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A note on the existence and construction of Dulac functions

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Abstract. In this paper, we construct Dulac functions for a family of planar differential equations. We provide some conditions on the components of a vector field, which ensure the existence of Dulac functions for such vector field. We also present some applications and examples in biomathematical models to illustrate our results.

Keywords: Bendixson–Dulac criterion, Dulac functions, limit cycles.

1 Introduction

In the study of ordinary differential equations, the analysis of periodic solutions is an important goal. But deciding whether an arbitrary differential equation has periodic orbits or not is a difficult question that remains open. For the two-dimensional case, the Bendixson–Dulac criterion gives a sufficient condition for the non-existence of periodic orbits. However, the Bendixson–Dulac criterion requires an auxiliary function with specific properties at Dulac function. Various techniques have been proposed to construct Dulac functions, which range from algebraic methods for special systems, methods for the construction of Lyapunov functions to techniques involving the solutions of certain partial differential equations (see [3, 4, 6, 10, 13]). The Bendixson–Dulac criterion also discards existence of polycycles making it useful in establishing global stability for certain systems.

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For convenience, we recall the Bendixson–Dulac criterion, see [5, p. 137].

Theorem 1 [Bendixson–Dulac criterion]. *Let $f_i(x_1, x_2)$, $i \in 1, 2$, and $h(x_1, x_2)$ be functions C^1 in a simply connected domain $\Omega \subset \mathbb{R}^2$ such that $\partial(f_1 h)/\partial x_1 + \partial(f_2 h)/\partial x_2$ does not change sign in Ω and vanishes at most on a set of measure zero. Then system*

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2), \quad (x_1, x_2) \in \Omega, \tag{1}$$

does not have periodic orbits in Ω.

The h function in the theorem is called a *Dulac function*. Even though Dulac functions are an important tool in many issues of differential equations, their determination is a difficult task. Dulac functions can be used to discard the existence of limit cycles or to estimate the number of limit cycles in some regions.

In this paper, we investigate the existence and construction of Dulac functions of planar vector fields. We give, as far as we known, some new conditions on the components of vector fields, which imply the existence of Dulac functions of these vector fields. Our methods are constructive. We give some consequences and examples to illustrate applications of these results.

2 Results

Consider the vector field $F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$, then system (1) can be rewritten in the form

$$\dot{x} = F(x), \quad x = (x_1, x_2) \in \Omega.$$

As usual, the divergence of the vector field F is defined by

$$\operatorname{div} F = \operatorname{div}(f_1, f_2) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}.$$

We consider $C^0(\Omega, \mathbb{R})$ the set of continuous functions and define the set

$$\mathcal{F}_\Omega = \bigl\{ f \in C^0(\Omega, \mathbb{R}) \colon f \text{ does not change sign and vanishes only on a measure zero set} \bigr\}.$$

Also for the simply connected region Ω, we introduce the sets

$$\mathcal{D}^+_\Omega(F) = \left\{ h \in C^1(\Omega, \mathbb{R}) \colon k := \frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} \geqslant 0,\ k \in \mathcal{F}_\Omega \right\}$$

and

$$\mathcal{D}^-_\Omega(F) = \left\{ h \in C^1(\Omega, \mathbb{R}) \colon k := \frac{\partial(hf_1)}{\partial x_1} + \frac{\partial(hf_2)}{\partial x_2} \leqslant 0,\ k \in \mathcal{F}_\Omega \right\}.$$

A Dulac function in system (1) of the Bendixson–Dulac theorem is an element in the set

$$\mathcal{D}_\Omega(F) := \mathcal{D}^+_\Omega(F) \cup \mathcal{D}^-_\Omega(F).$$

Our results are established with the help of the techniques developed in [10] and [11], let us recall the following result.

Theorem 2. (*See* [10].) *If there exist* $c \in \mathcal{F}_\Omega$ *such that* h *is a solution of the system*

$$f_1 \frac{\partial h}{\partial x_1} + f_2 \frac{\partial h}{\partial x_2} = h\big(c(x_1, x_2) - \operatorname{div} F\big) \tag{2}$$

with $h \in \mathcal{F}_\Omega$*, then* h *is a Dulac function for system* (1) *on* Ω*.*

In the next theorem, we obtain Dulac functions considering special cases such that (2) can be reduced to an ordinary differential equation, which can be solved explicitly.

Theorem 3. *Let* Ω *be a simply connected open set. Suppose a vector field*

$$F = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} \in C^1\big(\Omega, \mathbb{R}^2\big).$$

If there is $c \in \mathcal{F}_\Omega$ *such that any of the following conditions holds, then* $\mathcal{D}_\Omega(F) \neq \emptyset$:

(a) *The function* $\gamma := (c - \operatorname{div} F)/(f_1 g_2 g_1' + f_2 g_1 g_2')$ *depends on* $z := g_1(x_1) g_2(x_2)$ *and is continuous in* Ω;

(b) *The function* $\eta := (c - \operatorname{div} F)/(f_1 g_1 + f_2 g_2)$ *depends on* $z := k_1(x_1) + k_2(x_2)$ (*with* $k_i'(x_i) = g_i(x_i)$ *for* $i = 1, 2$) *and is continuous* Ω;

(c) *The function* $\sigma := (c - \operatorname{div} F)/(f_1 \partial z/\partial x_1 + f_2 \partial z/\partial x_2)$ *depends on* $z := z(x_1, x_2)$ *and is continuous* Ω*.*

Proof. We consider case (a), the others are analogous. We seek a Dulac function using the associated equation (2).

First, assume that h depends only on $z := g_1(x_1) g_2(x_2)$. Thus, equation (2) reduces to

$$\begin{aligned} &f_1(x_1, x_2) g_2(x_2) g_1'(x_1) \frac{\partial h}{\partial z} + f_2(x_1, x_2) g_1(x_1) g_2'(x_2) \frac{\partial h}{\partial z} \\ &\quad = h(z)\big(c(x_1, x_2) - \operatorname{div} F\big), \end{aligned}$$

which is rewritten as

$$\frac{\partial \log h}{\partial z} = \frac{c - \operatorname{div} F}{f_1 g_2 g_1' + f_2 g_1 g_2'} = \gamma(z).$$

From our hypothesis $h := \exp(\int^z \gamma(s)\, \mathrm{d}s$ is a solution of the previous equation. Now, it is easy to verify that $h = \exp(\int^z \gamma(s)\, \mathrm{d}s)$ is indeed a Dulac function. The proof is complete. □

Note that, by the continuity of the functions in the proof of the previous theorem, the constructed Dulac function is everywhere different from zero and can be represented by means of the exponential function, but this does not mean that the constructed Dulac function is an exponential function. The following result is a direct consequence of Theorem 3 and mainly gives some particular cases.

Corollary 1. *Under the conditions of Theorem 3, if there exist a $c \in \mathcal{F}_\Omega$ such that any of the following conditions holds, then $\mathcal{D}_\Omega(F) \neq \emptyset$:*

(i) *The function $\alpha_i := (c - \operatorname{div} F)/f_i$ depends only on x_i for some $i \in \{1, 2\}$ and is continuous;*
(ii) *The function $\beta := (c - \operatorname{div} F)/(x_2 f_1 + x_1 f_2)$ depends on $z := x_1 x_2$ and is continuous;*
(iii) *The function $\delta := (c - \operatorname{div} F)/(f_1 + f_2)$ depends on $z := x_1 + x_2$ and is continuous;*
(iv) *The function $\epsilon := (c - \operatorname{div} F)/(c_1 f_1 + c_2 f_2)$ depends on $z := c_1 x_1 + c_2 x_2$ and is continuous;*
(v) *The function $\kappa := x_2[c(x_1, x_2) - \operatorname{div} F]/(f_1(x_1, x_2) - (x_1/x_2) f_2(x_1, x_2))$ depends on $z := x_1/x_2$ and is continuous.*

3 Applications and examples

In this section, we shall construct Dulac functions to some biomathematical models. We present these results through propositions, and immediately we present examples illustrating this fact.

We propose a family of epidemic systems that supports a Dulac function. This fact is proved in the following proposition.

Proposition 1. *Let $\omega : \mathbb{R}_+^2 \to \mathbb{R}_+$ be continuous functions, then the planar system*

$$\dot{x}_1 = \lambda_1 - \mu_1 x_1 - \omega(x_1, x_2) + \tau_2 x_2,$$
$$\dot{x}_2 = \lambda_2 + \omega(x_1, x_2) - \tau_1 x_2$$

with $\partial\omega(x_1, x_2)/\partial x_1 > 0$, $(\partial/\partial x_2)(\omega(x_1, x_2)/x_2) < 0$ and all parameters are positive, except τ_2, which can have either sign, then the above system supports a Dulac function on $\mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 > 0,\ x_2 > 0\}$.

Proof. Denote by $F = (f_1, f_2)$ the vector field associated to the equation, we get

$$-\operatorname{div}(f_1, f_2) = \mu_1 + \frac{\partial\omega(x_1, x_2)}{\partial x_1} - \frac{\partial\omega(x_1, x_2)}{\partial x_2} + (\mu_2 + \tau_1).$$

We choose

$$c(x_1, x_2) = -\mu_1 - \frac{\partial\omega(x_1, x_2)}{\partial x_1} + \frac{\partial\omega(x_1, x_2)}{\partial x_2} - \frac{\omega(x_1, x_2)}{x_2} - \frac{\lambda_2}{x_2}$$
$$= -\mu_1 - \frac{\partial\omega(x_1, x_2)}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \frac{\omega(x_1, x_2)}{x_2} - \frac{\lambda_2}{x_2} < 0.$$

We can write

$$\alpha_2 := \frac{c - \operatorname{div} F}{f_2} = \frac{-\frac{\lambda_2}{x_2} - \frac{\omega(x_1,x_2)}{x_2} + (\mu_2 + \tau_1)}{x_2[\frac{\lambda_2}{x_2} + \frac{\omega(x_1,x_2)}{x_2} - (\mu_2 + \tau_1)]} = -\frac{1}{x_2},$$

which is continuous and depends on $z := x_2$. Therefore by (i) of Corollary 1 we get $\mathcal{D}_{\mathbb{R}^2_+}(F) \neq \emptyset$. In fact, a direct calculation gives us that

$$h(x_2) = \exp\left(\int^{z} \alpha_2(s)\,\mathrm{d}s\right) = \exp\left(-\int^{x_2} \frac{1}{s}\,\mathrm{d}s\right) = \frac{1}{x_2}$$

is a Dulac function for the system. □

We present some examples that illustrate the Proposition 1.

Example 1. We consider the classic SIS epidemic model with nonmonotone incidence function and disease-induced death

$$\begin{aligned}\dot{x}_1 &= \lambda - \mu_1 x_1 - \frac{\beta_1 x_1 x_2}{1 + x_2^2} + \tau x_2,\\ \dot{x}_2 &= \frac{\beta_1 x_1 x_2}{1 + x_2^2} - \mu_2 x_2 - \tau x_2,\end{aligned}$$

where x_1 and x_2 are the population of susceptible and infectious, respectively. The non-monotone incidence function $\omega(x_1, x_2) = \beta_1 x_1 x_2/(1+x_2^2)$ is proposed in [14]. Note that SIS model satisfies the conditions of Proposition 1, therefore supports a Dulac function.

Example 2. We consider a polynomial differential system of degree n in biochemical reactions [9]

$$\begin{aligned}\dot{x}_1 &= k_1 x_0 - k_2 x_1 - k_3 x_1^p x_2^q,\\ \dot{x}_2 &= k_3 x_1^p x_2^q - k_4 x_2,\end{aligned}$$

where x_0, x_1 and x_2 denote the concentrations of chemical species. The kinetic constants k_i, $i = 1, \dots, 4$, are positive, and $p+q = n \in \mathbb{N}$. If $q = 1$, the conditions of Proposition 1 are satisfied, therefore the above planar system supports a Dulac function.

Example 3. We consider an SIRS epidemic model with constant population

$$\begin{aligned}\dot{x}_1 &= \mu - \mu x_1 - \omega(x_1, x_2) + \tau x_3,\\ \dot{x}_2 &= \omega(x_1, x_2) - (\mu + \sigma) x_2,\\ \dot{x}_3 &= \sigma x_2 - (\mu + \tau) x_3,\end{aligned} \tag{3}$$

where x_1, x_2 and x_3 are the population of susceptible, infectious and recovered, respectively. System (3) is subject to the restriction $x_1(t) + x_2(t) + x_3(t) = 1$, and using $x_3 = 1 - x_1 - x_2$ in the system, we can eliminate x_3 from the equations. This substitution gives the simpler model:

$$\begin{aligned}\dot{x}_1 &= \mu + \tau - (\mu + \tau) x_1 - \omega(x_1, x_2) - \tau x_2,\\ \dot{x}_2 &= \omega(x_1, x_2) - (\mu + \sigma) x_2.\end{aligned} \tag{4}$$

The feasible region of planar system (4) is given by $\{(x_1, x_2) \in \mathbb{R}^2_+\colon x_1 > 0,\ x_2 > 0,\ x_1 + x_2 \leqslant 1\}$. Note that SIRS model satisfies the conditions of Proposition 1, therefore supports a Dulac function.

Now, we analyze a family of population dynamics models with generalized harvest function that supports a Dulac function. We obtain the following proposition.

Proposition 2. *Let $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions and $a_i \in \mathbb{R}_+$ for $i = 1, 2$,*

$$\begin{aligned}\dot{x}_1 &= x_1\big(a_1 - x_1 g_1(x_2)\big) - k(x_1, x_2),\\ \dot{x}_2 &= x_2\big(a_2 - x_2 g_2(x_1)\big)\end{aligned}$$

with $a_2 \geqslant a_1$ and $\partial k(x_1, x_2)/\partial x_1 \geqslant 0$, then the above system supports a Dulac function on $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 > 0,\ x_2 > 0\}$.

Proof. We assume $a_2 \geqslant a_1$ and taking $c(x_1, x_2) := -(a_2-a_1)-2x_1g_1(x_2)-\partial k(x_1, x_2)/\partial x_1 < 0$ on $\mathbb{R}^2_+$, then condition (i) of Corollary 1 is written as

$$\alpha_2 = \frac{c - \operatorname{div} F}{f_2} = -\frac{2a_2 - 2x_2 g_2(x_1)}{x_2(a_2 - x_2 g_2(x_1))} = -\frac{2}{x_2},$$

which is continuous and depends on $z := x_2$, therefore by (i) of Corollary 1 we get that

$$h(x_2) = \exp\left(-2\int^{x_2} \frac{1}{s}\,\mathrm{d}s\right) = \frac{1}{x_2^2}$$

is a Dulac function on $\mathbb{R}^2_+$. □

We present the following example that illustrate the previous proposition.

Example 4. We include the sigmoid harvest function to two-species mutualism model [1]

$$\begin{aligned}\dot{x}_1 &= x_1\left(a_1 - \frac{x_1}{K_1 + a_{12}x_2}\right) - \frac{bx_1^2}{h + x_1^2},\\ \dot{x}_2 &= x_2\left(a_2 - \frac{x_2}{K_2 + a_{21}x_1}\right).\end{aligned}$$

Note that mutualistic model satisfies the conditions of Proposition 2, therefore supports a Dulac function.

In the following proposition, we study a family of two–species cooperative systems that supports a Dulac function.

Proposition 3. *Let $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous functions and $a_i \in \mathbb{R}_+$ for $i = 1, 2$, then the planar system*

$$\begin{aligned}\dot{x}_1 &= x_1\big(g_1(x_2) - a_1 x_1\big),\\ \dot{x}_2 &= x_2\big(g_2(x_1) - a_2 x_2\big)\end{aligned}$$

supports a Dulac function on $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 > 0,\ x_2 > 0\}$.

Proof. We verified that the conditions of Corollary 1(i) are satisfied, in effect

$$-\operatorname{div} F = -g_1(x_2) + 2a_1x_1 - g_2(x_1) + 2a_2x_2,$$

and taking $c = -(a_1x_1 + a_2x_2) < 0$, then

$$\beta := \frac{c - \operatorname{div} F}{x_2 f_1 + x_1 f_2} = -\frac{1}{x_1 x_2}.$$

Integrating this function, we get that

$$h(x_1, x_2) = \exp\left(\int^z \beta(s)\,\mathrm{d}s\right) = \frac{1}{x_1 x_2}$$

is a Dulac function on $\mathbb{R}^2_+$. □

Example 5. The following system is a model for mutualism [8]:

$$\begin{aligned}
\dot{x}_1 &= x_1\big[\big(r_1 + (r_{11} - r_1)\big(1 - \mathrm{e}^{-k_1 x_2}\big)\big) - a_1 x_1\big],\\
\dot{x}_2 &= x_2\big[\big(r_2 + (r_{22} - r_2)\big(1 - \mathrm{e}^{-k_2 x_1}\big)\big) - a_2 x_2\big],
\end{aligned}$$

where r_i, r_{ii}, k_i, $a_i \in \mathbb{R}_+$ are constants and $r_{ii} > r_i$, $i = 1, 2$. Note that mutualistic model satisfies the conditions of Proposition 3, therefore supports a Dulac function.

Example 6. Gopalsamy [7] had proposed the following model to describe the mutualism mechanism:

$$\dot{x}_1 = r_1 x_1\left[\frac{k_1 + a_1 x_2}{1 + x_2} - x_1\right], \qquad \dot{x}_2 = r_2 x_2\left[\frac{k_2 + a_2 x_1}{1 + x_1} - x_2\right],$$

where r_i, k_i, $a_i \in \mathbb{R}_+$ are constants and $a_i > k_i$, $i = 1, 2$. Depending on the nature of k_i, $i = 1, 2$, previous system can be classified as facultative, obligate or a combination of both, so by Proposition 3, therefore supports a Dulac function.

We analyze a polynomial planar system of type Lotka–Volterra that supports a Dulac function. This fact is proved in the following proposition.

Proposition 4. *We consider the planar system of differential equations*

$$\begin{aligned}
\dot{x}_1 &= (\alpha_1 x_1 + \alpha_2 x_2)\big(\beta_1 + \beta_2 x_2^n + \sigma_1 x_1^{2p+1}\big),\\
\dot{x}_2 &= (\alpha_1 x_1 + \alpha_2 x_2)\big(\beta_3 + \beta_4 x_1^m + \sigma_2 x_2^{2q+1}\big).
\end{aligned}$$

If $\alpha_1\alpha_2 > 0$ and $\sigma_1\sigma_2 \geqslant 0$ with σ_1, σ_2 are not both zero; and n, m, p and q are non-negative integers, then the above planar system supports a Dulac function on $\mathbb{R}^2_+$.

Proof. Denote by $F = (f_1, f_2)$ the vector field associated to the equation, we get

$$\begin{aligned}
-\operatorname{div} F = {}&-\alpha_1\big(\beta_1 + \beta_2 x_2^n + \sigma_1 x_1^{2p+1}\big) - \sigma_1(2p+1)x_1^{2p}(\alpha_1 x_1 + \alpha_2 x_2)\\
&- \alpha_2\big(\beta_3 + \beta_4 x_1^m + \sigma_2 x_2^{2q+1}\big) - \sigma_2(2q+1)x_1^{2q}(\alpha_1 x_1 + \alpha_2 x_2).
\end{aligned}$$

Taking $c(x_1, x_2) := (\sigma_1(2p+1)x_1^{2p} + \sigma_2(2q+1)x_1^{2q})(\alpha_1 x_1 + \alpha_2 x_2)$, we can write

$$\begin{aligned}\epsilon : &= \frac{c - \operatorname{div} F}{\alpha_1 f_1 + \alpha_2 f_2} \\ &= \frac{-\alpha_1(\beta_1 + \beta_2 x_2^n + \sigma_1 x_1^{2p+1}) - \alpha_2(\beta_3 + \beta_4 x_1^m + \sigma_2 x_2^{2q+1})}{(\alpha_1 x_1 + \alpha_2 x_2)(\alpha_1(\beta_1 + \beta_2 x_2^n + \sigma_1 x_1^{2p+1}) + \alpha_2(\beta_3 + \beta_4 x_1^m + \sigma_2 x_2^{2q+1}))} \\ &= \frac{-1}{\alpha_1 x_1 + \alpha_2 x_2},\end{aligned}$$

which is continuous and depends on $z := \alpha_1 x_1 + \alpha_2 x_2$, therefore by (iv) of Corollary 1 we have that $h(x_1, x_2) = 1/(\alpha_1 x_1 + \alpha_2 x_2)$ is a Dulac function. □

In the following two examples, we study ecological models that support a Dulac function.

Example 7. We consider following phytoplankton–zooplankton system with instantaneous toxin liberation (see [12]):

$$\begin{aligned}\dot{x}_1 &= r x_1 \left[1 - \frac{x_1}{K}\right] - a w(x_1) x_2, \\ \dot{x}_2 &= b w(x_1) x_2 - s x_2 - d h(x_1) x_2,\end{aligned}$$

where x_1 is the density of phytoplankton population and x_2 is the density of zooplankton population at any instant of time t. The parameters r, K, a, b, a and s are positive constants. The functions are $w(x_1) > 0$, $h(x_1) > 0$ and $w(x_1)(r x_1(1 - x_1/K)/w(x_1))' < 0$.

We get

$$-\operatorname{div}(f_1, f_2) = -r + 2x_1 \frac{r}{K} + a w'(x_1) x_2 - b f w(x_1) + s + d h(x_1).$$

We verified that condition (a) of Theorem 3 with $z = w(x_1) x_2$ is satisfied and choose

$$c(x_1, x_2) = w(x_1) \left(\frac{r x_1 (1 - \frac{x_1}{K})}{w(x_1)}\right)' = r - 2x_1 \frac{r}{K} - r x_1 \left(1 - \frac{x_1}{K}\right) \frac{w'(x_1)}{w(x_1)},$$

then

$$\begin{aligned}\gamma &= \frac{c - \operatorname{div}(f_1, f_2)}{w'(x_1) x_2 f_1 + w(x_1) f_2} \\ &= \frac{-r x_1 (1 - \frac{x_1}{K}) \frac{w'(x_1)}{w(x_1)} + a x_2 w'(x_1) - b w(x_1) + s + d h(x_1)}{w(x_1) x_2 [r x_1 (1 - \frac{x_1}{K}) \frac{w'(x_1)}{w(x_1)} - a x_2 w'(x_1) + b w(x_1) - s - d h(x_1)]} \\ &= -\frac{1}{w(x_1) x_2},\end{aligned}$$

therefore the system supports the Dulac function $h(x_1, x_2) = 1/(w(x_1) x_2)$.

Example 8. We propose a modified Leslie–Gower-type predator–prey model with Hassell–Varley-type functional response

$$\dot{x}_1 = r_1 x_1\left[1 - \frac{x_1}{K_1}\right] - \frac{a x_1 x_2}{x_1 + m x_2^{\phi}}, \qquad \dot{x}_2 = r_2 x_2\left[1 - \frac{x_2}{\tau x_1 + d}\right],$$

where r_1, K_1, a, m, ϕ, r_2, τ and d are positive constants, and $0 \leqslant \phi \leqslant 1$. When $\phi = 0$, the above system reduces to the predator–prey model with modified Leslie–Gower and Holling-type II schemes [2]. We consider $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2\colon\ x_1 > 0,\ x_2 > 0\}$. Note that

$$-\operatorname{div} F = -r_1 + 2\frac{r_1}{K_1}x_1 + \frac{a m x_2^{\phi+1}}{(x_1 + m x_2^{\phi})^2} - r_2 + \frac{2 r_2 x_2}{\tau x_1 + d}.$$

If $r_2 \geqslant r_1$, then

$$\begin{aligned} c(x_1, x_2) = &-\frac{r_2(1-\phi) m x_2^{\phi}}{x_1 + m x_2^{\phi}} - \frac{r_2 m \phi x_2^{\phi+1}}{(x_1 + m x_2^{\phi})(\tau x_1 + d)} \\ &- \frac{x_1}{x_1 + m x_2^{\phi}}\left(\frac{r_1 m}{K_1}x_2^{\phi} + 2\frac{r_1}{K_1}x_1 + r_2 - r_1\right) < 0. \end{aligned}$$

We can write

$$\sigma := \frac{c - \operatorname{div} F}{f_1\frac{\partial z}{\partial x_1} + f_2\frac{\partial z}{\partial x_2}} = \frac{-\Theta}{\frac{x_1 x_2^2}{x_1 + m x_2^{\phi}}\Theta} = -\frac{1}{\frac{x_1 x_2^2}{x_1 + m x_2^{\phi}}},$$

where

$$\begin{aligned} \Theta := &\frac{r_1 m x_2^{\phi}}{x_1 + m x_2^{\phi}} - \frac{r_1 m x_1 x_2^{\phi}}{K_1(x_1 + m x_2^{\phi})} - \frac{a m x_2^{\phi+1}}{(x_1 + m x_2^{\phi})^2} + 2r_2 \\ &- \frac{2 r_2 x_2}{\tau x_1 + d} - \frac{r_2 m \phi x_2^{\phi}}{x_1 + m x_2^{\phi}} + \frac{r_2 m \phi x_2^{\phi+1}}{(x_1 + m x_2^{\phi})(\tau x_1 + d)}. \end{aligned}$$

σ is continuous and depends on $z := z(x_1, x_2)$, therefore by (c) of Theorem 3 we get that $h(x_1, x_2) = (x_1 + m x_2^{\phi})/(x_1 x_2^2)$ is a Dulac function.

4 Concluding remarks

We extended the techniques for the construction of Dulac functions. In particular, we apply this technique to some types of biomathematical models. By using Bendixson–Dulac criterion, we can establish the non-existence of limit cycles in these models. It is important to note that the non-periodicity in epidemiological and ecological models reveals the non-recurrence outbreaks epidemic in a population and the absence of cyclical variations of animal populations, respectively.

Finally, the results of this work indicate that our method of Dulac functions construction can be especially useful to two-dimensional biological systems.

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