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Solution of Volterra integral inclusion in b-metric spaces via new fixed point theorem

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Abstract. An existence theorem for Volterra-type integral inclusion is establish in b-metric spaces. We first introduce two new F-contractions of Hardy–Rogers type and then establish fixed point theorems for these contractions in the setting of b-metric spaces. Finally, we apply our fixed point theorem to prove the existence theorem for Volterra-type integral inclusion. We also provide an example to show that our fixed point theorem is a proper generalization of a recent fixed point theorem by Cosentino et al.

Keywords: α_s-admissible mappings, α_s^*-admissible mappings, Hardy–Rogers-type F-contractions.

1 Introduction

The theory of differential equations are based on nonlinear functional analysis. Many existence theorems for the solution of differential equations are proved by means of fixed point theorems. The famous Banach contraction principle has a lot of applications in theory of integral equations. There are many generalizations of Banach contraction principle, see, for example, [1–37]. Wardowski [37] gave an interesting generalization of Banach contraction known as F-contraction. Several authors generalized F-contraction by combining it with some existing contractive conditions, see, for example, Acar and

Altun [1], Batra and Vashistha [6], Cosentino and Vetro [13], Mınak et al. [22], Paesano and Vetro [26], Piri and Kumam [29], Secelean [31], and Sgroi and Vetro [32].

The problem of the convergence of measurable functions with respect to a measure, lead to a generalization of notion of a metric. Using this idea, Czerwik [14] gave a generalization of the famous Banach fixed point theorem [14] in so-called b-metric spaces. For some important results on b-metric spaces, we refer the reader to [4, 9, 10, 15, 33]. Recently, Cosentino et al. [12] extended F-contraction in the setting of b-metric spaces and proved some fixed point theorems.

In this paper, we generalize the result of Cosentino et al. for new class of F-contractions in the setting of b-metric spaces. We also construct an example to show the generality of our result. Finally, we apply our result to obtain existence theorems for Volterra-type integral inclusion in b-metric spaces.

2 Preliminaries

Before going towards our findings, we need the following definitions, notions and results.

Definition 1. (See [14].) Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is said to be a b-metric on X if for each $x, y, z \in X$, we have a real number $s \geqslant 1$ such that

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leqslant s[d(x, y) + d(y, z)]$.

Then the triplet (X, d, s) is said to be a b-metric space.

Note that every metric space is a b-metric but converse is not true.

Example 1. Let $X = [0, \infty)$ and $d : X \times X \to [0, \infty)$, $d(x, y) = |x - y|^2$ for each $x, y \in X$. Clearly, $(X, d, 2)$ is a b-metric space, but not a metric space.

Following is one more interesting and very famous examples of b-metric, which is not a metric.

Example 2. (See [14].) Let $p \in (0, 1)$ and $l^p(\mathbb{R}) = \{\{x_n\} \subset \mathbb{R}\colon \sum_{n=1}^{\infty} |x_n|^p < \infty\}$ endowed with the functional $d : l^p(\mathbb{R}) \times l^p(\mathbb{R}) \to \mathbb{R}$,

$$d\big(\{x_n\}, \{y_n\}\big) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$$

for each $\{x_n\}, \{y_n\} \in l^p(\mathbb{R})$. This is a b-metric space with $s = 2^{1/p}$.

Recall that a sequence $\{x_n\}$ in a b-metric space (X, d, s) converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. A sequence $\{x_n\}$ in a b-metric space (X, d, s) is a Cauchy sequence if for each $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for each $m, n \geqslant N(\epsilon)$. A b-metric space (X, d, s) is a complete if each Cauchy sequence in X converges to some point of X.

Lemma 1. (*See* [14].) *Let* (X, d, s) *be a b-metric space, and let* $\{x_n\}$ *be a sequence in* X. *If* $\lim_{n\to\infty} x_n = y$ *and* $\lim_{n\to\infty} x_n = z$, *then* $y = z$.

Let (X, d, s) be a b-metric space. The closed and bounded sets in X are defined in a similar manner as for a metric space. We denote by $CB(X)$ the class of all nonempty closed and bounded subsets of X and by $CL(X)$ the class of all nonempty closed subsets of X.

Let $x \in X$ and $A \subset X$, $d(x, A) = \inf\{d(x, a)\colon a \in A\}$. For $A, B \in CB(X)$, the function $H : CB(X) \times CB(X) \to [0, \infty)$,

$$H(A, B) = \max\Bigl\{\sup_{a\in A} d(a, B),\ \sup_{b\in B} d(b, A)\Bigr\}$$

is said to be a Hausdorff b-metric [15] induced by the b-metric d. For $A, B \in CL(X)$, the function $H : CL(X) \times CL(X) \to [0, \infty)$, given by

$$H(A, B) = \begin{cases} \max\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\} & \text{if the maximum exists,} \\ \infty & \text{otherwise,} \end{cases}$$

is said to be a generalized Hausdorff b-metric induced by b-metric d.

Following properties based on b-metric are taken from [15].

Lemma 2. *Let* (X, d, s) *be a b-metric space. For any* $A, B, C \in CB(X)$ *and any* $x, y \in X$, *we have the following*:

(i) $d(x, A) \leqslant d(x, a)$ *for each* $a \in A$;
(ii) $d(x, B) \leqslant H(A, B)$ *for each* $x \in A$;
(iii) $H(A, A) = 0$;
(iv) $H(A, B) = H(B, A)$;
(v) $H(A, B) \leqslant s[H(A, C) + H(C, B)]$;
(vi) $d(x, A) \leqslant s[d(x, y) + d(y, A)]$.

Lemma 3. (*See* [15].) *Let* (X, d, s) *be a b-metric space. For any* $A, B \in CL(X)$ *and any* $x \in X$, *we have the following*:

(i) *For* $h > 1$ *and* $a \in A$, *there exists* $b \in B$ *such that* $d(a, b) \leqslant hH(A, B)$.
(ii) $d(x, A) = 0 \Leftrightarrow x \in \overline{A} = A$, *where* $\overline{A}$ *denotes the closure of the set* A.

Definition 2. (See [12].) Let $s \geqslant 1$ be a real number. Denote by $\mathfrak{F}_s$ the family of all functions $F : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

(F1) F is strictly increasing, that is, for each $a_1, a_2 \in (0, \infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$;
(F2) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers, we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$;
(F3) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers with $\lim_{n\to\infty} \mathfrak{d}_n = 0$, there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} \mathfrak{d}_n^k F(\mathfrak{d}_\mathfrak{n}) = 0$.

(F4) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers such that $\tau+F(s\mathfrak{d}_n) \leqslant F(\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$ and some $\tau > 0$, we have $\tau + F(s^n\mathfrak{d}_n) \leqslant F(s^{n-1}\mathfrak{d}_{n-1})$ for each $n \in \mathbb{N}$.

Cosentino et al. [12] also showed that the following functions belong to $\mathfrak{F}_s$.

- $F(x) = x + \ln x$ for each $x > 0$.
- $F(x) = \ln x$ for each $x > 0$.

3 Main results

We begin this section with the following definition.

Definition 3. Let (X, d, s) be a b-metric space, and let $\alpha : X \times X \to [0, \infty)$ be a function.

(i) A mapping $T : X \to CL(X)$ is α_s-admissible if for $x \in X$ and $y \in Tx$ such that $\alpha(x, y) \geqslant s^2$, we have $\alpha(y, z) \geqslant s^2$ for each $z \in Ty$.

(ii) A mapping $T : X \to CL(X)$ is α_s^*-admissible mapping if for $x, y \in X$ with $\alpha(x, y) \geqslant s^2$, we have $\alpha^*(Tx, Ty) \geqslant s^2$, where $\alpha^*(Tx, Ty) = \inf\{\alpha(u, v)\colon u \in Tx \text{ and } v \in Ty\}$.

Remark 1. Note that for $s = 1$, above definition reduces to α-admissible and α_*-admissible, as defined in [24] and [3], respectively.

Example 3. Let $X = [-1, 1]$ endowed with the b-metric $d(x, y) = |x - y|^2$ with $s = 2$. Define

$$T : X \to CL(X), \quad Tx = \begin{cases} \{0, 1\} & \text{if } x = -1, \\ \{1\} & \text{if } x = 0, \\ \{-x\} & \text{if } x \notin \{-1, 0\} \end{cases}$$

and

$$\alpha : X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 5 & \text{if } x \neq y. \end{cases}$$

It is straightforward to see that T is α_s-admissible, but not α_s^*-admissible.

Before proving our main results, we prove an auxiliary result.

Lemma 4. *Let (X, d, s) be a b-metric space, and let $\{x_n\}$ be any sequence in X for which there exist $\tau > 0$ and $F \in \mathfrak{F}_s$ such that*

$$\tau + F\big(sd(x_n, x_{n+1})\big) \leqslant F\big(d(x_{n-1}, x_n)\big), \quad n \in \mathbb{N}.$$

Then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Given that

$$\tau + F\big(sd(x_n, x_{n+1})\big) \leqslant F\big(d(x_{n-1}, x_n)\big), \quad n \in \mathbb{N}. \tag{1}$$

Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. Thus, by (1) and property (F4), we get

$$\tau + F\big(s^n d_n\big) \leqslant F\big(s^{n-1} d_{n-1}\big), \quad n \in \mathbb{N}.$$

Consequently, we get

$$F\big(s^n d_n\big) \leqslant F(d_0) - n\tau, \quad n \in \mathbb{N}. \tag{2}$$

Letting $n \to \infty$ in (2), we get $\lim_{n\to\infty} F(s^n d_n) = -\infty$. Then, by property (F2), we have $\lim_{n\to\infty} s^n d_n = 0$. From (F3) there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} \big(s^n d_n\big)^k F\big(s^n d_n\big) = 0.$$

From (2) we have

$$\big(s^n d_n\big)^k F\big(s^n d_n\big) - \big(s^n d_n\big)^k F(d_0) \leqslant -\big(s^n d_n\big)^k n\tau \leqslant 0, \quad n \in \mathbb{N}. \tag{3}$$

Letting $n \to \infty$ in (3), we get

$$\lim_{n\to\infty} n(s^n d_n)^k = 0. \tag{4}$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n(s^n d_n)^k \leqslant 1$ for each $n \geqslant n_1$. Thus, we have

$$s^n d_n \leqslant \frac{1}{n^{1/k}}, \quad n \geqslant n_1. \tag{5}$$

To prove that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (5), we have

$$d(x_n, x_m) \leqslant \sum_{i=n}^{m-1} s^i d_i \leqslant \sum_{i=n}^{\infty} s^i d_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}.$$

This implies $\{x_n\}$ is a Cauchy sequence since $\sum_{i=1}^{\infty} i^{-1/k}$ is convergent. $\square$

Now we define the notion of Hardy–Rogers-type (F,α)-contraction.

Definition 4. Let (X,d,s) be a b-metric space and $\alpha : X \times X \to [0,\infty)$ be a function. A mapping $T : X \to CL(X)$ is called Hardy–Rogers-type (F,α)-contraction if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F\big(\alpha(x,y)H(Tx,Ty)\big) \leqslant F\big(R(x,y)\big), \quad x, y \in X, \tag{6}$$

whenever $\min\{\alpha(x,y)H(Tx,Ty), R(x,y)\} > 0$, where

$$R(x,y) = a_1 d(x,y) + a_2 d(x,Tx) + a_3 d(y,Ty) + a_4 d(x,Ty) + L d(y,Tx)$$

with $a_1, a_2, a_3, a_4, L \geqslant 0$ satisfying $a_1 + a_2 + a_3 + 2s a_4 = 1$ and $a_3 \neq 1$.

Theorem 1. *Let (X,d,s) be a complete b-metric space with $s > 1$, and let $T : X \to CL(X)$ be a Hardy–Rogers-type (F,α)-contraction such that the following conditions hold*:

(i) *T is an α_s-admissible mapping*;

(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0,x_1) \geqslant s^2$*;

(iii) *For any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *and* $\alpha(x_n, x_{n+1}) \geqslant s^2$ *for each* $n \in \mathbb{N}$*, we have* $\alpha(x_n, x) \geqslant s^2$ *for each* $n \in \mathbb{N}$.

Then T *has a fixed point.*

Proof. By hypothesis (ii), there exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geqslant s^2$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$. As $\alpha(x_0, x_1) \geqslant s^2$, there exists $x_2 \in Tx_1$ such that

$$sd(x_1, x_2) \leqslant \alpha(x_0, x_1) H(Tx_0, Tx_1). \tag{7}$$

Since F is strictly increasing, we have

$$F\big(sd(x_1, x_2)\big) \leqslant F(\alpha(x_0, x_1) H\big(Tx_0, Tx_1)\big). \tag{8}$$

From (6), we have

$$\begin{aligned}
&\tau + F\big(sd(x_1, x_2)\big) \\
&\quad \leqslant \tau + F\big(\alpha(x_0, x_1) H(Tx_0, Tx_1)\big) \\
&\quad \leqslant F\big(a_1 d(x_0, x_1) + a_2 d(x_0, Tx_0) + a_3 d(x_1, Tx_1) + a_4 d(x_0, Tx_1) + L d(x_1, Tx_0)\big) \\
&\quad \leqslant F\big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + a_4 d(x_0, x_2) + L \cdot 0\big) \\
&\quad \leqslant F\big(a_1 d(x_0, x_1) + a_2 d(x_0, x_1) + a_3 d(x_1, x_2) + s a_4 \big(d(x_0, x_1) + d(x_1, x_2)\big)\big) \\
&\quad = F\big((a_1 + a_2 + s a_4) d(x_0, x_1) + (a_3 + s a_4) d(x_1, x_2)\big).
\end{aligned} \tag{9}$$

Since F is strictly increasing, we get from above that

$$sd(x_1, x_2) < (a_1 + a_2 + s a_4) d(x_0, x_1) + (a_3 + s a_4) d(x_1, x_2).$$

That is,

$$(1 - a_3 - s a_4) d(x_1, x_2) < (s - a_3 - s a_4) d(x_1, x_2) < (a_1 + a_2 + s a_4) d(x_0, x_1).$$

As $a_1 + a_2 + a_3 + 2 s a_4 = 1$, we have

$$d(x_1, x_2) < d(x_0, x_1).$$

Now, from (9), we obtain

$$\tau + F\big(sd(x_1, x_2)\big) \leqslant F\big(d(x_0, x_1)\big).$$

Since T is α_s-admissible, we have $\alpha(x_1, x_2) \geqslant s^2$. Continuing in the same way, we get a sequence $\{x_n\} \subset X$ such that

$$x_n \in Tx_{n-1}, \quad x_{n-1} \neq x_n \quad \text{and} \quad \alpha(x_{n-1}, x_n) \geqslant s^2, \quad n \in \mathbb{N}.$$

Furthermore,

$$\tau + F\big(sd(x_n, x_{n+1})\big) \leqslant F\big(d(x_{n-1}, x_n)\big), \quad n \in \mathbb{N}. \tag{10}$$

Thus, by Lemma 4, $\{x_n\}$ is a Cauchy sequence in X. As (X, d, s) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iii), we have $\alpha(x_n, x^*) \geqslant s^2$ for each $n \in \mathbb{N}$. We claim that $d(x^*, Tx^*) = 0$. On contrary suppose that $d(x^*, Tx^*) > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, Tx^*) > 0$ for each $n \geqslant n_0$. For each $n \geqslant n_0$, we have

$$\begin{aligned}
d(x^*, Tx^*) &\leqslant sd(x^*, x_{n+1}) + sd(x_{n+1}, Tx^*) \\
&< sd(x^*, x_{n+1}) + \alpha(x_n, x^*)H(Tx_n, Tx^*) \\
&< sd(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) + a_3 d(x^*, Tx^*) \\
&\quad + a_4 d(x_n, Tx^*) + Ld(x^*, x_{n+1}) \\
&\leqslant sd(x^*, x_{n+1}) + a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) + a_3 d(x^*, Tx^*) \\
&\quad + sa_4\bigl(d(x_n, x^*) + d(x^*, Tx^*)\bigr) + Ld(x^*, x_{n+1}).
\end{aligned} \tag{11}$$

Letting $n \to \infty$ in (11), we have

$$d(x^*, Tx^*) \leqslant (a_3 + sa_4)d(x^*, Tx^*) < d(x^*, Tx^*),$$

which is a contradiction. Thus, $d(x^*, Tx^*) = 0$. □

Example 4. Let $X = \mathbb{N} \cup \{0\}$ be endowed with a b-metric $d(x, y) = |x - y|^2$ for each $x, y \in X$ with $s = 2$. Define

$$T : X \to CL(X), \quad Tx = \begin{cases} \{0, 1\} & \text{if } x = 0, 1, \\ \{x, x + 1\} & \text{if } x > 1 \end{cases}$$

and

$$\alpha : X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} 4 & \text{if } x, y \in \{0, 1\}, \\ \frac{1}{3} & \text{if } x, y > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Take $F(x) = x + \ln x$ for each $x \in (0, \infty)$. Under this F, condition (6) reduces to

$$\frac{\alpha(x, y)H(Tx, Ty)}{R(x, y)} \mathrm{e}^{\alpha(x,y)H(Tx,Ty)-R(x,y)} \leqslant \mathrm{e}^{-\tau} \tag{12}$$

for each $x, y \in X$ with $\min\{\alpha(x, y)H(Tx, Ty), R(x, y)\} > 0$. Assume that $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $\tau = 1/3$. Clearly, $\min\{\alpha(x, y)H(Tx, Ty), d(x, y)\} > 0$ for each $x, y > 1$ with $x \neq y$. From (12), for each $x, y > 1$ with $x \neq y$, we have

$$\frac{1}{3}\mathrm{e}^{-2/3|x-y|^2} < \mathrm{e}^{-1/3}.$$

Thus, T is Hardy–Rogers-type (F, α)-contraction with $F(x) = x + \ln x$. For $x_0 = 1$, we have $x_1 = 0 \in Tx_0$ such that $\alpha(x_0, x_1) = 4$. Moreover, it is easy to see that T is α_s-admissible mapping and for any sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) = 4$ for each $n \in \mathbb{N}$. Hence, we have $\alpha(x_n, x) = 4$ for each $n \in \mathbb{N}$. Therefore, all conditions of Theorem 1 are satisfied, and T has a fixed point in X.

Remark 2. Note that [12, Thm. 3.4] is not applicable here with $F(x) = x + \ln x$. To see this, take $x = 0$ and $y = 3$. Thus, this example shows the importance of our result.

Corollary 1. *Let (X, d, s) be a complete b-metric space with $s > 1$, and let $T : X \to CL(X)$ be a mapping such that*

$$s^2 H(Tx, Ty) \leqslant k\big(a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ + a_4 d(x, Ty) + L d(y, Tx)\big), \quad x, y \in X,$$

where $k \in (0, 1)$, $a_1, a_2, a_3, a_4, L \geqslant 0$ satisfying $a_1 + a_2 + a_3 + 2sa_4 = 1$ and $a_3 \neq 1$. Then T has a fixed point.

Proof. Let $\alpha(x, y) = s^2$ for each $x, y \in X$, and let $\tau > 0$ such that $k = \mathrm{e}^{-\tau}$. Then for all $x, y \in X$ with $Tx \neq Ty$, the given inequality reduces to (6), where $F(x) = \ln x$. Thus, conclusion follows from Theorem 1. □

Definition 5. Let (X, d, s) be a b-metric space and $\alpha : X \times X \to [0, \infty)$ be a function. A mapping $T : X \to CL(X)$ is called Hardy–Rogers-type (F, α^*)-contraction if there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that

$$\tau + F\big(\alpha^*(Tx, Ty) H(Tx, Ty)\big) \leqslant F\big(R(x, y)\big) \tag{13}$$

for each $x, y \in X$, whenever $\min\{\alpha^*(Tx, Ty) H(Tx, Ty), R(x, y)\} > 0$, where

$$R(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)$$

with $a_1, a_2, a_3, a_4, L \geqslant 0$ satisfying $a_1 + a_2 + a_3 + 2sa_4 = 1$ and $a_3 \neq 1$.

Theorem 2. *Let (X, d, s) be a complete b-metric space with $s > 1$, and let $T : X \to CL(X)$ be a (F, α^*)-contraction of Hardy–Rogers type such that the following conditions hold*:

(i) *T is an α_s^*-admissible mapping*;
(ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geqslant s^2$*;
(iii) *For any sequence $\{x_n\}$ in X such that $x_n \to x$ and $\alpha(x_n, x_{n+1}) \geqslant s^2$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) \geqslant s^2$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Proof. The proof of this theorem runs along the same lines as the proof of Theorem 1. □

4 Consequences

In this section, we apply our results to obtain some new fixed point theorems for mappings on b-metric spaces endowed with a partial ordering/graphs. If we define

$$\alpha : X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} s^2 & \text{if } x \preccurlyeq y, \\ 0 & \text{otherwise,} \end{cases}$$

then the following result is a direct consequence of our results.

Theorem 3. *Let $(X, d, s, \preccurlyeq)$ be a complete ordered b-metric space with $s > 1$, and let $T : X \to CL(X)$ be a mapping for which there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that*

$$\tau + F\big(s^2 H(Tx, Ty)\big) \leqslant F\big(R(x, y)\big), \quad x, y \in X, \tag{14}$$

with $x \preccurlyeq y$, whenever $\min\{s^2 H(Tx, Ty), R(x, y)\} > 0$, where

$$R(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)$$

with $a_1, a_2, a_3, a_4, L \geqslant 0$ satisfying $a_1 + a_2 + a_3 + 2sa_4 = 1$ and $a_3 \neq 1$. Moreover, the following conditions hold:

- (i) *For each $x \in X$ and $y \in Tx$ such that $x \preccurlyeq y$, we have $y \preccurlyeq z$ for each $z \in Ty$; or if $x \preccurlyeq y$, then we have $Tx \prec_r Ty$, that is, for each $a \in Tx$ and $b \in Ty$, we have $a \preccurlyeq b$*;
- (ii) *There exist $x_0 \in X$ and $x_1 \in Tx_0$ with $x_0 \preccurlyeq x_1$*;
- (iii) *For any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$ and $x_n \preccurlyeq x_{n+1}$ for each $n \in \mathbb{N}$, we have $x_n \preccurlyeq x$ for each $n \in \mathbb{N}$.*

Then T has a fixed point.

Now, we drive a fixed point theorem for multivalued mappings from a metric spaces X, endowed with a graph, into the space of nonempty closed subsets of the metric space. Subsequently, we assume that $G = (V(G), E(G))$ is a directed graph such that the set of its vertices $V(G)$ coincides with X (i.e., $V(G) = X$) and the set of its edges $E(G)$ is such that $E(G) \supseteq \nabla$, where $\nabla = \{(x, x)\colon x \in X\}$. Moreover, G has no parallel edges. If we define

$$\alpha : X \times X \to [0, \infty), \quad \alpha(x, y) = \begin{cases} s^2 & \text{if } (x, y) \in E(G), \\ 0 & \text{otherwise}, \end{cases}$$

then the following result is a direct consequence of our main results.

Theorem 4. *Let (X, d, s) be a complete b-metric space endowed with the graph G, having $s > 1$, and let $T : X \to CL(X)$ be a mapping for which there exist $F \in \mathfrak{F}_s$ and $\tau > 0$ such that*

$$\tau + F\big(s^2 H(Tx, Ty)\big) \leqslant F\big(R(x, y)\big) \tag{15}$$

for each $x, y \in X$ with $(x, y) \in E(G)$, whenever $\min\{s^2 H(Tx, Ty), R(x, y)\} > 0$, where

$$R(x, y) = a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + L d(y, Tx)$$

with $a_1, a_2, a_3, a_4, L \geqslant 0$ satisfying $a_1 + a_2 + a_3 + 2sa_4 = 1$ and $a_3 \neq 1$. Moreover, the following conditions hold:

- (i) *For each $x \in X$ and $y \in Tx$ such that $(x, y) \in E(G)$, we have $(y, z) \in E(G)$ for each $z \in Ty$; or if $(x, y) \in E(G)$, then we have $(a, b) \in E(G)$ for each $a \in Tx$ and $b \in Ty$*;

(ii) *There exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *with* $(x_0, x_1) \in E(G)$;

(iii) *For any sequence* $\{x_n\}$ *in* X *such that* $x_n \to x$ *as* $n \to \infty$ *and* $(x_n, x_{n+1}) \in E(G)$ *for each* $n \in \mathbb{N}$*, we have* $(x_n, x) \in E(G)$ *for each* $n \in \mathbb{N}$.

Then T *has a fixed point.*

5 Applications

In this section, we give existence theorems for Volterra-type integral inclusion. For this purpose, let $X = C([a,b],\mathbb{R})$ be the space of all continuous realvalued functions on $[a,b]$. Note that X is complete b-metric space by considering $d(x,y) = \sup_{t\in[a,b]} |x(t) - y(t)|^2$ with $s = 2$.

Consider the Volterra-type integral inclusion as

$$x(t) \in \int_a^t M\bigl(t, s, x(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in [a,b], \tag{16}$$

where $M : [a,b] \times [a,b] \times \mathbb{R} \to P_{cv}(\mathbb{R})$, and $P_{cv}(\mathbb{R})$ denotes the family of nonempty compact and convex subsets of $\mathbb{R}$. For each $x \in C([a,b],\mathbb{R})$, the operator $M(\cdot,\cdot,x)$ is lower semi continuous. Further the function $g : [a,b] \to \mathbb{R}$ is continuous.

For the integral inclusion given above, we can define a multivalued operator $T : C([a,b],\mathbb{R}) \to CL(C([a,b],\mathbb{R}))$ as follows:

$$Tx(t) = \left\{ u \in C\bigl([a,b],\mathbb{R}\bigr)\colon\ u \in \int_a^t M\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t),\ t \in [a,b] \right\}.$$

Let $x \in C([a,b],\mathbb{R})$, and denote $M_x := M(t,s,x(s))$, $t,s \in [a,b]$. Now, for $M_x : [a,b]\times[a,b] \to P_{cv}(\mathbb{R})$, by Michael's selection theorem, there exists a continuous operator $m_x : [a,b] \times [a,b] \to \mathbb{R}$ with $m_x(t,s) \in M_x(t,s)$ for each $t,s \in [a,b]$. This shows that $\int_a^t m_x(t,s)\,\mathrm{d}s + g(t) \in Tx(t)$. Thus, the operator Tx is nonempty. Clearly, the operator Tx is closed following [35].

Theorem 5. *Let* $X = C([a,b],\mathbb{R})$*, and let the multivalued operator*

$$T : X \to CL(X), \quad Tx(t) = \left\{ u \in X\colon\ u \in \int_a^t M\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t),\ t \in [a,b] \right\},$$

where the function $g : [a,b] \to \mathbb{R}$ *is continuous and* $M : [a,b] \times [a,b] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ *is such that for each* $x \in C([a,b],\mathbb{R})$*, the operator* $M(\cdot,\cdot,x)$ *is lower semi continuous. Assume that the following conditions hold*:

(i) *There exists a continuous mapping* $q : X \to [0,\infty)$ *such that*

$$H\bigl(M(t,s,x(s)), M\bigl(t,s,y(s)\bigr)\bigr) \leqslant q(s)\bigl|x(s) - y(s)\bigr|$$

for each $t,s \in [a,b]$ *and* $x,y \in X$.

(ii) *There exist* $\tau > 0$ *and* $\alpha : X \times X \to (0, \infty)$ *such that for each* $x, y \in X$*, we have*

$$\int_a^t q(s)\,\mathrm{d}s \leqslant \sqrt{\frac{\mathrm{e}^{-\tau}}{\alpha(x,y)}}, \quad t \in [a,b];$$

(iii) *There exist* $x_0 \in X$ *and* $x_1 \in Tx_0$ *with* $\alpha(x_0, x_1) \geqslant 4$;

(iv) *If* $x \in X$ *and* $y \in Tx$ *such that* $\alpha(x,y) \geqslant 4$*, then we have* $\alpha(y,z) \geqslant 4$ *for each* $z \in Ty$;

(v) *For any sequence* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n, x_{n+1}) \geqslant 4$ *for each* $n \in \mathbb{N}$*, we have* $\alpha(x_n, x) \geqslant 4$ *for each* $n \in \mathbb{N}$.

Then the integral inclusion (16) *has a solution.*

Proof. We have to show that the operator T satisfies all conditions of Theorem 1. First, we check (6). Let $x, y \in X$ be such that $u \in Tx$. Then we have $m_x(t,s) \in M_x(t,s)$ for $t, s \in [a,b]$ such that $u(t) = \int_a^t m_x(t,s)\,\mathrm{d}s + g(t)$, $t \in [a,b]$. On the other hand, hypothesis (i) ensures that there exists $v(t,s) \in M_y(t,s)$ such that

$$\left|m_x(t,s) - v(t,s)\right| \leqslant q(s)\left|x(s) - y(s)\right| \quad \forall t, s \in [a,b].$$

Let us consider the multivalued operator S defined by

$$S(t,s) = M_y(t,s) \cap \bigl\{w \in \mathbb{R}\colon \left|m_x(t,s) - w\right| \leqslant q(s)\left|x(s) - y(s)\right|\bigr\}, \quad t, s \in [a,b].$$

Since the operator T is lower semi continuous, there exists $m_y : [a,b] \times [a,b] \to \mathbb{R}$ such that $m_y(t,s) \in S(t,s)$ for each $t, s \in [a,b]$. Thus, we get

$$r(t) = \int_a^t m_y(t,s)\,\mathrm{d}s + g(t) \in \int_a^t M\bigl(t, s, y(s)\bigr)\,\mathrm{d}s + g(t), \quad t \in [a,b],$$

and for each $t \in [a,b]$, we have

$$\begin{aligned}
\left|u(t) - r(t)\right|^2 &\leqslant \left(\int_a^t \left|m_x(t,s) - m_y(t,s)\right|\mathrm{d}s\right)^2 \leqslant \left(\int_a^t q(s)\left|x(s) - y(s)\right|\mathrm{d}s\right)^2 \\
&\leqslant \left(\sqrt{\sup_{s \in [a,b]} \left|x(s) - y(s)\right|^2} \int_a^t q(s)\,\mathrm{d}s\right)^2 = d(x,y)\left(\int_a^t q(s)\,\mathrm{d}s\right)^2 \\
&\leqslant \frac{\mathrm{e}^{-\tau}}{\alpha(x,y)} d(x,y).
\end{aligned}$$

Consequently, we have

$$\alpha(x,y) d(u,r) \leqslant \mathrm{e}^{-\tau} d(x,y).$$

Now, by just interchanging the role of x and y, we reach to

$$\alpha(x,y)H(Tx,Ty) \leqslant \mathrm{e}^{-\tau} d(x,y), \quad x,y \in X.$$

As natural logarithm belongs to $\mathfrak{F}_s$, applying it on above inequality and after some simplification, we get

$$\tau + \ln\bigl(\alpha(x,y)H(Tx,Ty)\bigr) \leqslant \ln\bigl(d(x,y)\bigr).$$

Thus, T is (F,α)-contraction of Hardy–Rogers type with $a_1 = 1$, $a_2 = a_3 = a_4 = L = 0$ and $F(x) = \ln x$. All other conditions of Theorem 1 immediately follows by the hypothesis. Therefore, the operator T has a fixed point, that is, the Volterra-type integral inclusion (16) has a solution. □

By using Corollary 1, we can prove the following existence theorem along the same lines as the proof of above is done.

Theorem 6. *Let* $X = C([a,b],\mathbb{R})$ *and let the multivalued operator*

$$T: X \to CL(X), \quad Tx(t) = \left\{ u \in X\colon u \in \int_a^t M\bigl(t,s,x(s)\bigr)\,\mathrm{d}s + g(t),\ t \in [a,b] \right\},$$

where the function $g: [a,b] \to \mathbb{R}$ *is continuous and* $M: [a,b] \times [a,b] \times \mathbb{R} \to P_{cv}(\mathbb{R})$ *is such that for each* $x \in C([a,b],\mathbb{R})$*, the operator* $M(\cdot,\cdot,x)$ *is lower semi continuous. Assume that the following conditions hold*:

(i) *There exists a continuous mapping* $q: X \to [0,\infty)$ *such that*

$$H\bigl(M\bigl(t,s,x(s)\bigr), M\bigl(t,s,y(s)\bigr)\bigr) \leqslant q(s)\bigl|x(s) - y(s)\bigr|$$

for each $t,s \in [a,b]$ *and* $x,y \in X$.

(ii) *There exists* $\tau > 0$ *such that*

$$\int_a^t q(s)\,\mathrm{d}s \leqslant \sqrt{\frac{\mathrm{e}^{-\tau}}{4}}, \quad t \in [a,b].$$

Then the integral inclusion (16) *has a solution.*

References

1. O. Acar, I. Altun, A fixed point theorem for multivalued mappings with δ-distance, *Abstr. Appl. Anal.*, **2014**:497092, 2014.
2. M.U. Ali, T. Kamran, N. Shahzad, Best proximity point for α-ψ-proximal contractive multimaps, *Abstr. Appl. Anal.*, **2014**:181598, 2014.

3. J.H. Asl, S. Rezapour, N. Shahzad, On fixed points of α-ψ-contractive multifunctions, *Fixed Point Theory Appl.*, **2012**(212), 2012.
4. H. Aydi, M.-F. Bota, E. Karapınar, S. Moradi, A common fixed point for weak-Φ-contractions on b-metric spaces, *Fixed Point Theory*, **13**(2):337–346, 2012.
5. H. Aydi, E. Karapınar, B. Samet, Fixed points for generalized (α, ψ)-contractions on generalized metric spaces, *J. Inequal. Appl.*, **2014**(229), 2014.
6. R. Batra, S. Vashistha, Fixed points of an F-contraction on metric spaces with a graph, *Int. J. Comput. Math.*, **91**(12):2483–2490, 2014.
7. M. Berinde, V. Berinde, On a general class of multi-valued weakly Picard mappings, *J. Math. Anal. Appl.*, **326**:772–782, 2007.
8. F. Bojor, Fixed points of Kannan mappings in metric spaces endowed with a graph, *An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat.*, **20**(1):31–40, 2012.
9. M.-F. Bota, E. Karapınar, A note on "Some results on multi-valued weakly Jungck mappings in b-metric space", *Cent. Eur. J. Math.*, **11**(9):1711–1712, 2013.
10. M.-F. Bota, E. Karapınar, O. Mleşniţe, Ulam–Hyers stability results for fixed point problems via α-ψ-contractive mapping in b-metric space, *Abstr. Appl. Anal.*, **2013**(4):825293, 2013.
11. S.H. Cho, Fixed point theorems for α-ψ-contractive type mappings in metric spaces, *Appl. Math. Sci.*, **7**:6765–6778, 2013.
12. M. Cosentino, M. Jleli, B. Samet, C. Vetro, Solvability of integrodifferential problem via fixed point theory in b-metric spaces, *Fixed Point Theory Appl.*, **2015**(70), 2015.
13. M. Cosentino, P. Vetro, Fixed point results for F-contractive mappings of Hardy–Rogers-type, *Filomat*, **28**(4):715–722, 2014.
14. S. Czerwik, Contraction mappings in b-metric spaces, *Acta Math. Inform. Univ. Ostrav.*, **1**: 5–11, 1993.
15. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, *Atti Semin. Mat. Fis. Univ. Modena*, **46**(2):263–276, 1998.
16. E. Karapınar, Discussion on $\alpha - \psi$ contractions on generalized metric spaces, *Abstr. Appl. Anal.*, **2014**:962784, 2014.
17. E. Karapınar, R. P. Agarwal, A note on 'Coupled fixed point theorems for α-ψ-contractive-type mappings in partially ordered metric spaces', *Fixed Point Theory Appl.*, **2013**(216), 2013.
18. E. Karapınar, B. Samet, Generalized α-ψ-contractive type mappings and related fixed point theorems with applications, *Abstr. Appl. Anal.*, **2012**:793486, 2012.
19. M.A. Miandaragh, M. Postolache, Sh. Rezapour, Approximate fixed points of generalized convex contractions, *Fixed Point Theory Appl.*, **2013**(255), 2013.
20. M.A. Miandaragh, M. Postolache, Sh. Rezapour, Some approximate fixed point results for generalized α-contractive mappings, *Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar.*, **75**(2):3–10, 2013.
21. G. Mınak, I. Altun, Some new generalizations of Mizoguchi–Takahashi type fixed point theorem, *J. Inequal. Appl.*, **2013**(493), 2013.
22. G. Mınak, A. Helvacı, I. Altun, Ćirić type generalized F-contractions on complete metric spaces and fixed point results, *Filomat*, **28**(6):1143–1151, 2014.

23. B. Mohammadi, S. Rezapour, On modified α-ϕ-contractions, *J. Adv. Math. Stud.*, **6**(2):162–166, 2013.
24. B. Mohammadi, S. Rezapour, N. Shahzad, Some results on fixed points of α-ψ-Ćirić generalized multifunctions, *Fixed Point Theory Appl.*, **2013**(24), 2013.
25. D. O'Regan, A. Petrusel, Fixed point theorems for generalized contractions in ordered metric spaces, *J. Math. Anal. Appl.*, **341**(1241–1252), 2008.
26. D. Paesano, C. Vetro, Multi-valued F-contractions in 0-complete partial metric spaces with application to Volterra type integral equation, *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, **108**(2):1005–1020, 2014.
27. H.K. Pathak, N. Shahzad, Fixed point results for set-valued contractions by altering distances in complete metric spaces, *Nonlinear Anal., Theory Methods Appl., Ser. A*, **70**(7):2634–2641, 2009.
28. A. Petruşel, Integral inclusions. Fixed point approaches, *Ann. Soc. Math. Pol., Ser. I, Commentat. Math.*, **40**:147–158, 2000.
29. H. Piri, P. Kumam, Some fixed point theorems concerning F-contraction in complete metric spaces, *Fixed Point Theory Appl.*, **2014**(210), 2014.
30. B. Samet, C. Vetro, P. Vetro, Fixed point theorems for α-ψ-contractive type mappings, *Nonlinear Anal., Theory Methods Appl.*, **75**(4):2154–2165, 2012.
31. N.-A. Secelean, Iterated function systems consisting of F-contractions, *Fixed Point Theory Appl.*, **2013**(277), 2013.
32. M. Sgroi, C. Vetro, Multi-valued F-contractions and the solution of certain functional and integral equations, *Filomat*, **27**(7):1259–1268, 2013.
33. W. Shatanawi, A. Pitea, R. Lazović, Contraction conditions using comparison functions on b-metric spaces, *Fixed Point Theory Appl.*, **2014**(135), 2014.
34. S.L. Singh, S.N. Mishra, S. Jain, Round-off stability for multi-valued maps, *Fixed Point Theory Appl.*, **2012**(12), 2012.
35. A. Sîntămărian, Integral inclusions of Fredholm type relative to multivalued φ-contraction, *Semin. Fixed Point Theory Cluj-Napoca*, **3**:361–368, 2002.
36. T. Sistani, M. Kazemipour, Fixed point theorems for α-ψ-contractions on metric spaces with a graph, *J. Adv. Math. Stud.*, **7**(1):65–79, 2014.
37. D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**(94), 2012.