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Nonlinear fractional equations with supercritical growth

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Abstract. We obtain existence of infinitely many solutions for a fractional differential equation with indefinite concave nonlinearities and supercritical growth.

Keywords: fractional problem, indefinite concave nonlinearity, variational methods, infinitely many solutions.

1 Introduction and main result

Recently, as observed in [16], a great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for the pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts such as, among the others, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultra-relativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, and water waves.

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We consider the non-local fractional Laplacian equation ($N > 1$)

$$\begin{cases} -\mathcal{L}_K u = b(x)|u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{$\mathcal{P}$}$$

where Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < q < 2$. Here $\mathcal{L}_K u$ is the non-local fractional Laplacian operator. The nonlocal operator $\mathcal{L}_K$ is defined as follows:

$$\mathcal{L}_K u(x) := 2 \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \big(u(x) - u(y)\big) K(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a measurable function with the following property:

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N), & \text{where } \gamma(x) = \min\{|x|^2, 1\}; \\ \text{there exists } k_0 > 0 \text{ such that } K(x) \geqslant k_0|x|^{-(N+2s)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}; \\ K(x) = K(-x) & \text{for any } x \in \mathbb{R}^N \setminus \{0\}. \end{cases} \tag{1}$$

A typical example for K is given by singular kernel $K(x) = |x|^{-(N+2s)}$. In this case, problem ($\mathcal{P}$) becomes

$$\begin{cases} (-\Delta)^s u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2}$$

where $(-\Delta)^s u$ is the fractional Laplacian operator with (up to normalization factors) may be defined as

$$(-\Delta)^s u(x) := 2 \lim_{\varepsilon \to 0^+} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y$$

for $x \in \mathbb{R}^N$, see [1, 2, 5–8, 12–30, 32–34] and the references therein for further details on the factional Laplacian operator.

The weight function b will be possibly sign-changing and the assumption for b is as follows:

(A1) $b(x) \in C(\bar{\Omega})$, and there is a nonempty open subset Ω' of Ω such that $b(x) > 0$ in Ω'.

A special case of our main result is the following theorem.

Theorem 1. *Assume* (A1)*,* $r \in (q,p) \cup (p,\infty)$*, and* $d(x) \in C(\bar{\Omega})$*. Then*

$$\begin{cases} -\mathcal{L}_K^p u = b(x)|u|^{q-2}u + d(x)|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases} \tag{3}$$

has a sequence of weak solutions (u_n) *such that* $\|u_n\|_{L^\infty(\Omega)} \to 0$ *as* $n \to \infty$*, where* $\Omega \subset \mathbb{R}^N$ *is a bounded domain,* $1 < p < \infty$*, and* $1 < q < p$*.*

Remark 1. This type of equations have been studied extensively [1, 2, 6–8, 12–34] in the subcritical and critical case. But these equations have not been well studied in the supercritical case, that is $r > 2N/(N-2s)$. Applying Theorem 3 to (3), our theorem includes results in the supercritical one. The exponent r in Theorem 1 can be critical or supercritical in the sense of Sobolev embedding because the solutions (u_n) we obtained are small solutions with $\|u_n\|_{L^\infty(\Omega)} \to 0$ and we only give the assumptions for f near zero. We use a suitable cut-off technique to overcome the exponent r is supercritical. This idea is from [30].

Now, we give the assumptions on f:

(A2) $f(x,u) = o(|u|)$ as $|u| \to 0$ uniformly for $x \in \Omega$;
(A3) $f(x,u) \in C(\Omega \times (-\delta,\delta), \mathbb{R})$ is odd in u for $\delta > 0$ small.

The main result is as follows.

Theorem 2. *Let* $1 < q < 2$ *and assume* (A1)–(A3) *are satisfied. Then* $(\mathcal{P})$ *has a sequence of solutions* (u_n) *such that* $\|u_n\|_{L^\infty(\Omega)} \to 0$ *as* $n \to \infty$.

Following the same idea, we can also consider the so-called fractional p-Laplacian equation

$$\begin{cases} -\mathcal{L}_K^p u = b(x)|u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \qquad (\mathcal{P}')$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\mathcal{L}_K^p u$ is the fractional p-Laplacian operator

$$\mathcal{L}_K^p u(x) := 2 \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \big|u(x) - u(y)\big|^{p-2}\big(u(x) - u(y)\big) K(x-y)\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a measurable function with the following property:

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N), & \text{where } \gamma(x) = \min\{|x|^p, 1\}; \\ \text{there exists } k_0 > 0 \text{ such that } K(x) \geqslant k_0|x|^{-(N+ps)} & \text{for any } x \in \mathbb{R}^N \setminus \{0\}; \\ K(x) = K(-x) & \text{for any } x \in \mathbb{R}^N \setminus \{0\}. \end{cases}$$

Moreover, $1 < p < \infty$ and $1 < q < p$. We need the following assumption for nonlinearity f:

(A4) $f(x,u) = o(|u|^{p-1})$ as $|u| \to 0$ uniformly for $x \in \Omega$.

Theorem 3. *Let* $1 < q < p$ *and assume* (A1), (A3) *and* (A4) *are satisfied. Then* $(\mathcal{P}')$ *has a sequence of solutions* (u_n) *such that* $\|u_n\|_{L^\infty(\Omega)} \to 0$ *as* $n \to \infty$.

Remark 2. For results on existence of multiple solutions for fractional Laplacian or p-Laplacian equations by using Nehari manifold, see, for example, [2, 9, 10].

2 Preliminarily

In this section, we first give some basic results and the functional space that will be used in the next section, which was introduced in [23].

Let $0 < s < 1$ be a real number and the fractional critical exponent 2_s^* be defined as

$$2_s^* := \begin{cases} \frac{2N}{N-2s} & \text{if } 2s < N, \\ \infty & \text{if } 2s \geqslant N. \end{cases}$$

In the following, we denote $Q = \mathbb{R}^N \setminus \mathcal{O}$, where

$$\mathcal{O} = \mathcal{C} \times \mathcal{C} \subset \mathbb{R}^{2N}$$

and $\mathcal{C} = \mathbb{R}^N \setminus \Omega$. W is a linear space of Lebesgue measurable function from $\mathbb{R}^N$ to $\mathbb{R}$ such that the restriction to Ω of any function u in W belongs to $L^2(\Omega)$ and

$$\int_Q \left|u(x) - u(y)\right|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

The space W is equipped with the norm

$$\|u\|_W := \|u\|_{L^2(\Omega)} + \left(\int_Q \left|u(x) - u(y)\right|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \tag{4}$$

We shall work in the closed linear subspace

$$W_0 := \left\{u \in W\colon\ u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\}. \tag{5}$$

According to the conditions of K, we have that $C_0^\infty(\Omega) \subset W_0$, and so W and W_0 are nonempty. The space W_0 is endowed with the norm defined by

$$\|u\|_{W_0} := \left(\int_Q \left|u(x) - u(y)\right|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \tag{6}$$

Since $u \in W_0$, then the integral in (6) can be extended to all $\mathbb{R}^{2N}$. Moreover, the norm on W_0 given in (6) is equivalent to the usual one defined in (4), by Lemma 6 in [23]. For the framework of fractional Sobolev space, we refer the reader to the survey of Di Nezza, Palatucci and Valdinoci [4].

In the following, we denote by $W^{s,2}(\Omega)$ the usual fractional Sobolev space endowed with the norm (the so-called Gagliardo norm)

$$\|u\|_{W^{s,2}(\Omega)} := \|u\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

Taking into account Lemma 5 in [23], we have the following result.

Lemma 1. *The embedding $W_0 \hookrightarrow L^q(\Omega)$ is continuous for any $q \in [1, 2_s^*]$, while it is compact whenever $q \in [1, 2_s^*[$. Moreover, there exists a positive constant $c(k_0)$ depending on k_0 (which is given in (1)) such that*

$$\|u\|_{W^{s,2}(\Omega)} \leqslant \|u\|_{W^{s,2}(\mathbb{R}^N)} \leqslant c(k_0)\|u\|_{W_0}.$$

Furthermore, there is a constant $c_q > 0$ such that for every $u \in W_0$,

$$\|u\|_{L^q(\Omega)} \leqslant c_q\|u\|_{W_0}.$$

We will use the following theorem, which is a variant of a result due to Clark [3], to prove our main result.

Theorem 4. *Let $\Phi \in C^1(X, \mathbb{R})$, where X is a Banach space. Assume Φ satisfies the Palais–Smale (PS) condition, is even and bounded from below, and $\Phi(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X^k and $\rho_k > 0$ such that*

$$\sup_{X^k \cap S_{\rho_k}} \Phi < 0,$$

where $S_\rho := \{u \in X\colon \|u\| = \rho\}$, then Φ has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.

Last, we show that the weak solutions of $(\mathcal{P})$ are bounded in $L^\infty(\Omega)$. This result was established in [31, Thm. 3.1] and proved by using the De Giorgi–Stampacchia iteration method.

Proposition 1. *Let $u \in W_0$ be a weak solution of problem $(\mathcal{P})$ and the nonlinearity is subcritical growth. Then $u \in L^\infty(\Omega)$, and there exists $C > 0$ possibly depending on N, s, Ω such that*

$$\|u\|_{L^\infty(\Omega)} \leqslant C\bigl(1 + \|u\|_{W_0}^{q-1}\bigr)$$

hold for some $q \in [1; 2_s^*[$.

3 Proof of Theorems 2 and 3

The proof is motivated by the arguments in [11, 30]. We shall only give the proof of Theorem 2 since the proof of Theorem 3 is similar. Denote by λ_1 the first eigenvalue of $-\mathcal{L}_K$ with Dirichlet boundary condition on Ω. As in [30], we first modify f so that the nonlinearity is defined for all $(x, u) \in \Omega \times \mathbb{R}$.

Lemma 2. *Let $f(x, u)$ be as in* (A2) *and* (A3). *Then for any $\lambda \in \mathbb{R}$, $0 < \lambda < \lambda_1$, there exist $\alpha \in (0, \delta/2)$ and $\tilde{f} \in C(\Omega \times \mathbb{R}, \mathbb{R})$ such that $\tilde{f}(x, u)$ is odd in u and satisfies*

$$\tilde{f}(x, u) = f(x, u) \quad \forall |u| \leqslant \alpha, \tag{7}$$

$$\tilde{f}(x, u)u - q\tilde{F}(x, u) \leqslant \frac{(2 - q)\lambda}{2}|u|^2 \quad \forall (x, u) \in \Omega \times \mathbb{R},$$

$$\left|\tilde{F}(x,u)\right| \leqslant \frac{\lambda}{2}|u|^2 \quad \forall (x,u) \in \Omega \times \mathbb{R}, \tag{8}$$

where $\tilde{F}(x,u) = \int_0^u \tilde{f}(x,s)ds$.

Proof. Fix $\lambda \in (0, \lambda_1)$ and denote $\theta = (2-q)\lambda/2$. Choose $\varepsilon \in (0, \theta/14)$. By (A2), there exists $\alpha \in (0, \delta/2)$ such that for $|u| \leqslant 2\alpha$,

$$\left|F(x,u)\right| \leqslant \varepsilon|u|^2, \qquad \left|f(x,u)u\right| \leqslant \varepsilon|u|^2.$$

Now we choose a cut-off function $\rho \in C^1(\mathbb{R}, \mathbb{R})$ so that it is even and satisfies

$$\rho(t) = 1 \quad \text{for } |t| \leqslant \alpha, \qquad \rho(t) = 0 \quad \text{for } |t| \geqslant 2\alpha,$$

and

$$\left|\rho'(t)\right| \leqslant \frac{2}{\alpha}, \qquad \rho'(t)t \leqslant 0.$$

Choose $\beta \in (0, \theta/16)$ and $F_\infty(u) = \beta|u|^2$. Using ρ and F_∞, we define

$$\tilde{F}(x,u) := \rho(u)F(x,u) + \big(1-\rho(u)\big)F_\infty(u)$$

and

$$\tilde{f}(x,u) := \tilde{F}'_u(x,u).$$

Then, for $|u| \leqslant 2\alpha$, we have

$$\tilde{f}(x,u) = \rho'(u)F(x,u) + \rho(u)f(x,u) + \big(1-\rho(u)\big)F'_\infty(u) - \rho'(u)F_\infty(u)$$

and

$$\begin{aligned}\tilde{f}(x,u) - q\tilde{F}(x,u) = {} & \rho'(u)uF(x,u) + \rho(u)f(x,u)u + 2\beta\big(1-\rho(u)\big)|u|^2 \\ & - \beta\rho'(u)u|u|^2 - q\rho(u)F(x,u) - q\beta\big(1-\rho(u)\big)|u|^2.\end{aligned}$$

It is easy to see that, for all $(x,u) \in \Omega \times \mathbb{R}$,

$$\left|\tilde{F}(x,u)\right| \leqslant (\varepsilon+\beta)|u|^2 \leqslant \frac{\lambda}{2}|u|^2$$

and

$$\tilde{f}(x,u)u - q\tilde{F}(x,u) \leqslant (7\varepsilon + 8\beta)|u|^2 \leqslant \theta|u|^2.$$

Therefore, α and $\tilde{f}$ defined above satisfy all the properties stated in the lemma. □

We now consider the modified problem

$$\begin{cases} -\mathcal{L}_K u = b(x)|u|^{q-2}u + \tilde{f}(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{9}$$

whose solutions correspond to critical points of the functional

$$\tilde{I}(u) = \frac{1}{2} \int_Q \left|u(x) - u(y)\right|^2 K(x-y) \,\mathrm{d}x \,\mathrm{d}y - \frac{1}{q} \int_\Omega b(x)|u|^q \,\mathrm{d}x - \int_\Omega \tilde{F}(x,u) \,\mathrm{d}x, \quad u \in W_0.$$

The construction of $\tilde{I}$ together with (8) shows that $\tilde{I}$ is C^1, even, bounded from below, and coercive, and therefore satisfies the (PS) condition.

Lemma 3. $\tilde{I}(u) = 0 = \langle \tilde{I}'(u), u\rangle$ *if and only if* $u = 0$.

Proof. Clearly, if $u = 0$, then $\tilde{I}(u) = 0 = \langle \tilde{I}'(u), u\rangle$. Next, we assume $\tilde{I}(u) = 0 = \langle \tilde{I}'(u), u\rangle$. Since

$$\frac{1}{2} \int_Q \left|u(x) - u(y)\right|^2 K(x-y) \,\mathrm{d}x \,\mathrm{d}y - \frac{1}{q} \int_\Omega b(x)|u|^q \,\mathrm{d}x - \int_\Omega \tilde{F}(x,u) \,\mathrm{d}x = 0$$

and

$$\int_Q \left|u(x) - u(y)\right|^2 K(x-y) \,\mathrm{d}x \,\mathrm{d}y - \int_\Omega b(x)|u|^q \,\mathrm{d}x - \int_\Omega \tilde{f}(x,u)u \,\mathrm{d}x = 0,$$

we obtain

$$\left(\frac{1}{q} - \frac{1}{2}\right) \int_Q \left|u(x) - u(y)\right|^2 K(x-y) \,\mathrm{d}x \,\mathrm{d}y = \int_\Omega \left(\frac{1}{q}\tilde{f}(x,u) - \tilde{F}(x,u)\right) \mathrm{d}x \leqslant \frac{(2-q)\lambda}{2q} \int_\Omega |u|^2 \,\mathrm{d}x,$$

where we have used (8) in Lemma 2. Then the fact that $0 < \lambda < \lambda_1$ implies $u = 0$. □

We are ready to prove Theorem 2.

Proof of Theorem 2. In order to apply Theorem 4 to $\tilde{I}$, we only need to find for any $k \in \mathbb{N}$ a subspace X^k and $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \tilde{I} < 0$. For any $k \in \mathbb{N}$, we find k linearly independent functions $e_1, \dots, e_k$ in $C_0^\infty(\Omega')$. We define $X^k := \operatorname{span}\{e_1, \dots, e_k\}$. By (A1), we may assume $b(x) > b_0 > 0$ in $\bigcup_{i=1}^k \operatorname{supp} e_i$ for some constant b_0. For $u \in X^k$, using (8) in Lemma 2, we have

$$\tilde{I}(u) \leqslant \frac{1}{2}\|u\|_{W^0}^2 - \frac{b_0}{q}\|u\|_{L^q(\Omega)}^q + \frac{\lambda}{2}\|u\|_{L^2(\Omega)}^2,$$

which implies the existence of $\rho_k > 0$ such that $\sup_{X^k \cap S_{\rho_k}} \tilde{I} < 0$ since the dimension of X^k is finite. According to Theorem 4, there exists a sequence of negative critical values c_k

of $\tilde{I}$ satisfying $c_k \to 0$ as $k \to \infty$. For any k, let u_k be a critical point of $\tilde{I}$ associated with c_k. Then u_k are solutions of (9) and they form a (PS) sequence. Without loss of generality, we may assume that $u_k \to u$ in W_0 as $k \to \infty$. Then u satisfies $\tilde{I}(u) = 0 = \langle \tilde{I}'(u), u\rangle$. Therefore, $u = 0$ according to Theorem 4, and $u_k \to 0$ in W_0 as $k \to \infty$. Proposition 1 shows that $u_k \to 0$ in $L^\infty(\Omega)$ as $k \to \infty$.

In view of (7) and (9), we see that u_k with k large are solutions of $(\mathcal{P})$. The proof is complete. □

Proof of Theorem 1. If $r \in (p, \infty)$, then the result is a consequence of Theorem 3. If $r \in (q, p)$, then we just apply Theorem 4 to the functional

$$J(u) = \frac{1}{2}\int\limits_Q \left|u(x) - u(y)\right|^2 K(x-y)\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{q}\int\limits_\Omega b(x)|u|^q\,\mathrm{d}x - \int\limits_\Omega |u|^r\,\mathrm{d}x, \quad u \in W_0.$$

to obtain the result. □

References

1. C. Bai, Existence results for non-local operators of elliptic type, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **83**:82–90, 2013.
2. W. Chen, S. Deng, The Nehari manifold for nonlocal elliptic operators involving concave-convex nonlinearities, *Z. Angew. Math. Phys.*, **66**(4):1387–1400, 2015.
3. D.C. Clark, A variant of the Lusternik–Schnirelman theory, *Indiana Univ. Math. J.*, **22**:65–74, 1972.
4. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136**(5):521–573, 2012.
5. S. Dipierro, M. Medina, I. Peral, E. Valdinoci, Bifurcation results for a fractional elliptic equation with critical exponent in $\mathbb{R}^n$, *Manusc. Math.*, pp. 1–48, 2016.
6. M. Ferrara, L. Guerrini, B. Zhang, Multiple solutions for perturbed non-local fractional Laplacian equations, *Electron. J. Differ. Equ.*, Paper No. 260, 2013.
7. M. Ferrara, G. Molica Bisci, B. Zhang, Existence of weak solutions for non-local fractional problems via Morse theory, *Discrete Contin. Dyn. Syst., Ser. B*, **19**(8):2483–2499, 2014.
8. A. Fiscella, R. Servadei, E. Valdinoci, A resonance problem for non-local elliptic operators, *Z. Anal. Anwend.*, **32**(4):411–431, 2013.
9. S. Goyal, K. Sreenadh, Existence of multiple solutions of p-fractional Laplace operator with sign-changing weight function, *Adv. Nonlinear Anal.*, **4**(1):37–58, 2015.
10. S. Goyal, K. Sreenadh, Nehari manifold for non-local elliptic operator with concave–convex nonlinearities and sign-changing weight functions, *Proc. Indian Acad. Sci., Math. Sci.*, **125**(4):545–558, 2015.

11. Z. Guo, Elliptic equations with indefinite concave nonlinearities near the origin, *J. Math. Anal. Appl.*, **367**(1):273–277, 2010.
12. A. Iannizzotto, M. Squassina, 1/2-Laplacian problems with exponential nonlinearity, *J. Math. Anal. Appl.*, **414**(1):372–385, 2014.
13. A. Iannizzotto, M. Squassina, Weyl-type laws for fractional p-eigenvalue problems, *Asymptotic Anal.*, **88**(4):233–245, 2014.
14. G. Molica Bisci, Fractional equations with bounded primitive, *Appl. Math. Lett.*, **27**:53–58, 2014.
15. G. Molica Bisci, Sequences of weak solutions for fractional equations, *Math. Res. Lett.*, **21**(2):241–253, 2014.
16. G. Molica Bisci, V.D. Rădulescu, R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encycl. Math. Appl., Vol. 162, Cambridge University Press, Cambridge, 2016.
17. G. Molica Bisci, D. Repovš, Existence and localization of solutions for nonlocal fractional equations, *Asymptotic Anal.*, **90**(3–4):367–378, 2014.
18. G. Molica Bisci, R. Servadei, A bifurcation result for non-local fractional equations, *Anal. Appl., Singap.*, **13**(4):371–394, 2015.
19. G. Molica Bisci, R. Servadei, A Brezis–Nirenberg splitting approach for nonlocal fractional equations, *Nonlinear Anal., Theory Methods, Ser. A., Theory Methods*, **119**:341–353, 2015.
20. R. Servadei, Infinitely many solutions for fractional Laplace equations with subcritical nonlinearity, in *Recent Trends in Nonlinear Partial Differential Equations. II: Stationary Problems*, Contemp. Math., Vol. 595, AMS, Providence, RI, 2013, pp. 317–340.
21. R. Servadei, The Yamabe equation in a non-local setting, *Adv. Nonlinear Anal.*, **2**(3):235–270, 2013.
22. R. Servadei, A critical fractional Laplace equation in the resonant case, *Topol. Methods Nonlinear Anal.*, **43**(1):251–267, 2014.
23. R. Servadei, E. Valdinoci, Mountain pass solutions for non-local elliptic operators, *J. Math. Anal. Appl.*, **389**(2):887–898, 2012.
24. R. Servadei, E. Valdinoci, A Brezis–Nirenberg result for non-local critical equations in low dimension, *Commun. Pure Appl. Anal.*, **12**(6):2445–2464, 2013.
25. R. Servadei, E. Valdinoci, Lewy–Stampacchia type estimates for variational inequalities driven by (non)local operators, *Rev. Mat. Iberoam.*, **29**(3):1091–1126, 2013.
26. R. Servadei, E. Valdinoci, Variational methods for non-local operators of elliptic type, *Discrete Contin. Dyn. Syst.*, **33**(5):2105–2137, 2013.
27. R. Servadei, E. Valdinoci, On the spectrum of two different fractional operators, *Proc. R. Soc. Edinb., Sect. A, Math.*, **144**(4):831–855, 2014.
28. R. Servadei, E. Valdinoci, The Brezis–Nirenberg result for the fractional Laplacian, *Trans. Am. Math. Soc.*, **367**(1):67–102, 2015.
29. R. Servadei, E. Valdinoci, Fractional Laplacian equations with critical Sobolev exponent, *Rev. Mat. Complut.*, **28**(3):655–676, 2015.
30. Z.-Q. Wang, Nonlinear boundary value problems with concave nonlinearities near the origin, *NoDEA Nonlinear Differ. Equ. Appl.*, **8**(1):15–33, 2001.

31. Y. Wei, X. Su, Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian, *Calc. Var. Partial Differ. Equ.*, **52**(1-2):95–124, 2015.

32. B. Zhang, M. Ferrara, Multiplicity of solutions for a class of superlinear non-local fractional equations, *Complex Var. Elliptic Equ.*, **60**(5):583–595, 2015.

33. B. Zhang, M. Ferrara, Two weak solutions for perturbed non-local fractional equations, *Appl. Anal.*, **94**(5):891–902, 2015.

34. Q.-M. Zhou, K.-Q. Wang, Existence and multiplicity of solutions for nonlinear elliptic problems with the fractional Laplacian, *Fract. Calc. Appl. Anal.*, **18**(1):133–145, 2015.