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Controllability of nonlinear fractional Langevin delay systems

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Abstract. In this paper, we discuss the controllability of fractional Langevin delay dynamical systems represented by the fractional delay differential equations of order $0 < \alpha, \beta \leqslant 1$. Necessary and sufficient conditions for the controllability of linear fractional Langevin delay dynamical system are obtained by using the Grammian matrix. Sufficient conditions for the controllability of the nonlinear delay dynamical systems are established by using the Schauders fixed-point theorem. The problem of controllability of linear and nonlinear fractional Langevin delay dynamical systems with multiple delays and distributed delays in control are studied by using the same technique. Examples are provided to illustrate the theory.

Keywords: Langevin equation, controllability, fractional delay differential equations, Mittag–Leffler matrix function.

1 Introduction

The concept of controllability plays a major role in both finite and infinite dimensional spaces for systems represented by ordinary differential equations and partial differential equations. So it is natural to study this concept for dynamical systems represented by fractional differential equations and fractional delay differential equations. The controllability of delay differential systems is studied by Wiess [33]. Chyung [10] studied the controllability of linear time-varying systems with delay. The controllability of nonlinear delay systems is discussed by Dauer and Gahl [12]. Balachandran and Somasundaram [6] studied the controllability of a class of nonlinear systems with distributed delay in control. The constrained controllability of semilinear delay systems was studied by Klamka [17]. Yi et al. [34] discussed the controllability and observability of systems of linear delay differential equations via the matrix Lambert W function. A sliding mode control for linear fractional systems with input and state delays is investigated by Si-Ammour et al. [27]. The controllability of differential equations with delayed and advanced arguments is investigated by Manzanilla et al. [22]. Bhalekar and Gejji [8] studied the fractional ordered Liu system with time delay. An application of delay differential equations in life sciences is discussed by Smith [29]. Pinning-controllability analysis of complex networks and M-matrix approach is discussed by Song et al. [30]. Balachandran et al. [3–5, 7] studied the relative controllability of fractional dynamical systems with multiple delay and distributed delay in control. A numerical method for delayed fractional-order differential equations is studied by Wang [32]. Zhang et al. [36] discussed the controllability criteria for linear fractional differential systems delay in state and impulse. Analysis and numerical methods for fractional differential equations with delay were analyzed by Morgado et al. [23]. Shu [26] studied the explicit representations of solutions of linear delay systems. Lu et al. [19, 20] studied the pinning controllability of Boolean control networks. Controllability of fractional damped dynamical systems with delay in control is investigated by He et al. [16]. Nirmala and Balachandran [24] investigated the controllability of nonlinear fractional delay dynamical systems. The control problems involving the delay in state variable are challenging and are not much developed. Controllability of impulsive neutral evolution integro-differential equations with state-dependent delay in Banach spaces is studied by Chalishajar et al. [9]. Sikora and Klamka [28] studied the constrained controllability of fractional linear systems with delays in control. Govindaraj and George [14] studied the controllability of fractional dynamical systems of functional analytic approach.

The theory of Brownian motion is perhaps the simplest approximate approach to treat the dynamics of nonequilibruim systems. The fundamental equation is called the Langevin equation; it contains both frictional forces and random forces. It is used to describe the evolution of physical phenomena in fluctuating environments. However, for the systems in complex media, an integer-order Langevin equation does not provide the correct description of the dynamics. One of the possible generalizations of a Langevin equation is to replace the integer-order derivative by a fractional-order derivative in it. This gives rise to a fractional Langevin equation [31]. The fractional Langevin equation with Brownian motion is revisited by Mainardi and Pironi [21]. Fa [13] investigated the fractional Langevin equation and Riemann–Liouville fractional derivative. Lim et al. [18] and Baghani [2] studied the Langevin equation with two fractional orders. Nonlinear Langevin equation involving two fractional orders in different intervals are discussed by Ahmad et al. [1]. Guo et al. [15] addressed the numerics for the fractional Langevin equation driven by the fractional Brownian motion. Yu et al. [35] discussed the existence and uniqueness of solutions of initial value problems for nonlinear Langevin equations involving two fractional orders. This motivates us to study the fractional Langevin delay differential equations.

The aim of this paper is to study the controllability of fractional Langevin delay dynamical systems. The necessary and sufficient conditions for the controllability for linear systems are derived using controllability Grammian matrix, which is defined by means of Mittag–Leffler matrix function. Sufficient conditions for the controllability of nonlinear fractional Langevin delay dynamical systems are established by using the Schauders fixed-point theorem. Controllability of linear and nonlinear fractional Langevin delay dynamical systems with multiple delays in control and distributed delays are studied by using the same technique.

2 Preliminaries

In this section, we introduce the definitions and preliminary results from fractional calculus, which are used throughout this paper.

Definition 1. The Riemann–Liouville fractional integral operator of order $\alpha > 0$ is defined by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s,$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2. The Caputo fractional derivative of order $\alpha \in \mathbb{C}$ with $n-1 < \alpha \leqslant n$, $n \in \mathbb{N}$, for a suitable function f is defined as

$$\left({}^{C}D^{\alpha}_{0+} f\right)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\,\mathrm{d}s,$$

where $f^{(n)}(s) = \mathrm{d}^n f/\mathrm{d}s^n$. In particular, if $0 < \alpha \leqslant 1$, then

$$\left({}^{C}D^{\alpha}_{0+} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s,$$

and, if $1 < \alpha \leqslant 2$, then

$$\left({}^{C}D^{\alpha}_{0+} f\right)(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)\,\mathrm{d}s.$$

For brevity, the Caputo fractional derivative ${}^{C}D^{\alpha}_{0+}$ is written as ${}^{C}D^{\alpha}$.

Definition 3. The Mittag–Leffler functions of various types are defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \qquad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k z^k}{k!\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,\ z \in \mathbb{C},$$

where $(\gamma)_n$ is a Pochhamer symbol, which is defined as $\gamma(\gamma+1)\cdots(\gamma+n-1)$, and $(\gamma)_n = \Gamma(\gamma+n)/\Gamma(\gamma)$. For an $n \times n$ matrix A,

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0,$$
$$E_{\alpha,1}(A) = E_{\alpha}(A) \quad \text{with } \beta = 1.$$

Definition 4. (See [25].) The formal definition of the Laplace transform of a function $f(t)$ of a real variable $t \in \mathbb{R}^+ = (0, \infty)$ is given by

$$\mathcal{L}\big\{f(t)\big\} = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad \mathcal{L}\big\{I^\alpha f(t)\big\} = s^{-\alpha} F(s), \quad s \in \mathbb{C}.$$

The convolution operator of two functions $f(t)$ and $g(t)$ given on $\mathbb{R}^+$ is defined for $x \in \mathbb{R}^+$ by the integral

$$(f * g)(t) = \int_0^t f(t-s) g(s)\,\mathrm{d}s.$$

The Laplace transform of a convolution is given by

$$\mathcal{L}\big\{f(t) * g(t)\big\} = \mathcal{L}\big\{f(t)\big\}\mathcal{L}\big\{g(t)\big\}.$$

Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. The inverse Laplace transform of product of two functions $F(s)$ and $G(s)$ is defined by

$$\mathcal{L}^{-1}\big\{F(s)G(s)\big\} = \mathcal{L}^{-1}\big\{F(s)\big\} * \mathcal{L}^{-1}\big\{G(s)\big\}.$$

The Laplace transforms of Mittag–Leffler functions are defined as

$$\begin{aligned} \mathcal{L}\big[E_{\alpha,1}\big(\pm\lambda t^\alpha\big)\big](s) &= \frac{s^{\alpha-1}}{s^\alpha \mp \lambda}, && \operatorname{Re}(\alpha) > 0, \\ \mathcal{L}\big[t^{\beta-1} E_{\alpha,\beta}\big(\pm\lambda t^\alpha\big)\big](s) &= \frac{s^{\alpha-\beta}}{s^\alpha \mp \lambda}, && \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0, \\ \mathcal{L}\big[t^{\beta-1} E^\gamma_{\alpha,\beta}\big(\pm\lambda t^\alpha\big)\big](s) &= \frac{s^{\alpha\gamma-\beta}}{(s^\alpha \mp \lambda)^\gamma}, && \operatorname{Re}(\alpha), \operatorname{Re}(\beta) > 0,\ \big|\lambda s^{-\alpha} < 1\big|. \end{aligned}$$

If $F(s) = \mathcal{L}[f(t)](s)$ for $\operatorname{Re}(s) > 0$, then

$$F(s-a) = \mathcal{L}\big[\mathrm{e}^{at} f(t)\big](s)$$

and

$$\mathcal{L}\big[u_a(t) f(t-a)\big](s) = \mathrm{e}^{-as} F(s), \quad a \geqslant 0,$$

and also we have

$$\mathcal{L}^{-1}\big[\mathrm{e}^{-as} F(s)\big](t) = u_a(t) f(t-a).$$

3 Linear delay systems

Consider the linear fractional Langevin delay dynamical system of the form

$$\begin{aligned} &{}^C D^\beta \big({}^C D^\alpha + A\big) x(t) = Bx(t-h) + Cu(t), \quad 0 < \alpha, \beta \leqslant 1,\ t \in J, \\ &x(t) = \phi(t), \quad -h < t \leqslant 0, \qquad {}^C D^\alpha x(t)\big|_{t=0} = q_0, \end{aligned} \tag{1}$$

where $\alpha+\beta>1$, $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, A,B are $n\times n$ real matrices, and C is an $n\times m$ real matrix with $n>m$; $\phi(t)$ is the initial function on $[-h,0]$. By taking Laplace and inverse Laplace transform on both sides of (1) and, by simple calculations using convolution of Laplace transform, we have

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\bigl[s^{\alpha-1}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t)\phi(0)\\
&\quad + A\mathcal{L}^{-1}\bigl[s^{-\alpha}s^{\alpha-1}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t)\phi(0)\\
&\quad + \mathcal{L}^{-1}\bigl[s^{-\alpha}s^{\alpha-1}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t)q_0\\
&\quad + B\mathcal{L}^{-1}\biggl[\frac{s^{\alpha-1}}{s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}}s^{1-\alpha-\beta}\biggr](t)*\mathcal{L}^{-1}\Biggl[\mathrm{e}^{-hs}\int_{-h}^{0}\mathrm{e}^{-s\tau}\phi(\tau)\,\mathrm{d}\tau\Biggr](t)\\
&\quad + C\mathcal{L}^{-1}\bigl[U(s)\bigr](t)*\mathcal{L}^{-1}\biggl[\frac{s^{\alpha-1}}{s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}}s^{1-\alpha-\beta}\biggr](t).
\end{aligned}$$

For simplicity of notation, let us take

$$\begin{aligned}
X_\alpha(t) &= \mathcal{L}^{-1}\bigl[s^{\alpha-1}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t),\\
X_{\alpha,1+\alpha}(t) &= \mathcal{L}^{-1}\bigl[s^{-\alpha}s^{\alpha-1}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t),\\
X_{\alpha,\alpha+\beta}(t) &= \mathcal{L}^{-1}\bigl[s^{\alpha-1}s^{1-\alpha-\beta}\bigl(s^\alpha I+A-Bs^{-\beta}\mathrm{e}^{-hs}\bigr)^{-1}\bigr](t).
\end{aligned}$$

The function $\phi(t)$ is extended to $(-h,\infty)$ by defining $\phi(t)=\phi(0)$ for $t\geqslant 0$; then the solution of (1) is given as [24]

$$x(t) = x(t;\phi,q_0) + \int_0^t (t-s)^{\alpha+\beta-1}X_{\alpha,\,\alpha+\beta}(t-s)Cu(s)\,\mathrm{d}s, \tag{2}$$

where

$$\begin{aligned}
x(t;\phi,q_0) &= X_\alpha(t)\phi(0)+t^\alpha AX_{\alpha,1+\alpha}(t)\phi(0)+t^\alpha X_{\alpha,1+\alpha}(t)q_0\\
&\quad + B\int_{-h}^{0}(t-s-h)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s-h)\phi(s)\,\mathrm{d}s.
\end{aligned}$$

Definition 5. System (1) is said to be completely controllable on J if, for every initial function $\phi(t)$ and $q_0,x_1\in\mathbb{R}^n$, there exists a continuous control function u such that the solutions of (1) satisfy $x(T)=x_1$.

Define the controllability Grammian matrix by

$$W = \int_0^T (T-s)^{2(\alpha+\beta-1)}\bigl[X_{\alpha,\alpha+\beta}(T-s)C\bigr]\bigl[X_{\alpha,\alpha+\beta}(T-s)C\bigr]^\mathsf{T}\,\mathrm{d}s, \tag{3}$$

where the $^\mathsf{T}$ denotes the matrix transpose.

Theorem 1. *The linear system* (1) *is completely controllable on* $[0,T]$ *if and only if the controllability Grammian matrix* W *is positive definite.*

Proof. Assume that W is positive definite. Let $\phi(t)$ be continuous on $[-h,0]$, and therefore its inverse is well defined. Define the control function as

$$u(t) = (T-t)^{\alpha+\beta-1}\big(X_{\alpha,\alpha+\beta}(T-t)C\big)^{\intercal} W^{-1}\big[x_1 - x(T;\phi,q_0)\big]. \tag{4}$$

Substituting $t=T$ in the solution (1) and inserting (4), we have

$$\begin{aligned} x(T) &= x(T;\phi,q_0) + \int_0^T (T-s)^{2\alpha+2\beta-2} X_{\alpha,\alpha+\beta}(t-s)CC^{\intercal}X_{\alpha,\alpha+\beta}(t-s)\,\mathrm{d}s \\ &\quad \times W^{-1}\big[x_1 - x(T;\phi,q_0)\big], \\ x(T) &= x_1. \end{aligned}$$

Thus, (1) is controllable. Now assume that W is not positive definite and there exists a vector $y \neq 0$ such that $y^{\intercal}Wy = 0$. It follows that

$$y^{\intercal}\int_0^T (T-s)^{2(\alpha+\beta-1)}\big(X_{\alpha,\alpha+\beta}(T-s)C\big)\big(CX_{\alpha,\alpha+\beta}(T-s)\big)^{\intercal} y\,\mathrm{d}s = 0,$$

that is,

$$y^{\intercal}(T-s)^{\alpha+\beta-1}\big(X_{\alpha,\alpha+\beta}(T-s)C\big) = 0 \quad \text{on } [0,T].$$

Consider the zero initial function $\phi, q_0 = 0$ and the final point $x_1 = y$. Since the system is controllable, there exists a control $u(t)$ on J that steers the response to $x_1 = y$ at $t = T$, that is,

$$y = \int_0^T (T-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(T-s)Cu(s)\,\mathrm{d}s.$$

It follows that

$$y^{\intercal}y = \int_0^T (T-s)^{\alpha+\beta-1} y^{\intercal}X_{\alpha,\alpha+\beta}(T-s)Cu(s)\,\mathrm{d}s,$$

and leading to the conclusion that $y^{\intercal}y = 0$. This is a contradiction to $y \neq 0$. Thus, W is positive definite. Hence, the proof is complete. □

4 Nonlinear delay systems

Consider the nonlinear fractional Langevin delay dynamical systems of the form

$$\begin{aligned} &{}^{C}D^{\beta}\big({}^{C}D^{\alpha} + A\big)x(t) = Bx(t-h) + Cu(t) + f\big(t, x(t), x(t-h), u(t)\big), \\ &0 < \alpha, \beta \leqslant 1,\ t \in J, \\ &x(t) = \phi(t), \quad -h < t \leqslant 0, \qquad {}^{C}D^{\alpha}x(t)\big|_{t=0} = q_0, \end{aligned} \tag{5}$$

where $\alpha+\beta > 1$, the state vector $x \in \mathbb{R}^n$, the control vector $u \in \mathbb{R}^m$, and A, B are $n \times n$ real matrices, C is an $n \times m$ real matrix with $n > m$, and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous. The solution of (5) is given by

$$\begin{aligned} x(t) = x(t;\phi,q_0) &+ \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s)Cu(s)\,\mathrm{d}s \\ &+ \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) f\bigl(s,x(s),x(s-h),u(s)\bigr)\,\mathrm{d}s \end{aligned} \tag{6}$$

for $t \in J$ and $y(t) = \phi(t)$, $t \in [-h, 0]$. For simplicity, let us take

$$\begin{gathered} a_1 = \sup\bigl\|X_{\alpha,\alpha+\beta}(T-t)C\bigr\|, \qquad a_2 = \bigl|W^{-1}\bigr|, \qquad a_3 = \sup\bigl\{x(t;\phi,q_0) + |x_1|\bigr\}, \\ a_4 = \sup\bigl|X_{\alpha,\alpha+\beta}(T-t)\bigr|, \qquad b = \max\bigl\{a_1T^{\alpha+\beta}(\alpha+\beta)^{-1},\, 1\bigr\}, \\ c_1 = 6ba_1a_2a_4T^{\alpha+\beta}(\alpha+\beta)^{-1}, \qquad c_2 = 6a_4T^{\alpha+\beta}(\alpha+\beta)^{-1}, \\ d_1 = 6ba_1a_2a_3, \qquad d_2 = 6a_3, \qquad c = \max\{c_1,c_2\}, \qquad d = \max\{d_1,d_2\}. \end{gathered}$$

Now we prove the main result of the paper.

Theorem 2. *Let the continuous function f satisfies the condition*

$$\lim_{|p|\to\infty} \frac{|f(t,p)|}{|p|} = 0 \tag{7}$$

uniformly in $t \in J$. Suppose that the linear system (1) *is completely controllable on J. Then the nonlinear system* (5) *is completely controllable on J.*

Proof. Let $\phi(t)$ be continuous on $[-h, 0]$. Let Q be the Banach space of all continuous functions of

$$(x,u) : [-h,T] \times [0,T] \to \mathbb{R}^n \times \mathbb{R}^n$$

with the norm

$$\bigl\|(x,u)\bigr\| = \|x\| + \|u\|,$$

where $\|x\| = \sup\{|x(t)| \text{ for } t \in [-h,T]\}$ and $\|u\| = \sup\{|u(t)| \text{ for } t \in [0,T]\}$.

Define the operator $P : Q \to Q$ by $P(x,u) = (z,v)$, where

$$\begin{aligned} v(t) = (T-t)^{\alpha+\beta-1} C^{\intercal} X_{\alpha,\alpha+\beta}(T-t) W^{-1} \Biggl[x_1 - x(T;\phi,q_0) \\ - \int_0^T (T-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(T-s) f\bigl(s,x(s),x(s-h),u(s)\bigr)\,\mathrm{d}s \Biggr], \end{aligned}$$

$$z(t) = x(t; \phi, q_0) + \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) C u(s)\,\mathrm{d}s$$
$$+ \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) f\bigl(s, x(s), x(s-h), u(s)\bigr)\,\mathrm{d}s.$$

For $t \in J$ and $z(t) = \phi(t)$, $t \in [-h, 0]$, we have

$$\begin{aligned}
\bigl|v(t)\bigr| &\leqslant a_1 a_2 \Bigl(a_3 + a_4 T^{\alpha+\beta}(\alpha+\beta)^{-1} \\
&\quad + \sup_{s\in J}\bigl|f\bigl(s, x(s), x(s-h), u(s)\bigr)\bigr|\Bigr) \\
&\leqslant \biggl[\frac{d_1}{6b} + \frac{c_1}{6b}\sup_{s\in J}\bigl|f\bigl(s, x(s), x(s-h), u(s)\bigr)\bigr|\biggr] \\
&\leqslant \frac{1}{6b}\Bigl[d + c\sup_{s\in J}\bigl|f\bigl(s, x(s), x(s-h), u(s)\bigr)\bigr|\Bigr]
\end{aligned}$$

and

$$\begin{aligned}
\bigl|z(t)\bigr| &\leqslant a_3 + \frac{a_1 T^{\alpha+\beta}\|v\|}{\alpha+\beta} \\
&\quad + \frac{a_4 T^{\alpha+\beta}}{\alpha+\beta}\sup_{s\in J}\bigl|f\bigl(s, x(s), x(s-h), u(s)\bigr)\bigr| \\
&\leqslant \frac{d}{6} + b\|v\| + \frac{c}{6}\sup_{s\in J}\bigl|f\bigl(s, x(s), x(s-h), u(s)\bigr)\bigr|.
\end{aligned}$$

By hypothesis, the function f satisfies the following conditions in [11]. For each pair of positive constants c and d, there is a positive constant r such that, if $|(x, u)| \leqslant r$, then

$$c\bigl|f(t, p)\bigr| + d \leqslant r \quad \text{for all } t \in J. \tag{8}$$

Also, for given c and d, if r is a constant such that inequality (8) is satisfied, then any r_1 such that $r < r_1$ will also satisfy (8). Now take c and d as given above, and let r be chosen so that the aforementioned inequality is satisfied and

$$\sup_{-1\leqslant t\leqslant 0}\bigl|\phi(t)\bigr| \leqslant \frac{r}{3}.$$

Therefore, if $\|x\| \leqslant r/3$, then $\|u\| \leqslant r/3$ for $s \in J$. It follows that $d + c\sup|f(s, x(s), x(s-h), u(s)) \leqslant r$ for $s \in J$. Therefore, $|v(t)| \leqslant r/(6b)$ for all $t \in J$ and hence $\|v(t)\| \leqslant r/(6b)$, which gives $\|z(t)\| \leqslant r/3$. Thus, we have proved that, if $Q(r) = \{(x, u) \in Q\colon \|x\| \leqslant r/3 \text{ and } \|u\| \leqslant r/3\}$, then P maps $Q(r)$ into itself. It is easy to see that P is completely continuous. Since $Q(r)$ is closed, bounded and convex, the Schauders fixed-point theorem implies that P has a fixed-point $(x, u) \in Q(r)$ such that

$(z, v) = P(x, u) = (x, u)$. It follows that

$$
\begin{aligned}
x(t) &= x(t;\phi,q_0) + \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) C u(s)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) f\bigl(s, x(s), x(s-h), u(s)\bigr)\,\mathrm{d}s
\end{aligned} \tag{9}
$$

for $t \in J$, and $x(t) = \phi(t)$ for $t \in [-h, 0]$. Hence, $x(t)$ is a solution of system (5), and $x(T) = x_1$. So system (5) is completely controllable on J. □

5 Systems with multiple delays

Consider the linear fractional Langevin delay dynamical systems with multiple delays in control of the form

$$
\begin{aligned}
&{}^C D^\beta \bigl({}^C D^\alpha + A\bigr) x(t) = Bx(t-h) + \sum_{i=0}^{M} C_i u\bigl(\sigma_i(t)\bigr), \quad 0 < \alpha, \beta \leqslant 1,\ t \in J, \\
&x(t) = \phi(t), \quad -h < t \leqslant 0, \qquad {}^C D^\alpha x(t)\big|_{t=0} = q_0,
\end{aligned} \tag{10}
$$

where $\alpha + \beta > 1$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A, B are $n \times n$ real matrices, and C_i are $n \times m$ matrices for $i = 0, 1, 2, 3 \ldots, M$, $\phi(t)$ is initial function on $[-h, 0]$. The solution of (10) is

$$
x(t) = x(t;\phi,q_0) + \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) \sum_{i=0}^{M} C_i u\bigl(\sigma_i(s)\bigr)\,\mathrm{d}s. \tag{11}
$$

Assume the following conditions as in [5]:

(H1) The functions $\sigma_i : J \to \mathbb{R}$, $i = 0, 1, 2, \ldots, M$, are twice continuously differentiable and strictly increasing in J. Moreover,

$$
\sigma_i(t) \leqslant t \quad \text{for } i = 0, 1, 2, \ldots, M, \text{ for all } t \in J. \tag{12}
$$

(H2) Introduce the time lead functions $r_i(t) : [\sigma_i(0), \sigma_i(T)] \to [0, T]$, $i = 0, 1, 2, \ldots, M$, such that $r_i(\sigma_i(t)) = t$ for $t \in J$. Further, let also $\sigma_0(T) = T$. Then the following inequality holds:

$$
\begin{aligned}
\sigma_M(T) \leqslant \sigma_{M_1}(T) \leqslant \cdots \leqslant \sigma_{m+1}(T) \leqslant 0 = \sigma_m(T) \\
< \sigma_{m-1}(T) = \cdots = \sigma_1(T) = \sigma_0(T) = T.
\end{aligned} \tag{13}
$$

The following definitions of complete state of system (10) at time t and relative controllability are assumed.

Definition 6. The set $y(t) = \{x(t), u_0(t,s)\}$, where $u_0(t,s) = u(s)$ for $s \in \min[\sigma_i(t), t)$ is said to be the complete state of system (10) at time t.

Definition 7. System (10) is said to be relatively controllable on $[0,T]$ if, for every complete state $y(t)$ and every $x_1 \in \mathbb{R}^n$, there exists a control $u(t)$ defined on $[0,T]$ such that the solution of system (10) satisfies $x(T) = x_1$.

The solution of system (10) can be written as

$$x(t) = x(t;\phi,q_0) + \int_0^t (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) \sum_{i=0}^{M} C_i u\bigl(\sigma_i(s)\bigr)\,\mathrm{d}s. \tag{14}$$

Using the time lead functions $r_i(t)$, we have

$$x(t) = x(t;\phi,q_0) + \sum_{i=0}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u(s)\,\mathrm{d}s.$$

By using inequality (14), we get

$$\begin{aligned}
x(t) = x(t;\phi,q_0) &+ \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s \\
&+ \sum_{i=0}^{m} \int_0^t (t - r_i(s))^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u(s)\,\mathrm{d}s \\
&+ \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s.
\end{aligned}$$

For simplicity, let us write the solution as

$$\begin{aligned}
x(t) &= x(t;\phi,q_0) + G(t) \\
&\quad + \sum_{i=0}^{M} \int_0^t \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u(s)\,\mathrm{d}s,
\end{aligned}$$

where

$$\begin{aligned}
G(t) &= \sum_{i=0}^{m} \int_{\sigma_i(0)}^{0} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s \\
&\quad + \sum_{i=m+1}^{M} \int_{\sigma_i(0)}^{\sigma_i(t)} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u_0(s)\,\mathrm{d}s.
\end{aligned}$$

Now let us define the controllability Grammian matrix as

$$W = \sum_{i=0}^{m} \int_{0}^{T} \bigl(T - r_i(s)\bigr)^{2(\alpha+\beta-1)} \bigl(X_{\alpha,\alpha+\beta}\bigl(T - r_i(s)\bigr)C_i \dot{r}_i(s)\bigr) \times \bigl(X_{\alpha,\alpha+\beta}\bigl(T - r_i(s)\bigr)C_i \dot{r}_i(s)\bigr)^{\intercal} \,\mathrm{d}s.$$

Theorem 3. *The linear system* (10) *is relatively controllable on* $[0, T]$ *if and only if the controllability Grammian matrix* W *is positive definite for some* $T > 0$*.*

Proof. The proof is similar to that of Theorem 1. Hence, it is omitted. □

Consider the nonlinear fractional Langevin delay dynamical systems with multiple delays in control of the form

$$\begin{aligned} &{}^{C}D^{\beta}\bigl({}^{C}D^{\alpha} + A\bigr)x(t) \\ &\quad = Bx(t-h) + \sum_{i=0}^{M} C_i u\bigl(h_i(t)\bigr) + f\bigl(t, x(t), x(t-h), u(t)\bigr), \quad t \in J, \qquad (15) \\ &x(t) = \phi(t), \quad -h < t \leqslant 0, \qquad {}^{C}D^{\alpha}x(t)\big|_{t=0} = q_0, \end{aligned}$$

where $0 < \alpha, \beta \leqslant 1$ and assume $\alpha + \beta > 1$, $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^m$ is a control vector, A, B are $n \times n$ real matrices, C_i for $i = 0, 1, 2, \dots, M$, are $n \times m$ real matrices and $f : J \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function. The solution of nonlinear system (15) using the time lead function $r_i(t)$ is given by

$$\begin{aligned} x(t) &= x(t; \phi, q_0) + G(t) \\ &\quad + \sum_{i=0}^{m} \int_{0}^{t} \bigl(t - r_i(s)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}\bigl(t - r_i(s)\bigr) C_i \dot{r}_i(s) u(s) \,\mathrm{d}s \\ &\quad + \int_{0}^{t} (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) f\bigl(s, x(s), x(s-h), u(s)\bigr) \,\mathrm{d}s. \qquad (16) \end{aligned}$$

Theorem 4. *Let the continuous function* f *satisfies the condition*

$$\lim_{|p| \to \infty} \frac{|f(t,p)|}{|p|} = 0, \qquad (17)$$

uniformly in $t \in J$ *and suppose that system* (10) *is relatively controllable on* J*. Then system* (15) *is relatively controllable on* J*.*

Proof. The proof is similar to that of Theorem 2. Hence, it is omitted. □

6 Systems with distributed delays

Consider the linear fractional Langevin delay dynamical system with distributed delays in control represented by the fractional differential equation of the form

$$\begin{aligned} &{}^{C}D^{\beta}\big({}^{C}D^{\alpha}+A\big)x(t)=Bx(t-h)+\int_{-h}^{0}d_{\tau}C(t,\tau)u(t+\tau),\\ &\quad 0<\alpha,\beta\leqslant 1,\ t\in J,\\ &x(t)=\phi(t),\quad -h<t\leqslant 0,\qquad {}^{C}D^{\alpha}x(t)\big|_{t=0}=q_0, \end{aligned}\tag{18}$$

where $\alpha+\beta>1$, $x\in\mathbb{R}^n$, and the second integral term is in the Lebesgue–Stieltjes sense with respect to τ. Let $h>0$ be given. For function $u:[-h,\tau]\to\mathbb{R}^m$ and $t\in J$, we use the symbol u_t to denote the function on $[-h,0]$ defined by $u_t(s)=u(t+s)$ for $s\in[-h,0)$. A and B are $n\times n$ real matrices, $C(t,\tau)$ is an $n\times m$ matrix continuous in t for fixed τ and of bounded variation in τ on $[-h,0]$ for each $t\in J$ and continuous from left in τ on the interval $(-h,0)$.

Definition 8. The set $y(t)=\{x(t),u_t\}$ is the complete state of system (18) at time t.

Definition 9. System (18) is said to be relatively controllable on J if, for every complete state $y(0)$ and every vector $x_1\in\mathbb{R}^n$, there exists a control $u(t)$ defined on J such that the corresponding trajectory of system (18) satisfies $x(T)=x_1$.

The solution of system (18) can be expressed as

$$\begin{aligned} x(t)&=x(t;\phi,q_0)\\ &\quad+\int_0^t(t-s)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)\left[\int_{-h}^{0}d_{\tau}C(s,\tau)u(s+\tau)\right]\mathrm{d}s. \end{aligned}\tag{19}$$

Now using the well-known unsymmetric Fubini theorem and changing the order of integration of the last term, we have

$$\begin{aligned} x(t)&=x(t;\phi,q_0)\\ &\quad+\int_{-h}^{0}dC_{\tau}\left[\int_0^t(t-s)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)C(s,\tau)u(s+\tau)\,\mathrm{d}s\right]\\ &=x(t;\phi,q_0)\\ &\quad+\int_{-h}^{0}dC_{\tau}\left[\int_{\tau}^{0}\big(t-(s-\tau)\big)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}\big(t-(s-\tau)\big)C(s-\tau,\,\tau)u_0(s)\,\mathrm{d}s\right]\\ &\quad+\int_0^t\left[\int_{-h}^{0}\big(t-(s-\tau)\big)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)C_{\tau}C_t(s-\tau,\,\tau)\right]u(s)\,\mathrm{d}s, \end{aligned}\tag{20}$$

where

$$C_t(s,\tau)=\begin{cases}C(s,\tau), & s\leqslant t,\\ 0, & s>t,\end{cases} \tag{21}$$

$\mathrm{d}C_\tau$ denotes the Lebesgue–Stieltjes integration with respect to τ in the function $C(t,\tau)$.

For convenience, let us introduce the notation

$$G(t,s)=\int_{-h}^{0}\bigl(t-(s-\tau)\bigr)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)d_\tau C_t(s-\tau,\,\tau). \tag{22}$$

Define the controllability Grammian matrix by

$$W=\int_0^T G(T,s)G^{\mathsf{T}}(T,s)\,\mathrm{d}s. \tag{23}$$

Theorem 5. *The linear fractional Langevin delay dynamical systems* (18) *with distributed delays in control are controllable on J if and only if the controllability Grammian matrix*

$$W=\int_0^T G(T,s)G^{\mathsf{T}}(T,s)\,\mathrm{d}s \tag{24}$$

is positive definite, for some $T>0$.

Proof. The proof is similar to that of Theorem 2. Hence, it is omitted. □

Consider the nonlinear fractional Langevin delay dynamical systems with distributed delays in control of the form

$$\begin{aligned}&{}^{C}D^{\beta}\bigl({}^{C}D^{\alpha}+A\bigr)x(t)\\&\quad=Bx(t-h)+\int_{-h}^{0}\mathrm{d}_\tau C(t,\tau)u(t+\tau)+f\bigl(t,x(t),x(t-h),u(t)\bigr),\\&x(t)=\phi(t),\quad -h<t\leqslant 0,\qquad {}^{C}D^{\alpha}x(t)\big|_{t=0}=q_0,\end{aligned} \tag{25}$$

where $0<\alpha,\beta\leqslant 1$, and assume $\alpha+\beta>1$, $x\in\mathbb{R}^n$, A, B and C are as above and $f: J\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$ is a continuous function. Now the solution of nonlinear system (25) can be expressed in the following form:

$$\begin{aligned}x(t)&=x(t;\phi,q_0)\\&\quad+\int_0^t(t-s)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)\Biggl[\int_{-h}^{0}d_\tau C(s,\tau)u(s+\tau)\Biggr]\mathrm{d}s\\&\quad+\int_0^t(t-s)^{\alpha+\beta-1}X_{\alpha,\alpha+\beta}(t-s)f\bigl(s,x(s),x(s-h),u(s)\bigr)\,\mathrm{d}s.\end{aligned} \tag{26}$$

It follows from the unsymmetric Fubini theorem that

$$\begin{aligned} x(t) = {} & x(t;\phi,q_0) \\ & + \int_{-h}^{0} \mathrm{d}C_\tau \left[\int_{\tau}^{0} \bigl(t-(s-\tau)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) C(s-\tau,\tau)\psi(s)\,\mathrm{d}s \right] \\ & + \int_{0}^{t} \left[\int_{-h}^{0} \bigl(t-(s-\tau)\bigr)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) C_\tau C_t(s-\tau,\tau) \right] u(s)\,\mathrm{d}s \\ & + \int_{0}^{t} (t-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(t-s) f\bigl(s,x(s),x(s-h),u(s)\bigr)\,\mathrm{d}s, \end{aligned}$$

where

$$C_t(s,\tau) = \begin{cases} C(s,\tau), & s \leqslant t, \\ 0, & s > t, \end{cases}$$

and $\mathrm{d}_\tau C_t$ denotes the Lebesgue–Stieltjes integration with respect to the variable τ in the function $C_t(s-\tau,\tau)$. For brevity, let us introduce the notation

$$\begin{aligned} & \eta\bigl(y(0),x_1;z,v\bigr) \\ & \quad = x_1 - x(T;\phi,q_0) \\ & \qquad - \int_{0}^{T} (T-s)^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(T-s) f\bigl(s,x(s),x(s-h),u(s)\bigr)\,\mathrm{d}s \\ & \qquad - \int_{-h}^{0} \mathrm{d}C_\tau \left[\int_{\tau}^{0} (T-(s-\tau))^{\alpha+\beta-1} X_{\alpha,\alpha+\beta}(T-s) C(s-\tau,\tau) u_0(s)\,\mathrm{d}s \right]. \end{aligned}$$

Define the control function by

$$u(t) = G^{\intercal}(T,t) W^{-1} \eta\bigl(y(0),x_1;x,u\bigr), \tag{27}$$

where the complete state $y(0)$ and the vector $x_1 \in \mathbb{R}^n$ are chosen arbitrary and $\intercal$ denotes the matrix transpose.

Theorem 6. *Let the continuous function f satisfies the condition*

$$\lim_{|p|\to\infty} \frac{|f(t,p)|}{|p|} = 0 \tag{28}$$

uniformly in $t \in J$, and suppose that the linear fractional system (18) *is relatively controllable on J. Then system* (25) *is relatively controllable on J.*

Proof. The proof is similar to that of Theorem 2. Hence, it is omitted. □

7 Examples

Example 1. Consider the linear fractional Langevin delay dynamical system

$$
\begin{aligned}
&{}^{C}D^{\beta}\big({}^{C}D^{\alpha}+A\big)x(t)=Bx(t-h)+Cu(t), \quad t\in J,\\
&x(t)=\phi(t), \quad -h<t\leqslant 0, \qquad {}^{C}D^{\alpha}x(t)\big|_{t=0}=q_0,
\end{aligned}
\tag{29}
$$

where $\alpha=1/2$, $\beta=2/3$, $h=1$,

$$
A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \qquad B=\begin{pmatrix}1 & 1\\ -1 & 0\end{pmatrix},
$$

$C=(1,\,0)^{\intercal}$, $x(t)=\phi(t)\in\mathbb{R}^2$ and $x(t)=(x_1(t),\,x_2(t))^{\intercal}$, with initial conditions $x(0)=(0,\,1)^{\intercal}$, $q_0=(0,\,0)^{\intercal}$ and final condition $x(1)=(1,\,0)^{\intercal}$. Here $x(t)$ is the state variable, and $u(t)$ is the control variable. In this example, the solution takes the form as in [24]:

$$
\begin{aligned}
x(t)={}&E_{1/2}\big(-At^{1/2}\big)\phi(0)+t^{1/2}AE_{1/2,3/2}\big(-At^{1/2}\big)\phi(0)\\
&+t^{1/2}E_{1/2,3/2}(-At^{1/2})q_0\\
&+B\int\limits_{-1}^{0}(t-s-1)^{1/6}E_{1/2,7/6}\big(-A(t-s-1)^{1/2}\big)\phi(s)\,\mathrm{d}s\\
&+\int\limits_{0}^{t}(t-s)^{1/6}E_{1/2,7/6}\big(-A(t-s)^{1/2}\big)Cu(s)\,\mathrm{d}s.
\end{aligned}
\tag{30}
$$

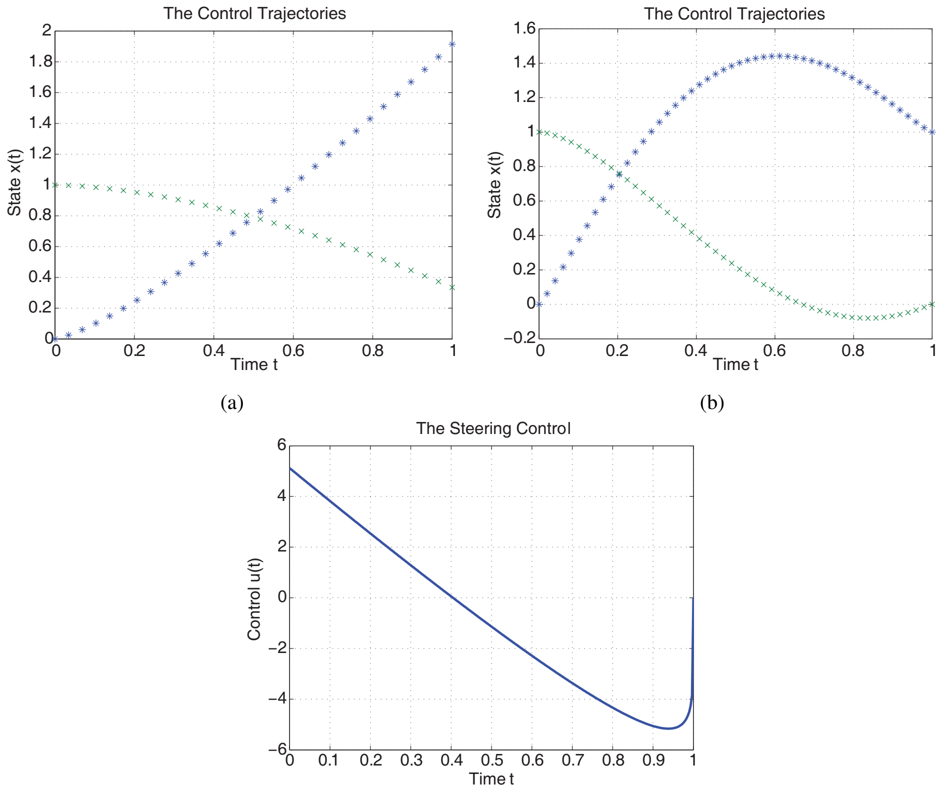
By simple matrix calculation, we have the controllability Grammian matrix as

$$
W=\begin{pmatrix}0.2739 & -0.2901\\ -0.2901 & 0.5259\end{pmatrix}>0,
$$

which is positive definite. Hence, system (29) is completely controllable on $[0,1]$. Next, we give the numerical simulation of the state and control variables for system (29) and the control

$$
\begin{aligned}
u(t)={}&(1-t)^{1/6}\big(E_{1/2,7/6}\big(-A(1-t)^{1/2}\big)C\big)^{\intercal}\\
&\times W^{-1}\big(x_1-E_{1/2}(-A)x(0)-AE_{1/2,3/2}(-A)x(0)\\
&-BE_{1/2,13/6}(-A)x(0)\big),
\end{aligned}
$$

which steers $x(0)=(0,\,1)^{\intercal}$ to $x(1)=(1,\,0)^{\intercal}$. Figure 1(a) represents the trajectory of equation (29) without control starting from the initial point $x(0)=(0,\,1)^{\intercal}$ and not reaching the final point $x(1)=(1,\,0)^{\intercal}$ in $[0,1]$. Figure 1(b) represents the trajectory of



(a) (b)

(c)

Figure 1

equation (29) with control starting from the initial vector $x(0) = (0,\, 1)^\intercal$ and reaching the final vector $x(1) = (1,\, 0)^\intercal$ in $[0, 1]$ and Figure 1(c) represents the steering control.

Example 2. Consider the nonlinear fractional Langevin delay dynamical system

$$\begin{aligned} &{}^{C}D^{\beta}\big({}^{C}D^{\alpha} + A\big)x(t) \\ &\quad = Bx(t-h) + Cu(t) + f\big(t, x(t), x(t-h), u(t)\big), \quad t \in J, \\ &x(t) = \phi(t), \quad -h < t \leqslant 0, \qquad {}^{C}D^{\alpha}x(t)\big|_{t=0} = q_0, \end{aligned} \tag{31}$$

where $\alpha = 2/3$, $\beta = 1/2$, $h = 1$,

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

$C = (1,\, 0)^\intercal$, $x(t) = \phi(t) \in \mathbb{R}^2$ and $x(t) = (x_1(t),\, x_2(t))^\intercal$, with initial conditions $x(0) = (0,\, 2)^\intercal$, $q_0 = (0,\, 0)^\intercal$ and final condition $x(1) = (2,\, 0)^\intercal$ and $f(t, x(t), x(t-h), u(t)) =$

$(0,\ (x_1(t)+x_2(t))/(1+x_2(t-1)+u(t)))^\intercal$. Here $x(t)$ is the state variable, and $u(t)$ is the control variable. The solution of the nonlinear system (31) is

$$\begin{aligned} x(t) &= E_{2/3}(-At^{2/3})\phi(0) + t^{2/3}AE_{2/3,5/3}\big(-At^{2/3}\big)\phi(0) \\ &\quad + t^{2/3}E_{2/3,5/3}\big(-At^{2/3}\big)q_0 \\ &\quad + B\int_{-1}^{0}(t-s-1)^{1/6}E_{2/3,7/6}\big(-A(t-s-1)^{2/3}\big)\phi(s)\,\mathrm{ds} \\ &\quad + \int_{0}^{t}(t-s)^{1/6}E_{2/3,7/6}\big(-A(t-s)^{2/3}\big)Cu(s)\,\mathrm{ds} \\ &\quad + \int_{0}^{t}(t-s)^{1/6}E_{2/3,7/6}\big(-A(t-s)^{2/3}\big)f\big(s,x(s),x(s-h),u(s)\big)\,\mathrm{ds}. \end{aligned} \tag{32}$$

By simple matrix calculation, we have the controllability Grammian matrix as

$$W = \begin{pmatrix} 0.4047 & -0.3330 \\ -0.3330 & 0.4238 \end{pmatrix} > 0,$$

which is positive definite. Hence, the linear system of (31) is completely controllable on $[0,1]$. The nonlinear function $f(t,x(t),x(t-h),u(t))$ satisfies the hypothesis of Theorem 2. Thus, the nonlinear system (31) is completely controllable on $[0,1]$. Next, we give the numerical simulation of the state and control variables for system (31) on $[0,1]$ with the initial points $x(0)=(0,\ 2)^\intercal$, $q_0=(0,\ 0)^\intercal$ and final point $x(1)=(2,\ 0)^\intercal$ approximated by the following algorithm:

$$\begin{aligned} u_n(t) &= (1-t)^{1/6}\big(E_{2/3,7/6}\big(-A(1-t)^{2/3}\big)C\big)^\intercal W^{-1}\big(x_1 - E_{2/3}(-A)\phi(0) \\ &\quad - AE_{2/3,5/3}(-A)\phi(0) - E_{2/3,5/3}(-A)q_0 - BE_{2/3,13/6}(-A)\phi(0) \\ &\quad - \int_{0}^{1}(1-t)^{1/6}E_{2/3,7/6}\big(-A(1-t)^{2/3}\big)f\big(t,x_n(t),x_n(t-h),u_n(t)\big)\,\mathrm{dt}\big), \\ x_{n+1}(t) &= E_{2/3}\big(-At^{2/3}\big)\phi(0) + t^{2/3}AE_{2/3,5/3}\big(-At^{2/3}\big)\phi(0) \\ &\quad + t^{2/3}E_{2/3,5/3}\big(-At^{2/3}\big)q_0 + t^{7/6}BE_{2/3,13/6}\big(-At^{2/3}\big)\phi(0) \\ &\quad + \int_{0}^{t}(t-s)^{1/6}E_{2/3,7/6}\big(-A(t-s)^{2/3}\big)Cu_n(s)\,\mathrm{ds} \\ &\quad + \int_{0}^{t}(t-s)^{1/6}E_{2/3,7/6}\big(-A(t-s)^{2/3}\big)f\big(s,x_n(s),x_n(s-h),u_n(s)\big)\,\mathrm{ds} \end{aligned}$$

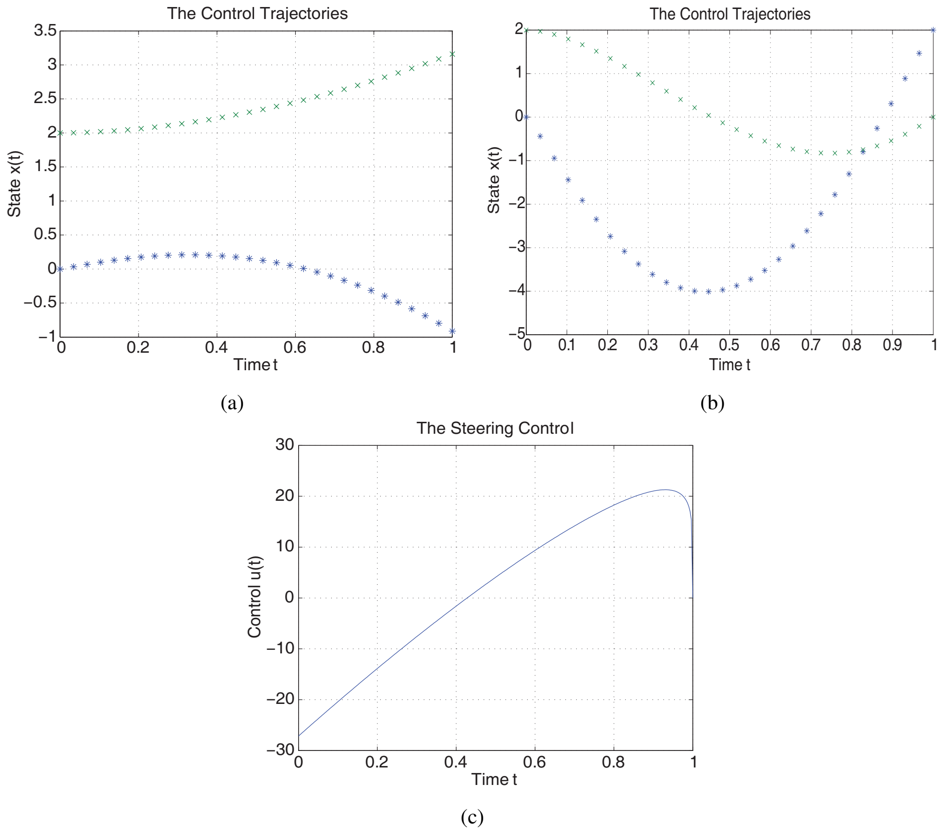


Figure 2

with $x_0(t) = x_0$, where $n = 0, 1, 2, \dots$. Using MATLAB, the controlled trajectories and steering control $u(t)$ are computed. Figure 2(a) represents the trajectory of equation (31) without control starting from the initial point $x(0) = (0,\ 2)^\intercal$ and not reaching the final point $x(1) = (2,\ 0)^\intercal$ in $[0, 1]$. Figure 2(b) represents the trajectory of equation (31) with control starting from the initial vector $x(0) = (0,\ 2)^\intercal$ and reaching the final vector $x(1) = (2,\ 0)^\intercal$ in $[0, 1]$, and Fig. 2(c) represents the steering control.

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