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Nonlocal initial value problems for implicit differential equations with Hilfer–Hadamard fractional derivative

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Abstract. In this paper, the Schaefer's fixed-point theorem is used to investigate the existence of solutions to nonlocal initial value problems for implicit differential equations with Hilfer–Hadamard fractional derivative. Then the Ulam stability result is obtained by using Banach contraction principle. An example is given to illustrate the applications of the main result.

Keywords: Hilfer–Hadamard fractional derivative, implicit differential equations, fixed point, generalized Ulam–Hyers stability.

1 Introduction

Fractional differential equations (FDEs) have been applied in many fields such as physics, mechanics, chemistry, engineering etc. There has been a significant development in ordinary differential equations involving fractional-order derivatives, one can see the monographs of Hilfer [19], Kilbas [16] and Podlubny [18] and the references therein. Moreover, Hilfer [19] studied applications of a generalized fractional operator having the Riemann–Liouville and the Caputo derivatives as specific cases. Hilfer fractional derivative has been receiving more and more attention in recent times; see, for example, [8–11, 14, 21, 24]. Benchohra et al. [4, 5] studied implicit differential equations(IDEs) of fractional order in various aspects. Recently, some mathematicians have considered FDEs depending on the Hadamard fractional derivative [2, 6, 7].

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In this paper, we consider the Hilfer–Hadamard-type IDE with nonlocal condition of the form

$$\begin{aligned} &{}_HD_{1^+}^{\alpha,\beta}x(t) = f\bigl(t, x(t), {}_HD_{1^+}^{\alpha,\beta}x(t)\bigr), \quad 0 < \alpha < 1,\ 0 \leqslant \beta \leqslant 1, \\ &\quad t \in J := [1, b], \\ &{}_HI_{1^+}^{1-\gamma}x(1) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \alpha \leqslant \gamma = \alpha + \beta - \alpha\beta < 1,\ \tau_i \in [1, b], \end{aligned} \tag{1}$$

where ${}_HD_{1^+}^{\alpha,\beta}$ is the Hilfer–Hadamard fractional derivative of order α and type β. Let X be a Banach space, $f : J \times X \times X \to X$ is a given continuous function and ${}_HI_{1^+}^{1-\gamma}$ is the left-sided mixed Hadamard integral of order $1 - \gamma$.

In passing, we remark that the application of nonlocal condition ${}_HI_{1^+}^{1-\gamma}x(1) = \sum_{i=1}^{m} c_i x(\tau_i)$ in physical problems yields better effect than the initial condition ${}_HI_{1^+}^{1-\gamma}x(1) = x_0$.

For sake of brevity, let us take

$${}_HD_{1^+}^{\alpha,\beta}x(t) := K_x(t) = f\bigl(t, x(t), K_x(t)\bigr).$$

A new and important equivalent mixed-type integral equation for our system (1) can be established. We adopt some ideas in [24] to establish an equivalent mixed-type integral equation

$$\begin{aligned} x(t) = {}& \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s} \\ & + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}, \end{aligned} \tag{2}$$

where

$$Z := \frac{1}{\Gamma(\gamma) - \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1}} \quad \text{if } \Gamma(\gamma) \neq \sum_{i=1}^{m} c_i (\log \tau_i)^{\gamma-1}.$$

In the theory of functional equations, there is some special kind of data dependence [3, 13, 17, 20]. For the advanced contribution on Ulam stability for FDEs, we refer the reader to [12, 23, 25, 26]. In this paper, we study different types of Ulam stability: Ulam–Hyers stability, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias stability and generalized Ulam–Hyers–Rassias stability for the IDEs with Hilfer–Hadamard fractional derivative. Moreover, the Ulam–Hyers stability for FDEs with Hilfer fractional derivative was investigated in [1, 22].

The paper is organized as follows. A brief review of the fractional calculus theory is given in Section 2. In Section 3, we will prove the existence and uniqueness of solutions for problem (1). In Section 4, we discuss the Ulam–Hyers stability results. Finally, an example is given in Section 5 to illustrate the usefulness of our main results.

2 Fundamental concepts

In this section, we introduce some definitions and preliminary facts, which are used in this paper.

Let $C[J,X]$ be the Banach space of all continuous functions from J into X with the norm

$$\|x\|_C = \max\{|x(t)|: t \in [1,b]\}.$$

For $0 \leqslant \gamma < 1$, we denote the space $C_{\gamma,\log}[J,X]$ as

$$C_{\gamma,\log}[J,X] := \{f(t) : [1,b] \to X \mid (\log t)^{\gamma} f(t) \in C[J,X]\},$$

where $C_{\gamma,\log}[J,X]$ is the weighted space of the continuous functions f on the finite interval $[1,b]$.

Obviously, $C_{\gamma,\log}[J,X]$ is the Banach space with the norm

$$\|f\|_{C_{\gamma,\log}} = \|(\log t)^{\gamma} f(t)\|_C.$$

Meanwhile, $C^n_{\gamma,\log}[J,X] := \{f \in C^{n-1}[J,X]: f^{(n)} \in C_{\gamma,\log}[J,X]\}$ is the Banach space with the norm

$$\|f\|_{C^n_{\gamma,\log}} = \sum_{i=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_{C_{\gamma,\log}}, \quad n \in \mathbb{N}.$$

Moreover, $C^0_{\gamma,\log}[J,X] := C_{\gamma,\log}[J,X]$.

Definition 1. (See [26].) The Hadamard fractional integral of order α for a continuous function f is defined as

$${}_H I^{\alpha}_{1^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s) \frac{\mathrm{d}s}{s}, \quad \alpha > 0,$$

provided the integral exists.

Definition 2. (See [26].) The Hadamard derivative of fractional order α for a continuous function $f : [1,\infty) \to X$ is defined as

$${}_H D^{\alpha}_{1^+} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_1^t \left(\log \frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{\mathrm{d}s}{s}, \quad n-1 < \alpha < n,$$

where $n = \lceil \alpha \rceil + 1$, $\lceil \alpha \rceil$ denotes the integer part of real number α, and $\log(\cdot) = \log_{\mathrm{e}}(\cdot)$.

We review some basic properties of Hilfer–Hadamard fractional derivative, which are used for this work. For details, see [8, 9, 11, 19, 24] and references therein.

Definition 3. (See [15].) The Hilfer–Hadamard fractional derivative of order $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$ of function $f(t)$ is defined by

$$ {}_H D_{1^+}^{\alpha,\beta} f(t) = \bigl({}_H I_{1^+}^{\beta(1-\alpha)} D\bigl({}_H I_{1^+}^{(1-\beta)(1-\alpha)} f\bigr)\bigr)(t), $$

where $D := \mathrm{d}/\mathrm{d}t$.

Remark 1. Clearly,

(i) The operator ${}_H D_{1^+}^{\alpha,\beta}$ also can be rewritten as

$$ {}_H D_{1^+}^{\alpha,\beta} = {}_H I_{1^+}^{\beta(1-\alpha)} D\, {}_H I_{1^+}^{(1-\beta)(1-\alpha)} = {}_H I_{1^+}^{\beta(1-\alpha)}\, {}_H D_{1^+}^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta. $$

(ii) Let $\beta = 0$, the Hadamard-type Riemann–Liouville fractional derivative can be presented as ${}_H D_{1^+}^{\alpha} := {}_H D_{1^+}^{\alpha,0}$.

(iii) Let $\beta = 1$, the Hadamard-type Caputo fractional derivative can be presented as ${}^c{}_H D_{1^+}^{\alpha} := {}_H I_{1^+}^{1-\alpha} D$.

Lemma 1. (*See* [15].) *If* $\alpha, \beta > 0$ *and* $1 < b < \infty$*, then*

$$ \bigl[{}_H I_{1^+}^{\alpha} (\log s)^{\beta-1}\bigr](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\log t)^{\beta+\alpha-1} $$

and

$$ \bigl[{}_H D_{1^+}^{\alpha} (\log s)^{\alpha-1}\bigr](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\log t)^{\beta-\alpha-1}. $$

In particular, if $\beta = 1$ *and* $\alpha \geqslant 0$*, then the Hadamard fractional derivative of a constant is not equal to zero:*

$$ \bigl({}_H D_{1^+}^{\alpha} 1\bigr)(t) = \frac{1}{\Gamma(1-\alpha)} (\log t)^{-\alpha}, \quad 0 < \alpha < 1. $$

Lemma 2. *If* $\alpha, \beta > 0$ *and* $f \in L^1(J)$ *for* $t \in [1, T]$ *there exist the following properties*:

$$ \bigl({}_H I_{1^+}^{\alpha}\, {}_H I_{1^+}^{\beta} f\bigr)(t) = \bigl({}_H I_{1^+}^{\alpha+\beta} f\bigr)(t) $$

and

$$ \bigl({}_H D_{1^+}^{\alpha}\, {}_H I_{1^+}^{\alpha} f\bigr)(t) = f(t). $$

In particular, if $f \in C_{\gamma,\log}[J, X]$ *or* $f \in C[J, X]$*, then these equalities hold at* $t \in (1, b]$ *or* $t \in [1, b]$*, respectively.*

Lemma 3. *Let* $0 < \alpha < 1$*,* $0 \leqslant \gamma < 1$*. If* $f \in C_{\gamma,\log}[J, X]$ *and* ${}_H I_{1^+}^{1-\alpha} f \in C_{\gamma,\log}^1[J, X]$*, then*

$$ {}_H I_{1^+}^{\alpha}\, {}_H D_{1^+}^{\alpha} f(t) = f(t) - \frac{({}_H I_{1^+}^{1-\alpha} f)(1)}{\Gamma(\alpha)} (\log t)^{\alpha-1} \quad \forall t \in (1, b]. $$

Lemma 4. *If* $0 \leqslant \gamma < 1$ *and* $f \in C_{\gamma,\log}[J, X]$*, then*

$$ \bigl({}_H I_{1^+}^{\alpha} f\bigr)(1) := \lim_{t \to 1^+} {}_H I_{1^+}^{\alpha} f(t) = 0, \quad 0 \leqslant \gamma < \alpha. $$

Lemma 5. *Let* $\alpha, \beta > 0$ *and* $\gamma = \alpha + \beta - \alpha\beta$. *If* $f \in C^{\gamma}_{1-\gamma,\log}[J, X]$, *then*

$$ {}_H I^{\gamma}_{1^+}\, {}_H D^{\gamma}_{1^+} h = {}_H I^{\alpha}_{1^+}\, {}_H D^{\alpha,\beta}_{1^+} f, \qquad {}_H D^{\gamma}_{1^+}\, {}_H I^{\alpha}_{1^+} f = {}_H D^{\beta(1-\alpha)}_{1^+} f(t). $$

Lemma 6. *Let* $f \in L^1(J)$ *and* ${}_H D^{\beta(1-\alpha)}_{1^+} f \in L^1(J)$ *exist, then*

$$ {}_H D^{\alpha,\beta}_{1^+}\, {}_H I^{\alpha}_{1^+} f = {}_H I^{\beta(1-\alpha)}_{1^+}\, {}_H D^{\beta(1-\alpha)}_{1^+} f. $$

3 Existence results

In this section, we introduce spaces that helps us to solve and reduce system (1) to an equivalent integral equation (2):

$$ C^{\alpha,\beta}_{1-\gamma,\log} = \bigl\{ f \in C_{1-\gamma,\log}[J, X],\ {}_H D^{\alpha,\beta}_{1^+} f \in C_{1-\gamma,\log}[J, X] \bigr\} $$

and

$$ C^{\gamma}_{1-\gamma,\log} = \bigl\{ f \in C_{1-\gamma,\log}[J, X],\ {}_H D^{\gamma}_{1^+} f \in C_{1-\gamma,\log}[J, X] \bigr\}. $$

It is obvious that

$$ C^{\gamma}_{1-\gamma,\log}[J, X] \subset C^{\alpha,\beta}_{1-\gamma,\log}[J, X]. $$

Lemma 7. *Let* $f : J \times X \times X \to X$ *be a function such that* $f(\cdot, x(\cdot), {}_H D^{\alpha,\beta}_{1^+} x(\cdot)) \in C_{1-\gamma,\log}[J, X]$ *for any* $x \in C_{1-\gamma,\log}[J, X]$. *A function* $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ *is a solution of Hilfer-type fractional IDE*:

$$ \begin{aligned} &{}_H D^{\alpha,\beta}_{1^+} x(t) = f\bigl(t, x(t), {}_H D^{\alpha,\beta}_{1^+} x(t)\bigr), \quad 0 < \alpha < 1,\ 0 \leqslant \beta \leqslant 1,\ t \in J, \\ &{}_H I^{1-\gamma}_{1^+} x(1) = x_0, \quad \gamma = \alpha + \beta - \alpha\beta, \end{aligned} $$

if and only if x *satisfies the following Volterra integral equation*:

$$ x(t) = \frac{x_0 (\log t)^{\gamma-1}}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} f\bigl(s, x(s), {}_H D^{\alpha,\beta}_{1^+} x(s)\bigr) \frac{\mathrm{d}s}{s}. $$

Further details can be found in [15]. From Lemma 7 we have the following result.

Lemma 8. *Let* $f : J \times X \times X \to X$ *be a function such that* $f(\cdot, x(\cdot), {}_H D^{\alpha,\beta}_{1^+} x(\cdot)) \in C_{1-\gamma,\log}[J, X]$ *for any* $x \in C_{1-\gamma,\log}[J, X]$. *A function* $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ *is a solution of system* (1) *if and only if* x *satisfies the mixed-type integral* (2).

Proof. According to Lemma 7, a solution of system (1) can be expressed by

$$ x(t) = \frac{{}_H I^{1-\gamma}_{1^+} x(1)}{\Gamma(\gamma)} (\log t)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}. \tag{3} $$

Next, we substitute $t = \tau_i$ into the above equation:

$$x(\tau_i) = \frac{{}_H I_{1^+}^{1-\gamma} x(1)}{\Gamma(\gamma)} (\log \tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}, \tag{4}$$

multiplying both sides of (4) by c_i, we can write

$$c_i x(\tau_i) = \frac{{}_H I_{1^+}^{1-\gamma} x(1)}{\Gamma(\gamma)} c_i (\log \tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}.$$

Thus, we have

$$\begin{aligned} {}_H I_{1^+}^{1-\gamma} x(1) &= \sum_{i=1}^m c_i x(\tau_i) \\ &= \frac{{}_H I_{1^+}^{1-\gamma} x(1)}{\Gamma(\gamma)} \sum_{i=1}^m c_i (\log \tau_i)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}, \end{aligned}$$

which implies

$${}_H I_{1^+}^{1-\gamma} x(1) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}. \tag{5}$$

Submitting (5) to (3), we derive that (2). It is probative that x is also a solution of the integral equation (2) when x is a solution of (1).

The necessity has been already proved. Next, we are ready to prove its sufficiency. Applying ${}_H I_{1^+}^{1-\gamma}$ to both sides of (2), we have

$$\begin{aligned} {}_H I_{1^+}^{1-\gamma} x(t) = {}&{}_H I_{1^+}^{1-\gamma} (\log t)^{\gamma-1} \frac{Z}{\Gamma(\alpha)} \sum_{i=1}^m c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s} \\ &+ {}_H I_{1^+}^{1-\gamma} \, {}_H I_{1^+}^{\alpha} K_x(t), \end{aligned}$$

using Lemmas 1 and 2,

$${}_H I_{1^+}^{1-\gamma} x(t) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s} + I_{1^+}^{1-\beta(1-\alpha)} K_x(t).$$

Since $1 - \gamma < 1 - \beta(1 - \alpha)$, Lemma 4 can be used when taking the limit as $t \to 1$:

$${}_H I_{1^+}^{1-\gamma} x(1) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z \sum_{i=1}^m c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}. \tag{6}$$

Substituting $t = \tau_i$ into (2), we have

$$x(\tau_i) = \frac{Z}{\Gamma(\alpha)}(\log\tau_i)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)}\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s}.$$

Then we derive

$$\begin{aligned}\sum_{i=1}^{m} c_i x(\tau_i) &= \frac{Z}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s}\sum_{i=1}^{m} c_i(\log\tau_i)^{\gamma-1} \\ &\quad + \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s} \\ &= \frac{1}{\Gamma(\alpha)}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s}\left(1 + Z\sum_{i=1}^{m} c_i(\log\tau_i)^{\gamma-1}\right) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s},\end{aligned}$$

that is,

$$\sum_{i=1}^{m} c_i x(\tau_i) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} Z\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s}. \tag{7}$$

It follows (6) and (7) that

$${}_H I_{1^+}^{1-\gamma} x(1) = \sum_{i=1}^{m} c_i x(\tau_i).$$

Now by applying ${}_H D_{1^+}^{\gamma}$ to both sides of (2), it follows from Lemmas 1 and 5 that

$${}_H D_{1^+}^{\gamma} x(t) = {}_H D_{1^+}^{\beta(1-\alpha)} K_x(t) = {}_H D_{1^+}^{\beta(1-\alpha)} f\left(t, x(t), {}_H D_{1^+}^{\alpha,\beta} x(t)\right). \tag{8}$$

Since $x \in C_{1-\gamma,\log}^{\gamma}[J,X]$ and by the definition of $C_{1-\gamma,\log}^{\gamma}[J,X]$, we have that ${}_H D_{1^+}^{\gamma} x \in C_{1-\gamma,\log}[J,X]$, then ${}_H D_{1^+}^{\beta(1-\alpha)} f = D\, {}_H I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}[J,X]$. For $f \in C_{1-\gamma,\log}[J,X]$, it is obvious that ${}_H I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}[J,X]$, then ${}_H I_{1^+}^{1-\beta(1-\alpha)} f \in C_{1-\gamma,\log}^{1}[J,X]$. Thus, f and ${}_H I_{1^+}^{1-\beta(1-\alpha)} f$ satisfy the conditions of Lemma 3.

Next, by applying ${}_H I_{1^+}^{\beta(1-\alpha)}$ to both sides of (8) and using Lemma 3, we can obtain

$${}_H D_{1^+}^{\alpha,\beta} x(t) = K_x(t) - \frac{({}_H I_{1^+}^{1-\beta(1-\alpha)} K_x)(1)}{\Gamma(\beta(1-\alpha))}(\log t)^{\beta(1-\alpha)-1},$$

where $(I_{1^+}^{\beta(1-\alpha)} K_x)(1) = 0$ is implied by Lemma 4.

Hence, it reduces to ${}_HD_{1^+}^{\alpha,\beta}x(t) = K_x(t) = f(t, x(t), {}_HD_{1^+}^{\alpha,\beta}x(t))$. The results are proved completely. □

First, we list the following hypotheses to study the existence and uniqueness results:

(H1) The function $f : J \times X \times X \to X$ is continuous.

(H2) There exist $l, p, q \in C_{1-\gamma,\log}[J, X]$ with $l^* = \sup_{t\in J} l(t) < 1$ such that

$$\left|f(t,u,v)\right| \leqslant l(t) + p(t)|u| + q(t)|v|, \quad t \in J,\ u, v \in X.$$

(H3) There exist positive constants $K, L > 0$ such that

$$\left|f(t,u,v) - f(t,\overline{u},\overline{v})\right| \leqslant K|u-\overline{u}| + L|v-\overline{v}|, \quad u, v, \overline{u}, \overline{v} \in X,\ t \in J.$$

The existence result for problem (1) will be proved by using the Schaefer's fixed-point theorem.

Theorem 1. *Assume that* (H1) *and* (H2) *are satisfied. Then system* (1) *has at least one solution in* $C_{1-\gamma,\log}^{\gamma}[J,X] \subset C_{1-\gamma,\log}^{\alpha,\beta}[J,X]$.

Proof. For sake of clarity, we split the proof into a sequence of steps.

Consider the operator $N : C_{1-\gamma,\log}[J, X] \to C_{1-\gamma,\log}[J, X]$.

$$Nx)(t) = \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} K_x(s)\frac{\mathrm{d}s}{s}.$$

It is obvious that the operator N is well defined.

Step 1. N is continuous.

Let x_n be a sequence such that $x_n \to x$ in $C_{1-\gamma,\log}[J, X]$. Then for each $t \in J$,

$$\begin{aligned}
&\left|(\log t)^{1-\gamma}\big((Nx_n)(t) - (Nx)(t)\big)\right| \\
&\quad\leqslant \frac{|Z|}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log\frac{\tau_i}{s}\right)^{\alpha-1} \left|K_{x_n}(s) - K_x(s)\right| \frac{\mathrm{d}s}{s} \\
&\qquad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)} \int_1^{t} \left(\log\frac{t}{s}\right)^{\alpha-1} \left|K_{x_n}(s) - K_x(s)\right| \frac{\mathrm{d}s}{s} \\
&\quad\leqslant \frac{|Z|B(\gamma,\alpha)\sum_{i=1}^{m} c_i(\log\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)} \left\|K_{x_n}(\cdot) - K_{x_n}(\cdot)\right\|_{C_{1-\gamma,\log}} \\
&\qquad + \frac{(\log b)^{\alpha}B(\gamma,\alpha)}{\Gamma(\alpha)} \left\|K_{x_n}(\cdot) - K_x(\cdot)\right\|_{C_{1-\gamma,\log}} \\
&\quad\leqslant \frac{B(\gamma,\alpha)}{\Gamma(\alpha)} \left(|Z| \sum_{i=1}^{m} c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right) \left\|K_{x_n}(\cdot) - K_x(\cdot)\right\|_{C_{1-\gamma,\log}},
\end{aligned}$$

where we use the formula

$$\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left|x(s)\right|\frac{\mathrm{d}s}{s} \leqslant \left(\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left(\log\frac{s}{a}\right)^{\gamma-1}\frac{\mathrm{d}s}{s}\right)\|x\|_{C_{1-\gamma}}$$
$$= \left(\log\frac{t}{a}\right)^{\alpha+\gamma-1} B(\gamma,\alpha)\|x\|_{C_{1-\gamma,\log}}.$$

Since K_x is continuous (i.e., f is continuous), then we have

$$\|Nx_n - Nx\|_{C_{1-\gamma,\log}} \to 0 \quad \text{as } n \to \infty.$$

Step 2. N maps bounded sets into bounded sets in $C_{1-\gamma,\log}[J,X]$.

Indeed, it is enough to show that $\eta > 0$, there exists a positive constant l such that $x \in B_\eta = \{x \in C_{1-\gamma,\log}[J,X]\colon \|x\| \leqslant \eta\}$, we have $\|Nx\|_{C_{1-\gamma,\log}} \leqslant l$.

$$\begin{aligned}\left|(Nx)(t)(\log t)^{1-\gamma}\right| &\leqslant \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^m c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\left|K_x(s)\right|\frac{d}{\mathrm{d}s} \\ &\quad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left|K_x(s)\right|\frac{d}{\mathrm{d}s} \\ &:= A_1 + A_2. \end{aligned} \tag{9}$$

For computational work, we set

$$A_1 = \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^m c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\left|K_x(s)\right|\frac{\mathrm{d}}{\mathrm{d}s},$$
$$A_2 = \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left|K_x(s)\right|\frac{\mathrm{d}}{\mathrm{d}s},$$

and by (H2),

$$\begin{aligned}\left|K_x(t)\right| &= \left|f\big(t,x(t),K_x(t)\big)\right| \leqslant l(t) + p(t)\left|x(t)\right| + q(t)\left|K_x(t)\right| \\ &\leqslant l^* + p^*\left|x(t)\right| + q^*\left|K_x(t)\right| \leqslant \frac{l^* + p^*\left|x(t)\right|}{1-q^*}.\end{aligned} \tag{10}$$

We estimate A_1, A_2 terms separately. By (10) we have

$$A_1 \leqslant \frac{|Z|}{1-q^*}\sum_{i=1}^m c_i\left(\frac{l^*(\log\tau_i)^\alpha}{\Gamma(\alpha+1)} + p^*\frac{(\log\tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)}B(\gamma,\alpha)\|x\|_{C_{1-\gamma,\log}}\right), \tag{11}$$
$$A_2 \leqslant \frac{1}{1-q^*}\left(\frac{l^*(\log b)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + p^*\frac{(\log b)^\alpha}{\Gamma(\alpha)}B(\gamma,\alpha)\|x\|_{C_{1-\gamma,\log}}\right). \tag{12}$$

Bringing inequalities (11) and (12) into (9), we have

$$\begin{aligned}
&\left|(Nx)(t)(\log t)^{1-\gamma}\right| \\
&\quad\leqslant \frac{l^*}{(1-q^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^{m} c_i(\log \tau_i)^{\alpha} + (\log b)^{\alpha+\gamma-1}\right) \\
&\qquad + \frac{p^*}{(1-q^*)\Gamma(\alpha)}\left(|Z|\sum_{i=1}^{m} c_i(\log \tau_i)^{\alpha+\gamma-1} + (\log b)^{\alpha}\right)B(\gamma,\alpha)\|x\|_{C_{1-\gamma}} \\
&\quad := l.
\end{aligned}$$

Step 3. N maps bounded sets into equicontinuous set of $C_{1-\gamma,\log}[J,X]$.

Let $t_1, t_2 \in J$, $t_2 \leqslant t_1$ and $x \in B_\eta$. Using the fact f is bounded on the compact set $J \times B_\eta$ (thus $\sup_{(t,x)\in J\times B_\eta}\|K_x(t)\| := C_0 < \infty$), we will get

$$\begin{aligned}
&\left|(Nx)(t_1) - (Nx)(t_2)\right| \\
&\quad\leqslant \frac{C_0|Z|B(\gamma,\alpha)\sum_{i=1}^{m} c_i(\log \tau_i)^{\alpha+\gamma-1}}{\Gamma(\alpha)}\left|(\log t_1)^{\gamma-1} - (\log t_2)^{\gamma-1}\right| \\
&\qquad + \frac{C_0 B(\gamma,\alpha)}{\Gamma(\alpha)}\left|(\log t_1)^{\alpha+\gamma-1} - (\log t_2)^{\alpha+\gamma-1}\right|
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1–3 together with Arzela–Ascoli theorem, we can conclude that $N : C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$ is continuous and completely continuous.

Step 4. A priori bounds.

Now it remains to show that the set

$$\omega = \left\{x \in C_{1-\gamma,\log}[J,X]\colon\ x = \delta(Nx),\ 0 < \delta < 1\right\}$$

is bounded set.

Let $x \in \omega$, $x = \delta(Nx)$ for some $0 < \delta < 1$. Thus, for each $t \in J$, we have

$$\begin{aligned}
x(t) = \delta\Bigg[&\frac{|Z|}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^{m} c_i \int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)K_x(s)\frac{\mathrm{d}s}{s} \\
&+ \frac{1}{\Gamma(\alpha)}\int_1^{t}\left(\log\frac{t}{s}\right)^{\alpha-1}K_x(s)\frac{\mathrm{d}s}{s}\Bigg].
\end{aligned}$$

This implies by (H2) that for each $t \in J$, we have

$$\left|x(t)(\log t)^{1-\gamma}\right| \leqslant \left|(Nx)(t)(\log t)^{1-\gamma}\right|$$

$$\begin{aligned} &\leqslant \frac{l^*}{(1-q^*)\Gamma(\alpha+1)}\left(|Z|\sum_{i=1}^m c_i(\log\tau_i)^\alpha + (\log b)^{\alpha+\gamma-1}\right) \\ &\quad + \frac{p^*}{(1-q^*)\Gamma(\alpha)}\left(|Z|\sum_{i=1}^m c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^\alpha\right)B(\gamma,\alpha)\|x\|_{C_{1-\gamma}} \\ &:= R. \end{aligned}$$

This shows that the set ω is bounded. As a consequence of Schaefer's fixed-point theorem, we deduce that N has a fixed point, which is a solution of system (1). The proof is completed. □

Our second theorem is based on the Banach contraction principle.

Theorem 2. *Assume that* (H1) *and* (H3) *are satisfied. If*

$$\frac{K}{(1-L)\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^m c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^\alpha\right) < 1, \tag{13}$$

then system (1) *has a unique solution.*

Proof. Let the operator $N: C_{1-\gamma,\log}[J,X] \to C_{1-\gamma,\log}[J,X]$.

$$\begin{aligned} (Nx)(t) &= \frac{Z}{\Gamma(\alpha)}(\log t)^{\gamma-1}\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}K_x(s)\frac{\mathrm{d}s}{s} \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}K_x(s)\frac{\mathrm{d}s}{s}. \end{aligned}$$

By Lemma 8 it is clear that the fixed points of N are solutions of system (1).

Let $x_1, x_2 \in C_{1-\gamma,\log}[J,X]$ and $t \in J$, then we have

$$\begin{aligned} &\left|\big((Nx_1)(t)-(Nx_2)(t)\big)(\log t)^{1-\gamma}\right| \\ &\quad \leqslant \frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^m c_i\int_1^{\tau_i}\left(\log\frac{\tau_i}{s}\right)^{\alpha-1}\left|K_{x_1}(s)-K_{x_2}(s)\right|\frac{\mathrm{d}s}{s} \\ &\qquad + \frac{(\log t)^{1-\gamma}}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\left|K_{x_1}(s)-K_{x_2}(s)\right|\frac{\mathrm{d}s}{s} \end{aligned} \tag{14}$$

and

$$\begin{aligned} \left|K_{x_1}(t)-K_{x_2}(t)\right| &= \left|f\big(t,x_1(t),K_{x_1}(t)\big) - f\big(t,x_2(t),K_{x_2}(t)\big)\right| \\ &\leqslant K\left|x_1(t)-x_2(t)\right| + L\left|K_{x_1}(t)-K_{x_2}(t)\right| \\ &\leqslant \frac{K}{1-L}\left|x_1(t)-x_2(t)\right|. \end{aligned} \tag{15}$$

By replacing (15) in inequality (14) we get

$$\begin{aligned}&\big|\big((Nx_1)(t)-(Nx_2)(t)\big)(\log t)^{1-\gamma}\big|\\&\quad\leqslant\frac{|Z|}{\Gamma(\alpha)}\sum_{i=1}^{m}c_i\left(\frac{K}{1-L}B(\gamma,\alpha)(\log\tau)^{\alpha+\gamma-1}\|x_1-x_2\|_{C_{1-\gamma,\log}}\right)\\&\qquad+\frac{(\log b)^{1-\gamma}}{\Gamma(\alpha)}(\log b)^{\alpha+\gamma-1}\frac{K}{1-L}B(\gamma,\alpha)\|x_1-x_2\|_{C_{1-\gamma,\log}}\\&\quad\leqslant\frac{K}{(1-L)\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha+\gamma-1}+(\log b)^{\alpha}\right)\|x_1-x_2\|_{C_{1-\gamma,\log}}.\end{aligned}$$

Hence,

$$\begin{aligned}&\big\|Nx_1(t)-Nx_2(t)\big\|_{C_{1-\gamma,\log}}\\&\quad\leqslant\frac{K}{(1-L)\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^{m}c_i(\log\tau_i)^{\alpha+\gamma-1}+(\log b)^{\alpha}\right)\|x_1-x_2\|_{C_{1-\gamma,\log}}.\end{aligned}$$

From (13) it follows that N has a unique fixed point, which is solution of system (1). The proof of Theorem 2 is completed. □

4 Ulam–Hyers–Rassias stability

In this section, we consider the Ulam stability of Hilfer–Hadamard-type IDE (1). The following observations are taken from [4, 20].

Definition 4. Equation (1) is Ulam–Hyers stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of the inequality

$$\big|{}_HD^{\alpha,\beta}_{1^+}z(t)-f\big(t,z(t),{}_HD^{\alpha,\beta}_{1^+}z(t)\big)\big|\leqslant\epsilon,\quad t\in J,\tag{16}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of equation (1) with

$$\big|z(t)-x(t)\big|\leqslant C_f\epsilon,\quad t\in J.$$

Definition 5. Equation (1) is generalized Ulam–Hyers stable if there exists $\psi_f \in C_{1-\gamma}([1,\infty),[1,\infty))$, $\psi_f(1) = 0$, such that for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of the inequality

$$\big|{}_HD^{\alpha,\beta}_{1^+}z(t)-f\big(t,z(t),{}_HD^{\alpha,\beta}_{1^+}z(t)\big)\big|\leqslant\epsilon,\quad t\in J,$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ of equation (1) with

$$\big|z(t)-x(t)\big|\leqslant\psi_f\epsilon,\quad t\in J.$$

Definition 6. Equation (1) is Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of the inequality

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f\bigl(t, z(t), {}_H D^{\alpha,\beta}_{1^+} z(t)\bigr)\right| \leqslant \epsilon\varphi(t), \quad t \in J, \tag{17}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of equation (1) with

$$\left|z(t) - x(t)\right| \leqslant C_f \epsilon\varphi(t), \quad t \in J.$$

Definition 7. Equation (1) is generalized Ulam–Hyers–Rassias stable with respect to $\varphi \in C_{1-\gamma,\log}[J,X]$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of the inequality

$$\left|{}_H D^{\alpha,\beta}_{1^+} z(t) - f\bigl(t, z(t), {}_H D^{\alpha,\beta}_{1^+} z(t)\bigr)\right| \leqslant \varphi(t), \quad t \in J, \tag{18}$$

there exists a solution $x \in C^{\gamma}_{1-\gamma,\log}[J,X]$ of equation (1) with

$$\left|z(t) - x(t)\right| \leqslant C_{f,\varphi}\varphi(t), \quad t \in J.$$

Remark 2. A function $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ is a solution of inequality (16) if and only if there exist a function $g \in C^{\gamma}_{1-\gamma,\log}[J,X]$ (which depends on solution z) such that

(i) $|g(t)| \leqslant \epsilon$ for all $t \in J$;

(ii) ${}_H D^{\alpha,\beta}_{1^+} z(t) = f(t, z(t), {}_H D^{\alpha,\beta}_{1^+} z(t)) + g(t)$, $t \in J$.

Remark 3. It is clear that

(i) Definition 4 $\Longrightarrow$ Definition 5.

(ii) Definition 6 $\Longrightarrow$ Definition 7.

(iii) Definition 6 for $\varphi(t) = 1 \Longrightarrow$ Definition 4.

One can have similar remarks for inequalities (17) and (18).

Lemma 9. *Let* $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$. *If a function* $z \in C^{\gamma}_{1-\gamma,\log}[J,X]$ *is a solution of inequality* (16)*, then* x *is a solution of the following integral inequality*:

$$\left|z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} K_z(s)\frac{\mathrm{d}s}{s}\right|$$
$$\leqslant \left(\frac{Z(mc)(\log b)^{\alpha+\gamma-1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)}\right)\epsilon,$$

where

$$A_z = \frac{Z}{\Gamma(\alpha)}(\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log\frac{\tau_i}{s}\right)^{\alpha-1} K_z(s)\frac{\mathrm{d}s}{s}.$$

Proof. Indeed, by Remark 2 we have that

$$ {}_H D_{1^+}^{\alpha,\beta} z(t) = f\bigl(t, z(t), {}_H D_{1^+}^{\alpha,\beta} z(t)\bigr) + g(t) = K_z(t) + g(t). $$

Then

$$ z(t) = \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \left(\int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} + \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g(s) \frac{\mathrm{d}s}{s} \right) + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{\mathrm{d}s}{s}. $$

From this it follows that

$$ \begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} K_z \frac{\mathrm{d}s}{s} \right| \\ &\quad = \left| \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} g(s) \frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} g(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad \leqslant \frac{Z(\log t)^{\gamma-1}}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} \bigl|g(s)\bigr| \frac{\mathrm{d}s}{s} + \frac{1}{\Gamma(\alpha)} \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \bigl|g(s)\bigr| \frac{\mathrm{d}s}{s} \\ &\quad \leqslant \left(\frac{Z(mc)(\log b)^{\alpha+\gamma-1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon. \end{aligned} $$

□

We have similar remark for the solutions of inequalities (17) and (18).

The following generalized Gronwall inequalities will be used to deal with our system in the sequence.

Lemma 10. (*See* [26].) *Let* $v, w : [1, b] \to [1, +\infty)$ *be continuous functions. If* w *is nondecreasing and there are constants* $k \geqslant 0$ *and* $0 < \alpha < 1$ *such that*

$$ v(t) \leqslant w(t) + k \int_1^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} v(s) \frac{\mathrm{d}s}{s}, \quad t \in J = [1, b], $$

then

$$ v(t) \leqslant w(t) + \int_1^{t} \left[\sum_{n=1}^{\infty} \frac{(k\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\log \frac{t}{s}\right)^{n\alpha-1} w(s) \right] \frac{\mathrm{d}s}{s}, \quad t \in J. $$

Remark 4. Under the assumptions of Lemma 10, let $w(t)$ be a nondecreasing function on J. Then we have

$$ v(t) \leqslant w(t) E_{\alpha,1}\bigl(k\Gamma(\alpha)(\log t)^{\alpha}\bigr), $$

where $E_{\alpha,1}$ is the Mittag–Leffler function defined by

$$E_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in \mathbb{C}.$$

Now we give the Ulam–Hyers and Ulam–Hyers–Rassias results in this sequel.

Theorem 3. *Assume that* (H3) *and* (13) *are satisfied. Then system* (1) *is Ulam–Hyers stable.*

Proof. Let $\epsilon > 0$, and let $z \in C^{\gamma}_{1-\gamma,\log}[J, X]$ be a function, which satisfies inequality (16), and let $x \in C^{\gamma}_{1-\gamma,\log}[J, X]$ is the unique solution of the following Hilfer–Hadamard-type IDE:

$$\begin{aligned}
&{}_H D^{\alpha,\beta}_{1^+} x(t) = f\bigl(t, x(t), {}_H D^{\alpha,\beta}_{1^+} x(t)\bigr), \quad t \in J := [1, b],\\
&{}_H I^{1-\gamma}_{1^+} x(1) = {}_H I^{1-\gamma}_{1^+} z(1) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [1, b],\ \gamma = \alpha + \beta - \alpha\beta,
\end{aligned}$$

where $0 < \alpha < 1,\ 0 \leqslant \beta \leqslant 1$.

Using Lemma 8, we obtain

$$x(t) = A_x + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s},$$

where

$$A_x = \frac{Z}{\Gamma(\alpha)} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}.$$

On the other hand, if $\sum_{i=1}^{m} c_i x(\tau_i) = \sum_{i=1}^{m} c_i z(\tau_i)$ and ${}_H I^{1-\gamma}_{1^+} z(1) = {}_H I^{1-\gamma}_{1^+} x(1)$, then $A_x = A_z$.

Indeed,

$$\begin{aligned}
|A_x - A_z| &\leqslant \frac{|Z|}{\Gamma(\alpha)} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} \bigl|K_x(s) - K_z(s)\bigr| \frac{\mathrm{d}s}{s}\\
&\leqslant \frac{|Z|}{\Gamma(\alpha)} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} \frac{K}{1-L} \bigl|x(s) - z(s)\bigr| \frac{\mathrm{d}s}{s}\\
&\leqslant \frac{K|Z|}{1-L} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i\, {}_H I^{\alpha}_{1+} \bigl|x(\tau_i) - z(\tau_i)\bigr| = 0.
\end{aligned}$$

Thus,

$$A_x = A_z.$$

Then we have

$$x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}.$$

By integration of inequality (16) and applying Lemma 9, we obtain

$$\begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad \leqslant \left(\frac{Z(mc)(\log b)^{\alpha+\gamma-1}}{\Gamma(\alpha+1)} + \frac{(\log b)^{\alpha}}{\Gamma(\alpha+1)} \right) \epsilon. \end{aligned} \tag{19}$$

For sake of brevity, we take $U = Z(mc)(\log b)^{\alpha+\gamma-1}/\Gamma(\alpha+1) + (\log b)^{\alpha}/\Gamma(\alpha+1)$.

We have for any $t \in J$,

$$\begin{aligned} \left| z(t) - x(t) \right| &\leqslant \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left| K_z(s) - K_x(s) \right| \frac{\mathrm{d}s}{s} \\ &\leqslant \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad + \frac{K}{(1-L)\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left| z(s) - x(s) \right| \frac{\mathrm{d}s}{s}. \end{aligned}$$

By using (19) we have

$$\left| z(t) - x(t) \right| \leqslant U\epsilon + \frac{K}{(1-L)\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \left| z(s) - x(s) \right| \frac{\mathrm{d}s}{s},$$

and applying Lemma 10 and Remark 4, we obtain

$$\begin{aligned} \left| z(t) - x(t) \right| &\leqslant U E_{\alpha,1}\left(\frac{K}{1-L} (\log b)^{\alpha} \right) \cdot \epsilon \\ &:= C_f \epsilon, \end{aligned}$$

where

$$C_f = U E_{\alpha,1}\left(\frac{K}{1-L} (\log b)^{\alpha} \right).$$

Thus, system (1) is Ulam–Hyers stable. □

Theorem 4. *Assume that* (H3) *and* (13) *are satisfied. Suppose that there exists an increasing function* $\varphi \in C_{1-\gamma,\log}[J, X]$ *and there exists* $\lambda_\varphi > 0$ *such that for any* $t \in J$*,*

$$ {}_H I_{1^+}^{\alpha} \varphi(t) \leqslant \lambda_\varphi \varphi(t). $$

Then system (1) *is generalized Ulam–Hyers–Rassias stable.*

Proof. Let $\epsilon > 0$, and let $z \in C_{1-\gamma,\log}^{\gamma}[J, X]$ be a function, which satisfies inequality (17), and let $x \in C_{1-\gamma,\log}^{\gamma}[J, X]$ be the unique solution of the following Hilfer–Hadamard IDE:

$$ \begin{aligned} &{}_H D_{1^+}^{\alpha,\beta} x(t) = f\big(t, x(t), {}_H D_{1^+}^{\alpha,\beta} x(t)\big), \quad t \in J := [1, b], \\ &{}_H I_{1^+}^{1-\gamma} x(1) = {}_H I_{1^+}^{1-\gamma} z(1) = \sum_{i=1}^{m} c_i x(\tau_i), \quad \tau_i \in [1, b],\ \gamma = \alpha + \beta - \alpha\beta, \end{aligned} $$

where $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$.

Using Lemma 8, we obtain

$$ x(t) = A_z + \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_x(s) \frac{\mathrm{d}s}{s}, $$

where

$$ A_z = \frac{Z}{\Gamma(\alpha)} (\log t)^{\gamma-1} \sum_{i=1}^{m} c_i \int_1^{\tau_i} \left(\log \frac{\tau_i}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s}. $$

By integration of (17) we obtain

$$ \begin{aligned} &\left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad \leqslant \frac{\epsilon}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \varphi(s) \frac{\mathrm{d}s}{s} \leqslant \big(Z (\log b)^{\gamma-1} (mc) + 1\big) \epsilon \lambda_\varphi \varphi(t). \qquad (20) \end{aligned} $$

For sake of brevity, we take $\overline{U} = Z(\log b)^{\gamma-1}(mc) + 1$.

On the other hand, we have

$$ \begin{aligned} \big|z(t) - x(t)\big| &\leqslant \left| z(t) - A_z - \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} K_z(s) \frac{\mathrm{d}s}{s} \right| \\ &\quad + \frac{K}{(1-L)\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \big|z(s) - x(s)\big| \frac{\mathrm{d}s}{s}. \end{aligned} $$

By using (20) we have

$$\left|z(t) - x(t)\right| \leqslant \overline{U}\epsilon\lambda_\varphi\varphi(t) + \frac{K}{(1-L)\Gamma(\alpha)}\int\limits_1^t \left(\log\frac{t}{s}\right)^{\alpha-1}\left|z(s)-x(s)\right|\frac{\mathrm{d}s}{s},$$

and applying Lemma 10 and Remark 4, we obtain

$$\left|z(t) - x(t)\right| \leqslant \overline{U}\epsilon\lambda_\varphi\varphi(t)E_{\alpha,1}\left(\frac{K}{1-L}(\log b)^\alpha\right), \quad t \in [1,b].$$

Thus, system (1) is generalized Ulam–Hyers–Rassias stable. The proof is completed. □

5 An example

As an application of our results, we consider the following problem of Hilfer–Hadamard IDE:

$$ {}_HD_{1^+}^{\alpha,\beta}x(t) = \frac{\mathrm{e}^{-(\log t)}}{(9+\mathrm{e}^{\log t})}\left[\frac{|x(t)|}{1+|x(t)|} + \frac{|{}_HD_{1^+}^{\alpha,\beta}x(t)|}{1+|{}_HD_{1^+}^{\alpha,\beta}x(t)|}\right], \quad t \in J := [1,\mathrm{e}], \tag{21}$$

$$ {}_HI_{1^+}^{1-\gamma} = 2x\left(\frac{3}{2}\right), \quad \gamma = \alpha+\beta-\alpha\beta. \tag{22}$$

Notice that this problem is a particular case of (1), where $\alpha = 2/3$, $\beta = 1/2$, and choose $\gamma = 5/6$.

Set

$$f(t,u,v) = \frac{\mathrm{e}^{-(\log t)}}{(9+\mathrm{e}^{\log t})}\left[\frac{u}{1+u} + \frac{v}{1+v}\right] \quad \text{for any } u,v \in X.$$

Clearly, the function f satisfies the condition of Theorem 1.

For any $u, v, \overline{u}, \overline{v} \in X$ and $t \in J$,

$$\left|f(t,u,v) - f(t,\overline{u},\overline{v})\right| \leqslant \frac{1}{10}\left|u-\overline{u}\right| + \frac{1}{10}|v-\overline{v}|.$$

Hence, condition (H3) is satisfied with $K = L = 1/10$.

Thus, from (13) we have

$$\frac{K}{(1-L)\Gamma(\alpha)}B(\gamma,\alpha)\left(|Z|\sum_{i=1}^m c_i(\log\tau_i)^{\alpha+\gamma-1} + (\log b)^\alpha\right) < 1,$$

where $|Z| = 0.8959$. It follows from Theorem 2 that problem (21)–(22) has a unique solution.

From the above example the existence and Ulam–Hyers stability of Hilfer–Hadamard-type IDE with nonlocal condition are verified by Theorems 1 and 3.

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