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Analysis and control of a nonlinear boundary value problem

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Abstract. We consider a nonlinear two-dimensional boundary value problem which models the frictional contact of a bar with a rigid obstacle. The weak formulation of the problem is in the form of an elliptic variational inequality of the second kind. We establish the existence of a unique weak solution to the problem, then we introduce a regularized version of the variational inequality for which we prove existence, uniqueness, and convergence results. We proceed with an optimal control problem for which we prove the existence of an optimal pair. Finally, we consider the corresponding optimal control problem associated to the regularized variational inequality and prove a convergence result.

Keywords: nonlinear problem, variational inequality, optimal pair.

1 Introduction

In this paper, we deal with the variational analysis and the optimal control of a nonlinear elliptic boundary value problem. The problem is formulated in the two dimensional rectangular $\Omega = (0, L) \times (-h, h)$, where L and h are given positive constants. We use x and y for the spatial variables, and the subscripts will represent partial derivatives, i.e., $u_x = \partial u/\partial x$, $u_y = \partial u/\partial y$, and $u_{xy} = \partial^2 u/(\partial x \partial y)$. Moreover, E, G, and μ are given constants, and f, q are given real-valued functions defined on $[0, L]$. Then the problem under consideration is the following.

Problem $\mathcal{P}$: Find two functions $u = u(x, y) : [0, L] \times [-h, h] \to \mathbb{R}$ and $w = w(x) : [0, L] \to \mathbb{R}$ such that

$$Eu_{xx}(x, y) + Gu_{yy}(x, y) = 0 \quad \forall (x, y) \in \Omega, \tag{1}$$

$$Gw_{xx}(x) + (E - G)u_{xy}(x, y) = 0 \quad \forall (x, y) \in \Omega, \tag{2}$$

$$u(0, y) = w(0) = 0 \quad \forall y \in [-h, h], \tag{3}$$

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$$G\big(u_y(x,h) + w_x(x)\big) = q(x) \quad \forall x \in [0, L], \tag{4}$$

$$(E - 2G)u_x(x,h) = f(x) \quad \forall x \in [0, L], \tag{5}$$

$$u(L,y) = 0 \quad \forall y \in [-h, h], \tag{6}$$

$$\begin{cases} |u_y(L,y) + w_x(L)| \leqslant \mu, \\ u_y(L,y) + w_x(L) = -\mu\frac{w(L)}{|w(L)|} \quad \text{if } w(L) \neq 0 \end{cases} \quad \forall y \in [-h, h], \tag{7}$$

$$u_x(x,-h) = 0 \quad \forall x \in [0, L], \tag{8}$$

$$u_y(x,-h) + w_x(x) = 0 \quad \forall x \in [0, L]. \tag{9}$$

Problem $\mathcal{P}$ represents a mathematical model which describes the deformation of an elastic bar in frictional contact with a rigid foundation. Here Ω represents the cross section of the bar, u is the horizontal displacement, and w denotes the vertical displacement of the central axis of the bar. The bar is clamped on $\{0\} \times [-h, h]$, and its top $[0, L] \times \{h\}$ is submitted to the traction $\boldsymbol{f} = (q, f)$. Moreover, the bar is in frictional contact with a rigid obstacle on $\{L\} \times [-h, h]$ and could be either in slip or stick status on this part of its boundary. In addition, its bottom $[0, L] \times \{-h\}$ is traction free. The physical setting is depicted in Fig. 1.

The equations and boundary conditions (1)–(9) can be derived from the physical setting described above by using the arguments used in [23]. Here we restrict ourselves to provide the following brief description of them. First, equations (1) and (2) represent the equilibrium equations in which E and G are positive elastic coefficients, the Young modulus and the shear modulus, respectively. Note that in these equations, the body forces are neglected for simplicity. Condition (3) is the displacement condition which shows that the bar is fixed on the boundary $x = 0$. Next, conditions (4), (5) represent the traction conditions where, recall, the functions $q : [0, L] \to \mathbb{R}$ and $f : [0, L] \to \mathbb{R}$ denote the horizontal and the vertical components of the traction which acts on the top $y = h$ of the bar. Condition (6) represents the bilateral contact condition at $x = L$, and (7) represents the Tresca friction law. Here μ denotes a positive constant, and μG represents the friction bound. Finally, conditions (8), (9) show that the bottom of the bar, $x = -h$, is free of traction.

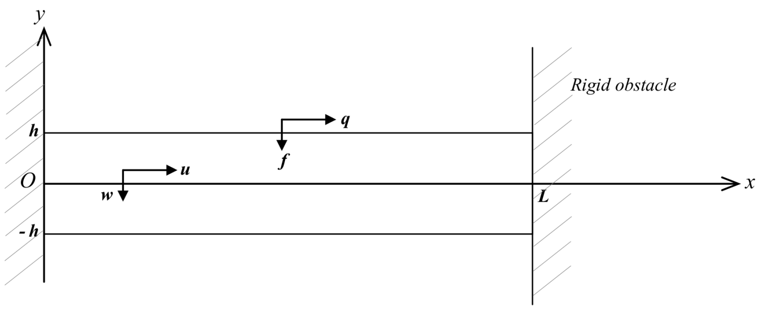

Figure 1. The cross section of the bar.

Note that similar models of contact have been considered in [4, 23] in the quasistatic case and in [21] in the static case. The results in [23] concern the derivation of the governing equations and boundary conditions, starting from the corresponding fully three-dimensional problem. They also concern the weak solvability of the problem based on arguments of evolutionary quasivariational inequalities. This study was completed in [4], where numerical simulations which illustrate the behavior of the solution were provided. The model considered in [21] was frictionless, and the contact was with both the normal compliance and the Signorini condition. There the unique weak solvability of the problem was proved together with the existence of a solution to an associated optimal control problem; various convergence results which show the continuous dependence of the weak solution with respect the set of constraints were also provided. General results on modeling and analysis of contact problems can be found in [8, 13, 15, 16, 19, 22] in the three-dimentional case. Existence and uniqueness results for contact problems with thin structures like beams and bars were obtained, for instance, in [1, 3, 9, 10, 20]. Results on optimal control for various contact problems with elastic materials could be find in [2, 5, 6, 11, 12, 24] and the references therein. The literature on nonlinear problems also includes the papers [17] and [18]. They concern the analytic solution of the three-dimensional Navier–Stokes equation for the flow near an infinite rotating disk and the study of nonlinear vibration of Von Karman rectangular plates, respectively.

The rest of paper is structured as follows. In Section 2, we list the assumptions on the data and derive the variational formulation of Problem $\mathcal{P}$, denoted $\mathcal{P}_V$. Then we prove an existence and uniqueness result, Theorem 1. In Section 3, we introduce a regularized version of the problem, denoted $\mathcal{P}_V^\varepsilon$, for which we prove an existence, uniqueness, and convergence result, Theorem 2. In Section 4, we state and prove the solvability of an optimal control problem, Theorem 3, associated to Problem $\mathcal{P}_V$. Finally, in Section 5, we consider the corresponding optimal control problem associated to the regularized problem (Problem $\mathcal{P}_V^\varepsilon$) for which we prove an existence and convergence result, Theorem 4.

2 An existence and uniqueness result

Everywhere below, we use standard notation for Sobolev and Lebesgue spaces. In addition, we consider the spaces

$$V = \{u \in H^1(\Omega)\colon\ u(0,\cdot) = 0,\ u(L,\cdot) = 0\},$$
$$W = \{w \in H^1(0,L)\colon\ w(0) = 0\},$$

which are real Hilbert spaces with the cannonical inner products given by

$$(u,\psi)_V = \iint\limits_{\Omega} (u\psi + u_x\psi_x + u_y\psi_y)\,\mathrm{d}x\,\mathrm{d}y \quad \forall u,\psi \in V, \tag{10}$$

$$(w,\varphi)_W = \int\limits_0^L (w\varphi + w_x\varphi_x)\,\mathrm{d}x \quad \forall w,\varphi \in W. \tag{11}$$

Note that, in (10)–(11) and below, when no confusion arises, we skip the dependence of various functions on the spatial variables x and y.

Let $X = V \times W$ be the product Hilbert space, i.e., X is endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_X = (u, \psi)_V + (w, \varphi)_W \quad \forall \boldsymbol{u} = (u, w),\ \boldsymbol{v} = (\psi, \varphi) \in X. \tag{12}$$

The corresponding norms on the spaces V, W, and X are denoted by $\|\cdot\|_V$, $\|\cdot\|_W$, and $\|\cdot\|_X$, respectively. Therefore,

$$\|\boldsymbol{u}\|_X^2 = \|u\|_V^2 + \|w\|_W^2 \quad \forall \boldsymbol{u} = (u, w) \in X. \tag{13}$$

The following elementary inequalities are valid for all $\boldsymbol{u} = (u, w) \in X$ and will be repeatedly used in various places below:

$$\|w\|_{L^2(0,L)} \leqslant \|\boldsymbol{u}\|_X, \tag{14}$$

$$\big|w(L)\big| \leqslant d_0 \|\boldsymbol{u}\|_X \quad \text{with } d_0 > 0. \tag{15}$$

Note that inequality (15) represents a consequence of the Sobolev trace theorem.

Next, we consider the product Hilbert space $Y = L^2(0, L) \times L^2(0, L)$ endowed with the cannonical inner product $(\cdot, \cdot)_Y$ and the associated norm $\|\cdot\|_Y$. We denote by $\pi : X \to Y$ the operator defined by

$$\pi \boldsymbol{v} = (\psi^h, \varphi) \quad \forall \boldsymbol{v} = (\psi, \varphi) \in X, \tag{16}$$

where ψ^h represents the trace of the function $\psi \in H^1(\Omega)$ to the boundary $y = h$, i.e., $\psi^h(x) = \psi(x, h)$, a.e. $x \in (0, L)$. Note that π is a linear continuous operator, and therefore, there exists a constant $c_0 > 0$ such that

$$\|\pi \boldsymbol{v}\|_Y \leqslant c_0 \|\boldsymbol{v}\|_X \quad \forall \boldsymbol{v} \in X. \tag{17}$$

Moreover, the compactness of the trace operator combined with the compactness of the embedding $H^1(0, L) \subset L^2(0, L)$ imply that π is a weakly-strongly continuous operator, i.e.,

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \quad \text{in } X \quad \Longrightarrow \quad \pi \boldsymbol{v}_n \to \pi \boldsymbol{v} \quad \text{in } Y. \tag{18}$$

Here and everywhere in the rest of the paper, we denote by $\to$ and $\rightharpoonup$ the strong and weak convergence on various Hilbert spaces, respectively.

We now consider the following assumptions on the data of Problem $\mathcal{P}$:

$$E > 0, \quad G > 0, \quad \mu \geqslant 0, \tag{19}$$

$$\boldsymbol{f} = (q, f) \in Y. \tag{20}$$

Under these assumptions, we denote $g = 2h\mu G$, and we define the operator $A : X \to X$ and the functional $j : X \to \mathbb{R}$ by equalities

$$(A\boldsymbol{u}, \boldsymbol{v})_X = E \iint_{\Omega} u_x \psi_x \,\mathrm{d}x\,\mathrm{d}y + G \iint_{\Omega} (u_y + w_x)(\psi_y + \varphi_x) \,\mathrm{d}x\,\mathrm{d}y, \tag{21}$$

$$j(\boldsymbol{v}) = g\big|\varphi(L)\big| \tag{22}$$

for all $\boldsymbol{u} = (u, w)$, $\boldsymbol{v} = (\psi, \varphi) \in X$. Moreover, we note that definition (16) yields

$$(\boldsymbol{f}, \pi\boldsymbol{v})_Y = \int_0^L q\psi^h \,\mathrm{d}x + \int_0^L f\varphi \,\mathrm{d}x \quad \forall \boldsymbol{v} = (\psi, \varphi) \in X. \tag{23}$$

Note that, in general, Problem $\mathcal{P}$ does not have solution. Therefore, as usual in the study of contact problems, we shall replace it with a new formulation, the so-called variational formulation. The importance of variational formulations is widely recognized in contact mechanics since, besides their unique solvability, they lead directly to finite element approximations for the corresponding contact models.

To derive the variational formulation of Problem $\mathcal{P}$, we assume in what follows that $\boldsymbol{u} = (u(x, y), w(x))$ represents a regular solution to this problem, and we consider an arbitrary element $\boldsymbol{v} = (\psi(x, y), \varphi(x)) \in X$. We multiply (1) by $(\psi - u)$, then we integrate the result over Ω to obtain

$$\iint_\Omega E u_{xx}(x, y)\big(\psi(x, y) - u(x, y)\big) \,\mathrm{d}x \,\mathrm{d}y + \iint_\Omega G u_{yy}(x, y)\big(\psi(x, y) - u(x, y)\big) \,\mathrm{d}x \,\mathrm{d}y = 0.$$

Next, we use Green's formula, the boundary condition (3), (6), and the definition of the space V to deduce that

$$\begin{aligned} &E \iint_\Omega u_x(x, y)\big(\psi_x(x, y) - u_x(x, y)\big) \,\mathrm{d}x \,\mathrm{d}y \\ &\quad + G \iint_\Omega u_y(x, y)\big(\psi_y(x, y) - u_y(x, y)\big) \,\mathrm{d}x \,\mathrm{d}y \\ &= -G \int_0^L u_y(x, -h)\big(\psi(x, -h) - u(x, -h)\big) \,\mathrm{d}x \\ &\quad + G \int_0^L u_y(x, h)\big(\psi(x, h) - u(x, h)\big) \,\mathrm{d}x. \end{aligned} \tag{24}$$

To proceed, we multiply equality (2) by $(\varphi - w)$ and integrate the result over Ω to find that

$$\iint_\Omega G w_{xx}(x)\big(\varphi(x) - w(x)\big) \,\mathrm{d}x \,\mathrm{d}y + \iint_\Omega (E - G) u_{xy}(x, y)\big(\varphi(x) - w(x)\big) \,\mathrm{d}x \,\mathrm{d}y = 0.$$

We now perform integration by parts, and we use the boundary condition (3) to obtain that

$$G\iint_{\Omega} w_x(x)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y + \iint_{\Omega}(E-G)u_y(x,y)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y - G\int_{-h}^{h} w_x(L)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y - (E-G)\int_{-h}^{h} u_y(L,y)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y = 0. \tag{25}$$

Next, we use the boundary condition (7) to see that

$$G\big(u_y(L,y)+w_x(L)\big)\big(\varphi(L)-w(L)\big) \geqslant \mu G\big|w(L)\big| - \mu G\big|\varphi(L)\big| \quad \forall y\in[-h,h].$$

Thus, using notation $g = 2h\mu G$ and (22) yields

$$G\int_{-h}^{h}\big(u_y(L,y)+w_x(L)\big)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y \geqslant j(\boldsymbol{u})-j(\boldsymbol{v}),$$

which implies that

$$G\int_{-h}^{h} u_y(L,y)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y + j(\boldsymbol{v})-j(\boldsymbol{u}) \geqslant -G\int_{-h}^{h} w_x(L)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y. \tag{26}$$

Therefore, combining (25) and (26), we obtain that

$$G\iint_{\Omega}\big(w_x(x)+u_y(x,y)\big)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y + (E-2G)\iint_{\Omega} u_y(x,y)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y - (E-2G)\int_{-h}^{h} u_y(L,y)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y + j(\boldsymbol{v})-j(\boldsymbol{u}) \geqslant 0. \tag{27}$$

We now add inequality (24) and equality (27) to find that

$$
\begin{aligned}
E &\iint\limits_{\Omega} u_x(x,y)\big(\psi_x(x,y)-u_x(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\iint\limits_{\Omega} \big(w_x(x)+u_y(x,y)\big)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\iint\limits_{\Omega} u_y(x,y)\big(\psi_y(x,y)-u_y(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\int\limits_0^L u_y(x,-h)\big(\psi(x,-h)-u(x,-h)\big)\,\mathrm{d}x \\
&\quad + (E-2G)\iint\limits_{\Omega} u_y(x,y)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - (E-2G)\int\limits_{-h}^{h} u_y(L,y)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y + j(\boldsymbol{v}) - j(\boldsymbol{u}) \\
&\geqslant G\int\limits_0^L u_y(x,h)\big(\psi(x,h)-u(x,h)\big)\,\mathrm{d}y.
\end{aligned} \tag{28}
$$

Next, we use the boundary condition (4) to see that

$$
\begin{aligned}
G&\int\limits_0^L u_y(x,h)\big(\psi(x,h)-u(x,h)\big)\,\mathrm{d}x \\
&= \int\limits_0^L \big(q(x)-Gw_x(x)\big)\big(\psi(x,h)-u(x,h)\big)\,\mathrm{d}x.
\end{aligned} \tag{29}
$$

In addition, we remark that

$$
\begin{aligned}
(E-2G)&\iint\limits_{\Omega} u_y(x,y)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - (E-2G)\int\limits_{-h}^{h} u_y(L,y)\big(\varphi(L)-w(L)\big)\,\mathrm{d}y \\
&= -(E-2G)\iint\limits_{\Omega} u_{xy}(x,y)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned} \tag{30}
$$

We now substitute equalities (29) and (30) in (28) to deduce that

$$\begin{aligned}
&E\iint_{\Omega} u_x(x,y)\big(\psi_x(x,y)-u_x(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\iint_{\Omega} \big(w_x(x)+u_y(x,y)\big)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\iint_{\Omega} u_y(x,y,t)\big(\psi_y(x,y)-u_y(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + G\int_0^L u_y(x,-h)\big(\psi(x,-h)-u(x,-h)\big)\,\mathrm{d}x \\
&\quad + G\int_0^L w_x(x)\big(\psi(x,h)-u(x,h)\big)\,\mathrm{d}x \\
&\quad - (E-2G)\iint_{\Omega} u_{xy}(x,y)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x\,\mathrm{d}y + j(\boldsymbol{v})-j(\boldsymbol{u}) \\
&\geqslant \int_0^L q(x)\big(\psi(x,h)-u(x,h)\big)\,\mathrm{d}x. \qquad (31)
\end{aligned}$$

On the other hand, an elementary calculus shows us that

$$\begin{aligned}
&(E-2G)\iint_{\Omega} u_{xy}(x,y)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = (E-2G)\int_0^L u_x(x,h)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x \\
&\qquad - (E-2G)\int_0^L u_x(x,-h)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x.
\end{aligned}$$

Then we use the boundary conditions (5) and (8) to see that

$$\begin{aligned}
&\iint_{\Omega} (E-2G)u_{xy}(x,y)\big(\varphi_x(x)-w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \int_0^L f(x)\big(\varphi(x)-w(x)\big)\,\mathrm{d}x. \qquad (32)
\end{aligned}$$

We now combine inequality (31) and equalities (32), (23) to obtain that

$$\begin{aligned} &E \iint_{\Omega} u_x(x,y)\big(\psi_x(x,y) - u_x(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + G \iint_{\Omega} \big(w_x(x) + u_y(x,y)\big)\big(\varphi_x(x) - w_x(x)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + G \iint_{\Omega} u_y(x,y)\big(\psi_y(x,y) - u_y(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + G \int_0^L u_y(x,-h)\big(\psi(x,-h) - u(x,-h)\big)\,\mathrm{d}x \\ &\quad + G \int_0^L w_x(x)\big(\psi(x,h) - u(x,h)\big)\,\mathrm{d}x + j(\boldsymbol{v}) - j(\boldsymbol{u}) \\ &\geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\boldsymbol{u})_Y. \end{aligned} \tag{33}$$

Next, we use the boundary condition (9) to see that

$$\begin{aligned} &G \int_0^L w_x(x)\big(\psi(x,h) - u(x,h)\big)\,\mathrm{d}x \\ &\quad = G \iint_{\Omega} w_x(x)\big(\psi_y(x,y) - u_y(x,y)\big)\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad - G \int_0^L u_y(x,-h)\big(\psi(x,-h) - u(x,-h)\big)\,\mathrm{d}x. \end{aligned} \tag{34}$$

We now substitute (34) in (33), then we use definition (21) of the operator A to deduce that

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_X + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\boldsymbol{u})_Y. \tag{35}$$

Finally, we use the boundary conditions (3), (6) and the definition of the spaces V, W, and X to see that $\boldsymbol{u} \in X$. We combine this inclusion with inequality (35) to obtain the following variational formulation of Problem $\mathcal{P}$.

Problem $\mathcal{P}_V$: Find $\boldsymbol{u} = (u, w) \in X$ such that

$$(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_X + j(\boldsymbol{v}) - j(\boldsymbol{u}) \geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\boldsymbol{u})_Y \quad \forall \boldsymbol{v} \in X. \tag{36}$$

The unique solvability of Problem $\mathcal{P}_V$ is provided by the following result.

Theorem 1. *Assume that* (19) *and* (20) *hold. Then Problem* $\mathcal{P}_V$ *has a unique solution.*

Proof. Let $\boldsymbol{u} = (u, w) \in X$. Then, using definition (21), it follows that

$$(A\boldsymbol{u}, \boldsymbol{u})_X \geqslant \min(E, 2G) \iint_{\Omega} \left(u_x^2 + \frac{1}{2}(u_y + w_x)^2 \right) \mathrm{d}x\, \mathrm{d}y. \tag{37}$$

Next, we claim that there exists a constant $c_K > 0$, which depends on Ω, such that

$$\iint_{\Omega} \left(u_x^2 + \frac{1}{2}(u_y + w_x)^2 \right) \mathrm{d}x\, \mathrm{d}y \geqslant c_K \iint_{\Omega} \left(u^2 + u_x^2 + u_y^2 + w^2 + w_x^2 \right) \mathrm{d}x\, \mathrm{d}y. \tag{38}$$

This inequality is a direct consequence of the well-known Korn's inequality. Nevertheless, for the convenience of the reader, we prove the claim, and, to this end, we consider in what follows an arbitrary element $\boldsymbol{u} = (u(x, y), w(x)) \in X$. Then the linearized strain tensor associated with the two-dimensional displacement field $\boldsymbol{u}$ is given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \begin{pmatrix} u_x & \frac{1}{2}(u_y + w_x) \\ \frac{1}{2}(u_y + w_x) & 0 \end{pmatrix}.$$

Denote by $\cdot$ and $\|\cdot\|$ the inner product and the Euclidean norm in the space of the second-order symmetric tensors on $\mathbb{R}^2$. Then

$$\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2 = \boldsymbol{\varepsilon}(\boldsymbol{u}) \cdot \boldsymbol{\varepsilon}(\boldsymbol{u}) = u_x^2 + \frac{1}{2}(u_y + w_x)^2 \quad \text{a.e. on } \Omega. \tag{39}$$

Note also that the function $\boldsymbol{u}$ vanishes on the part of Γ characterized by $x = 0$ which is, obviously, of positive one-dimensional measure. Therefore, we are in a position to use Korn's inequality, which states that there exists a constant $c_K > 0$, which depends on h, such that

$$\iint_{\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|^2 \,\mathrm{d}x\, \mathrm{d}y \geqslant c_K \|\boldsymbol{u}\|^2_{H^1(\Omega)^2}. \tag{40}$$

For a proof of Korn's inequality (40), see, for instance, [14, p. 79]). We now combine (39) and (40) to deduce that (38) holds. Therefore, using (10)–(13), we obtain that

$$\iint_{\Omega} \left(u_x^2 + \frac{1}{2}(u_y + w_x)^2 \right) \mathrm{d}x\, \mathrm{d}y \geqslant \widetilde{c}_K \|\boldsymbol{u}\|_X^2, \tag{41}$$

where $\widetilde{c}_K = c_K \min(2h, 1)$. We now combine (37) and (41) to find that

$$(A\boldsymbol{u}, \boldsymbol{u})_X \geqslant m\|\boldsymbol{u}\|_X^2 \tag{42}$$

with $m = \widetilde{c}_K \min(E, 2G) > 0$. On the other hand, using again definition (21), it follows that

$$(A\boldsymbol{u}, \boldsymbol{v})_X \leqslant M\|\boldsymbol{u}\|_X \|\boldsymbol{v}\|_X \quad \forall \boldsymbol{v} \in X, \tag{43}$$

where M is a positive constant, i.e., $M > 0$. We now take $\boldsymbol{v} = A\boldsymbol{u}$ in (43) to find that the operator A satisfies condition

$$\|A\boldsymbol{u}\|_X \leqslant M\|\boldsymbol{u}\|_X \quad \forall \boldsymbol{u} \in X. \tag{44}$$

Recall that A is a linear operator. Then inequalities (42) and (44) show that

$$A : X \to X \text{ is a positively defined linear continuous operator.} \tag{45}$$

Next, we use assumption (19) to see that $g = 2h\mu G > 0$. Therefore, by definition (22) and the continuity of trace, (15), we deduce that

$$j : X \to \mathbb{R}_+ \quad \text{is a continuous seminorm.} \tag{46}$$

Finally, using assumption (20) and the Riesz representation theorem, we deduce that there exists a unique element $\widetilde{\boldsymbol{f}} \in X$ such that

$$(\widetilde{\boldsymbol{f}}, \boldsymbol{v})_X = (\boldsymbol{f}, \pi\boldsymbol{v})_Y \quad \forall \boldsymbol{v} \in X. \tag{47}$$

Theorem 1 is a direct consequence of (45)–(47) which allow to apply a standard existence and uniqueness result for elliptic variational inequalities (see [22, p. 40] for instance). □

A pair of functions $\boldsymbol{u} = (u, w) \in X$ which solves Problem $\mathcal{P}_V$ is called a weak solution for the contact problem. We conclude from here that Theorem 1 provides the unique weak solvability of Problem $\mathcal{P}$.

3 Regularization

We assume in what follows that (19) and (20) hold, and we denote by $\boldsymbol{u} = (u, w)$ the solution of Problem $\mathcal{P}_V$ obtained in Theorem 1. Let ε denote a positive parameter which will converge to zero. We define the functional $j_\varepsilon : X \to \mathbb{R}$ by

$$j_\varepsilon(\boldsymbol{v}) = g\bigl(\sqrt{\varphi^2(L) + \varepsilon^2} - \varepsilon\bigr) \quad \forall \boldsymbol{v} = (\psi, \varphi) \in X, \tag{48}$$

and we consider the following regularized version of Problem $\mathcal{P}_V$.

Problem $\mathcal{P}_V^\varepsilon$: Find $\boldsymbol{u}_\varepsilon = (u_\varepsilon, w_\varepsilon) \in X$ such that

$$(A\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_X + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) \geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\boldsymbol{u}_\varepsilon)_Y \quad \forall \boldsymbol{v} \in X. \tag{49}$$

Note that, from mechanical point of view, Problem $\mathcal{P}_V^\varepsilon$ represents the variational formulation of the contact problem (1)–(9) in which the Tresca friction law (7) was replaced by its regularization

$$u_y(L, y) + w_x(L) = -\mu\bigl(\sqrt{w(L)^2 + \varepsilon^2} - \varepsilon\bigr).$$

Such regularization are commonly used in the study of frictional contact problems mainly for numerical reasons. Indeed, since functional (48) is differentiable on X, inequality (49) is equivalent with the nonlinear variational equation

$$(A\boldsymbol{u}_\varepsilon, \boldsymbol{v})_X + \big(\nabla j_\varepsilon(\boldsymbol{u}), \boldsymbol{v}\big)_X = (\boldsymbol{f}, \pi\boldsymbol{v})_Y \quad \forall \boldsymbol{v} \in X, \tag{50}$$

where $\nabla j_\varepsilon : X \to X$ denotes the gradient of j_ϵ. The numerical treatment of (50) could be done by using various methods as explained in [7] and the references therein.

Our main result in this section is the following.

Theorem 2. *Assume that* (19) *and* (20). *Then*:

(i) *For each $\varepsilon > 0$, Problem $\mathcal{P}_V^\varepsilon$ has a unique solution $\boldsymbol{u}_\varepsilon = (u_\varepsilon, w_\varepsilon)$.*
(ii) *The solution $\boldsymbol{u}_\varepsilon$ of Problem $\mathcal{P}_V^\varepsilon$ converges to the solution $\boldsymbol{u}$ of Problem $\mathcal{P}_V$, i.e.,*

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{in } X \text{ as } \varepsilon \to 0. \tag{51}$$

Proof. (i) Let $\varepsilon > 0$. We note that functional (48) is convex. Moreover, using the elementary inequality

$$\left|\sqrt{a^2+\varepsilon^2} - \sqrt{b^2+\varepsilon^2}\right| \leqslant |a-b| \quad \forall a, b \in \mathbb{R}$$

combined with the trace inequality (15), we deduce that

$$\left|j_\varepsilon(\boldsymbol{u}) - j_\varepsilon(\boldsymbol{v})\right| \leqslant d_0 g \|\boldsymbol{u}-\boldsymbol{v}\|_X \quad \forall \boldsymbol{u}, \boldsymbol{v} \in X,$$

which shows that the functional j_ε is continuous. In addition, recall that A satisfies condition (45), and moreover, (47) holds. The unique solvability of Problem $\mathcal{P}_V^\varepsilon$ follows from a standard result on elliptic variational inequalities (see, for instance, [22, p.40]).

(ii) Let $\varepsilon > 0$. We take $\boldsymbol{v} = \boldsymbol{u}_\varepsilon$ in (36) and $\boldsymbol{v} = \boldsymbol{u}$ in (49), then we add the resulting inequalities to obtain that

$$(A\boldsymbol{u}_\varepsilon - A\boldsymbol{u}, \boldsymbol{u}_\varepsilon - \boldsymbol{u})_X \leqslant j_\varepsilon(\boldsymbol{u}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) + j(\boldsymbol{u}_\varepsilon) - j(\boldsymbol{u}).$$

Next, using inequality (42), it follows that

$$m\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}\|_X^2 \leqslant j_\varepsilon(\boldsymbol{u}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) + j(\boldsymbol{u}_\varepsilon) - j(\boldsymbol{u}). \tag{52}$$

On the other hand, the elementary inequality

$$\left|\sqrt{a^2+\varepsilon^2} - |a|\right| \leqslant \varepsilon \quad \forall a \in \mathbb{R}$$

combined with the trace inequality (15) yields

$$j_\varepsilon(\boldsymbol{u}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) + j(\boldsymbol{u}_\varepsilon) - j(\boldsymbol{u}) \leqslant 2d_0 g\varepsilon. \tag{53}$$

The convergence result (51) is now a consequence of inequalities (52) and (53) which concludes the proof. □

Remark 1. A carefully examination of the proof of Theorem 2 reveals that the convergence result (51) can be extended in the following way: let $\boldsymbol{f}_\varepsilon \in Y$, and let $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(\boldsymbol{f}_\varepsilon)$, $\widetilde{\boldsymbol{u}}_\varepsilon = \boldsymbol{u}(\boldsymbol{f}_\varepsilon) \in X$ be the solutions of the variational inequalities

$$(A\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_X + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) \geqslant (\boldsymbol{f}_\varepsilon, \pi\boldsymbol{v} - \pi\boldsymbol{u}_\varepsilon)_Y \quad \forall \boldsymbol{v} \in X,$$
$$(A\widetilde{\boldsymbol{u}}_\varepsilon, \boldsymbol{v} - \widetilde{\boldsymbol{u}}_\varepsilon)_X + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\widetilde{\boldsymbol{u}}_\varepsilon) \geqslant (\boldsymbol{f}_\varepsilon, \pi\boldsymbol{v} - \pi\widetilde{\boldsymbol{u}}_\varepsilon)_Y \quad \forall \boldsymbol{v} \in X.$$

Then

$$\|\boldsymbol{u}_\varepsilon - \widetilde{\boldsymbol{u}}_\varepsilon\|_X^2 \to 0 \quad \text{as } \varepsilon \to 0.$$

We shall use this result in Section 5 in the following form:

$$\boldsymbol{u}_\varepsilon(\boldsymbol{f}_\varepsilon) - \boldsymbol{u}(\boldsymbol{f}_\varepsilon) \to \boldsymbol{0}_X \quad \text{in } X \text{ as } \varepsilon \to 0 \ \forall \boldsymbol{f}_\varepsilon \in Y. \tag{54}$$

In addition to the mathematical interest in the convergence result (51), it is important from the mechanical point of view since is shows that the weak solution of the contact problem (Problem $\mathcal{P}$) can be approached by the weak solution of a regularized frictional contact problem as the regularization parameter converges to zero.

4 An optimal control problem

In this section, we consider an optimal control problem associated to Problem $\mathcal{P}_V$, and, to this end, everywhere below, we denote by $X \times Y$ the product Hilbert space equipped with the canonical inner product. Moreover, we assume that (19) and (20) hold. Let $w_0 \in \mathbb{R}$ be given, and let $\alpha, \beta, \gamma > 0$. We consider the cost functional $\mathcal{L} : X \times Y \to \mathbb{R}$ defined by

$$\mathcal{L}(\boldsymbol{u}, \boldsymbol{f}) = \alpha\bigl|w(L) - w_0\bigr|^2 + \beta\|q\|^2_{L^2(0,L)} + \gamma\|f\|^2_{L^2(0,L)} \tag{55}$$

for all $\boldsymbol{u} = (u, w) \in X$, $\boldsymbol{f} = (q, f) \in Y$. Using standard arguments, it is easy to see that $\mathcal{L}$ is a weakly lower semicontinuous function on $X \times Y$. Next, we define the set of admissible pairs by

$$\begin{aligned}\mathcal{V}_{\text{ad}} = \bigl\{(\boldsymbol{u}, \boldsymbol{f}) \in X \times Y\colon\ &(A\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})_X + j(\boldsymbol{v}) - j(\boldsymbol{u}) \\ &\geqslant (\boldsymbol{f}, \pi bv - \pi\boldsymbol{u})_Y \ \forall \boldsymbol{v} \in X\bigr\}.\end{aligned} \tag{56}$$

Then the optimal control problem we are interested in can be formulated as follows.

Problem $\mathcal{Q}$: Find $(\boldsymbol{u}^*, \boldsymbol{f}^*) \in \mathcal{V}_{\text{ad}}$ such that

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{f}^*) = \min_{(\boldsymbol{u}, \boldsymbol{f}) \in \mathcal{V}_{\text{ad}}} \mathcal{L}(\boldsymbol{u}, \boldsymbol{f}). \tag{57}$$

An element $(\boldsymbol{u}^*, \boldsymbol{f}^*) \in \mathcal{V}_{\text{ad}}$ which solves Problem $\mathcal{Q}$ is called an *optimal pair*, and the corresponding traction $\boldsymbol{f}^*$ is called an *optimal control*. The mechanical interpretation of Problem $\mathcal{Q}$ is the following: we are looking for admissible traction $\boldsymbol{f} \in Y$ such that the associate slip on the contact surface, $w(L)$, is as close as possible to a given slip w_0. Furthermore, this choice has to fulfill a minimum expenditure condition which is taken into account by the last term in (55).

Our main result in this section is the following existence result.

Theorem 3. *Assume that* (19) *and* (20) *hold. Then there exists at least one solution* $(\boldsymbol{u}^*, \boldsymbol{f}^*) \in \mathcal{V}_{\rm ad}$ *of Problem* $\mathcal{Q}$.

In order to provide the proof of Theorem 3, we need the following auxiliary result.

Lemma 1. *Assume that* (19) *and* (20) *hold, let* $\{\boldsymbol{f}_n\} \subset Y$ *be a sequence of functions, and, for all* $n \in \mathbb{N}$*, let* $\boldsymbol{u}_n$ *be the solution of the variational inequality* (36) *with* $\boldsymbol{f} = \boldsymbol{f}_n$*. Assume that*

$$\boldsymbol{f}_n \rightharpoonup \boldsymbol{f} \quad \text{in } Y \text{ as } n \to \infty \tag{58}$$

and denote by $\mathbf{u}$ *the solution of Problem* $\mathcal{P}_V$*. Then*

$$\boldsymbol{u}_n \to \boldsymbol{u} \quad \text{in } X \text{ as } n \to \infty. \tag{59}$$

Proof. Let $n \in \mathbb{N}$. We use inequality (36) to see that

$$(A\boldsymbol{u}_n, \boldsymbol{v} - \boldsymbol{u}_n)_X + j(\boldsymbol{v}) - j(\boldsymbol{u}_n) \geqslant (\boldsymbol{f}_n, \pi\boldsymbol{v} - \pi\boldsymbol{u}_n)_Y \quad \forall \boldsymbol{v} \in X, \tag{60}$$

then we take $\boldsymbol{v} = \mathbf{0}_X$ in (60) to obtain that

$$(A\boldsymbol{u}_n, \boldsymbol{u}_n)_X + j(\boldsymbol{u}_n) \leqslant (\boldsymbol{f}_n, \pi\boldsymbol{u}_n)_Y.$$

Next, using (42), inequality $j(\boldsymbol{u}_n) \geqslant 0$, and the continuity of the operator π, (17), it follows that

$$\|\boldsymbol{u}_n\|_X \leqslant \frac{c_0}{m}\|\boldsymbol{f}_n\|_Y. \tag{61}$$

We now combine convergence (58) and inequality (61) to deduce that there exists a positive constant $c > 0$, which does not depend on n, such that

$$\|\boldsymbol{u}_n\|_X \leqslant c. \tag{62}$$

Inequality (62) shows that the sequence $\{\boldsymbol{u}_n\}$ is bounded in X. Therefore, from a standard argument of compactness we deduce that there exists $\widetilde{\boldsymbol{u}} \in X$ such that

$$\boldsymbol{u}_n \rightharpoonup \widetilde{\boldsymbol{u}} \quad \text{in } X \text{ as } n \to \infty. \tag{63}$$

On the other hand, the compactness of the trace implies that

$$j(\boldsymbol{u}_n) \to j(\widetilde{\boldsymbol{u}}) \quad \text{as } n \to \infty. \tag{64}$$

We now take $\boldsymbol{v} = \widetilde{\boldsymbol{u}}$ in (60) to obtain that

$$(A\boldsymbol{u}_n, \boldsymbol{u}_n - \widetilde{\boldsymbol{u}})_X \leqslant (\boldsymbol{f}_n, \pi\boldsymbol{u}_n - \pi\widetilde{\boldsymbol{u}})_Y + j(\boldsymbol{u}_n) - j(\widetilde{\boldsymbol{u}}),$$

then we pass to the upper limit as $n \to \infty$ in this inequality and use convergences (58), (63), (64), (18). As a result, we deduce that

$$\limsup_{n\to\infty}\, (A\boldsymbol{u}_n, \boldsymbol{u}_n - \widetilde{\boldsymbol{u}})_X \leqslant 0.$$

Therefore, using the pseudomonotonicity of the operator A guaranteed by (45) and the convergence (63), we deduce that

$$\liminf_{n\to\infty} (A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{v})_X \geqslant (A\widetilde{\boldsymbol{u}}, \widetilde{\boldsymbol{u}} - \boldsymbol{v})_X \quad \forall \boldsymbol{v} \in X. \tag{65}$$

On the other hand, using (60), (63), (64), and (18) yields

$$\limsup_{n\to\infty} (A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{v})_X \leqslant (\boldsymbol{f}, \pi\widetilde{\boldsymbol{u}} - \pi\boldsymbol{v})_Y + j(\boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}) \quad \forall \boldsymbol{v} \in X. \tag{66}$$

We combine now inequalities (65) and (66) to see that

$$(A\widetilde{\boldsymbol{u}}, \boldsymbol{v} - \widetilde{\boldsymbol{u}})_X + j(\boldsymbol{v}) - j(\widetilde{\boldsymbol{u}}) \geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\widetilde{\boldsymbol{u}})_Y \quad \forall \boldsymbol{v} \in X. \tag{67}$$

Inequality (67) shows that $\widetilde{\boldsymbol{u}}$ is a solution to Problem $\mathcal{P}_V$. Therefore, by the uniqueness part of Theorem 1 we deduce that

$$\widetilde{\boldsymbol{u}} = \boldsymbol{u}. \tag{68}$$

A carefully analysis based on the arguments above shows that any weakly convergent subsequence of the sequence $\{\boldsymbol{u}_n\} \subset X$ converges weakly to $\boldsymbol{u}$, where, recall, $\boldsymbol{u}$ is the element of X which solves the variational inequality (36). Moreover, estimate (61) shows that the sequence $\{\boldsymbol{u}_n\}$ is bounded in X. Thus, a standard compactness argument allows us to conclude that the whole sequence $\{\boldsymbol{u}_n\} \subset X$ converges weakly to $\boldsymbol{u}$, i.e.,

$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u} \quad \text{in } X \text{ as } n \to \infty.$$

We now prove the strong convergence (59). To this end, let $n \in \mathbb{N}$. We take $\boldsymbol{v} = \boldsymbol{u}$ in (60) to obtain that

$$(A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})_X \leqslant (\boldsymbol{f}_n, \pi\boldsymbol{u}_n - \pi\boldsymbol{u})_Y + j(\boldsymbol{u}) - j(\boldsymbol{u}_n). \tag{69}$$

Next, we use (69) and (42) to find that

$$\begin{aligned} m\|\boldsymbol{u}_n - \boldsymbol{u}\|_X^2 &\leqslant (A\boldsymbol{u}_n - A\boldsymbol{u}, \boldsymbol{u}_n - \boldsymbol{u})_X \\ &= (A\boldsymbol{u}_n, \boldsymbol{u}_n - \boldsymbol{u})_X - (A\boldsymbol{u}, \boldsymbol{u}_n - u)_X \\ &\leqslant (\boldsymbol{f}_n, \pi\boldsymbol{u}_n - \pi\boldsymbol{u})_Y - (A\boldsymbol{u}, \widetilde{\boldsymbol{u}}_n - \boldsymbol{u})_X + j(\boldsymbol{u}) - j(\boldsymbol{u}_n). \end{aligned}$$

We now pass to the limit in this inequality and use convergences (63), (64), (18), (58) and equality (68). As a result, we deduce that

$$\|\boldsymbol{u}_n - \boldsymbol{u}\|_X^2 \to 0 \quad \text{as } n \to \infty,$$

which conclude the proof. □

Remark 2. Lemma 1 shows that the map $\boldsymbol{f} \mapsto \boldsymbol{u}(\boldsymbol{f}) : Y \to X$ which associates to each element $\boldsymbol{f} \in Y$ the solution $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{f}) \in X$ of the variational inequality (36) is weakly-strongly continuous. We shall use this result in Section 5 in the following form:

$$\boldsymbol{f}_\varepsilon^* \rightharpoonup \boldsymbol{f}^* \quad \text{in } Y \quad \Longrightarrow \quad \boldsymbol{u}(\boldsymbol{f}_\varepsilon^*) \to \boldsymbol{u}(\boldsymbol{f}^*) \quad \text{in } X \text{ as } \varepsilon \to 0. \tag{70}$$

We now have all the ingredients to provide the proof of the Theorem 3.

Proof. Denote

$$\theta = \inf_{(\boldsymbol{u},\boldsymbol{f})\in\mathcal{V}_{\mathrm{ad}}} \mathcal{L}(\boldsymbol{u},\boldsymbol{f}), \tag{71}$$

and let $\{(\boldsymbol{u}_n,\boldsymbol{f}_n)\} \subset \mathcal{V}_{\mathrm{ad}}$ be a minimizing sequence for the functional $\mathcal{L}$, i.e.,

$$\lim_{n\to\infty} \mathcal{L}(\boldsymbol{u}_n,\boldsymbol{f}_n) = \theta. \tag{72}$$

Assume that the sequence $\{\boldsymbol{f}_n\}$ is not bounded in Y. Then, passing to a subsequence still denoted $\{\boldsymbol{f}_n\}$, we have

$$\|\boldsymbol{f}_n\|_Y \to +\infty \quad \text{as } n \to +\infty. \tag{73}$$

We now use definition (55) of the functional $\mathcal{L}$ and equality $\boldsymbol{f}_n = (q_n, f_n)$ to see that

$$\mathcal{L}(\boldsymbol{u}_n,\boldsymbol{f}_n) \geqslant \min(\beta,\gamma)\|\boldsymbol{f}_n\|_Y.$$

Therefore, passing to the limit as $n \to +\infty$ in this inequality and using (73), we deduce that

$$\lim_{n\to+\infty} \mathcal{L}(\boldsymbol{u}_n,\boldsymbol{f}_n) = +\infty. \tag{74}$$

Equalities (72) and (74) imply that $\theta = +\infty$ which represents a contradiction, since (71) shows that $\theta \in \mathbb{R}$.

We conclude from above that the sequence $\{\boldsymbol{f}_n\}$ is bounded in Y, and therefore, there exists $\boldsymbol{f}^* \in Y$ such that, passing to a subsequence still denoted $\{\boldsymbol{f}_n\}$, we have

$$\boldsymbol{f}_n \rightharpoonup \boldsymbol{f}^* \quad \text{in } Y \quad \text{as } n \to +\infty. \tag{75}$$

Let $\boldsymbol{u}^*$ be the solution of the variational inequality (36) for $\boldsymbol{f} = \boldsymbol{f}^*$, i.e., $\boldsymbol{u}^* = \boldsymbol{u}(\boldsymbol{f}^*)$. Then by definition (56) of the set $\mathcal{V}_{\mathrm{ad}}$ we have

$$(\boldsymbol{u}^*,\boldsymbol{f}^*) \in \mathcal{V}_{\mathrm{ad}}. \tag{76}$$

Moreover, using (75) and (70), it follows that

$$\boldsymbol{u}_n \to \boldsymbol{u}^* \quad \text{in } X \text{ as } n \to +\infty. \tag{77}$$

We now use convergences (75), (77) and the weakly lower semicontinuity of the functional $\mathcal{L}$ to deduce that

$$\liminf_{n\to+\infty} \mathcal{L}(\boldsymbol{u}_n,\boldsymbol{f}_n) \geqslant \mathcal{L}(\boldsymbol{u}^*,\boldsymbol{f}^*). \tag{78}$$

Equality (72) and inequality (78) yield

$$\theta \geqslant \mathcal{L}(\boldsymbol{u}^*,\boldsymbol{f}^*). \tag{79}$$

In addition, (76) and (71) imply that

$$\theta \leqslant \mathcal{L}(\boldsymbol{u}^*,\boldsymbol{f}^*). \tag{80}$$

We now combine inequalities (79), (80) with equality (71) to see that (57) holds which concludes the proof. □

5 An convergence result

We now turn to the optimal control problem associated to Problem $\mathcal{P}_V^\varepsilon$, and, to this end, we assume that (19) and (20) hold. Let $w_0 \in \mathbb{R}$ be given, and let $\alpha, \beta, \gamma > 0$. We consider the cost functional $\mathcal{L} : X \times Y \to \mathbb{R}$ defined by (55), and, for each $\varepsilon > 0$, we define the set of admissible pairs by

$$\mathcal{V}_{\mathrm{ad}}^\varepsilon = \bigl\{(\boldsymbol{u}_\varepsilon, \boldsymbol{f}) \in X \times Y\colon\ (A\boldsymbol{u}_\varepsilon, \boldsymbol{v} - \boldsymbol{u}_\varepsilon)_X + j_\varepsilon(\boldsymbol{v}) - j_\varepsilon(\boldsymbol{u}_\varepsilon) \\ \geqslant (\boldsymbol{f}, \pi\boldsymbol{v} - \pi\boldsymbol{u}_\varepsilon)_Y \ \forall \boldsymbol{v} \in X\bigr\}.$$

Then the optimal control problem we study in this section is the following.

Problem $\mathcal{Q}^\varepsilon$: Find $(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) \in \mathcal{V}_{\mathrm{ad}}^\varepsilon$ such that

$$\mathcal{L}(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) = \min_{(\boldsymbol{u}_\varepsilon, \boldsymbol{f}) \in \mathcal{V}_{\mathrm{ad}}^\varepsilon} \mathcal{L}(\boldsymbol{u}_\varepsilon, \boldsymbol{f}).$$

Using Theorem 3, it follows that the optimal control problem (Problem $\mathcal{Q}^\varepsilon$) has at least one solution for each $\varepsilon > 0$. Moreover, we have the following convergence result.

Theorem 4. *Assume that* (19) *and* (20) *hold, and let* $\{(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)\}$ *be a sequence of solutions of Problem* $\mathcal{Q}^\varepsilon$. *Then there exists a subsequence of the sequence* $\{(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)\}$ *again denoted* $\{(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)\}$ *and an element* $(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) \in X \times Y$ *such that*

$$\boldsymbol{f}_\varepsilon^* \rightharpoonup \boldsymbol{f}^* \quad \text{in } Y \text{ as } \varepsilon \to 0, \tag{81}$$

$$\boldsymbol{u}_\varepsilon^* \to \boldsymbol{u}^* \quad \text{in } X \text{ as } \varepsilon \to 0, \tag{82}$$

$$(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) \quad \text{is a solution of Problem } \mathcal{Q}. \tag{83}$$

Proof. Let $\varepsilon > 0$, and denote $\boldsymbol{u}_\varepsilon^* = (u_\varepsilon^*, w_\varepsilon^*)$ and $\boldsymbol{f}_\varepsilon^* = (q_\varepsilon^*, f_\varepsilon^*)$. We have

$$\begin{aligned}\|\boldsymbol{f}_\varepsilon^*\|_Y^2 &= \|q_\varepsilon^*\|_{L^2(0,L)}^2 + \|f_\varepsilon^*\|_{L^2(0,L)}^2 \\ &\leqslant \frac{1}{\min(\beta,\gamma)}\bigl(\beta\|q_\varepsilon^*\|_{L^2(0,L)}^2 + \gamma\|f_\varepsilon\|_{L^2(0,L)}^2\bigr) \\ &\leqslant \frac{1}{\min(\beta,\gamma)}\,\mathcal{L}(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*),\end{aligned}$$

and, since $(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)$ is a solution of Problem $\mathcal{Q}^\varepsilon$, we deduce that

$$\|\boldsymbol{f}_\varepsilon^*\|_Y^2 \leqslant \frac{1}{\min(\beta,\gamma)}\,\mathcal{L}(\boldsymbol{u}_\varepsilon, \boldsymbol{f}) \quad \forall (\boldsymbol{u}_\varepsilon, \boldsymbol{f}) \in \mathcal{V}_{\mathrm{ad}}^\varepsilon. \tag{84}$$

Next, since A is linear and j is a positive functional, it follows that $\boldsymbol{u}_\varepsilon = \boldsymbol{0}_X$ is the solution of Problem $\mathcal{P}_V^\varepsilon$ for $\boldsymbol{f} = \boldsymbol{0}_Y$. Moreover, it is easy to see that

$$\mathcal{L}(\boldsymbol{0}_X, \boldsymbol{0}_Y) = \alpha|w_0|^2. \tag{85}$$

We now take $(\boldsymbol{u}_\varepsilon, \boldsymbol{f}) = (\boldsymbol{0}_X, \boldsymbol{0}_Y)$ in (84), then use (85) to see that the sequence $\{\boldsymbol{f}_\varepsilon^*\}$ is bounded in Y. Therefore, passing to a subsequence again denoted $\{\boldsymbol{f}_\varepsilon^*\}$, it follows that there exists $\boldsymbol{f}^* \in Y$ such that (81) holds. Denote by $\boldsymbol{u}^*$ the solution of Problem $\mathcal{P}_V$ for $\boldsymbol{f} = \boldsymbol{f}^*$, i.e., $\boldsymbol{u}^* = \boldsymbol{u}(\boldsymbol{f}^*)$. This implies that

$$(\boldsymbol{u}^*, \boldsymbol{f}^*) \in \mathcal{V}_{\mathrm{ad}}. \tag{86}$$

Moreover, using (81), (54), and (70), we have

$$\|\boldsymbol{u}_\varepsilon(\boldsymbol{f}_\varepsilon^*) - \boldsymbol{u}(\boldsymbol{f}^*)\|_X \leqslant \|\boldsymbol{u}_\varepsilon(\boldsymbol{f}_\varepsilon^*) - \boldsymbol{u}(\boldsymbol{f}_\varepsilon^*)\|_X + \|\boldsymbol{u}(\boldsymbol{f}_\varepsilon^*) - \boldsymbol{u}(\boldsymbol{f}^*)\|_X \to 0$$

as $\varepsilon \to 0$, and, since $\boldsymbol{u}_\varepsilon(\boldsymbol{f}_\varepsilon^*) = \boldsymbol{u}_\varepsilon^*$, $\boldsymbol{u}(\boldsymbol{f}_\varepsilon^*) = \boldsymbol{u}^*$, we deduce that (82) holds, too.

We now use convergences (81), (82) and the weakly lower semicontinuity of the functional $\mathcal{L}$ to see that

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{f}^*) \leqslant \liminf_{\varepsilon\to 0} \mathcal{L}(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*). \tag{87}$$

Next, we fix a solution $(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*)$ of Problem $\mathcal{Q}$, and therefore,

$$\mathcal{L}(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*) = \min_{(\boldsymbol{u},\boldsymbol{f})\in\mathcal{V}_{\mathrm{ad}}} \mathcal{L}(\boldsymbol{u}, \boldsymbol{f}). \tag{88}$$

In addition, for each $\varepsilon > 0$, we denote by $\widetilde{\boldsymbol{u}}_\varepsilon$ the solution of Problem $\mathcal{P}_V^\varepsilon$ for $\boldsymbol{f} = \widetilde{\boldsymbol{f}}^*$. This implies that $(\widetilde{\boldsymbol{u}}_\varepsilon, \widetilde{\boldsymbol{f}}^*) \in \mathcal{V}_{\mathrm{ad}}^\varepsilon$, and by the optimality of the pair $(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)$ we have that

$$\mathcal{L}(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) \leqslant \mathcal{L}(\widetilde{\boldsymbol{u}}_\varepsilon, \widetilde{\boldsymbol{f}}^*).$$

We pass to the upper limit in this inequality to see that

$$\limsup_{\varepsilon\to 0} \mathcal{L}(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*) \leqslant \limsup_{\varepsilon\to 0} \mathcal{L}(\widetilde{\boldsymbol{u}}_\varepsilon, \widetilde{\boldsymbol{f}}^*). \tag{89}$$

Now, remember that $\widetilde{\boldsymbol{u}}^*$ is the solution of Problem $\mathcal{P}_V$ for $\boldsymbol{f} = \widetilde{\boldsymbol{f}}^*$ and is the $\widetilde{\boldsymbol{u}}_\varepsilon$ solution of Problem $\mathcal{P}_V^\varepsilon$ for $\boldsymbol{f} = \widetilde{\boldsymbol{f}}^*$. Therefore, using part (ii) of Theorem 2, we deduce that

$$\widetilde{\boldsymbol{u}}_\varepsilon \to \widetilde{\boldsymbol{u}}^* \quad \text{in } X \text{ as } \varepsilon \to 0,$$

and then the continuity of the functional $\boldsymbol{u} \mapsto \mathcal{L}(\boldsymbol{u}, \widetilde{\boldsymbol{f}}^*) : X \to \mathbb{R}$ yields

$$\lim_{\varepsilon\to 0} \mathcal{L}(\widetilde{\boldsymbol{u}}_\varepsilon, \widetilde{\boldsymbol{f}}^*) = \mathcal{L}(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*). \tag{90}$$

We now combine (87), (89), and (90) to see that

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{f}^*) \leqslant \mathcal{L}(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*). \tag{91}$$

On the other hand, since $(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*)$ is a solution of Problem $\mathcal{Q}$, inclusion (86) implies that

$$\mathcal{L}(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*) \leqslant \mathcal{L}(\boldsymbol{u}^*, \boldsymbol{f}^*). \tag{92}$$

Inequalities (91) and (92) imply that

$$\mathcal{L}(\boldsymbol{u}^*, \boldsymbol{f}^*) = \mathcal{L}(\widetilde{\boldsymbol{u}}^*, \widetilde{\boldsymbol{f}}^*). \tag{93}$$

We now combine (86), (93), (88) to see that (83) holds which concludes the proof. □

We end the section with the remark that if the solution $(\boldsymbol{u}^*, \boldsymbol{f}^*)$ of Problem $\mathcal{Q}$ is unique, then the whole sequence $\{(\boldsymbol{u}_\varepsilon^*, \boldsymbol{f}_\varepsilon^*)\}$ converge to this solution in the sense given by (81), (82). In addition to the mathematical interest in this convergence result, it is important from the mechanical point of view since it proves that the solution of the optimal control problem for the frictional contact problem can be approached by the solutions of the optimal control problem for the regularized frictional contact problem as the regularization parameter converges to zero.

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