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Some notes on a second-order random boundary value problem*

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Abstract. We consider a two-point boundary value problem of second-order random differential equation. Using a variant of the α-ψ-contractive type mapping theorem in metric spaces, we show the existence of at least one solution.

Keywords: α-ψ-contractive type mapping, measurable space, random differential equation.

1 Introduction

In this paper, we consider the following two-point boundary value problem of second-order random differential equation:

$$\begin{aligned} &-\frac{\mathrm{d}^2u}{\mathrm{d}t^2}(\omega,t) = f\bigl(\omega,t,u(\omega,t)\bigr), \quad t\in[0,1],\\ &u(\omega,0)=u(\omega,1)=0 \end{aligned} \tag{1}$$

for all $\omega\in\Omega$, where $f:\Omega\times[0,1]\times\mathbb{R}\to\mathbb{R}$ has certain regularities and Ω is a nonempty set.

By a random solution of system (1), we mean a measurable mapping $u:\Omega\to C([0,1],\mathbb{R})$ satisfying (1), where $C([0,1],\mathbb{R})$ denote the space of all continuous functions defined on $[0,1]$. The interest for the random version of well-known ordinary differential equations is motivated by the necessity to model and understand certain nonspecific dynamic processes of natural phenomena arising in the applied sciences; see the books of Bharucha-Reid [2] and Skorohod [13]. For some interesting contributions to this problem,

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see Itoh [4], Li and Duan [5], Papageorgiou [9], Sinacer et al. [12], Tchier et al. [14]. Clearly, in absence of ω, system (1) reduces to

$$\begin{aligned} &-\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(t) = f\bigl(t, u(t)\bigr), \quad t \in [0,1], \\ &u(0) = u(1) = 0. \end{aligned} \tag{2}$$

The approach developed in this paper uses a combination of classical tools based on Green's functions theory and fixed-point theorems for operators in Banach spaces. Indeed, we recall that the Green's function associated to (2) is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1, \\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases} \tag{3}$$

The space $C([0,1],\mathbb{R})$, endowed with the metric

$$d_\infty(x,y) = \|x-y\|_\infty = \max_{t\in[0,1]} \bigl|x(t) - y(t)\bigr|,$$

is a complete metric space. In this setting, Samet et al. [11] investigated the solvability of system (2) by using a new concept of α-ψ-contractive type mapping, which generalizes the Banach contraction in [1] and many others fixed-point theorems in the literature (see, for example, Nieto and Rodríguez-López [7] and Ran and Reurings [10]). Motivated by [11], we propose a study of system (1). Precisely, we extend the original notion of α-admissible mapping to work with measurable mappings, then we give a random version of the main result in [11], finally, we establish the existence of at least one solution for system (1). The interesting feature of our work is that we do not impose contractive conditions to all points of the involved space, but just to the ones satisfying a specific inequality relation (defined by using a given function α; see Definition 2 below). This means that we enlarge the class of operators such that our results apply. Also, by appropriate choices of the function α, we are able to control the whole process (as shown by the proofs of the results).

2 Mathematical background

Here we give some concepts and notations from the existing literature. We denote the Borel σ-algebra on a metric space X by $\mathcal{B}(X)$. Let (Ω,Σ) be a measurable space so that by $\Sigma \otimes \mathcal{B}(X)$ we mean the smallest σ-algebra on $\Omega \times X$ containing all the sets $M \times B$ (with $M \in \Sigma$ and $B \in \mathcal{B}(X)$).

We recall a definition that we need in the statement of the main theorem.

Definition 1. Let (Ω,Σ) be a measurable space, X and Y be two metric spaces. A mapping $\widehat{h} : \Omega \times X \to Y$ is called Carathéodory if, for all $x \in X$, the mapping $\omega \to \widehat{h}(\omega,x)$ is $(\Sigma,\mathcal{B}(Y))$-measurable (Σ-measurable for short) and, for all $\omega \in \Omega$, the mapping $x \to \widehat{h}(\omega,x)$ is continuous.

The interest for this definition is related to the following facts.

Theorem 1. (*See* [3, *Thm.* 2.5.22].) *If (Ω, Σ) is a measurable space, X is a separable metric space, Y is a metric space, and $\widehat{h} : \Omega \times X \to Y$ is a Carathéodory mapping, then $\widehat{h}$ is $\Sigma \otimes \mathcal{B}(X)$-measurable.*

Corollary 1. (*See* [3, *Cor.* 2.5.24].) *If (Ω, Σ) is a measurable space, X is a separable metric space, Y is a metric space, $\widehat{h} : \Omega \times X \to Y$ is a Carathéodory mapping, and $u : \Omega \to X$ is Σ-measurable, then $\omega \to \widehat{h}(\omega, u(\omega))$ is a Σ-measurable mapping from Ω into Y.*

Let (Ω, Σ) be a measurable space, X be a separable metric space, and Y be a metric space. A mapping $\widetilde{h} : \Omega \times X \to Y$ is said to be superpositionally measurable (sup-measurable for short) if, for all Σ-measurable mapping $u : \Omega \to X$, the mapping $\omega \to \widetilde{h}(\omega, u(\omega))$ is Σ-measurable from Ω into Y.

From Corollary 1 we deduce that a Carathéodory mapping is sup-measurable. Also, every $\Sigma \otimes \mathcal{B}(X)$-measurable mapping is sup-measurable (see Remark 2.5.26 of Denkowski et al. [3]).

Moreover, a mapping $f : \Omega \times X \to X$ is called random operator whenever, for any $x \in X$, $f(\cdot, x)$ is Σ-measurable. So, a random fixed point of f is a Σ-measurable mapping $z : \Omega \to X$ such that $z(\omega) = f(\omega, z(\omega))$ for all $\omega \in \Omega$.

Lemma 1. *Let X, Y be two locally compact metric spaces. A mapping $f : \Omega \times X \to Y$ is Carathéodory if and only if the mapping $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is Σ-measurable from Ω to $C(X, Y)$ (i.e., the space of all continuous functions from X into Y endowed with the compact-open topology).*

Let Ψ be the family of all nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n denote the nth iterate of ψ.

Lemma 2. *For every nondecreasing function $\psi : [0, +\infty) \to [0, +\infty)$, the following implication holds*:

$$\forall t > 0, \quad \lim_{n \to +\infty} \psi^n(t) = 0 \quad \Longrightarrow \quad \psi(t) < t.$$

Now, we have sufficient elements to generalize the concepts of α-ψ-contractive mapping and α-admissible mapping introduced in [11]. So, we give the following new definitions.

Definition 2. Let (Ω, Σ) be a measurable space, (X, d) be a metric space, and $T : \Omega \times X \to X$ be a given mapping. We say that T is a random α-ψ-contractive mapping if there exist functions $\alpha : \Omega \times X \times X \to [0, +\infty)$ and $\psi_\omega \in \Psi$, $\omega \in \Omega$, such that

$$d\big(T(\omega, u), T(\omega, v)\big) \leqslant \psi_\omega\big(d(u, v)\big) \tag{4}$$

for all $u, v \in X$ and $\omega \in \Omega$ such that $\alpha(\omega, u, v) \geqslant 1$.

Definition 3. Let $T : \varOmega \times X \to X$ and $\alpha : \varOmega \times X \times X \to [0, +\infty)$. We say that T is random α-admissible if

$$u, v \in X,\ \omega \in \varOmega, \quad \alpha(\omega, u, v) \geqslant 1 \quad \Longrightarrow \quad \alpha\bigl(\omega, T(\omega, u), T(\omega, v)\bigr) \geqslant 1.$$

Remark 1. From (4) we retrieve the random version of the Banach contraction condition whenever

(i) $\alpha(\omega, u, v) = 1$ for all $u, v \in X$ and $\omega \in \varOmega$;
(ii) $\psi_\omega(t) = k_\omega t$ for all $t \geqslant 0$ and some $0 \leqslant k_\omega < 1$.

To show the role of function $\alpha : \varOmega \times X \times X \to [0, +\infty)$, we give the following example (see also Examples 2.1 and 2.2 in [11]).

Example 1. In both the following cases, $T : \varOmega \times X \to X$ is a random α-admissible mapping.

(i) $\varOmega = \{\omega_1, \omega_2\}$, $X = (0, +\infty)$, $T(\omega_1, u) = \ln u$, and $T(\omega_2, u) = 1$ for all $u \in X$,

$$\alpha(\omega, u, v) = \begin{cases} 2 & \text{if } u \geqslant v, \\ 0 & \text{if } u < v \end{cases}$$

for all $\omega \in \varOmega$;

(ii) $\varOmega = \{\omega_1, \omega_2\}$, $X = [0, +\infty)$, $T(\omega_1, u) = \sqrt{u}$, and $T(\omega_2, u) = u$ for all $u \in X$,

$$\alpha(\omega, u, v) = \begin{cases} \mathrm{e}^{u-v} & \text{if } u \geqslant v, \\ 0 & \text{if } u < v \end{cases}$$

for all $\omega \in \varOmega$.

3 Random fixed point results

In this section, we prove the existence of a random fixed point for a given mapping. Let $(\varOmega, \varSigma)$ be a measurable space, (X, d) be a separable complete metric space, $T : \varOmega \times X \to X$ and $\alpha : \varOmega \times X \times X \to [0, +\infty)$. The hypotheses are the following:

(H1) T is a random α-admissible mapping;
(H2) There exists a measurable mapping $u_0 : \varOmega \to X$ such that, for all $\omega \in \varOmega$, $\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geqslant 1$;
(H3) T is a Carathéodory mapping;
(H4) T is a random α-ψ-contractive mapping;
(H5) If $\{u_n\}$ is a sequence in X such that $\alpha(\omega, u_n, u_{n+1}) \geqslant 1$ for some $\omega \in \varOmega$, all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u \in X$ as $n \to +\infty$, then $\alpha(\omega, u_n, u) \geqslant 1$ for all $n \in \mathbb{N} \cup \{0\}$.

Remark 2. Let $\varOmega$, X, T, and α be as in (ii) of Example 1. If $u_0 : \varOmega \to X$ is defined by $u_0(\omega_1) = u_0(\omega_2) = 1$, then hypothesis (H2) holds true for each $\varSigma$.

Remark 3. Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $X = \mathbb{R}$, and

$$\alpha(\omega, u, v) = \begin{cases} 1 & \text{if } u, v \in [0, 1], \\ 0 & \text{otherwise} \end{cases}$$

for all $\omega \in \Omega$. If $u_n : \Omega \to X$ is defined by $u_n(\omega_1) = u_n(\omega_3) = 0$ and $u_n(\omega_2) = u_n(\omega_4) = 1$ for all $n \in \mathbb{N} \cup \{0\}$, it follows trivially that hypothesis (H5) holds true for all $\omega \in \Omega$.

Theorem 2. *If hypotheses* (H1)–(H4) *hold, then T has a random fixed point, that is, there exists $\xi : \Omega \to X$ measurable such that $T(\omega, \xi(\omega)) = \xi(\omega)$ for all $\omega \in \Omega$.*

Proof. Hypothesis (H2) ensures that there exists a measurable mapping $u_0 : \Omega \to X$ such that $\alpha(\omega, u_0(\omega), T(\omega, u_0(\omega))) \geqslant 1$ for all $\omega \in \Omega$. Define the sequence $\{u_n(\omega)\}$ in X by

$$u_{n+1}(\omega) = T\big(\omega, u_n(\omega)\big) \quad \text{for all } n \in \mathbb{N} \cup \{0\},\ \omega \in \Omega.$$

If $u_n(\omega) = u_{n+1}(\omega)$ for some $n \in \mathbb{N} \cup \{0\}$ and for all $\omega \in \Omega$, then $\xi = u_n$ is a random fixed point of T. Assume that there exists some $\omega_n \in \Omega$ such that $u_n(\omega_n) \neq u_{n+1}(\omega_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is random α-admissible (hypothesis (H1), we have

$$\begin{aligned} &\alpha\big(\omega, u_0(\omega), u_1(\omega)\big) = \alpha\big(\omega, u_0(\omega), T\big(\omega, u_0(\omega)\big)\big) \geqslant 1 \\ &\quad \Longrightarrow \quad \alpha\big(\omega, T\big(\omega, u_0(\omega)\big), T\big(\omega, u_1(\omega)\big)\big) = \alpha\big(\omega, u_1(\omega), u_2(\omega)\big) \geqslant 1. \end{aligned}$$

Iterating this process, we get

$$\alpha\big(\omega, u_n(\omega), u_{n+1}(\omega)\big) \geqslant 1 \quad \text{for all } n \in \mathbb{N} \cup \{0\},\ \omega \in \Omega. \tag{5}$$

So, by (5) and hypothesis (H4), we deduce that the contractive condition (4) applies for $u = u_{n-1}(\omega)$ and $v = u_n(\omega)$. Thus, we have

$$\begin{aligned} d\big(u_n(\omega), u_{n+1}(\omega)\big) &= d\big(T\big(\omega, u_{n-1}(\omega)\big), T\big(\omega, u_n(\omega)\big)\big) \\ &\leqslant \psi_\omega\big(d\big(u_{n-1}(\omega), u_n(\omega)\big)\big). \end{aligned}$$

We have to iterate this process in order to obtain

$$d\big(u_n(\omega), u_{n+1}(\omega)\big) \leqslant \psi_\omega^n\big(d\big(u_0(\omega), u_1(\omega)\big)\big) \quad \text{for all } n \in \mathbb{N},\ \omega \in \Omega.$$

Fix $\varepsilon > 0$, and let $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geqslant n(\varepsilon)} \psi_\omega^n(d(u_0(\omega), u_1(\omega))) < \varepsilon$. Also, let $n, m \in \mathbb{N}$ with $m > n \geqslant n(\varepsilon)$. By repeated application of the triangle inequality, we get

$$\begin{aligned} d\big(u_n(\omega), u_m(\omega)\big) &\leqslant \sum_{k=n}^{m-1} d\big(u_k(\omega), u_{k+1}(\omega)\big) \\ &\leqslant \sum_{k=n}^{m-1} \psi_\omega^k\big(d\big(u_0(\omega), u_1(\omega)\big)\big) \\ &\leqslant \sum_{n \geqslant n(\varepsilon)} \psi_\omega^n\big(d\big(u_0(\omega), u_1(\omega)\big)\big) \\ &< \varepsilon. \end{aligned}$$

This argument shows that the sequence $\{u_n(\omega)\}$ is Cauchy in the metric space (X, d) for all $\omega \in \Omega$. Since (X, d) is complete, there exists $\xi : \Omega \to X$ such that $u_n(\omega) \to \xi(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. Since T is a Carathéodory mapping (hypothesis (H3)), it follows that u_n is measurable for all $n \in \mathbb{N}$ and that $u_{n+1}(\omega) = T(\omega, u_n(\omega)) \to T(\omega, \xi(\omega))$ as $n \to +\infty$ for all $\omega \in \Omega$. By the uniqueness of the limit, we get $\xi(\omega) = T(\omega, \xi(\omega))$, that is, ξ is a random fixed point of T. Note that ξ is a measurable mapping since it is a limit of a sequence of measurable mappings. □

Example 2. Let $X := \{u \in C([0,1], \mathbb{R})\colon \|u\|_\infty \leqslant \mu \text{ with } \mu > 0\}$, and let (Ω, Σ) be a measurable space, where $\Omega = [0,1]$ and Σ the σ-algebra of Borel. Let $\alpha(\omega, u, v) = 1$ for all $u, v \in X$ and $\omega \in \Omega$. Assume that $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function such that

$$\bigl|h(s,u) - h(s,v)\bigr| \leqslant \psi\bigl(|u-v|\bigr) \quad \text{for all } u, v \in \mathbb{R},\ s \in [0,1],\ \text{some } \psi \in \Psi.$$

Define $T : \Omega \times X \to X$ by

$$T(\omega, u)(t) = \int_0^t h\bigl(s, u(s)\bigr)\,\mathrm{d}s + u_0(\omega), \quad t \in [0,1],\ \omega \in \Omega,$$

where $u_0 : \Omega \to C([0,1], \mathbb{R})$ is Σ-measurable. Suppose that the following condition holds:

$$\int_0^1 \bigl|h(s,0)\bigr|\,\mathrm{d}s + \bigl|u_0(\omega)\bigr| \leqslant \mu - \psi(\mu) \quad \text{for all } \omega \in \Omega.$$

So, since $\psi(\|u\|_\infty) \leqslant \psi(\mu)$, we have

$$\begin{aligned}
\bigl|T(\omega, u)(t)\bigr| &\leqslant \int_0^t \bigl|h\bigl(s, u(s)\bigr)\bigr|\,\mathrm{d}s + \bigl|u_0(\omega)\bigr| \\
&\leqslant \int_0^t \bigl|h\bigl(s, u(s)\bigr) - h(s,0)\bigr|\,\mathrm{d}s + \int_0^t \bigl|h(s,0)\bigr|\,\mathrm{d}s + \bigl|u_0(\omega)\bigr| \\
&\leqslant \int_0^t \psi\bigl(\bigl|u(s)\bigr|\bigr)\,\mathrm{d}s + \mu - \psi(\mu) \\
&\leqslant \psi\bigl(\|u\|_\infty\bigr) + \mu - \psi(\mu) \\
&\leqslant \mu,
\end{aligned}$$

and hence, $T(\omega, \cdot) \in X$ for all $\omega \in \Omega$. Obviously, T is a random α-admissible and Carathéodory mapping. Now, we show that T is a random α-ψ-contractive mapping. So,

(X, d_∞) is a separable complete metric space, and we have

$$\begin{aligned}\bigl|T(\omega,u)(t)-T(\omega,v)(t)\bigr| &\leqslant \int_0^t \bigl|h\bigl(s,u(s)\bigr)-h\bigl(s,v(s)\bigr)\bigr|\,\mathrm{d}s\\ &\leqslant \int_0^t \psi\bigl(\bigl|u(s)-v(s)\bigr|\bigr)\,\mathrm{d}s\\ &\leqslant \psi\bigl(\|u-v\|_\infty\bigr).\end{aligned}$$

It follows that

$$\bigl\|T(\omega,u)-T(\omega,v)\bigr\|_\infty \leqslant \psi\bigl(\|u-v\|_\infty\bigr) \quad \text{for all } \omega\in\Omega,\ u,v\in X.$$

By Theorem 2, we deduce that there exists $\xi : \Omega \to X$ such that $\xi(\omega) = T(\omega, \xi(\omega))$, that is, ξ is a random fixed point of T on X. In this case, also the random fixed point is unique (see hypothesis (H6) below).

In the next theorem, we replace hypothesis (H3) (T is Carathéodory) by hypothesis (H5), which is a regularity condition on the metric space (X, d).

Theorem 3. *If hypotheses* (H1), (H2), (H4), (H5) *hold and T is sup-measurable, then T has a random fixed point.*

Proof. A similar reasoning as in the proof of Theorem 2 gives us that the sequence $\{u_n(\omega)\}$ is Cauchy in the complete metric space (X, d) for all $\omega \in \Omega$. This means that there exists $\xi : \Omega \to X$ such that $u_n(\omega) \to \xi(\omega)$ as $n \to +\infty$ for all $\omega \in \Omega$. On the other hand, from (5) and hypothesis (H5), we have

$$\alpha\bigl(\omega, u_n(\omega), \xi(\omega)\bigr) \geqslant 1 \quad \text{for all } n \in \mathbb{N}\cup\{0\},\ \omega\in\Omega. \tag{6}$$

Now, using the triangle inequality, (4), and (6), we get

$$\begin{aligned}d\bigl(T\bigl(\omega,\xi(\omega)\bigr),\xi(\omega)\bigr) &\leqslant d\bigl(T\bigl(\omega,\xi(\omega)\bigr),T\bigl(\omega,u_n(\omega)\bigr)\bigr)+d\bigl(u_{n+1}(\omega),\xi(\omega)\bigr)\\ &\leqslant \psi_\omega\bigl(d\bigl(u_n(\omega),\xi(\omega)\bigr)\bigr)+d\bigl(u_{n+1}(\omega),\xi(\omega)\bigr).\end{aligned}$$

Taking the limit as $n \to +\infty$ and since ψ_ω is continuous at $t = 0$, we have $d(T(\omega, \xi(\omega)), \xi(\omega)) = 0$, that is, $T(\omega, \xi(\omega)) = \xi(\omega)$ for all $\omega \in \Omega$. The hypothesis that T is sup-measurable implies that u_n is measurable for all $n \in \mathbb{N}$ and hence ξ is measurable. Thus, ξ is a random fixed point of T. □

We have discussed the existence of a random fixed point for a given mapping under suitable hypotheses. Next, the question of uniqueness also applies to our study. So, to obtain a unique random fixed point, we consider the following hypothesis:

(H6) For all $u, v \in X$ and $\omega \in \Omega$, there exists $z(\omega) \in X$ such that $\alpha(\omega, u, z(\omega)) \geqslant 1$ and $\alpha(\omega, v, z(\omega)) \geqslant 1$.

Theorem 4. *Adding hypothesis* (H6) *to the ones in the statement of Theorem 2* (*resp. Theorem* 3)*, we obtain uniqueness of the random fixed point of* T.

Proof. Arguing by contradiction, we assume that there exist ξ^* and ξ_*, with $\xi^* \neq \xi_*$, such that $T(\omega, \xi^*(\omega)) = \xi^*(\omega)$ and $T(\omega, \xi_*(\omega)) = \xi_*(\omega)$ for all $\omega \in \Omega$. By hypothesis (H6), there exists a mapping $z_0 : \Omega \to X$ such that

$$\alpha\big(\omega, \xi^*(\omega), z_0(\omega)\big) \geqslant 1 \quad \text{and} \quad \alpha\big(\omega, \xi_*(\omega), z_0(\omega)\big) \geqslant 1, \quad \omega \in \Omega. \tag{7}$$

Now, let $z_n(\omega) = T(\omega, z_{n-1}(\omega))$ for all $n \in \mathbb{N}$, all $\omega \in \Omega$. From (7), using hypothesis (H1), we get

$$\alpha\big(\omega, \xi^*(\omega), z_n(\omega)\big) \geqslant 1 \quad \text{and} \quad \alpha\big(\omega, \xi_*(\omega), z_n(\omega)\big) \geqslant 1 \tag{8}$$

for all $n \in \mathbb{N}$, $\omega \in \Omega$. Combining (8) with the contractive condition (4), we have

$$\begin{aligned} d\big(\xi^*(\omega), z_n(\omega)\big) &= d\big(T\big(\omega, \xi^*(\omega)\big), T\big(\omega, z_{n-1}(\omega)\big)\big) \\ &\leqslant \psi_\omega\big(d(\xi^*(\omega), z_{n-1}(\omega))\big). \end{aligned}$$

It follows that

$$\begin{aligned} &d\big(\xi^*(\omega), z_n(\omega)\big) \leqslant \psi_\omega^n\big(d\big(\xi^*(\omega), z_0(\omega)\big)\big) \quad \text{for all } n \in \mathbb{N},\ \omega \in \Omega \\ &\implies \quad z_n(\omega) \to \xi^*(\omega) \quad \text{as } n \to +\infty \text{ for all } \omega \in \Omega. \end{aligned} \tag{9}$$

Similarly, changing ξ^* with ξ_* and repeating the above reasoning, we conclude that

$$z_n(\omega) \to \xi_*(\omega) \quad \text{as } n \to +\infty. \tag{10}$$

From (9) and (10) the uniqueness of the limit gives us $\xi^* = \xi_*$, which is a contradiction. This shows that T has a unique random fixed point. □

An interesting feature of the function α is the fact that it is suitable to obtain easily the ordered counterparts of many fixed-point theorems without requiring any change to the proofs of previous theorems. We recall that the study of fixed points in partially ordered spaces has been considered in Ran and Reurings [10] and further investigated in a lot of papers (see, for example, Nieto and Rodríguez-López [7, 8], Vetro [15], and references therein). The ordered approach is significant and largely motivated by nice applications to matrix equations (see [10]) and boundary value problems (see [7, 8]).

So, our results can be immediately read in an ordered context by defining $\alpha : \Omega \times X \times X \to [0, +\infty)$ as

$$\alpha(\omega, u, v) = \begin{cases} 1 & \text{if } u \preccurlyeq v, \\ 0 & \text{otherwise} \end{cases}$$

for all $\omega \in \Omega$, where $\preccurlyeq$ denote an order relation on the set X (two elements $u, v \in X$ are called comparable if $u \preccurlyeq v$ or $v \preccurlyeq u$). In this case, hypotheses (H1)–(H6) reduce to

- (O1) T is a nondecreasing mapping w.r.t. $\preccurlyeq$;
- (O2) There exists a measurable mapping $u_0 : \Omega \to X$ such that $u_0(\omega) \preccurlyeq T(\omega, u_0(\omega))$ for all $\omega \in \Omega$;

(O3) T is a Carathéodory mapping;

(O4) There exists function $\psi_\omega \in \Psi$, $\omega \in \Omega$ such that $d(T(\omega,u),T(\omega,v)) \leqslant \psi_\omega(d(u,v))$ for all $u,v \in X$ such that $u \preccurlyeq v$;

(O5) If $\{u_n\}$ is a sequence in X such that $u_n \preccurlyeq u_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$ and $u_n \to u \in X$ as $n \to +\infty$, then $u_n \preccurlyeq u$ for all $n \in \mathbb{N} \cup \{0\}$;

(O6) For all $u,v \in X$, there exists $z \in X$ such that $u \preccurlyeq z$ and $v \preccurlyeq z$.

By using hypotheses (O1)–(O6), our Theorems 2–4 are generalized random versions of well-known theorems in Ran and Reurings [10] and Nieto and Rodríguez-López [7] (see also Samet et al. [11]).

4 Solution of system (1)

In this section, we prove a theorem producing the existence of a unique random solution of system (1); see also Nieto et al. [6] and Samet et al. [11]. Let (Ω,Σ) be a measurable space. Let $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which means that $\omega \mapsto f(\omega,t,u)$ is measurable for all $(t,u) \in [0,1] \times \mathbb{R}$ and $(t,u) \mapsto f(\omega,t,u)$ is continuous for all $\omega \in \Omega$.

Then consider the integral operator $F : \Omega \times C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$F(\omega,u)(t) = \int_0^1 G(t,s) f\bigl(\omega,s,u(s)\bigr)\,\mathrm{d}s + g\bigl(\omega,t,u(t)\bigr) \tag{11}$$

for all $u \in C([0,1],\mathbb{R})$ and $\omega \in \Omega$, where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and $g : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.

Remark 4. F is a random operator from $\Omega \times C([0,1],\mathbb{R})$ into $C([0,1],\mathbb{R})$. In fact, given $u \in C([0,1],\mathbb{R})$, since f is a Carathéodory function for $s \in [0,1]$ fixed, the function $h : \Omega \times [0,1] \to \mathbb{R}$, defined by $h(\omega,t) = G(t,s)f(\omega,s,u(s))$, is Carathéodory. By Lemma 1, the integral in (11) is limit of a finite sum of measurable functions. So, the mapping $\omega \to F(\omega,u)$ is measurable, and hence F is a random operator.

Remark 5. Let $h : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, $u \in C([0,1],\mathbb{R})$, and let $\{u_n\} \subset C([0,1],\mathbb{R})$ be a sequence convergent to u. Then there exists an interval $[a,b] \subset \mathbb{R}$ such that $u_n(s), u(s) \in [a,b]$ for all $s \in [0,1]$. The continuity of the function $h(\omega,\cdot,\cdot)$ in $[0,1] \times \mathbb{R}$ for fixed $\omega \in \Omega$ ensures that the function $h(\omega,\cdot,\cdot)$ is uniformly continuous in $[0,1] \times [a,b]$.

The hypotheses are the following:

(F1) For each $\omega \in \Omega$, there exist a function $\beta : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and a nondecreasing function $r_\omega : \mathbb{R} \to \mathbb{R}$ such that, for all $t \in [0,1]$, for all $a,b \in \mathbb{R}$ with $\beta(\omega,a,b) \geqslant 0$, we have

$$\bigl|f(\omega,t,a) - f(\omega,t,b)\bigr| \leqslant \lambda r_\omega\bigl(|a-b|\bigr),$$

where $0 < \lambda^{-1} = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|\,\mathrm{d}s$;

(F2) There exists a measurable mapping $u_0 : \Omega \to C([0,1],\mathbb{R})$ such that, for all $\omega \in \Omega$, we have

$$\beta\big(\omega, u_0(\omega)(t), F\big(\omega, u_0(\omega)\big)(t)\big) \geqslant 0 \quad \text{for all } t \in [0,1];$$

(F3) For each $\omega \in \Omega$ and all $t \in [0,1]$, $u, v \in C([0,1],\mathbb{R})$, we have

$$\beta\big(\omega, u(t), v(t)\big) \geqslant 0 \quad \Longrightarrow \quad \beta\big(\omega, F(\omega,u)(t), F(\omega,v)(t)\big) \geqslant 0;$$

(F4) For each $\omega \in \Omega$, there exists a function $\psi_\omega \in \Psi$ such that

$$r_\omega\big(\|u-v\|_\infty\big) + \big|g\big(\omega,t,u(t)\big) - g\big(\omega,t,v(t)\big)\big| \leqslant \psi_\omega\big(\|u-v\|_\infty\big)$$

for each $t \in [0,1]$ and all $u, v \in C([0,1],\mathbb{R})$ with $\beta(\omega, u(t), v(t)) \geqslant 0$ for all $t \in [0,1]$.

Now, we can have the main theorem of this section.

Theorem 5. *If hypotheses* (F1)–(F4) *hold, then the random integral operator F has a random fixed point.*

Proof. For fixed $\omega \in \Omega$, we show that $F(\omega,\cdot)$ is continuous. Indeed, consider a sequence $\{u_n\}$ in $C([0,1],\mathbb{R})$ with $u_n \to u \in C([0,1],\mathbb{R})$ as $n \to +\infty$. By Remark 5, there exists $[a,b] \subset \mathbb{R}$ such that $u_n(s), u(s) \in [a,b]$ for all $s \in [0,1]$. In addition, the functions $f(\omega,\cdot,\cdot)$ and $g(\omega,\cdot,\cdot)$ are uniformly continuous in $[0,1] \times [a,b]$. Thus, for fixed $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
\big|f(\omega,s_1,u_1) - f(\omega,s_2,u_2)\big| &< \lambda\varepsilon,\\
\big|g(\omega,s_1,u_1) - g(\omega,s_2,u_2)\big| &< \varepsilon
\end{aligned}$$

for all $s_1, s_2 \in [0,1]$ and $u_1, u_2 \in [a,b]$ such that $|s_1 - s_2| + |u_1 - u_2| < \delta$.

Now, let $n(\delta) \in \mathbb{N}$ such that $\|u_n - u\|_\infty < \delta$ whenever $n \geqslant n(\delta)$. Then, for every $n \geqslant n(\delta)$, we have

$$\begin{aligned}
\big|f\big(\omega,s,u_n(s)\big) - f\big(\omega,s,u(s)\big)\big| &< \lambda\varepsilon,\\
\big|g\big(\omega,s,u_n(s)\big) - g\big(\omega,s,u(s)\big)\big| &< \varepsilon.
\end{aligned}$$

Consequently, for $t \in [0,1]$ and $n \geqslant n(\delta)$, we have

$$\begin{aligned}
\big|F(\omega,u_n)(t) - F(\omega,u)(t)\big| &\leqslant \int_0^1 \big|G(t,s)\big|\big|f\big(\omega,s,u_n(s)\big) - f\big(\omega,s,u(s)\big)\big|\,\mathrm{d}s\\
&\quad + \big|g\big(\omega,t,u_n(t)\big) - g\big(\omega,t,u(t)\big)\big|\\
&\leqslant 2\varepsilon \quad \Bigg(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\,\mathrm{d}s = \lambda^{-1}\Bigg)\\
\Longrightarrow \quad \big\|F(\omega,u_n) - F(\omega,u)\big\|_\infty &\leqslant 2\varepsilon.
\end{aligned}$$

So,

$$\begin{aligned}&d_\infty\big(F(\omega, u_n), F(\omega, u)\big) \to 0 \quad \text{as } n \to +\infty\\ &\quad\Longrightarrow \quad F(\omega, \cdot) \text{ is a continuous operator for each fixed } \omega \in \Omega.\end{aligned}$$

Thus, by Remark 4, $F : \Omega \times C([0,1], \mathbb{R}) \to C([0,1], \mathbb{R})$ is a Carathéodory mapping.

Next step is to show that the integral operator F satisfies a random α-ψ-contractive type condition as in (H4). So, for each $\omega \in \Omega$ and all $u, v \in C([0,1], \mathbb{R})$ such that $\beta(\omega, u(t), v(t)) \geqslant 0$ for all $t \in [0,1]$, we prove that

$$d_\infty\big(F(\omega, u), F(\omega, v)\big) \leqslant \psi_\omega\big(d(u, v)\big).$$

Indeed, let $\omega \in \Omega$ be fixed, and $u, v \in C([0,1], \mathbb{R})$ be such that $\beta(\omega, u(t), v(t)) \geqslant 0$ for all $t \in [0,1]$, then

$$\begin{aligned}&\big|F(\omega, u)(t) - F(\omega, v)(t)\big|\\ &\quad\leqslant \int_0^1 \big|G(t,s)\big|\big|f\big(\omega, s, u(s)\big) - f\big(\omega, s, v(s)\big)\big|\,\mathrm{d}s + \big|g\big(\omega, t, u(t)\big) - g\big(\omega, t, v(t)\big)\big|\\ &\quad\leqslant \lambda \int_0^1 \big|G(t,s)\big| r_\omega\big(\big|u(s) - v(s)\big|\big)\,\mathrm{d}s + \big|g\big(\omega, t, u(t)\big) - g\big(\omega, t, v(t)\big)\big| \quad \big(\text{by (F1)}\big)\\ &\quad\leqslant r_\omega\big(\|u - v\|_\infty\big) + \big|g\big(\omega, t, u(t)\big) - g\big(\omega, t, v(t)\big)\big| \quad \big(\text{by (F4)}\big)\\ &\Longrightarrow \quad \big\|F(\omega, u) - F(\omega, v)\big\|_\infty \leqslant \psi_\omega\big(\|u - v\|_\infty\big).\end{aligned}$$

Let $\alpha : \Omega \times C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R}) \to [0, +\infty)$ be a function given as

$$\alpha(\omega, u, v) = \begin{cases} 1 & \text{if } \beta(\omega, u(t), v(t)) \geqslant 0 \text{ for all } t \in [0,1],\\ 0 & \text{otherwise}\end{cases}$$

for all $\omega \in \Omega$. So, for all $u, v \in C([0,1], \mathbb{R})$ with $\alpha(\omega, u, v) \geqslant 1$, we get

$$\big\|F(\omega, u) - F(\omega, v)\big\|_\infty \leqslant \psi_\omega\big(\|u - v\|_\infty\big),$$

which means that F is a random α-ψ-contractive integral operator.

Note that, for each $\omega \in \Omega$ and all $t \in [0,1]$, $u, v \in C([0,1], \mathbb{R})$, we have

$$\begin{aligned}&\alpha(\omega, u, v) \geqslant 1\\ &\quad\Longrightarrow \quad \beta\big(\omega, u(t), v(t)\big) \geqslant 0 \quad \text{for all } t \in [0,1]\\ &\quad\Longrightarrow \quad \beta\big(\omega, F(\omega, u)(t), F(\omega, v)(t)\big) \geqslant 0 \quad \text{for all } t \in [0,1] \quad \big(\text{by (F3)}\big)\\ &\quad\Longrightarrow \quad \alpha\big(\omega, F(\omega, u), F(\omega, v)\big) \geqslant 1,\end{aligned}$$

which means that F is a random α-admissible integral operator. Moreover, hypothesis (F2) ensures that there exists a measurable mapping $u_0 : \Omega \to C([0,1], \mathbb{R})$ such that $\alpha(\omega, u_0(\omega), F(\omega, u_0(\omega))) \geqslant 1$ for all $\omega \in \Omega$. We conclude that the integral operator F satisfies all the hypotheses of Theorem 2 and so admits a fixed point. □

Now, we can have the theorem producing a random solution of problem (1).

Theorem 6. *If hypotheses* (F1)–(F4) *hold with* $g : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ *being the null function and* $\lambda = 8$*, then problem* (1) *has at least one random solution.*

Proof. The assumption that $g : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is the null function reduces the random integral operator (11) to the following:

$$\widetilde{F}(\omega, u)(t) = \int_0^1 G(t,s) f\bigl(\omega, s, u(s)\bigr) \,\mathrm{d}s.$$

Now, if $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by (3), then, for all $t \in [0,1]$, we have

$$\int_0^1 G(t,s)\,\mathrm{d}s = -\frac{t^2}{2} + \frac{t}{2} \quad \Longrightarrow \quad \sup_{t\in[0,1]} \int_0^1 G(t,s)\,\mathrm{d}s = \frac{1}{8}.$$

Note that the random fixed points of $\widetilde{F}$ are solutions to (1) and conversely. So, given a random variable $u : \Omega \to C([0,1],\mathbb{R})$, we have that

$$\widetilde{F}\bigl(\omega, u(\omega)\bigr) = u(\omega) \quad \text{for all } \omega \in \Omega,$$

is equivalent to

$$u(\omega)(t) = \int_0^1 G(t,s) f\bigl(\omega, s, u(\omega)(s)\bigr)\,\mathrm{d}s, \quad t \in (0,1).$$

This means that the corresponding solution of (1) is given by $u(\omega,t) = u(\omega)(t)$ for $t \in [0,1]$ and $\omega \in \Omega$. By an application of Theorem 5, we deduce that problem (1) admits a random solution. □

Example 3. Let (Ω, Σ) be a measurable space, where $\Omega = [0,+\infty)$ and Σ is the σ-algebra of Borel on $[0,+\infty)$. Consider the two-point boundary value problem

$$\begin{aligned} &-\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}(\omega,t) = \frac{1}{7\mathrm{e}^{\omega^2 t+1}(1+|u(\omega,t)|)}, \quad t \in [0,1], \\ &u(\omega,0) = u(\omega,1) = 0 \end{aligned} \tag{12}$$

for all $u \in C([0,1],\mathbb{R})$ and $\omega \in \Omega$. Solving this problem is equivalent to finding a random fixed point of the integral operator $F : \Omega \times C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ defined by

$$F(\omega,u)(t) = \int_0^1 \frac{G(t,s)}{7\mathrm{e}^{\omega^2 t+1}(1+|u(t)|)}\,\mathrm{d}s, \tag{13}$$

where $G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the Green's function in (3).

Clearly $f : \Omega \times [0,1] \times \mathbb{R} \to \mathbb{R}$, given by $f(\omega,t,u) = 1/(7\mathrm{e}^{\omega^2 t+1}(1+|u|))$, is a Carathéodory function. Hypotheses (F2) and (F3) hold true by defining $\beta : \Omega\times\mathbb{R}\times\mathbb{R} \to [0,+\infty)$ as $\beta(\omega,a,b) = 1$ for all $\omega \in \Omega$, all $a,b \in \mathbb{R}$. Consequently, hypotheses (F1) and (F4) are satisfied with $r_\omega(t) = \psi_\omega(t) = t/(7\mathrm{e})$. So, by Theorem 5, the integral operator (13) has a random fixed point. On the other hand, by Theorem 6, the two-point boundary value problem (12) has at least one random solution.

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