 1. (*See* [2].) *Let Ψ be a MNC on a Banach space X. Let f be (Ψ, Ψ)-condensing operator. If f maps a nonempty, convex, closed subset M of the Banach space X into itself, then f has atleast one fixed point in M.*

Definition 4. The set given by $\mathcal{R}(f) = \{x(T) \in X \colon x \text{ is a mild solution of (1)}\}$ is called reachable set of system (1) for some $T > 0$. $\mathcal{R}(0)$ is the reachable set of the corresponding linear control system (2).

Definition 5. System (1) is said to be approximately controllable if $\mathcal{R}(f)$ is dense in X. The corresponding linear system is approximately controllable if $\mathcal{R}(0)$ is dense in X.

Lemma 3. (*See* [26].) *Let X be Hilbert space, and X_1, X_2 closed subspaces such that $X = X_1 + X_2$. Then there exists a bounded linear operator $P : X \to X_2$ such that for each $x \in X$, $x = x - Px \in X_1$ and $\|x_1\| = \min\{\|y\|\colon y \in X_1, (1 - Q)(y) = (1 - Q)(x)\}$, where Q denotes the orthogonal projection on X_2.*

We state the corresponding linear control system

$$
\begin{aligned}
x''(t) &= Ax(t) + Bu(t), \quad t \in J, \\
x(0) &= x^0, \quad x'(0) = x^1.
\end{aligned}
\tag{2}
$$

Lemma 4. (*See* [13].) *Under the assumption that $h : [0, a] \to X$ is an integrable function such that*

$$
\begin{aligned}
x''(t) &= Ax(t) + h(t), \quad t \in J, \\
x(0) &= x^0, \quad x'(0) = x^1
\end{aligned}
$$

and h is a function continuously differentiable,

$$
\int_0^t C(t - s)h(s)\,\mathrm{d}s = S(t)h(0) + \int_0^t S(t - s)h'(s)\,\mathrm{d}s.
$$

3 Main result

We define mild solution of problem (1) as follows:

Definition 6. An $\mathcal{F}_t$-adapted process $x : (-\infty, a] \to X$ is a mild solution of problem (1) if $x_0 = \phi$, $x'(0) = \psi x(\cdot) \in C^1(J, L_2(\Omega, X))$, the functions $f(s, x_{\rho(s, x_s)})$, $G(s, x_s)$ and $g(s, x_s)$ are integrable, and for $t \in [0, a]$, the following integral equation is satisfied:

$$
x(t) = C(t)\phi(0) + S(t)\big[\psi + g(0, \phi)\big] - \int\limits_0^t C(t-s)g(s, x_s)\,\mathrm{d}s
$$

$$
+ \int\limits_0^t S(t-s)\big[f(s, x_{\rho(s, x_s)}) + Bu(s)\big]\,\mathrm{d}s + \int\limits_0^t S(t-s)G(s, x_s)\,\mathrm{d}W(s).
$$

To prove our result, we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function. The following hypotheses are used:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s, \psi)\colon \rho(s, \psi) \leqslant 0\}$ into $\mathfrak{B}$, and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \leqslant J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(H_f) $f : J \times \mathfrak{B} \to X$ satisfies the following:

   (i) For every $x : (-\infty, a] \to X$, $x_0 \in \mathfrak{B}$ and $x|_J \in PC$, the function $f(\cdot, \psi) : J \to X$ is strongly measurable for every $\psi \in \mathfrak{B}$ and $f(\cdot, t)$ is continuous for a.e. $t \in J$.

   (ii) There exists an integrable function $\alpha_f : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Upsilon_f : [0, +\infty) \to (0, +\infty)$ such that $\|f(t, v)\| \leqslant \alpha_f(t)\Upsilon_f(\|v\|_{\mathfrak{B}})$ for all $t \in J$ and $v \in \mathfrak{B}$.

(H_G) The function G satisfies the following conditions:

   (i) For almost all $t \in J$, the function $G(t, \cdot) : \mathfrak{B} \to L_Q(K, X)$ is continuous. For all $z \in \mathfrak{B}$, the function $G(\cdot, z) : J \to L_Q(K, X)$ is strongly $\mathcal{F}_t$-measurable.

   (ii) There exists integrable function $\alpha_G : J \to [0, \infty)$ and a monotone continuous nondecreasing function $\Upsilon_G : [0, \infty) \to (0, \infty)$ such that

$$
\big\|G(t, z)\big\|_Q^2 \leqslant \alpha_G(t)\Upsilon_G\big(\|z\|_{\mathfrak{B}}^2\big).
$$

(H_g) $g : J \times \mathfrak{B} \to X$ satisfies the following:

   (i) For every $x : (-\infty, a] \to X$, $x_0 \in \mathfrak{B}$ and $x|_J \in PC$, the function $g(\cdot, \psi) : J \to X$ is strongly measurable for every $\psi \in \mathfrak{B}$ and $g(\cdot, t)$ is continuous for a.e. $t \in J$.

   (ii) There exists an integrable function $\alpha_g : J \to [0, +\infty)$ and a monotone continuous nondecreasing function $\Upsilon_g : [0, +\infty) \to (0, +\infty)$ such that $\|g(t, v)\| \leqslant \alpha_g(t)\Upsilon_g(\|v\|_{\mathfrak{B}})$ for all $t \in J$ and $v \in \mathfrak{B}$.

(H$_l$) There exists a function $H : [0, \infty) \times [0, \infty) \to [0, \infty)$, which is locally integrable in t. H is a continuous, monotone, nondecreasing in second variable, also $H(t, 0) \equiv 0$ and

$$
\mathbf{E}\big(\|f(t, m_1) - f(t, m_2)\|^2\big) + \mathbf{E}\big(\|G(t, m_1) - G(t, m_2)\|^2\big)
$$
$$
\leqslant H\big(t, \mathbf{E}(\|m_1 - m_2\|^2)\big),
$$
$$
\mathbf{E}\big(\|g(t, m_1) - g(t, m_2)\|^2\big) \leqslant H\big(t, \mathbf{E}(\|m_1 - m_2\|^2)\big)
$$

for all $t \in [0, a]$ and $m_1, m_2 \in L_2(\Omega, \mathcal{F}, X)$.

(H$_1$)
$$
\lim_{\tau \to \infty} \inf \frac{\Upsilon(\tau)}{\tau} = 0, \quad \Upsilon = \max\{\Upsilon_g, \Upsilon_G, \Upsilon_f\}.
$$

Lemma 5. (*See* [2].) *Let m be a nonnegative, continuous function, and $\mathfrak{A} > 0$ such that*

$$
m(t) \leqslant \mathfrak{A} \int_{t_0}^{t} H\big(s, m(s)\big) \,\mathrm{d}s, \quad t \in [0, T_1],
$$

then m has no nonzero nonnegative solution.

3.1 Existence and uniqueness of mild solution

In this section, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi$ and $y(t) = C(t)\phi(0) + S(t)(\psi + g(0, \phi))$ on J. Clearly, $\|u\|_{\mathfrak{B}} \leqslant K_a \mathbf{E}\|y\|_a + M_a \mathbf{E}\|\phi\|_{\mathfrak{B}}$, where $\mathbf{E}\|y\|_a = \sup_{0 \leqslant t \leqslant a}\{\mathbf{E}\|y(t)\|, 0 \leqslant s \leqslant a\}$.

Theorem 2. *If hypotheses* (H$_f$), (H$_g$), (H$_\phi$), (H$_G$), (H$_1$) *and* (H$_l$) *are satisfied, then the initial value problem* (1) *has atleast one mild solution.*

Proof. Let $S(a)$ be the space $S(a) = \{x \in C(J, L_2(\Omega; X)): x(0) = 0\}$ endowed with the norm of uniform convergence. $x \in C_0$ is identified with its extension to $(-\infty, a]$ by assuming $x(\theta) = 0$ for $\theta < 0$.

Let $\Gamma : S(a) \to S(a)$ be the map defined by

$$
(\Gamma x)(t) = \int_0^t C(t - s)g(s, \overline{x_s}) \,\mathrm{d}s + \int_0^t S(t - s)f(s, \overline{x}_{\rho(s, \overline{x_s})}) \,\mathrm{d}s
$$
$$
+ \int_0^t S(t - s)G(s, x_s) \,\mathrm{d}W(s),
$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J. It is easy to see that

$$
\|\overline{x}_t\|_{\mathfrak{B}} \leqslant K_a \mathbf{E}\|y\|_a + K_a \mathbf{E}\|x\|_a, \qquad \|\overline{x}_{\rho(s, \overline{x_s})}\|_{\mathfrak{B}} \leqslant K_a \mathbf{E}\|y\|_a + K_a \mathbf{E}\|x\|_a.
$$

Thus, Γ is well defined and has values in $S(a)$. Also, by axioms of phase space, the Lebesgue-dominated convergence theorem and conditions (H$_f$), (H$_G$), (H$_g$) it can be shown that Γ is continuous.

Step 1. We prove that there exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a): \mathbf{E}\|x\|^2 \leqslant k\}$. In fact, if we assume that the assertion is false, then for $k > 0$, there exist $x_k \in B_k$ and $t \in (0, a]$ such that $k < \|\mathbf{E}(\Gamma x_k(t_k))\|^2$,

$$
\mathbf{E}\big\|\Gamma x_k(t_k)\big\|^2
$$

$$
\leqslant 3\bigg\{ \mathbf{E}\bigg\| \int_0^t C(t-s)g(s, \overline{x}_{k_s})\,\mathrm{d}s \bigg\|^2 + \mathbf{E}\bigg\| \int_0^t S(t-s)f(s, \overline{x}_{k\,\rho(s, \overline{x}_{k_s})})\,\mathrm{d}s \bigg\|^2
$$

$$
+ \mathbf{E}\bigg\| \int_0^t S(t-s)G(s, \overline{x}_{k_s})\,\mathrm{d}W(S) \bigg\|^2 \bigg\}
$$

$$
\leqslant 3\bigg\{ Na \int_0^t \big(\alpha_g(s)\Upsilon_g\big(\|\overline{x_{k_s}}\|_{\mathfrak{B}}\big)\big)\,\mathrm{d}s + \widetilde{N}a \int_0^t \alpha_f(s)\Upsilon_f\big(\|\overline{x_{k\rho(s,\quad)}}\|_{\mathfrak{B}}\big)\,\mathrm{d}s
$$

$$
+ \widetilde{N}\operatorname{Tr}(Q)\int_0^t \mathbf{E}\big\|G(s, \overline{x_{k_s}})\big\|^2\,\mathrm{d}s \bigg\}
$$

$$
\leqslant 3\bigg\{ Na \int_0^t \Upsilon_g\big(c + K_a^2 k\big) \int_0^t \alpha_g(s)\,\mathrm{d}s + \widetilde{N}a\Upsilon_f\big(c + K_a^2 k\big) \int_0^t \alpha_f(s)\,\mathrm{d}s
$$

$$
+ \widetilde{N}\operatorname{Tr}(Q)\Upsilon_G\big(c + K_a^2\big) \int_0^t \alpha_G(s)\,\mathrm{d}s \bigg\}\,\mathrm{d}s.
$$

Hence,

$$
1 < 3\bigg(Na \int_0^a \alpha(s)\,\mathrm{d}s \lim_{k\to\infty} \inf \frac{\Upsilon_G(c + K_a^2 k)}{k} + \widetilde{N}a \int_0^a \alpha(s)\,\mathrm{d}s \lim_{k\to\infty} \inf \frac{\Upsilon_f(c + K_a^2 k)}{k}
$$

$$
+ \widetilde{N}\operatorname{Tr}(Q) \int_0^a \alpha(s)\,\mathrm{d}s \lim_{k\to\infty} \inf \frac{\Upsilon_G(c + K_a^2 k)}{k} \bigg)
$$

$$
\leqslant 3\big(Na + \widetilde{N}a + \widetilde{N}\operatorname{Tr}(Q)\big) \int_0^a \alpha(s)\,\mathrm{d}s \lim_{\tau\to\infty} \inf \frac{\Upsilon(\tau)}{\tau}, \tag{3}
$$

where $\alpha = \max\{\alpha_g, \alpha_f, \alpha_G\}$. Thus, (3) is a contradiction to hypothesis (H$_1$). Hence, $\Gamma(B_k) \subset B_k$.

Step 2. We prove that Γ is a condensing map on any bounded subset of the space $C(J, L_2(\Omega; X))$. Let O be a bounded subset of $C(J, L_2(\Omega; X))$. Let $\mathfrak{M}[0, a]$ be the partially ordered linear space of all real monotone nondecreasing functions on $[0, a]$, and

we define a Measure of Noncompactness (MNC), $\Psi : C(J, L_2(\Omega; X)) \to \mathfrak{M}[0, a]$ by

$$[\Psi(O)](t) = \chi_t[O_t],$$

where χ_t is the Hausdorff MNC in $C(J, L_2(\Omega; X))$ and $O_t = \{x_t = x|_{[0,t]}, \, x \in O\} \subset C([0, t], L_2(\Omega; X))$. If $\Psi(O) \leqslant \Psi(\Gamma O)$, then it is proved that $\Psi(O) = 0$. Since the function $t \to [\Psi(O)](t)$ is nondecreasing and bounded, so for all $\epsilon > 0$, it has only a finite number of jumps of magnitude greater than ϵ. The disjoint δ_1 neighborhoods of the points corresponding to these jumps are removed from $[0, a]$. Using points β_j, $j = 1, 2, \ldots, m$, divide the remaining part into intervals on which the oscillations of $\Psi(O)$ is less than ϵ. These points β_j are surrounded by disjoint δ_2 neighborhoods. Now consider the family $o = \{o_k, \, k = 1, \ldots, l\}$ of all functions continuous with probability one such that o_k coincides with an arbitrary element of $[(\Psi(O))(\beta_j) + 1]$ net of the set O_{β_j} on the segment $\sigma_j = [\beta_{j-1} + \delta_2, \beta_j - \delta_2]$, $j = 1, \ldots, m$, and linear on the complementary segments.

Suppose $p \in (\Gamma O)_t$. This implies $p = \Gamma o$ for some $o \in O$ and

$$\left\| o - o_r^{\beta_j} \right\|_{C([0,t], L_2(\Omega; X))}^2 \leqslant \left[(\Psi(O))(\beta_j) + \epsilon \right],$$

where $o_r^{\beta_j}$ is some element of $[(\Psi(O))(\beta_j) + \epsilon]$ net of the set O_{β_j}, i.e. $o_r^{\beta_j} = o_k|_{\sigma_j}$. This implies that for $s \in \sigma_j$,

$$
\begin{aligned}
\mathbf{E}\left\| o(s) - o_k(s) \right\|^2 &\leqslant \mathbf{E} \sup_{\beta_{j-1} + \delta_2 \leqslant s \leqslant \beta_j - \delta_2} \mathbf{E}\left\| o(s) - o_k(s) \right\|^2 \\
&\leqslant \left\| o - o^{\beta_j} \right\|_{C([0,t], L_2(\Omega; X))}^2 \leqslant \left\| o - o_r^{\beta_j} \right\|_{C([0,t], L_2(\Omega; X))}^2 \\
&\leqslant \left[(\Psi(O))(s) + 2\epsilon \right]^2.
\end{aligned}
$$

Then

$$
\mathbf{E} \sup_{0 \leqslant s \leqslant t} \left\| (\Gamma o)(s) - (\Gamma o_k)(s) \right\|^2
$$

$$
\begin{aligned}
&\leqslant 3 \Bigg\{ \mathbf{E}\left\| \int_0^t C(t - s)\big(g(s, o_s + y_s) - g(s, o_{k_s} + y_s)\big)\, \mathrm{d}s \right\|^2 \\
&\quad + \mathbf{E}\left\| \int_0^t S(t - s)\big(f(s, o_{\rho(s, o_s)} + y_{\rho(s, y_s)}) - f(s, o_{k_{\rho(s, o_{k_s})}} + y_{\rho(s, y_s)})\big)\, \mathrm{d}s \right\|^2 \\
&\quad + \mathbf{E}\left\| \int_0^t S(t - s)\big(G(s, o_s + y_s) - G(s, o_{k_s} + y_s)\big)\, \mathrm{d}W(S) \right\|^2 \Bigg\} \\
&\leqslant 3 \Bigg\{ Na \int_0^t \mathbf{E}\left\| g(s, o_s + y_s) - g(s, o_{k_s} + y_s) \right\|^2 \mathrm{d}s \\
&\quad + \widetilde{N}a \int_0^t \mathbf{E}\left\| f(s, o_{\rho(s, o(s))} + y_{\rho(s, y(s))}) - f(s, o_{k_{\rho(s, o_k(s))}} + y_{\rho(s, y(s))}) \right\|^2 \mathrm{d}s
\end{aligned}
$$

$$
+ \widetilde{N} \operatorname{Tr}(Q) \int_0^t \mathbf{E} \big\| G(s, o_s + y_s) - G(s, o_{k_s} + y_s) \big\|^2 \, \mathrm{d}s \Bigg\}
$$

$$
\leqslant 3 \Bigg\{ Na \int_0^t H\big(s, \mathbf{E}\|o_s - o_{k_s}\|^2\big) \, \mathrm{d}s + \widetilde{N}a \int_0^t H\big(s, \mathbf{E}\big\|o_{\rho(s, o(s))} - o_{k_{\rho(s, o_k(s))}}\big\|^2\big) \, \mathrm{d}s
$$

$$
+ \widetilde{N} \operatorname{Tr}(Q) \int_0^t H\big(s, \mathbf{E}\|o_s - o_{k_s}\|^2\big) \, \mathrm{d}s \Bigg\}
$$

$$
\leqslant 3K_a^2 \Bigg\{ Na \int_0^t H\big(s, \mathbf{E}\|o(s) - o_k(s)\|^2\big) \, \mathrm{d}s + \widetilde{N}a \int_0^t H\big(s, \mathbf{E}\|o(s) - o_k(s)\|^2\big) \, \mathrm{d}s
$$

$$
+ C_T \int_0^t H\big(s, \mathbf{E}\|o(s) - o_k(s)\|^2\big) \, \mathrm{d}s \Bigg\}
$$

$$
\leqslant \mathfrak{A} \sum_{j=1}^m \int_{\sigma_j} H\big(s, \mathbf{E}\|o(s) - o_k(s)\|^2\big) \, \mathrm{d}s + \mathfrak{A} \sum_{j=1}^m \int_{[0,t] - \bigcup_{i=1}^m \sigma_j} H\big(s, \mathbf{E}\|o(s) - o_k(s)\|^2\big) \, \mathrm{d}s.
$$

By choosing $\delta_1 > 0$ and $\delta_2 > 0$ sufficiently small, we can make sure that

$$
\big[(\Psi(O))(t) \big]^2 \leqslant \big[(\Psi(\Gamma O))(t) \big] < \epsilon + A \int_0^t H\big(s, (\Psi(O))(s) + 2\epsilon\big) \, \mathrm{d}s.
$$

Together with Lemma 5, we get that $\Psi(O) \equiv 0$. Similarly, we can prove that Γ is continuous. The MNC $\mathfrak{U}$ possess all required properties. The operator Γ is condensing. Then from Theorem 1 it is implied that there exist a mild solution to problem (1).

The uniqueness of mild solution follows from Lemma 5. Let $m_1, m_2 \in C(J, L_2(\Omega; X))$ be two mild solution of Γ. Then it follows that

$$
\mathbf{E} \sup_{0 \leqslant s \leqslant t} \|m_1 - m_2\|^2
$$

$$
\leqslant 3 \Bigg\{ \mathbf{E} \bigg\| \int_0^t C(t - s) \big(g(s, m_{1s} + y_s) - g(s, m_{2s} + y_s) \big) \, \mathrm{d}s \bigg\|^2
$$

$$
+ \mathbf{E} \bigg\| \int_0^t S(t - s) \big(f(s, m_{1\rho(s, m_{1s})} + y_{\rho(s, y_s)}) - f(s, m_{2\rho(s, m_{2s})} + y_{\rho(s, y_s)}) \big) \, \mathrm{d}s \bigg\|^2
$$

$$
+ \mathbf{E} \bigg\| \int_0^t S(t - s) \big(G(s, m_{1s} + y_s) - G(s, m_{2s} + y_s) \big) \, \mathrm{d}W(S) \bigg\|^2 \Bigg\}
$$

$$\leqslant 3\Bigg\{ Na \int_0^t \mathbf{E}\big\| g(s, m_{1s} + y_s) - g(s, m_{2s} + y_s) \big\|^2 \, \mathrm{d}s$$

$$+ \widetilde{N}a \int_0^t \mathbf{E}\big\| f(s, m_{1\rho(s, m_1(s))} + y_{\rho(s, y(s))}) - f(s, m_{2\rho(s, o_k(s))} + y_{\rho(s, y(s))}) \big\|^2 \, \mathrm{d}s$$

$$+ \widetilde{N}\,\mathrm{Tr}(Q) \int_0^t \mathbf{E}\big\| G(s, m_{1s} + y_s) - G(s, m_{2s} + y_s) \big\|^2 \, \mathrm{d}s \Bigg\}$$

$$\leqslant 3\Bigg\{ N \int_0^t H\big(s, \mathbf{E}\|m_{1s} - m_{2s}\|^2\big) \, \mathrm{d}s + \widetilde{N}a \int_0^t H\big(s, \mathbf{E}\|m_{1\rho(s, m_{1s})} - m_{2\rho(s, m_{2s})}\|^2\big) \, \mathrm{d}s$$

$$+ \widetilde{N}\,\mathrm{Tr}(Q) \int_0^t H\big(s, \mathbf{E}\|m_{1s} - m_{2s}\|^2\big) \, \mathrm{d}s \Bigg\}$$

$$\leqslant 3K_a^2 \Bigg\{ N \int_0^t H\big(s, \mathbf{E}\|m_1(s) - m_2(s)\|^2\big) \, \mathrm{d}s + \widetilde{N}a \int_0^t H\big(s, \mathbf{E}\|m_1(s) - m_2(s)\|^2\big) \, \mathrm{d}s$$

$$+ C_T \int_0^t H\big(s, \mathbf{E}\|m_1(s) - m_2(s)\|^2\big) \, \mathrm{d}s \Bigg\}$$

$$\leqslant \mathfrak{A} \int_0^t H\big(s, \mathbf{E}\|m_1(s) - m_2(s)\|^2\big) \, \mathrm{d}s$$

Thus, from Lemma 5 it follows that $\|m_1 - m_2\|^2_{C(J, L_2(\Omega; X))} \equiv 0$. Hence, $m_1 = m_2$ $\quad\square$

3.2 Approximate controllability

In this section, the approximate controllability of the distributed control system (1) is studied as an extension of co-author N. Sukavanam's method in [26]. Assume that f, g, G satisfy the following conditions:

(C1) The function $f, g : J \times \mathfrak{B} \to X$ are continuous. For all $t \in J$ and for all $z_1, z_2 \in L_2(J; \mathfrak{B})$, there exists constants $L_f, L_g > 0$ such that

$$\big\| f(t, z_1) - f(t, z_2) \big\| \leqslant L_f \|z_1 - z_2\|_{\mathfrak{B}},$$
$$\big\| g(t, z_1) - g(t, z_2) \big\| \leqslant L_g \|z_1 - z_2\|_{\mathfrak{B}}.$$

(C2) The function $G : J \times \mathfrak{B} \to L_Q(K, X)$ is Lipschitz continuous with constant $L_G > 0$ such that

$$\big\| G(t, z_1) - G(t, z_2) \big\| \leqslant L_G \|z_1 - z_2\|_{\mathfrak{B}}.$$

Also, $y : (-\infty, a] \to X$ is the function defined by $y_0 = \phi$ and $y(t) = C(t)\phi(0) + S(t)(z + g(0, \phi))$ on J. Clearly, $\|y_t\|_{\mathfrak{B}} \leqslant K_a E\|y\|_a + M_a E\|\phi\|_{\mathfrak{B}}$, where $\|y\|_a = \sup_{0 \leqslant t \leqslant a} \|y(t)\|$.

The operators $\Lambda_i : L_2(J, X) \to X$, $i = 1, 2$, and $\Lambda_3 : L_2(J, X) \to C_0(J, L_2(\Omega, (L_Q(K, X))))$ are defined as

$$\Lambda_1 x(t) = \int_0^a S(t - s)x(s)\,\mathrm{d}s, \qquad \Lambda_2 x(t) = \int_0^a C(t - s)x(s)\,\mathrm{d}s,$$

$$\Lambda_3 x(t) = \int_0^a S(t - s)x(s)\,\mathrm{d}W(s).$$

Clearly, Λ_i are bounded linear operators. We set $\mathfrak{N}_i = \ker(\Lambda_i)$, $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$ and $\mathfrak{N} = \ker(\Lambda)$. Let $C_0(J, X)$ denote the space consisting of continuous functions $x : J \to X$ such that $x(0) = 0$ endowed with the norm of uniform convergence. Let $J_i : L_2(J, X) \to C_0(J, X)$, $i = 1, 2$, and $J_3 : L_2(J, X) \to C_0(J, L_2(\Omega, L_Q(K, X)))$ be maps defined as follows:

$$J_1 x(t) = \int_0^t S(t - s)x(s)\,\mathrm{d}s, \qquad J_2 x(t) = \int_0^t C(t - s)x(s)\,\mathrm{d}s,$$

$$J_3 x(t) = \int_0^t S(t - s)x(s)\,\mathrm{d}W(s).$$

So, $J_i x(a) = \Lambda_i(x)$, $i = 1, 2$. For a fixed $\phi \in \mathfrak{B}$ and $x \in C(J, X)$ such that $x(0) = \phi(0)$, we define maps $F, g : C_0(J, X) \to L^2(J, X)$ by $F(z)(t) = f(t, z_t + x_t)$ and $g(z)(t) = g(t, z_t + x_t)$. We also define maps $G(z)(t) = G(t, z_t + x_t)$. Here $x_t(\theta) = x(t + \theta)$ for $t + \theta \geqslant 0$, $x_t(\theta) = \phi(t + \theta)$ for $t + \theta \leqslant 0$; $z_t(\theta) = z(t + \theta)$ for $t + \theta \geqslant 0$ and $z_t(\theta) = 0$ for $t + \theta \leqslant 0$. Clearly, F, g, G are continuous maps. We also assume that $L^2(J, X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1, 2$, and $L_2(J, L_2(U_0, X)) = \mathfrak{N}_3 + \overline{R(B)}$. We denote P_i, $i = 1, 2, 3$, the map associated to this decomposition and constructs $X_2 = \mathfrak{N}_i$, $i = 1, 2, 3$, and $X_1 = \overline{R(B)}$. Also, set $c_i = \|P_i\|^2$. We introduce the space

$$Z = \big\{ z \in C_0(J, X) \colon z = J_1(n_1) - J_2(n_2) + J_3(n_3),\ n_i \in \mathfrak{N}_i,\ i = 1, 2, 3 \big\},$$

and we define the map $\Gamma_c : \overline{Z} \to C_0$ by

$$\Gamma_c = J_1 \circ P_1 \circ F - J_2 \circ P_2 \circ g + J_3 \circ P_3 \circ G.$$

Lemma 6. *If hypothesis* (H_ϕ)–(H_g) *and conditions* (C1)–(C2) *hold for* $\widetilde{f}$, g, G *and* $aK_a(c_1 \widetilde{N} L_f + c_2 N L_g) < \sqrt{2}$, *then* Γ *has a fixed point.*

Proof. For $z^1, z^2 \in \overline{Z}$, let $\Delta f(s) = f(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}) - f(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))})$, $\Delta g(s) = g(s, z^2_s + x_s) - g(s, z^1_s + x_s)$, and $\Delta G(s) = G(s, z^2_s + x_s) - G(s, z^1_s + x_s)$ for all $0 \leqslant t \leqslant a$, then

$$\mathbf{E}\big\|\big(\varGamma_c z^1 - \varGamma_c z^2\big)(t)\big\|^2$$

$$\leqslant 3\bigg\{\mathbf{E}\bigg\|\int_0^t S(t-s)\big[P_1(\Delta f)\big](s)\,\mathrm{d}s\bigg\|^2 + \mathbf{E}\bigg\|\int_0^t C(t-s)\big[P_2(\Delta g)\big](s)\,\mathrm{d}s\bigg\|^2$$

$$+ \mathbf{E}\bigg\|\int_0^t S(t-s)\big[P_3(\Delta G)\big](s)\,\mathrm{d}s\bigg\|^2\bigg\}$$

$$\leqslant 3\bigg\{\widetilde{N}a\int_0^t \mathbf{E}\big\|\big[P_1(\Delta f)\big](s)\big\|\,\mathrm{d}s + Na\int_0^t \mathbf{E}\big\|\big[P_2(\Delta g)\big](s)\big\|\,\mathrm{d}s$$

$$+ \widetilde{N}\operatorname{Tr}(Q)\int_0^t \mathbf{E}\big\|\big[P_3(\Delta G)\big](s)\big\|\,\mathrm{d}s\bigg\}$$

$$\leqslant 3\bigg\{\widetilde{N}ac_1\int_0^t \mathbf{E}\|\Delta f\|^2 + Nac_2\int_0^t \mathbf{E}\|\Delta g\|^2 + \widetilde{N}\operatorname{Tr}(Q)c_3\int_0^t \mathbf{E}\|\Delta G\|^2\,\mathrm{d}s\bigg\}.$$

Now

$$\int_0^a \mathbf{E}\|\Delta f\|^2\,\mathrm{d}s$$

$$= \int_0^a \mathbf{E}\big\|f\big(s, z^2_{\rho(s,z^2(s))} + x_{\rho(s,x(s))}\big) - f\big(s, z^1_{\rho(s,z^1(s))} + x_{\rho(s,x(s))}\big)\big\|^2\,\mathrm{d}s$$

$$\leqslant L_f^2\int_0^a \big\|z^2_{\rho(s,z^2(s))} - z^1_{\rho(s,z^1(s))}\big\|^2_{\mathfrak{B}}\,\mathrm{d}s \leqslant L_f^2\int_0^a \big\|z^2_s - z^1_s\big\|^2_{\mathfrak{B}}\,\mathrm{d}s$$

$$\leqslant aL_f^2 K_a^2\big\|z^2 - z^1\big\|^2\,\mathrm{d}s.$$

Similarly, we find for g, G,

$$\int_0^a \mathbf{E}\|\Delta g\|^2\,\mathrm{d}s = \int_0^a \mathbf{E}\big\|g\big(s, z^2_s + x_s\big) - g\big(s, z^1_s + x_s\big)\big\|^2\,\mathrm{d}s$$

$$\leqslant L_g^2\int_0^a \big\|z^2_s - z^1_s\big\|^2_{\mathfrak{B}}\,\mathrm{d}s \leqslant aL_g^2 K_a^2\big\|z^2 - z^1\big\|^2\,\mathrm{d}s,$$

764 S. Das, D.N. Pandey, N. Sukavanam

$$\int\limits_0^a \mathbf{E}\|\Delta G\|^2\,\mathrm{d}s = \int\limits_0^a \mathbf{E}\big\|G\big(s, z_s^2 + x_s\big) - G\big(s, z_s^1 + x_s\big)\big\|^2\,\mathrm{d}s$$

$$\leqslant L_G^2 \int\limits_0^a \big\|z_s^2 - z_s^1\big\|_\mathfrak{B}^2\,\mathrm{d}s \leqslant a L_G^2 K_a^2 \big\|z^2 - z^1\big\|^2\,\mathrm{d}s.$$

So,

$$\big\|\big(\Gamma z^2 - \Gamma z^1\big)(t)\big\| \leqslant bt\big\|z^2 - z^1\big\|^2,$$

where $b = 3aK_a^2(c_1\widetilde{N}aL_f^2 + c_2NaL_g^2 + \widetilde{N}\operatorname{Tr}(Q)c_3L_G)$. Repeating this, we get

$$\big\|\big(\Gamma^n z^2 - \Gamma^n z^1\big)(t)\big\|_\infty \leqslant \frac{(ba)^n}{2^{(n-1)/(2n)}}\big\|z^2 - z^1\big\|^2.$$

As $ab < 1$ and $2^{(n-1)/(2n)} \to \sqrt{2}$ as $n \to \infty$, the map Γ^n is a contraction for sufficiently large n, and therefore, Γ has a fixed point. $\square$

Theorem 3. *Assume that the associated linear control system* (2) *is approximately controllable on* $J = [0, a]$. *Let the space* $L^2([0, a], X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1, 2$, $L_2(J, L_Q(K, X)) = \mathfrak{N}_3 + \overline{R(B)}$ *and condition of the preceding lemma hold, then the semilinear control system with state-dependent delay is approximately controllable on* J.

Proof. Assume $x(\cdot)$ to be the mild solution and $u(\cdot)$ to be an admissible control function of system (2) with initial conditions $x(0) = \phi(0)$ and $x'(0) = \psi + g(0, \phi)$. Let z be the fixed point of Γ. So, $z(0) = 0$ and $z(a) = \Lambda_1(P_1(F(z))) - \Lambda_2(P_2(g(z))) + \Lambda_3(P_3(G(z))) = 0$. By Lemma 3 we can split the functions $F(z), g(z)$ with respect to the decomposition $L_2(J, X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1, 2$, and $L_2(J, L_Q(K, X)) = \mathfrak{N}_3 + \overline{R(B)}$ by setting, respectively,

$$q_1 = F(z) - P_1\big(F(z)\big), \qquad q_2 = g(z) - P_2\big(g(z)\big), \qquad q_3 = G(z) - P_3\big(G(z)\big).$$

We define the function $y(t) = z(t) + x(t)$ for $t \in J$ and $y_0 = \phi$. So, $x(a) = y(a)$. Thus, by the properties of x and z,

$$y(t) = C(t)x(0) + S(t)x'(0) - \int\limits_0^t C(t-s)\big(g(s, y_s) - q_2(s)\big)\,\mathrm{d}s$$

$$+ \int\limits_0^t S(t-s)\big(f(s, y_{\rho(s, y(s))}) - q_1(s) + Bu(s)\big)\,\mathrm{d}s$$

$$+ \int\limits_0^t S(t-s)\big(G(s, y_s) - q_3(s)\big)\,\mathrm{d}W(s).$$

As $C^1(J, L_2(\Omega, U))$ is dense in $L_2^{\mathcal{F}}(J, U)$, we can choose a sequence $v_n^2 \in C^1(J, L_2(\Omega, U))$ and a sequence $v_n^1, v_n^3 \in L_2^{\mathcal{F}}(J, U)$ such that $Bv_n^1 \to q_1$ and $Bv_n^2 \to q_2$ as $n \to \infty$. By Lemma 4 we get

$$y^n(t) = \int_0^t S(t-s)\big(f\big(s, y_{\rho(s,y(s))}^n\big) - Bv_n^1(s) + Bu(s)\big)\,\mathrm{d}s$$

$$- \int_0^t C(t-s)\big(g\big(s, y_s^n\big) - Bv_n^2(s)\big)\,\mathrm{d}s + C(t)\phi(0) + S(t)\big(w - g(0,\phi)\big)$$

$$+ \int_0^t S(t-s)\big(G\big(s, y_s^n\big) - Bv_n^3(s)\big)\,\mathrm{d}W(s)$$

$$= \int_0^t S(t-s)f\big(s, y_{\rho(s,y(s))}^n\big)\,\mathrm{d}s$$

$$+ \int_0^t S(t-s)\left(-Bv_n^1(s) - B\frac{\mathrm{d}}{\mathrm{d}s}v_n^2(s) - \sqrt{s}Bv_n^3 + Bu(s)\right)\mathrm{d}s$$

$$- \int_0^t C(t-s)g\big(s, y_s^n\big)\,\mathrm{d}s + C(t)\phi(0) + S(t)\big(w + g(0,\phi)\big)$$

$$+ \int_0^t S(t-s)G\big(s, y_s^n\big)\,\mathrm{d}W(s).$$

Hence, by Definition 5 and the last expression we conclude that y^n is the mild solution of the following equation:

$$\mathrm{d}\big(p'(t) - g(t, p_t)\big) = \bigg(Ap(t) + f(t, p_{\rho(t,p(t))}) + G(t, p_t)\,\mathrm{d}W(t)$$

$$+ B\left(-v_n^1(t) - \frac{\mathrm{d}}{\mathrm{d}t}v_n^2(t) - \sqrt{t}v_n^3(t) + u(t)\right)\bigg)\,\mathrm{d}t$$

$$x(0) = \phi \in \mathfrak{B}, \quad x'(0) = \psi.$$

Hence, $y^n(a) \in \mathcal{R}(a, f, g, G, \phi, \psi)$. Since the solution map is generally continuous, $y^n \to y$ as $n \to \infty$. Thus, $y(a) \in \overline{\mathcal{R}(a, f, g, G, \phi, \psi)}$. Therefore, $\mathcal{R}_0(a, \phi(0), \psi + g(0, \phi)) \subset \overline{\mathcal{R}(a, f, g, G, \phi, \psi)}$, which means $\mathcal{R}(a, f, g, G, \phi, \psi)$ is dense in X. Thus, system (1) is controllable. $\qquad\square$

Remark 1. (See [15].) The condition $L_2([0, a], X) = \mathfrak{N}_i + \overline{R(B)}$, $i = 1, 2$, implies $L_2([0, a], X) = \overline{\mathfrak{N}_i + R(B)}$, $i = 1, 2$, which in turn implies the approximate controllability of the linear control system (2).

4 Example

In this section, we discuss a concrete partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$, see [19].

Consider the second-order neutral differential equation

$$
\frac{\partial}{\partial t}\left(\frac{\partial u(t,\xi)}{\partial t} + \int\limits_{-\infty}^{t}\int\limits_{0}^{\pi} b(t-s,\eta,\xi)u(s,\eta)\,\mathrm{d}\eta\,\mathrm{d}s\right)
$$

$$
= \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int\limits_{-\infty}^{t} a(t-s)u\big(s - \rho_1(t)\rho_2(\|u(t)\|),\xi\big)\,\mathrm{d}s + Bv(t,\xi)
$$

$$
+ s\left(t, \int\limits_{0}^{\infty} q_2(s)y(t-s,\xi)\,\mathrm{d}s\right)\partial\beta(t), \quad t \in [0,a],\ \xi \in [0,\pi], \tag{4}
$$

$$
u(t,0) = u(t,\pi) = 0, \quad t \in [0,a],
$$

$$
u(t,\xi) = \phi(t,\xi), \quad \tau \leqslant 0,\ 0 \leqslant \xi \leqslant \pi,
$$

where $\phi \in C_0 \times L^2(h, X)$, $0 < t_1 < \cdots < t_n < a$. By defining the maps $\rho, g, F : [0,a] \times \mathfrak{B} \to X$

$$
\rho(t,\psi) := \rho_1(t)\rho_2(\|\psi(0)\|),
$$

$$
g(\psi)(\xi) := \int\limits_{-\infty}^{0}\int\limits_{0}^{\pi} b(s,\upsilon,\xi)\psi(s,\upsilon)\,\mathrm{d}\upsilon\,\mathrm{d}s, \qquad F(\psi)(\xi) := \int\limits_{-\infty}^{0} a(s)\psi(s,\xi)\,\mathrm{d}s
$$

system (4) can be transformed into system (1). Assume that the functions $\rho_i : \mathbb{R} \to [0,\infty)$, $a : \mathbb{R} \to \mathbb{R}$ are piecewise continuous.

(a) The functions $b(s,\eta,\xi)$, $\partial b(s,\eta,\xi)/\partial \xi$ are measurable, $b(s,\eta,\pi) = b(s,\eta,0) = 0$, and

$$
L_g := \max\left\{\left(\int\limits_{0}^{\pi}\int\limits_{-\infty}^{0}\int\limits_{0}^{\pi} \frac{1}{h(s)}\left(\frac{\partial^i b(s,\eta,\xi)}{\partial \xi^i}\right)^2\,\mathrm{d}\eta\,\mathrm{d}s\,\mathrm{d}\xi\right)^{1/2},\ i = 0,1\right\} < \infty.
$$

(b) The function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and there is continuous function μ such that $\int_{-\infty}^{0} \mu(s)^2/h(s)\,\mathrm{d}s < \infty$ and $\|F(t,\xi)\| \leqslant \mu(s)\|\xi\|$.

(c) The functions $a_i^j \in C([0,\infty);\mathbb{R})$, and

$$
L_i^j := \left(\int\limits_{-\infty}^{0} \frac{(a_i^j(s))^2}{h(s)}\,\mathrm{d}s\right)^{1/2} < \infty, \quad i = 1,2,\ldots,n,\ j = 1,2.
$$

Moreover, $g(t,\cdot)$ is bounded linear operators.

Hence, by assumptions (a)–(c) and Theorem 2 it is ensured that problem (4) has a unique mild solution.

Now we check the approximate controllability of (4). For $y \in D(A)$, $y = \sum_{n=1}^{\infty} \langle y, \phi_n \rangle \phi_n$ and $Ay = -\sum_{n=1}^{\infty} n^2 \langle y, n \rangle \phi_n$, where $\phi_n(x) = \sqrt{2/\pi} \sin nx$, $0 \leqslant x \leqslant \pi$, $n = 1, 2, 3 \ldots$, is the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator A. ϕ_n is an orthonormal base. A will generate the operators $S(t)$, $C(t)$ such that $S(t)y = \sum_{n=1}^{\infty} (\sin(nt)/n) \langle y, \phi_n \rangle \phi_n$, $n = 1, 2, \ldots$, for all $y \in X$, and the operator $C(t)y = \sum_{n=1}^{\infty} \cos(nt) \langle y, \phi_n \rangle \phi_n$, $n = 1, 2, \ldots$, for all $y \in X$. Let the infinite dimensional control space be defined as $U = \{u : u = \sum_{n=2}^{\infty} u_n \phi_n, \sum_{n=2}^{\infty} u_n^2 < \infty\}$ with norm $\|u\|_U = (\sum_{n=2}^{\infty} u_n^2)^{1/2}$. Thus, U is a Hilbert space.

Let $\widetilde{B} : U \to X$ be defined as $\widetilde{B}u = 2u_2 \phi_1 + \sum_{n=2}^{\infty} u_n \phi_n$ for $u = \sum_{n=2}^{\infty} u_n \phi_n \in U$. The bounded linear operator $B : L_2([0, T]; U) \to L_2([0, T]; X)$ is defined by $(Bu)(t) = \widetilde{B}u(t)$.

Let $a \in N \subset L_2(0, T; X)$, N is the null space of Γ. Also, $a = \sum_1^{\infty} a_n(s) \phi_n$. Therefore,

$$\int_0^T S(T - s)a(s)\, \mathrm{d}s = 0. \tag{5}$$

This implies that

$$\int_0^T \frac{\sin n(T - s)}{n} a_n(s)\, \mathrm{d}s = 0, \quad n \in \mathcal{N}.$$

The Hilbert space $L_2(0, T)$ can be written as

$$L_2(0, T) = \mathrm{Sp}\{\sin s\}^{\perp} + \mathrm{Sp}\{\sin 4s\}^{\perp}.$$

Thus, for $h_1, h_2 \in L_2(0, T)$, there exists $a_1 \in \{\sin s\}^{\perp}$, $a_2 \in \{\sin 4s\}^{\perp}$ such that $h_1 - 2h_2 = a_1 - 2a_2$. So, let $u_2 = h_2 - a_2$, then $h_1 = a_1 + 2u_2$, $h_2 = a_2 + u_2$. Also, let $u_n = h_n$, $n = 3, 4, \ldots$, and $a_n = 0$, $n = 3, 4, \ldots$. Thus, we see that Lemma 3 is satisfied as $U = \{u : u = \sum_{n=2}^{\infty} u_n \phi_n, \sum_{n=2}^{\infty} u_n^2 < \infty\}$ and $\widetilde{B} : U \to X : \widetilde{B}u = 2u_2 \phi_1 + \sum_{n=2}^{\infty} u_n \phi_n$. The approximate controllability is deduced from Theorem 3.

Conclusion. We prove existence and uniqueness of the system by assuming simple growth conditions on the operators and Hausdorff measure of noncompactness. Our method to prove approximate controllability, removes the need to assume the invertibility of a controllability operator used by authors in [5], which fails to exist in infinite dimensional spaces if the associated semigroup is compact. Our approach also removes the need to check the invertibility of the controllability Gramian operator and associated limit condition used by the authors in [24], which are practically difficult to verify and apply.

However, the case when the operator A is nondense in the Hilbert space X is still not widely investigated. If the nondense case is similarly studied by using integrated semigroups, then the choice of operator A is not limited to only dense operators. Therefore, a large class of stochastic partial differential equations belonging to this prototype of

neutral functional differential equations can be studied. We considered the white noise as a Wiener process, but this work can be extended to incorporate other disturbances in the form of Poisson processes, etc.

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