Application of M-matrices theory to numerical investigation of a nonlinear elliptic equation with an integral condition[*]

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Abstract. The iterative methods to solve the system of the difference equations derived from the nonlinear elliptic equation with integral condition are considered. The convergence of these methods is proved using the properties of M-matrices, in particular, the regular splitting of an M-matrix. To our knowledge, the theory of M-matrices has not ever been applied to convergence of iterative methods for system of nonlinear difference equations. The main results for the convergence of the iterative methods are obtained by considering the structure of the spectrum of the two-dimensional difference operators with integral condition.

Keywords: elliptic equation, finite-difference method, integral boundary condition, iterative method, eigenvalue problem, M-matrix, regular splitting.

1 Introduction

Over the last few decades, in both the theory of differential equations and numerical analysis, much attention is paid to various types of differential equations with nonlocal conditions. In particular, a lot of articles appeared on numerical methods for elliptic equations with integral or other nonlocal conditions.

In one of the first papers written on this topic [10], the finite difference method for a linear second order elliptic equation with Bitsadze–Samarskii nonlocal condition was considered. In this paper, an iterative method for solving a system of difference equations with a nonlocal condition has been considered possibly for the first time.

Late on, many papers were written in order to justify the finite difference method for elliptic equations with various types of nonlocal conditions [1, 2, 4–6, 13, 18]. In papers [25, 30], an elliptic equation with an integral condition is solved by the fourth-order finite difference method.

For a long time, iterative methods for elliptic equations with nonlocal conditions were not intensively investigated. A certain stimulus to elaborate this topic was given by research of the spectrum structure of difference operators with nonlocal conditions. Even in the first articles [24, 28] of the investigation of the spectrum for such operators, it has been noted that the spectrum structure of both differential and difference operators with rather simple nonlocal conditions can be quite complex. This structure very sensitively depends on the parameters and functions under nonlocal conditions.

The results of the spectrum structure analysis have been applied to investigate the convergence of iterative methods for elliptic equations [24, 29, 30] as well as stability of difference schemes for parabolic [11, 12, 14, 16, 27] and hyperbolic equations [15, 20].

Other numerical methods for elliptic equations with various types of nonlocal conditions were explored in [17, 19, 21, 22, 35] (see also the references therein).

Many references to the subject matter of numerical methods for elliptic equations with nonlocal conditions are found in the review article [32].

Iterative methods for systems of nonlinear difference equations with nonlocal conditions were considered in [9, 31] (also see paper [36] close to the subject area).

In many cases, the matrix of system of difference equations approximating an elliptic equation with nonlocal conditions has the properties typical to M-matrices. Convergence of iterative method to some systems of linear difference equations using the M-matrix properties was analysed in [33].

In this paper, we develop further this idea and, according to the M-matrix methodology, we investigate the convergence of iterative method for a system of nonlinear difference equations with a nonlocal condition. M-matrix theory has not been applied to solution of such systems before. The main aim of this paper is the investigation of convergence of iterative methods. It is worth noting that M-matrix theory allows us to create and analyze not a single, but a whole family of iterative methods with different conditions.

The remaining part of this paper is organised as follows. In Section 2, a boundary value problem for a nonlinear differential equation with an integral condition is stated, and a corresponding system of difference equations is derived. In Section 3, we describe the main properties of M-matrices. The basic result of the paper – three theorems on the convergence of iterative methods – are presented in Section 4. In Section 5, the previously obtained results are generalized, in addition taking into account the structure of spectrum of difference operators with nonlocal conditions. This enables us to expand the convergence area of iterative methods. In Section 6, we shortly discuss how the obtained results can be generalized for elliptic equations with variable coefficients.

2 Statement of the problem

Let us solve a nonlinear elliptic equation in a rectangular domain $D = \{(x,y) \in (0,1) \times (0,1) \subset \mathbb{R}^2\}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y,u), \quad (x,y) \in D, \tag{1}$$

with the integral condition

$$u(0,y) = \gamma \int\limits_0^1 u(x,y)\,\mathrm{d}x + \mu_1(y), \quad 0 < y < 1, \tag{2}$$

and Dirichlet boundary conditions at the remaining three sides of the rectangle:

$$\begin{aligned}
u(1,y) &= \mu_2(y), \quad 0 \leqslant y \leqslant 1, \\
u(x,0) &= \mu_3(x), \quad u(x,1) = \mu_4(x), \quad 0 \leqslant x \leqslant 1,
\end{aligned} \tag{3}$$

where γ is the given real parameter. Functions μ_2, μ_3 and μ_4 satisfy the compatibility conditions $\mu_2(0) = \mu_3(1)$ and $\mu_2(1) = \mu_4(1)$.

Now we can write a difference problem corresponding to the differential problem (1)–(3)

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} = f_{ij}(U_{ij}), \quad i,j = 1, \ldots, N-1, \tag{4}$$

$$U_{0j} = \gamma h \left(\frac{U_{0j} + U_{Nj}}{2} + \sum_{i=1}^{N-1} U_{ij} \right) + (\mu_1)_j, \quad j = 1, \ldots, N-1, \tag{5}$$

$$U_{Nj} = (\mu_2)_j, \quad U_{i0} = (\mu_3)_i, \quad U_{iN} = (\mu_4)_i, \quad i,j = 0, \ldots, N, \tag{6}$$

where $h = 1/N$, N is a positive integer, $f_{ij}(U_{ij}) = f(x_i, y_j, U_{ij})$,

$$\delta_x^2 U_{ij} := \frac{U_{i-1,j} - 2U_{ij} + U_{i+1,j}}{h^2}, \qquad \delta_y^2 U_{ij} := \frac{U_{i,j-1} - 2U_{ij} + U_{i,j+1}}{h^2}.$$

If the solution $u(x,y)$ of boundary value problem (1)–(3) is smooth enough, then (4)–(6) approximates differential problem with truncation error $\mathcal{O}(h^2)$.

We rewrite the obtained system of difference equations (4)–(6) in the matrix form. To this end, we express, as usual [26], U_{0j} from equation (5)

$$U_{0j} = \alpha \sum_{i=1}^{N-1} U_{ij} + (\tilde{\mu}_1)_j, \quad j = 1, \ldots, N-1, \tag{7}$$

where

$$\alpha = \frac{\gamma h}{1 - \gamma h/2}, \quad (\tilde{\mu}_1)_j = \frac{(\mu_1)_j + \gamma h(\mu_2)_j/2}{1 - \gamma h/2}. \tag{8}$$

By putting (7) into equation (4) as $i = 1$, we rewrite system (4), (5) in the following way:

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} = f_{ij}(U_{ij}), \quad i = 2, \ldots, N-1, \; j = 1, \ldots, N-1, \tag{9}$$

$$\frac{\alpha \sum_{i=1}^{N-1} U_{ij} - 2U_{1j} + U_{2j}}{h^2} + \delta_y^2 U_{1j} = f_{1j}(U_{1j}) - \frac{(\tilde{\mu}_1)_j}{h^2}, \tag{10}$$

$$j = 1, \ldots, N-1.$$

Difference equations (9), (10) together with boundary conditions (6) define a system of equations the order of which and the number of unknowns U_{ij} are $(N-1)^2$. After finding the solution of this system $U_{ij}, i,j = 1, \ldots, N-1$, the remaining unknown values of the solution $U_{0j} \; j = 1, \ldots, N-1$, are obtained by (7). So, we have reduced the system of equations (4)–(6) with nonlocal conditions to the system of equations (9), (10) with the classical boundary condition (6).

We can write the system of equations (9), (10), (6) in the matrix form

$$\mathbf{A}\mathbf{U} + f(\mathbf{U}) = \mathbf{0}, \tag{11}$$

where $\mathbf{A}$ is a matrix of order $(N-1)^2$, $\mathbf{U}$ and $f(\mathbf{U})$ are vectors of order $(N-1)^2$, $\mathbf{U} = \{U_{ij}\}$, $f(\mathbf{U}) = \{f_{ij}(U_{ij})\}$, $i,j = 1, \ldots, N-1$, $\mathbf{A} = \boldsymbol{\Lambda} - \mathbf{C}$, $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_1 + \boldsymbol{\Lambda}_2$ is a matrix corresponding to the difference operator $-\delta_x^2 - \delta_y^2$ in the rectangular domain with the Dirichlet boundary conditions; $\mathbf{C}$ is a matrix composed of the coefficients of equations (10). More exactly, $\mathbf{C}$ is a block matrix

$$\mathbf{C} = \mathrm{diag}(\mathbf{C}_1, \mathbf{C}_1, \ldots, \mathbf{C}_1),$$

where

$$\mathbf{C}_1 = \frac{1}{h^2} \begin{pmatrix} \alpha & \alpha & \ldots & \alpha \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix},$$

the number of blocks of matrix $\mathbf{C}$ and order of matrix $\mathbf{C}_1$ are $N-1$.

Note that $\mathbf{C} = \mathbf{0}$ if $\gamma = 0$. If $0 \leqslant \gamma \leqslant 1$, then $0 \leqslant (N-1)\alpha \leqslant 1$, i.e. the matrix $\mathbf{A} = \boldsymbol{\Lambda} - \mathbf{C}$ is diagonally dominant in a weak sense, and $\mathbf{A}$ is irreducible for all γ. We can write system (11) also in the following form:

$$\boldsymbol{\Lambda}\mathbf{U} - \mathbf{C}\mathbf{U} + f(\mathbf{U}) = \mathbf{0}. \tag{12}$$

3 M-matrices and their main properties

We consider several iterative methods for solving the system of difference equations (12). To prove the convergence of these methods we will use the properties of M-matrices.

Definition 1. A square matrix $\mathbf{A} = \{a_{kl}\}$, $k, l = 1, \ldots, n$, is called an M-matrix if $a_{kl} \leqslant 0$ when $k \neq l$ and the inverse $\mathbf{A}^{-1}$ exists, whose all elements are non-negative ($\mathbf{A}^{-1} \geqslant 0$).

We point out several typical properties of M-matrices that will be used to investigate the convergence of iterative methods. More details can be found in [7], [34].

The definition of M-matrix yields $a_{kk} > 0$.

Property 1. *If* $\mathbf{A}_1$ *is an M-matrix and* $\mathbf{A}_2 \geqslant \mathbf{A}_1$*, additionally, all nondiagonal elements of the matrix* $\mathbf{A}_2$ *are nonpositive, then* $\mathbf{A}_2$ *is also an M-matrix and*

$$\mathbf{A}_2^{-1} \leqslant \mathbf{A}_1^{-1}.$$

Property 2. *If* $a_{kl} \leqslant 0$ *when* $k \neq l$*, then two next statements are equivalent*:

  (i) $\mathbf{A}$ *is an M-matrix*;
 (ii) *The real part of each eigenvalue of* $\mathbf{A}$ *is positive*: $\mathrm{Re}\lambda(\mathbf{A}) > 0$.

Property 3. *If* $a_{kk} > 0$*,* $a_{kl} \leqslant 0$ *as* $k \neq l$*, and* $\mathbf{A}$ *is diagonally dominant in weak sense and irreducible, then* $\mathbf{A}$ *is M-matrix and* $\mathbf{A}^{-1} > 0$*.*

Property 4. *If M-matrix* $\mathbf{A}$ *has a regular splitting, i.e.* $\mathbf{A} = \mathbf{B} - \mathbf{C}$*, where* $\mathbf{B}^{-1} \geqslant 0$*,* $\mathbf{C} \geqslant 0$*, then*

$$\varrho(\mathbf{B}^{-1}\mathbf{C}) = \max\left|\lambda(\mathbf{B}^{-1}\mathbf{C})\right| < 1.$$

Property 5. *If* $\mathbf{A} = \mathbf{B}_1 - \mathbf{C}_1$ *and* $\mathbf{A} = \mathbf{B}_2 - \mathbf{C}_2$ *are two regular splittings of an M-matrix* $\mathbf{A}$ *and* $\mathbf{C}_1 \leqslant \mathbf{C}_2$ *(besides* $\mathbf{C}_1 \neq \mathbf{C}_2$*), then*

$$\varrho\left(\mathbf{B}_1^{-1}\mathbf{C}_1\right) < \varrho\left(\mathbf{B}_2^{-1}\mathbf{C}_2\right) < 1.$$

Now let us return to system of difference equations (12).

Lemma 1. *The matrix* $\mathbf{A} = \Lambda - \mathbf{C}$ *of system* (12) *has the following properties*:

  1. $\mathbf{A}$ *is an M-matrix if* $0 \leqslant \gamma \leqslant 1$;
  2. Λ *is M-matrix independently of* γ;
  3. $\mathbf{C} \geqslant 0$ *as* $h < 2/\gamma$.

Proof. The diagonal elements of matrix $\mathbf{A}$ as $0 \leqslant \gamma \leqslant 1$ and matrix Λ are positive, nondiagonal elements of them are nonpositive and both matrices are diagonally dominant, in weak sense, and irreducible. So, according to Property 3, $\mathbf{A}$ and Λ are M-matrices.

The conclusion $\mathbf{C} \geqslant 0$, follows from definition of this matrix. $\qquad\square$

In Section 5, it will be proved that the condition $0 \leqslant \gamma \leqslant 1$ is not necessary for $\mathbf{A}$ to be an M-matrix. This condition guarantees that $\mathbf{A}$ is diagonally dominant.

4 Iterative methods

In this section, we prove several propositions on the convergence of iterative methods for a system of nonlinear difference equations by using the properties of M-matrices.

Let us consider system (12). First, we suppose that this system is not necessarily obtained from boundary value problem (1)–(3), but simply satisfies certain properties. Namely, assume that for system of equations (12) the following two hypotheses are true:

(H1) $\mathit{\Lambda}$ and $\mathbf{A} = \mathit{\Lambda} - \mathbf{C}$, where $\mathbf{C} \geqslant \mathbf{0}$ are the M-matrices; moreover, $\mathit{\Lambda}$ is diagonally dominant in a weak sense a irreducible M-matrix.

(H2) For the vector $f(\mathbf{U}) = \{f_{ij}(U_{ij})\}$, the inequality

$$0 \leqslant \alpha \leqslant \frac{\partial f_{ij}(U_{ij})}{\partial U_{ij}} < \beta < \infty, \quad i, j = 1, \ldots, N-1,$$

is true with all real values U_{ij}.

Lemma 2. *If hypothesis* (H1) *and* (H2) *are true, then there exists a unique solution of system* (12).

Proof. We rewrite (12) in equivalent form

$$\mathit{\Lambda}\mathbf{U} + \beta\mathbf{U} = \mathbf{CU} - f(\mathbf{U}) + \beta\mathbf{U} \tag{13}$$

or

$$\mathbf{U} = P(\mathbf{U}), \quad \text{where } P(\mathbf{U}) := (\mathit{\Lambda} + \beta\mathbf{I})^{-1}(\mathbf{CU} + \beta\mathbf{U} - f(\mathbf{U})). \tag{14}$$

We will prove that $P(\mathbf{U})$ is a contraction operator on a linear vector space $\mathbb{R}^m$, $m = (N-1)^2$, i.e. the inequality

$$\big\|P(\mathbf{U}) - P(\mathbf{V})\big\| \leqslant q\|\mathbf{U} - \mathbf{V}\| \tag{15}$$

is true for all vectors $\mathbf{U}, \mathbf{V} \in \mathbb{R}^m$, $0 \leqslant q < 1$. We have

$$P(\mathbf{U}) - P(\mathbf{V}) = \mathbf{S}(\mathbf{U} - \mathbf{V}),$$

where

$$\mathbf{S}(\mathbf{U} - \mathbf{V}) = (\mathit{\Lambda} + \beta\mathbf{I})^{-1}(\mathbf{C} + \beta\mathbf{I} - \mathbf{D})(\mathbf{U} - \mathbf{V}),$$

$\mathbf{D}$ is a diagonal matrix with diagonal elements d_{ij} (according to hypothesis (H2)):

$$0 \leqslant \alpha \leqslant d_{ij} = \frac{\partial f_{ij}(\tilde{U}_{ij})}{\partial \tilde{U}_{ij}} < \beta < \infty, \quad i, j = 1, \ldots, N-1,$$

$\tilde{\mathbf{U}}$ is some intermediate vector depending on $\mathbf{U}$ and $\mathbf{V}$. Hence, it follows

$$\mathbf{0} \leqslant \mathbf{C} + \beta\mathbf{I} - \mathbf{D} \leqslant \mathbf{C} + \beta\mathbf{I},$$

besides, all the diagonal elements of matrices $\mathbf{C} + \beta\mathbf{I} - \mathbf{D}$ and $\mathbf{C} + \beta\mathbf{I}$ are positive. From (H1) and Properties 1 and 3 we derive

$$(\mathit{\Lambda} + \beta\mathbf{I})^{-1} > \mathbf{0}.$$

Let us define

$$\mathbf{S}_1 = (\mathit{\Lambda} + \beta\mathbf{I})^{-1}(\mathbf{C} + \beta\mathbf{I}).$$

Therefore,

$$\mathbf{0} < \mathbf{S} < \mathbf{S}_1.$$

According to Perron–Frobenius theorem for positive matrices [7, 34],

$$\varrho(\mathbf{S}) < \varrho(\mathbf{S}_1).$$

So, according to Property 4,

$$\mathbf{A} = \boldsymbol{\Lambda} + \beta\mathbf{I} - (\mathbf{C} + \beta\mathbf{I})$$

is a regular splitting, so

$$\varrho(\mathbf{S}_1) < 1.$$

Let us choose arbitrary $\varepsilon > 0$ and determine the norm $\|\mathbf{S}\|_*$ of matrix $\mathbf{S}$ so that inequality

$$\|\mathbf{S}\|_* \leqslant \varrho(\mathbf{S}) + \varepsilon$$

be true (see [23, Chap. II.2, §3.4] or [3, Chap. 7.3]). Since $\varrho(\mathbf{S}) < \varrho(\mathbf{S}_1) < 1$, we can choose $\varepsilon > 0$ so that

$$\|\mathbf{S}\|_* \leqslant q < 1.$$

So, inequality (15) has proved. According to fixed point theorem, the system of equations has a unique solution $\mathbf{U}^*$. □

Remark 1. If $\mathbf{C} = \mathbf{0}$, the proposition of Lemma 2 is known (see, e.g. [8]). It is worth to note that in this case, $\mathbf{A}$ is symmetric matrix, so $\|\mathbf{S}\|_* = \varrho(\mathbf{S})$.

Theorem 1. *If hypothesis* (H1) *and* (H2) *are true, then the iterative method*

$$\boldsymbol{\Lambda}\mathbf{U}^{n+1} + \mathbf{f}\big(\mathbf{U}^{n+1}\big) = \mathbf{C}\mathbf{U}^n \tag{16}$$

converges.

Proof. Let us denote by $\mathbf{U}^*$ the exact solution of (12). By subtracting (16) from the identity

$$\boldsymbol{\Lambda}\mathbf{U}^* + \mathbf{f}(\mathbf{U}^*) = \mathbf{C}\mathbf{U}^*$$

we obtain, for the error $\mathbf{Z}^n = \mathbf{U}^* - \mathbf{U}^n$, the following system of linear equations:

$$\boldsymbol{\Lambda}\mathbf{Z}^{n+1} + \mathbf{D}_n(\mathbf{Z}^{n+1}) = \mathbf{C}\mathbf{Z}^n,$$

in which $\mathbf{D}_n$ is a diagonal matrix with diagonal elements

$$d_{ij}^n = \frac{\partial f_{ij}(\tilde{U}_{ij}^{n+1})}{\partial U_{ij}}.$$

Thus,

$$\mathbf{Z}^{n+1} = \mathbf{S}_n\mathbf{Z}^n,$$

where

$$\mathbf{S}_n = (\boldsymbol{\varLambda} + \mathbf{D}_n)^{-1}\mathbf{C}.$$

Now we define a new vector

$$\big|\mathbf{Z}^n\big| = \big\{\big|Z_{ij}^n\big|\big\}.$$

It follows from hypothesis (H1) and (H2) and Properties 1 and 3 that

$$(\boldsymbol{\varLambda} + \mathbf{D}_n)^{-1} > \mathbf{0}, \qquad \mathbf{D}_n \geqslant \alpha \mathbf{I}, \qquad (\boldsymbol{\varLambda} + \mathbf{D}_n)^{-1} \leqslant (\boldsymbol{\varLambda} + \alpha \mathbf{I})^{-1} \leqslant \boldsymbol{\varLambda}^{-1}.$$

Then

$$\big|\mathbf{Z}^{n+1}\big| \leqslant \big|(\boldsymbol{\varLambda} + \mathbf{D}_n)^{-1}\mathbf{C}\big|\big|\mathbf{Z}^n\big| = (\boldsymbol{\varLambda} + \mathbf{D}_n)^{-1}\mathbf{C}\big|\mathbf{Z}^n\big| \leqslant \boldsymbol{\varLambda}^{-1}\mathbf{C}\big|\mathbf{Z}^n\big|.$$

The splitting $\mathbf{A} = \boldsymbol{\varLambda} - \mathbf{C}$ is a regular splitting, therefore, according to Property 4,

$$\varrho(\boldsymbol{\varLambda}^{-1}\mathbf{C}) < 1$$

and

$$\big|\mathbf{Z}^{n+1}\big| \leqslant \varrho\big(\boldsymbol{\varLambda}^{-1}\mathbf{C}\big)^{n+1}\big|\mathbf{Z}^0\big| \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Remark 2. We can present the interpretation of iterative method (16). System of difference equations (12) can be solved by convergent iterative method (16), at each step of which it is necessary to solve the same system of nonlinear equations with a Dirichlet condition instead of nonlocal. In this sense, (16) can be interpreted as an external iteration that needs internal iterations to be realized.

We can rewrite (16) in equivalent form

$$\delta_x^2 U_{ij}^{n+1} + \delta_y^2 U_{ij}^{n+1} = f_{ij}\big(U_{ij}^{n+1}\big), \quad i,j = 1, \ldots, N-1,$$

$$U_{0j}^{n+1} = \gamma h\bigg(\frac{U_{0j}^{n+1} + U_{Nj}^n}{2} + \sum_{i=1}^{N-1} U_{ij}^n\bigg) + (\mu_1)_j, \quad j = 1, \ldots, N-1,$$

$$U_{Nj}^{n+1} = (\mu_2)_j, \quad U_{i0}^{n+1} = (\mu_3)_i, \quad U_{iN}^{n+1} = (\mu_4)_i, \quad i,j = 0, \ldots, N.$$

Remark 3. In Theorem 1, the regular splitting of matrix $\mathbf{A}$ is taken in a natural way, i.e. $\mathbf{A} = \boldsymbol{\varLambda} - \mathbf{C}$. We can choose several other regular splittings and, based on them, to write similar nonlinear iterative methods. For example, using the regular splitting

$$\mathbf{A} = \frac{4}{h^2}\mathbf{I} - \bigg(\frac{4}{h^2}\mathbf{I} - \boldsymbol{\varLambda} + \mathbf{C}\bigg), \quad \frac{4}{h^2}\mathbf{I} - \boldsymbol{\varLambda} + \mathbf{C} \geqslant \mathbf{C} \geqslant \mathbf{0},$$

we derive the following iterative method:

$$\mathbf{U}^{n+1} + \frac{h^2}{4}\mathbf{f}\big(\mathbf{U}^{n+1}\big) = \mathbf{U}^n + \frac{h^2}{4}\big(-\boldsymbol{\varLambda}\mathbf{U}^n + \mathbf{C}\mathbf{U}^n\big). \tag{17}$$

The equivalent coordinate form of the iterative method (17) is as follows:

$$\frac{U_{i-1,j}^n - 2U_{ij}^{n+1} + U_{i+1,j}^n}{h^2} + \frac{U_{i,j-1}^n - 2U_{ij}^{n+1} + U_{i,j+1}^n}{h^2} = f_{ij}\big(U_{ij}^{n+1}\big),$$

$$i, j = 1, N - 1,$$

$$U_{0j}^n = \gamma h\left(\frac{U_{0j}^n + U_{Nj}^n}{2} + \sum_{i=1}^{N-1} U_{ij}^n\right) + (\mu_1)_j, \quad j = 1, \dots, N - 1,$$

$$U_{Nj}^{n+1} = (\mu_2)_j, \quad U_{i0}^{n+1} = (\mu_3)_i, \quad U_{iN}^{n+1} = (\mu_4)_i, \quad i, j = 0, \dots, N.$$

After calculating U_{ij}^{n+1}, $i, j = 1, \dots, N - 1$, from this system, we must find U_{0j}^{n+1}, $j = 1, \dots, N-1$, only from nonlocal conditions. The convergence of this iterative method is proved quite analogously like in Theorem 1. This method can be interpreted as explicit nonlinear iterative method.

Now we will create an iterative method of another type to solve system of difference equations (12), where at each step of iteration it will be necessary to solve a system of linear equations with a nonlocal condition.

Theorem 2. *If hypothesis* (H1) *and* (H2) *are true, then the iterative method*

$$\boldsymbol{\Lambda}\mathbf{U}^{n+1} - \mathbf{C}\mathbf{U}^{n+1} + \beta\mathbf{U}^{n+1} = -\mathbf{f}\big(\mathbf{U}^n\big) + \beta\mathbf{U}^n \tag{18}$$

converges.

Proof. The structure of proof is close to that of Theorem 1. In this case, $\mathbf{Z}^n = \mathbf{U}^* - \mathbf{U}^n$ is the solution of the linear system

$$(\boldsymbol{\Lambda} - \mathbf{C} + \beta\mathbf{I})\mathbf{Z}^{n+1} = (\beta\mathbf{I} - \mathbf{D}_n)\mathbf{Z}^n, \tag{19}$$

where $\mathbf{D}_n$ is the same matrix as in the proof of Theorem 1.

Hence,

$$\big|\mathbf{Z}^{n+1}\big| \leqslant \big|(\boldsymbol{\Lambda} - \mathbf{C} + \beta\mathbf{I})^{-1}(\beta\mathbf{I} - \mathbf{D}_n)\big|\big|\mathbf{Z}^n\big| \leqslant (\boldsymbol{\Lambda} - \mathbf{C} + \beta\mathbf{I})^{-1}\beta\mathbf{I}\big|\mathbf{Z}^n\big|.$$

The splitting $\mathbf{A} = (\boldsymbol{\Lambda} - \mathbf{C} + \beta\mathbf{I}) - \beta\mathbf{I}$ is a regular splitting, therefore,

$$\varrho\big((\boldsymbol{\Lambda} - \mathbf{C} + \beta\mathbf{I})^{-1}\beta\mathbf{I}\big) < 1.$$

It follows that $|\mathbf{Z}^{n+1}| \to 0$ as $n \to 0$. $\qquad\square$

We can rewrite iterative method (18) as early in the other form

$$\delta_x^2 U_{ij}^{n+1} + \delta_y^2 U_{ij}^{n+1} - \beta U_{ij}^{n+1} = f_{ij}\big(U_{ij}^n\big) - \beta U_{ij}^n, \quad i, j = 1, N - 1,$$

$$U_{0j}^{n+1} = \gamma h\left(\frac{U_{0j}^{n+1} + U_{Nj}^{n+1}}{2} + \sum_{i=1}^{N-1} U_{ij}^{n+1}\right) + (\mu_1)_j, \quad j = 1, \dots, N - 1,$$

$$U_{Nj}^{n+1} = (\mu_2)_j, \quad U_{i0}^{n+1} = (\mu_3)_i, \quad U_{iN}^{n+1} = (\mu_4)_i, \quad i, j = 0, \dots, N.$$

In the next theorem, we even more simplify the iterative method, in each step of iteration of which we will have to solve a system of linear equations without a nonlocal condition.

Theorem 3. *If hypothesis* (H1) *and* (H2) *are true for system* (12)*, then the iterative method*

$$\boldsymbol{\Lambda}\mathbf{U}^{n+1} + \beta\mathbf{U}^{n+1} = -\mathbf{f}\big(\mathbf{U}^n\big) + \beta\mathbf{U}^n + \mathbf{C}\mathbf{U}^n \tag{20}$$

converges.

Proof. The proof is analogously to the proof of previous two theorems. $\qquad\square$

Remark 4. Analogously to Remark 3, we select another regular splitting of matrix $\mathbf{A}$

$$\mathbf{A} = \frac{4}{h^2}\mathbf{I} + \beta\mathbf{I} - \left(\frac{4}{h^2}\mathbf{I} + \beta\mathbf{I} - \boldsymbol{\Lambda} + \mathbf{C}\right),$$

where

$$\frac{4}{h^2}\mathbf{I} + \beta\mathbf{I} - \boldsymbol{\Lambda} + \mathbf{C} \geqslant \beta\mathbf{I} + \mathbf{C} \geqslant \mathbf{0}.$$

On the basis of this regular splitting, we can develop converged explicit iterative method

$$\mathbf{U}^{n+1} + \frac{\beta h^2}{4}\mathbf{U}^{n+1} = \mathbf{U}^n + \frac{h^2}{4}\big(\beta\mathbf{U}^n - \boldsymbol{\Lambda}\mathbf{U}^n + \mathbf{C}\mathbf{U}^n - \mathbf{f}\big(\mathbf{U}^n\big)\big). \tag{21}$$

In order to compare the convergence rates of various iterative methods, we use Property 5 of M-matrices.

Conclusion 1. Based on Property 5, it is possible to assert that

1. Iterative method (20) of Theorem 3 converges asymptotically slower than method (16) of Theorem 1 as well as than method (18) of Theorem 2;
2. Iterative method (17) converges asymptotically slower than method (16), and method (21) slower than method (20).

5   M-matrices and eigenvalue problem

As it was indicated in Section 3, the matrix $\mathbf{A} = \boldsymbol{\Lambda} - \mathbf{C}$ of system (12) is an M-matrix if the condition $0 \leqslant \gamma \leqslant 1$ is true. Under this condition, the matrix $\mathbf{A}$ is diagonally dominant, however, diagonal domination is not a necessary condition of M-matrices.

In this Section, we refuse the restriction $0 \leqslant \gamma \leqslant 1$ and investigate when the matrix $\mathbf{A}$ of the system of difference equations (4)–(6) is an M-matrix under condition $\gamma > 1$. To this end, we need some spectral properties of matrix $\mathbf{A}$.

Let us write the eigenvalue problem for matrix $\mathbf{A}$

$$\mathbf{A}\mathbf{U} = \lambda\mathbf{U}$$

in the form closer to system (4)–(6) (see, e.g. [26, 33])

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} + \lambda U_{ij} = 0, \quad i, j = 1, \dots, N - 1,$$

$$U_{0j} = \gamma h \left(\frac{U_{0j} + U_{Nj}}{2} + \sum_{i=1}^{N-1} U_{ij} \right), \quad j = 1, \dots, N - 1, \tag{22}$$

$$U_{Nj} = U_{i0} = U_{iN} = 0, \quad i, j = 0, \dots, N.$$

It has been proved [33] that all the eigenvalues of (22), in the case $\gamma > 0$, are positive if and only if $0 \leqslant \gamma < \gamma_0 \approx 3.42$. The exact value of γ_0 is

$$\gamma_0 = \frac{2 \tanh(\beta_0 h/2)}{h \tanh(\beta_0/2)}, \tag{23}$$

where

$$\beta_0 = \frac{2}{h} \ln \left(\sin \frac{\pi h}{2} + \sqrt{\sin^2 \frac{\pi h}{2} + 1} \right).$$

As $\gamma = \gamma_0$, one eigenvalue of matrix $\mathbf{A}$ is equal to zero, and if $\gamma > \gamma_0$, there exists a negative eigenvalue. Thus the following statement is true.

Lemma 3. *The matrix $\mathbf{A}$ of system (4)–(6) is an M-matrix if and only if $0 \leqslant \gamma < \gamma_0$, where γ_0 defined by (23) is approximately $\gamma_0 \approx 3.42$ when h is sufficiently small.*

Proof. Indeed, if $\gamma \geqslant \gamma_0$, there exists the negative eigenvalue, which contradicts the definition of an M-matrix according to Property 2. If $\gamma < 0$, then $\mathbf{C} \leqslant \mathbf{0}$, and some nondiagonal elements are positive. This fact also contradicts the definition of M-matrices. For $0 \leqslant \gamma < \gamma_0$, we have $\mathbf{C} \geqslant \mathbf{0}$ and $\lambda(\mathbf{A}) > 0$, and $\mathbf{A}$ is an M-matrix according to Property 2. $\qquad \square$

Conclusion 2. If $0 \leqslant \gamma < \gamma_0$ and hypothesis (H2) is true, then iterative methods (16), (17), (18), (20) and (21) for system (4)–(6) expressed by form (12) converge. Conclusion 1 is valid for these iterative methods.

6   The case of variable coefficients

The results of Section 4 can be generalized to a differential problem with variable coefficients. Let us consider the following boundary value problem:

$$\bigl(p(x,y)u_x\bigr)_x + \bigl(p(x,y)u_y\bigr)_y = f(x,y,u), \quad (x,y) \in D = \{0 < x, y < 1\}, \tag{24}$$

$$u(0,y) = \gamma \int_0^1 \alpha(x)u(x,y)\,\mathrm{d}x + \mu(y), \quad y \in (0,1), \tag{25}$$

with boundary conditions (3). We write for this differential problem the system of difference equations

$$\delta_x(p_{i-1/2,j}\delta_{\bar{x}}U_{ij}) + \delta_y(p_{i,j-1/2}\delta_{\bar{y}}U_{ij}) = f_{ij}(U_{ij}), \quad i,j = 1,\dots,N-1, \qquad (26)$$

$$U_{0j} = h\left(\frac{\alpha_0 U_{0j} + \alpha_N U_{Nj}}{2} + \sum_{i=1}^{N-1}\alpha_i U_{ij}\right) + (\mu_1)_j, \quad j = 1, N-1, \qquad (27)$$

where

$$\delta_x U_{ij} := \frac{U_{i+1,j} - U_{ij}}{h}, \qquad \delta_{\bar{x}}U_{ij} := \frac{U_{ij} - U_{i-1,j}}{h},$$

$$\delta_y U_{ij} := \frac{U_{i,j+1} - U_{ij}}{h}, \qquad \delta_{\bar{y}}U_{ij} := \frac{U_{ij} - U_{i,j-1}}{h}.$$

Suppose that the following hypothesis is true:

(H3) $\alpha(x) \geqslant 0$ for $x \in [0,1]$ and $\int_0^1 \alpha(x)\,dx \leqslant \varrho < 1$.

Note that if $|\alpha''| \leqslant M_2 < \infty$ as $x \in [0,1]$, then it follows from (H3) that, for all sufficiently small $h > 0$, the inequality

$$h\left(\frac{\alpha_0 + \alpha_N}{2} + \sum_{i=1}^{N-1}\alpha_i\right) \leqslant \varrho_1 < 1$$

is true. Now we will write system (26), (27), (6) in a matrix form. We express U_{0j} from (27)

$$U_{0j} = \sum_{i=1}^{N-1}\tilde{\alpha}_j U_{ij} + (\tilde{\mu}_1)_j, \quad j = 1,\dots,N-1,$$

where

$$\tilde{\alpha}_j = \frac{\alpha_j h}{1 - \alpha_0 h/2}, \quad (\tilde{\mu}_1)_j = \frac{(\mu_1)_j + \alpha_N h(\mu_1)_j/2}{1 - \alpha_0 h/2}.$$

By substituting U_{0j} into (26) as $i = 1$, analogously as in Section 2, we express system (26), (27), (6) in a matrix form

$$\mathbf{A}_1\mathbf{U} + f(\mathbf{U}) = \mathbf{0} \qquad (28)$$

or

$$\Lambda_1\mathbf{U} - \mathbf{C}_1\mathbf{U} + f(\mathbf{U}) = \mathbf{0}. \qquad (29)$$

The following lemma analogous to Lemma 1 is true.

Lemma 4. *If hypothesis* (H3) *is true, then the matrix* $\mathbf{A}_1 = \Lambda_1 - \mathbf{C}_1$ *of system* (29) *has the following properties*:

  1. $\mathbf{A}_1$ *is an M-matrix*;
  2. Λ_1 *is an M-matrix independent of* $\alpha(x)$;

3. $\mathbf{C}_1 \geqslant 0$ *as h is sufficiently small* ($h < 2/\alpha_0$ *if* $\alpha_0 \neq 0$; *there is any restriction for h if* $\alpha_0 = 0$).

Proof. Lemma 4 is proved just like Lemma 1. □

Consequently, if (H2) and (H3) are satisfied, Theorems 1–3 on the convergence of iterative methods are true for system of difference equations (26), (27), (6), approximating the boundary value problem (24), (25), (3) with variable coefficients $\alpha(x)$ and $p(x, y) > 0$.

Now we consider briefly a particular case of system (4)–(6) when

$$f(x, y, u) = c(x, y)u + g(x, y), \quad c(x, y) \geqslant 0.$$

Let us write an eigenvalue problem corresponding to (4)–(6)

$$\delta_x^2 U_{ij} + \delta_y^2 U_{ij} - c_{ij}U_{ij} + \lambda U_{ij} = 0, \quad i, j = 1, \ldots, N - 1, \tag{30}$$

$$U_{0j} = \gamma h \left(\frac{U_{0j} + U_{Nj}}{2} + \sum_{i=1}^{N-1} U_{ij} \right), \quad j = 1, \ldots, N - 1, \tag{31}$$

$$U_{Nj} = U_{i0} = U_{iN} = 0, \quad i, j = 0, \ldots, N. \tag{32}$$

When $c_{ij} = 0$, this problem is coincident with problem (22). We express again U_{0j} from nonlocal conditions (31) (see formula (7)) and substitute these values in difference equations (30) as $i = 1$. Thus, we can rewrite the eigenvalue problem (30)–(32) in matrix form

$$(\mathbf{A} + \mathbf{D})\mathbf{U} = \lambda \mathbf{U}, \tag{33}$$

where $\mathbf{A}$ is the same matrix as in (11), i.e. $\mathbf{A} = \Lambda - \mathbf{C}$ and $\mathbf{D} \geqslant \mathbf{0}$ is diagonal matrix with nonnegative diagonal elements.

If $\mathbf{D} = \mathbf{0}$, according to Lemma 3, $\mathbf{A}$ is an M-matrix if and only if $0 \leqslant \gamma \leqslant \gamma_0$. The matrix $\mathbf{A} + \mathbf{D}$ also is an M-matrix (see Property 1). According to Property 2, $\mathrm{Re}\lambda(\mathbf{A} + \mathbf{D}) > 0$. Thus, we have the following statement.

Conclusion 3. The condition $0 \leqslant \gamma \leqslant \gamma_0 \approx 3.42$ (see (23)) is a sufficient condition for the inequality

$$\mathrm{Re}\,\lambda(\mathbf{A}) \geqslant 0 \tag{34}$$

to be true for all eigenvalues of the eigenvalue problem (30)–(32).

As far as it is known for the authors, property (34) has not be noticed earlier when considering the eigenvalue problem for difference operators with nonlocal conditions.

7 Conclusions and remarks

Many of the iterative methods for systems of linear equations can be justified on the basis of M-matrix theory (see, for example, [7]). One of M-matrices advantage is that

by using this approach, we do not require the symmetry of the matrix. Symmetry is often one of the main characteristics of the matrix of the system when we solve the differential equations with classical boundary conditions (26), (3). But the matrix of the system is usually nonsymmetric for boundary value problems with nonlocal conditions, except in very rare cases. Therefore, application of M-matrices theory looks quite natural for such problems.

The main results of the paper are presented in Sections 4–6. Sufficient conditions of application of the theory of M-matrices for systems of nonlinear equations are specified in Lemmas 1, 3, 4.

We use the standard method of finite differences for elliptic equations with integral condition. For the resulting system of nonlinear equations, the various iterative methods (implicit and explicit, linear and nonlinear) were proposed. Convergence of all methods was proved in accordance with a one methodology using the concept of a regular splitting. To our knowledge, the application of the theory of M-matrices to convergence of iterative methods was applied for the first time to nonlinear system.

It is important to note that the differential equation (1) and the nonlocal condition (2) can be interpreted as a model problem, for which we are using a new methodology. This technique can be applied to much wider class of problems, both taking a more general equation and other nonlocal conditions or nonstandard finite difference approximation. In each case, we need to examine the conditions under which the matrix of finite difference problem be an M-matrix. This methodology can be applied to multidimensional elliptic and parabolic equations or to more general form of function $f(x, y, u)$ or high accuracy finite difference scheme. The requirement that domain is rectangular is also not necessary.

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