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Infinitely many solutions for the p-fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity

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Abstract. In this paper, we consider the fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity. By means of the concentration–compactness principle in fractional Sobolev space and the Kajikiya's new version of the symmetric mountain pass lemma, we obtain the existence of infinitely many solutions, which tend to zero for suitable positive parameters.

Keywords: fractional Kirchhoff equations, fractional magnetic operator, critical nonlinearity, variational methods.

1 Introduction

The main purpose of this paper is to study the existence and multiplicity of solutions for the p-fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity

$$M\big([u]_{s,A}^p\big)(-\Delta)_{p,A}^s u = \alpha|u|^{p_s^*-2}u + \beta k(x)|u|^{q-2}u, \quad x \in \mathbb{R}^N, \tag{1}$$

where $\varepsilon > 0$ is a positive parameter, $N > ps$, $0 < s < 1$,

$$[u]_{s,A}^p := \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y,$$

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where $p_s^* = pN/(N-ps)$ is the critical Sobolev exponent, $A \in C(\mathbb{R}^N, \mathbb{R}^N)$ is a magnetic potential, $k(x) \in L^r(\mathbb{R}^N)$ with $r = p_s^*/(p_s^* - q)$, α and β are real parameters.

Nonlocal operators can be seen as the infinitesimal generators of Lévy stable diffusion processes [2]. Moreover, they allow us to develop a generalization of quantum mechanics and also to describe the motion of a chain or an array of particles that are connected by elastic springs as well as unusual diffusion processes in turbulent fluid motions and material transports in fractured media (for more details, see, for example, [2, 11, 12] and the references therein). Indeed, the literature on nonlocal fractional operators and on their applications is quite large, see, for example, the recent monograph [10], the extensive paper [8], and the references cited there.

The important reason for studying problem (1) lies in the new feature of the Kirchhoff problems. More precisely, in 1883, Kirchhoff proposed the following model

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{\lambda} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \right) \frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

as a generalization of the well-known d'Alembert's wave equation for free vibrations of elastic strings. Here L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density, and p_0 is the initial tension. Essentially, Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Recently, Fiscella and Valdinoci in [17] first deduced a stationary fractional Kirchhoff model, which considered the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string, see the Appendix of [17] for more details. Moreover, the authors in [17] studied the following Kirchhoff type problem involving critical exponent:

$$\begin{aligned} &M\big([u]_s^2\big)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^*-2}u \quad \text{in } \Omega, \\ &u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{aligned} \tag{3}$$

where Ω is an open bounded domains in $\mathbb{R}^N$. By using the mountain pass theorem and the concentration–compactness principle together with a truncation technique, they obtained the existence of nonnegative solutions for problem (3). For more recent results, we refer the readers to [1, 4, 6, 24, 32] and references therein.

If the magnetic field $A \equiv 0$, the operator $(-\Delta)_{p,A}^s$ can be reduced to the p-fractional Laplacian operator $(-\Delta)_p^s$, which is defined as

$$(-\Delta)_p^s u(x) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, \mathrm{d}y$$

along any $u \in C_0^\infty(\mathbb{R}^N)$, where $B_\varepsilon(x)$ denotes the ball of $\mathbb{R}^N$ centered at $x \in \mathbb{R}^N$ and radius $\varepsilon > 0$. There are also some interesting results obtained by using some different approaches under various hypotheses on the potential and the nonlinearity. In [18], the authors obtained the existence and multiplicity results by using Morse theory. In [38], the authors investigated the existence of solutions for Kirchhoff-type problem involving the fractional p-Laplacian via variational methods, where the nonlinearity is subcritical, and the Kirchhoff function is nondegenerate. In [31], the authors studied a nonlocal equation involving the fractional p-Laplacian

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) + \lambda h \quad \text{in } \mathbb{R}^n.$$

When the nonlinearity f is assumed to have exponential growth, by using a fixed point method, the authors established an existence result on weak solutions. By using the mountain pass theorem and Ekeland's variational principle, the authors in [40] studied the multiplicity of solutions to a nonhomogeneous Kirchhoff-type problem driven by the fractional p-Laplacian, where the nonlinearity is convex-concave, and the Kirchhoff function is degenerate. Using the same methods as in [40], Pucci et al. in [28] obtained the existence of multiple solutions for the nonhomogeneous fractional p-Laplacian equations of Schrödinger–Kirchhoff type in the whole space. Indeed, there is a wide literature concerning the study of multiplicity results for critical Kirchhoff problems under a non-degenerate setting, see, for example, [3, 7, 8, 14, 15, 21–23, 26, 27, 30, 33, 35, 43] for the recent advances in this direction.

When $A \neq 0$ and $p = 2$, Xiang [37] first studied the following Schrödinger–Kirchhoff-type equation involving the fractional p-Laplacian and the magnetic operator

$$M\big([u]_{s,A}^2\big)(-\Delta)_A^s u + V(x)u = f\big(x,|u|\big)u \quad \text{in } \mathbb{R}^N, \tag{4}$$

where the right-hand term in (4) satisfies the subcritical growth. By using variational methods, they obtained several existence results for problem (4). Following similar methods, for $M(t) = a + bt$ with $a \in \mathbb{R}_0^+$ and $p = 2$, Wang and Xiang in [34] proved the existence of two solutions and infinitely many solutions for fractional Schrödinger–Choquard–Kirchhoff-type equations with external magnetic operator and critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality. In [41], the authors first considered the following fractional Schrödinger equations:

$$\varepsilon^{2s}(-\Delta)_{A_\varepsilon}^s u + V(x)u = f\big(x,|u|\big)u + K(x)|u|^{2_\alpha^*-2}u \quad \text{in } \mathbb{R}^N, \tag{5}$$

the existence of ground state solution (mountain pass solution) u_ε, which tends to the trivial solution as $\varepsilon \to 0$, is obtained by using variational methods. Moreover, they proved the existence of infinitely many solutions and sign-changing solutions for problem (5) under some additional assumptions. But for the case $p \neq 2$, to our best knowledge, there is no results about p-fractional Schrödinger–Kirchhoff equations with electromagnetic fields.

In this paper, we consider infinitely many solutions for the p-fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity. Here we use the fractional version of Lions' second concentration–compactness principle and concentration–compactness principle at infinity to prove that the Palais–Smale condition $(\mathrm{PS})_{\mathrm{c}}$ holds. Some difficulties arise when dealing with this problem because of the appearance of the magnetic field and the critical frequency and of the nonlocal nature of the fractional Laplacian. Therefore, we need to develop new techniques to overcome difficulties induced by these new features. As far as we know, this is the first time that the fractional version of the concentration–compactness principle and variational methods have been combined to get the multiplicity of solutions for the p-fractional Schrödinger–Kirchhoff equations with electromagnetic fields and critical nonlinearity.

The paper is organized as follows. In Section 2, we will introduce the working space and give some necessary definitions and properties, which will be used in the sequel. In Section 3, we will use the fractional version of Lions' second concentration–compactness principle and concentration–compactness principle at infinity to prove the $(\mathrm{PS})_{\mathrm{c}}$ condition. In Section 4, using symmetric mountain pass lemma together with some delicate estimates, we will prove the main result.

2 Preliminaries

For the convenience of the reader, we recall in this part some definitions and basic properties of fractional Sobolev spaces. For a deeper treatment of the (magnetic) fractional Sobolev spaces and their applications to fractional Laplacian problems of elliptic type, we refer to [25, 37, 41] and the references therein.

For any $s \in (0,1)$, the fractional Sobolev space $W_A^{s,p}(\mathbb{R}^N,\mathbb{C})$ is defined by

$$W_A^{s,p}\big(\mathbb{R}^N,\mathbb{C}\big) = \big\{u \in L^p\big(\mathbb{R}^N,\mathbb{C}\big)\colon [u]_{s,A} < \infty\big\},$$

where $[u]_{s,A}$ denotes the so-called Gagliardo seminorm, that is,

$$[u]_{s,A} = \bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/p},$$

and $W_A^{s,p}(\mathbb{R}^N,\mathbb{C})$ is endowed with the norm

$$\|u\|_{W_A^{s,p}(\mathbb{R}^N,\mathbb{C})} = \big([u]_{s,A}^p + \|u\|_{L^p}^p\big)^{1/p}.$$

If $A=0$, then $W_A^{s,p}(\mathbb{R}^N,\mathbb{C})$ reduces to the well-known space $W^{s,p}(\mathbb{R}^N,\mathbb{C})$. Furthermore, the space $D_A^{s,p}(\mathbb{R}^N)$ is defined as

$$D_A^{s,p}\big(\mathbb{R}^N,\mathbb{C}\big) = \big\{u \in L^{p_s^*}\big(\mathbb{R}^N,\mathbb{C}\big)\colon [u]_{s,A} < \infty\big\}$$

and endowed with the norm $[u]_{s,A}$. We have the following diamagnetic inequality:

Lemma 1. *For every $u \in D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$, we get $|u| \in D^{s,p}(\mathbb{R}^N)$. More precisely,*

$$\big[|u|\big]_s \leqslant [u]_{s,A}.$$

Proof. The assertion follows directly from the pointwise diamagnetic inequality

$$\big||u(x)| - |u(y)|\big| \leqslant \big|u(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u(y)\big|,$$

for a.e. $x, y \in \mathbb{R}^N$, see [13, Lemma 3.1, Remark 3.2].

We recall the following embedding theorem, the proof of which is similar to [13, Lemma 3.5] and [25].

Proposition 1. *Let $A \in C(\mathbb{R}^N, \mathbb{R}^N)$. Then the embedding*

$$D_A^{s,p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{p_s^*}(\mathbb{R}^N, \mathbb{C}), \qquad W_A^{s,p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^{\theta}(\mathbb{R}^N, \mathbb{C})$$

is continuous for any $\theta \in [p, p_s^*]$. Moreover, the embedding

$$W_A^{s,p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow\hookrightarrow L^{\theta}_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{C})$$

is compact for any $\theta \in [p, p_s^*)$.

For our problem, we first assume that the Kirchhoff function $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ and the weight function $k(x)$ satisfy the following assumptions:

(A1) $M \in C(\mathbb{R}_0^+, \mathbb{R}^+)$ satisfies $\inf_{t \in \mathbb{R}_0^+} M(t) \geqslant m_0 > 0$, where m_0 is a constant.

(A2) There exists $\theta \in [1, N/(N - ps))$ such that $\theta \widetilde{M}(t) := \theta \int_0^t M(\tau)\, \mathrm{d}\tau \geqslant M(t)t$ for any $t \in \mathbb{R}_0^+$.

(A3) $0 \leqslant k(x) \in L^r(\mathbb{R}^N)$, where $r = p_s^*/(p_s^* - q)$.

A typical example for M is $M(t) = m_0 + b_1 t^{\theta - 1}$ with $\theta \geqslant 1$, $m_0 \in \mathbb{R}^+$, and $b_1 \in \mathbb{R}_0^+$. When M is of this type, the Kirchhoff problem is said to be *nondegenerate* if $m_0 > 0$, while it is called *degenerate* if $m_0 = 0$.

The energy functional $J : D_A^{s,p}(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ associated with problem (1)

$$J(u) := \frac{1}{p} \widetilde{M}\big([u]_{s,A}^p\big) - \frac{\alpha}{p_s^*} \int_{\mathbb{R}^N} |u|^{p_s^*}\, \mathrm{d}x - \frac{\beta}{q} \int_{\mathbb{R}^N} k(x)|u|^q\, \mathrm{d}x$$

is well defined. Under the assumptions, it is easy to check that, as shown in [29, 36], $J \in C^1(D_A^{s,p}(\mathbb{R}^N, \mathbb{C}), \mathbb{R})$ and its critical points are weak solutions of problem (1).

Now we first give the definition of weak solutions for problem (1).

Definition 1. We say that $u \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$ is a weak solution of problem (1) if

$$M\big([u]_{s,A_\varepsilon}^p\big) \operatorname{Re} L(u, v) = \operatorname{Re} \int_{\mathbb{R}^N} \big(\alpha |u|^{2_s^* - 2} u + \beta k(x) |u|^{q-2} u\big) \bar{v}\, \mathrm{d}x,$$

where

$$\begin{aligned} L(u, v) = \iint_{\mathbb{R}^{2N}} \frac{1}{|x - y|^{N+ps}} &\big(\big|u(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)} u(y)\big|^{p-2} \\ &\times \big(u(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)} u(y)\big) \\ &\times \overline{\big(v(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)} v(y)\big)}\,\big)\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

and $v \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$.

In the sequel, we will omit the term weak when referring to solutions that satisfy the conditions of Definition 1. Our main result of this paper is stated as follows.

Theorem 1. *Let* (A1)–(A3) *and* $1 < q < p$ *hold. Then*:

- (i) *For all* $\alpha > 0$*, there exists* $\beta_0 > 0$ *such that if* $0 < \beta < \beta_0$*, then* (1) *has a sequence of solutions* $\{u_n\}_n$ *with* $J(u_n) < 0$*,* $J(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n \to 0$.
- (ii) *For all* $\beta > 0$*, there exists* $\alpha_0 > 0$ *such that if* $0 < \alpha < \alpha_0$*, then* (1) *has a sequence of solutions* $\{u_n\}_n$ *with* $J(u_n) < 0$*,* $J(u_n) \to 0$ *and* $\lim_{n\to\infty} u_n \to 0$.

Remark 1. Unlike solutions with concentration phenomena constructed in some earlier works without the magnetic field. We obtain the existence of infinitely many solutions for the p-fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity, and our nontrivial solutions are closed to the trivial solution.

Remark 2. It should be mentioned that our result also extends the result in [5, 17, 19, 35] in which the authors considered the case $A = 0$ and $p = 2$. To our best knowledge, it seems that there is no result on the existence of solutions for the p-fractional Kirchhoff equations with electromagnetic fields and critical nonlinearity.

Remark 3. The proof of Theorem 1 is mainly based on the application of the symmetric mountain pass lemma introduced by Kajikiya in [19]. For this, we need a truncation argument, which allow us to control from below functional J. Furthermore, as usual in elliptic problems involving critical nonlinearities, the main difficulties is to prove the $(\mathrm{PS})_\mathrm{c}$ condition, because of the appearance of the magnetic field and the critical nonlinearity, and of the nonlocal nature of the fractional Laplacian. To overcome this difficulty, we fix parameters α and β under a suitable threshold strongly depending on assumptions (A1) and (A2).

3 The Palais–Smale condition

In this section, we recall the concentration–compactness principle in the setting of the fractional p-Laplacian, see [39, Def. 2.1, Thms. 2.1 and 2.2] and [16].

Definition 2. Let $\widetilde{\mathcal{M}}(\mathbb{R})$ denote the finite nonnegative Borel measure space on $\mathbb{R}^N$. For any $\mu \in \widetilde{\mathcal{M}}(\mathbb{R}^N)$, $\mu(\mathbb{R}^N) = \|\mu\|$ holds. We say that $\mu \rightharpoonup \mu$ $*$-weakly in $\widetilde{\mathcal{M}}(\mathbb{R}^N)$ if $(\mu_n, \eta) \to (\mu, \eta)$ holds for all $\eta \in C_0(\mathbb{R}^N)$ as $n \to \infty$.

Proposition 2. *Let* $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ *with upper bound* $C > 0$ *for all* $n \geqslant 1$ *and suppose that*

$$u_n \rightharpoonup u \quad \text{weakly in } D^{s,p}\bigl(\mathbb{R}^N\bigr),$$

$$\int\limits_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y \rightharpoonup \mu \quad *\text{-weakly in } \widetilde{\mathcal{M}}\bigl(\mathbb{R}^N\bigr),$$

$$\bigl|u_n(x)\bigr|^{p_s^*} \rightharpoonup \nu \quad *\text{-weakly in } \widetilde{\mathcal{M}}\bigl(\mathbb{R}^N\bigr).$$

Then

$$\mu = \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y + \sum_{j\in I}\mu_j\delta_{x_j} + \tilde{\mu}, \quad \mu(\mathbb{R}^N) \leqslant C^p,$$

$$\nu = |u|^{p_s^*} + \sum_{j\in I}\nu_j\delta_{x_j}, \quad \nu(\mathbb{R}^N) \leqslant S^{p_s^*}C^p,$$

where I is at most countable, sequences $\{\mu_j\}_j, \{\nu_j\}_j \subset \mathbb{R}_0^+$, $\{x_j\}_j \subset \mathbb{R}^N, \delta_{x_j}$ is the Dirac mass centered at x_j, $\tilde{\mu}$ is a nonatomic measure. Furthermore,

$$\nu(\mathbb{R}^N) \leqslant S^{-p_s^*/p}\mu(\mathbb{R}^N)^{p_s^*/p}, \qquad \nu_j \leqslant S^{-p_s^*/p}\mu_j^{p_s^*/p} \quad \forall j \in I,$$

here $S > 0$ is the best constant of $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$.

Proposition 3. *Let $\{u_n\}_n \subset D^{s,p}(\mathbb{R}^N)$ be a bounded sequence such that*

$$\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y \rightharpoonup \mu \quad *\text{-weakly in } \widetilde{\mathcal{M}}(\mathbb{R}^N),$$

$$|u_n(x)|^{p_s^*} \rightharpoonup \nu \quad *\text{-weakly in } \widetilde{\mathcal{M}}(\mathbb{R}^N),$$

and define

$$\mu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N\colon |x|>R\}}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y\,\mathrm{d}x,$$

$$\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty} \int_{\{x\in\mathbb{R}^N\colon |x|>R\}} |u_n|^{p_s^*}\,\mathrm{d}x.$$

Then the quantities μ_∞ and ν_∞ are well defined and satisfy

$$\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}y\,\mathrm{d}x = \int_{\mathbb{R}^N}\mathrm{d}\mu + \mu_\infty,$$

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{p_s^*}\,\mathrm{d}x = \int_{\mathbb{R}^N}\mathrm{d}\nu + \nu_\infty.$$

Moreover,

$$S\nu_\infty^{p/p_s^*} \leqslant \mu_\infty.$$

Next, we perform a careful analysis of the behavior of the minimizing sequences with the aid of the concentration–compactness principle in fractional Sobolev space stated above, which allows us to recover compactness below some critical threshold.

Lemma 2. *Let* (A1)–(A3), $1 < q \leqslant p$ *and* $c < 0$ *hold. Then*:

(i) *There exists* $C > 0$ *such that, for all* $n \in \mathbb{N}$, $\|u_n\| \leqslant C$;

(ii) *For each* $\alpha > 0$, *there exists* $\beta_* > 0$ *such that if* $0 < \beta < \beta_*$, *then* J *satisfies* $(\mathrm{PS})_{\mathrm{c}}$;

(iii) *For each* $\beta > 0$, *there exists* $\alpha_* > 0$ *such that if* $0 < \alpha < \alpha_*$, *then* J *satisfies* $(\mathrm{PS})_{\mathrm{c}}$.

Proof. We first prove that $\{u_n\}_n$ is bounded in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$. Let $\{u_n\}_n$ be a $(\mathrm{PS})_{\mathrm{c}}$-sequence in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$. Then

$$c + o_n(\|u_n\|) = J(u_n) = \frac{1}{p}\widetilde{M}([u_n]_{s,A}^p) - \frac{\alpha}{p_s^*}\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\int_{\mathbb{R}^N}k(x)|u_n|^q\,\mathrm{d}x,$$

$$\begin{aligned}\langle J'_\varepsilon(u_n), v\rangle &= \mathrm{Re}\left\{M([u_n]_{s,A}^p)L(u_n,v) - \int_{\mathbb{R}^N}\left(\alpha|u|^{p_s^*-2}u + \beta k(x)|u|^{q-2}u\right)\bar{v}\,\mathrm{d}x\right\}\\ &= o(1)\|u_n\|.\end{aligned}$$

Therefore,

$$\begin{aligned}0 > c + o_n(\|u_n\|) &= J(u_n) - \frac{1}{p_s^*}\langle J'(u_n), u_n\rangle\\
&= \frac{1}{p}\widetilde{M}([u_n]_{s,A}^p) - \frac{1}{p_s^*}M([u_n]_{s,A}^p)[u_n]_{s,A}^p - \beta\left(\frac{1}{q} - \frac{1}{p_s^*}\right)\int_{\mathbb{R}^N}k(x)|u_n|^q\,\mathrm{d}x\\
&\geqslant \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)M([u_n]_{s,A}^p)[u_n]_{s,A}^p - \beta\left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|k(x)\|_r\left(\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,\mathrm{d}x\right)^{q/p_s^*}\\
&\geqslant \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)m_0[u_n]_{s,A}^p - \beta\left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|k(x)\|_r S^{-q/p}[u_n]_{s,A}^q\\
&\geqslant \left(\frac{1}{p\theta} - \frac{1}{p_s^*}\right)m_0\|u_n\|^p - \beta\left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|k(x)\|_r S^{-q/p}\|u_n\|^q.\end{aligned}$$

Since $\theta \in [1, N/(N-ps))$ and $q < p$, it follows that $\{u_n\}_n$ is bounded in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$. Hence, by diamagnetic inequality, $\{|u_n|\}_n$ is bounded in $D^{s,p}(\mathbb{R}^N,\mathbb{C})$. Then, for some subsequence, there is $u_0 \in E$ such that $u_n \rightharpoonup u_0$ in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$. We claim that, as $n \to \infty$,

$$\int_{\mathbb{R}^N}|u_n|^{p_s^*}\,\mathrm{d}x \to \int_{\mathbb{R}^N}|u_0|^{p_s^*}\,\mathrm{d}x.$$

In order to prove this claim, we invoke Prokhorov's theorem (see [9, Thm. 8.6.2]) to conclude that there exist $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N}\frac{||u_n(x)| - |u_n(y)||^p}{|x-y|^{N+ps}}\,\mathrm{d}y \rightharpoonup \mu \quad (*\text{-weak-sense of measures}),$$

$$|u_n|^{p_s^*} \rightharpoonup \nu \quad (*\text{-weak-sense of measures}),$$

where μ and ν are a nonnegative bounded measures on $\mathbb{R}^N$. It follows from Proposition 2 that either $u_n \to u$ in $L^{p_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or $\nu = |u|^{p_s^*} + \sum_{j\in I}\delta_{x_j}\nu_j$ as $n \to \infty$, where I is a countable set, $\{\nu_j\}_j \subset [0,\infty)$, $\{x_j\}_j \subset \mathbb{R}^N$.

Take $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \phi \leqslant 1$; $\phi \equiv 1$ in $B(x_j,\rho)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_j, 2\rho)$. For any $\rho > 0$, define $\phi_\rho = \phi((x-x_j)/\rho)$, where $j \in I$. It follows that $\{u_n\phi_\rho\}_n$ is bounded in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$ since $\{u_n\}_n$ is bounded in $D_A^{s,p}(\mathbb{R}^N,\mathbb{C})$. Then $\langle J'(u_n), u_n\phi_\rho\rangle \to 0$, which implies

$$\begin{aligned} M\big([u_n]_{s,A}^p\big) \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)|^p\phi_\rho(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ + \operatorname{Re}\big\{M\big([u_n]_{s,A}^p\big)L(u_n,u_n\phi_\rho)\big\} \\ = \alpha\int\limits_{\mathbb{R}^N}|u_n|^{p_s^*}\phi_\rho\,\mathrm{d}x + \beta\int\limits_{\mathbb{R}^N}k(x)|u_n|^q\phi_\rho\,\mathrm{d}x + o_n(1), \end{aligned} \tag{6}$$

where

$$\begin{aligned} L(u_n,u_n\phi_\rho) = \iint\limits_{\mathbb{R}^{2N}} \frac{1}{|x-y|^{N+ps}}\big(\big|u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)\big|^{p-2} \\ \times \big(u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)\big)\overline{u_n(x)\big(\phi_\rho(x)-\phi_\rho(y)\big)}\,\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

It is easy to verify that

$$\iint\limits_{\mathbb{R}^{2N}} \frac{||u_n(x)| - |u_n(y)||^p\phi_\rho(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \to \int\limits_{\mathbb{R}^N}\phi_\rho\,\mathrm{d}\mu$$

as $n \to \infty$ and

$$\int\limits_{\mathbb{R}^N}\phi_\rho\,\mathrm{d}\mu \to \mu\big(\{x_j\}\big)$$

as $\rho \to 0$. Note that the Hölder inequality implies

$$\begin{aligned} &\big|\operatorname{Re}\big\{M\big([u_n]_{s,A}^p\big)L(u_n,u_n\phi_\rho)\big\}\big| \\ &\quad\leqslant C\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)|^{p-1}|\phi_\rho(x)-\phi_\rho(y)||u_n(x)|}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad\leqslant C\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)|^p}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{(p-1)/p} \\ &\qquad\times\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/p} \\ &\quad\leqslant C\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/p}. \end{aligned} \tag{7}$$

In a way similar to the proof of Lemma 3.4 in [42], we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y=0. \tag{8}$$

In the following, we just give a sketch of the proof for reader's convenience.

On the one hand, we notice that

$$\begin{aligned}\mathbb{R}^N\times\mathbb{R}^N&=\big(\big(\mathbb{R}^N\setminus B(x_i,2\rho)\big)\cup B(x_i,2\rho)\big)\times\big(\big(\mathbb{R}^N\setminus B(x_i,2\rho)\big)\cup B(x_i,2\rho)\big)\\&=\big(\big(\mathbb{R}^N\setminus B(x_i,2\rho)\big)\times\big(\mathbb{R}^N\setminus B(x_i,2\rho)\big)\big)\cup\big(B(x_i,2\rho)\times\mathbb{R}^N\big)\\&\quad\cup\big(\big(\mathbb{R}^N\setminus B(x_i,2\rho)\big)\times B(x_i,2\rho)\big).\end{aligned}$$

Then we have

$$\begin{aligned}&\iint_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad=\iint_{B(x_i,2\rho)\times\mathbb{R}^N}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\&\qquad+\iint_{(\mathbb{R}^N\setminus B(x_i,2\rho))\times B(x_i,2\rho)}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\\&\quad\leqslant C\rho^{-ps}\int_{B(x_i,K\rho)}\big|u_n(x)\big|^p\,\mathrm{d}x+CK^{-N}\bigg(\int_{\mathbb{R}^N\setminus B(x_i,K\rho)}\big|u_n(x)\big|^{p_s^*}\,\mathrm{d}x\bigg)^{p/p_s^*}\\&\quad\leqslant C\rho^{-ps}\int_{B(x_i,K\rho)}\big|u_n(x)\big|^p\,\mathrm{d}x+CK^{-N}.\end{aligned}$$

Note that $u_n\rightharpoonup u$ in E and $u_n\to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$, $p\leqslant t<p_s^*$, which implies

$$C\rho^{-ps}\int_{B(x_i,K\rho)}\big|u_n(x)\big|^p\,\mathrm{d}x+CK^{-N}\to C\rho^{-ps}\int_{B(x_i,K\rho)}\big|u(x)\big|^p\,\mathrm{d}x+CK^{-N}$$

as $n\to\infty$. Then the Hölder inequality yields

$$\begin{aligned}&C\rho^{-ps}\int_{B(x_i,K\rho)}\big|u(x)\big|^p\,\mathrm{d}x+CK^{-N}\\&\quad\leqslant CK^{ps}\bigg(\int_{B(x_i,K\rho)}\big|u(x)\big|^{p_s^*}\,\mathrm{d}x\bigg)^{p/p_s^*}+CK^{-N}\to CK^{-N}\end{aligned}$$

as $\rho \to 0$. Furthermore, we have

$$\begin{aligned}
&\limsup_{\rho\to 0}\limsup_{n\to\infty}\iint\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad=\lim_{K\to\infty}\limsup_{\rho\to 0}\limsup_{n\to\infty}\iint\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)|^p|\phi_\rho(x)-\phi_\rho(y)|^p}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad=0.
\end{aligned}$$

This proves (8). By assumption (A3), we arrive at

$$\begin{aligned}
\lim_{\rho\to 0}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}k(x)|u_n|^q\phi_\rho\,\mathrm{d}x &= \lim_{\rho\to 0}\lim_{n\to\infty}\int\limits_{B_{2\rho}(x_j)}k(x)|u_n|^q\phi_\rho\,\mathrm{d}x \\
&\leqslant \lim_{\rho\to 0}\lim_{n\to\infty}\big\|k(x)\big\|_{L^r(B_{2\rho}(x_j))}\|u_n\|^q_{L^{p_s^*}(B_{2\rho}(x_j))} \\
&= 0. \qquad (9)
\end{aligned}$$

By using the diamagnetic inequality and (6), we have

$$\begin{aligned}
&m_0\big([u_n]^p_{s,A}\big)\iint\limits_{\mathbb{R}^{2N}}\frac{||u_n(x)|-|u_n(y)||^p\phi_\rho(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+\mathrm{Re}\big\{M\big([u_n]^p_{s,A}\big)L(u_n,u_n\phi_\rho)\big\} \\
&\quad\leqslant M\big([u_n]^p_{s,A}\big)\iint\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)-\mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)|^p\phi_\rho(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+\mathrm{Re}\big\{M\big([u_n]^p_{s,A}\big)L(u_n,u_n\phi_\rho)\big\} \\
&\quad=\alpha\int\limits_{\mathbb{R}^N}|u_n|^{p_s^*}\phi_\rho\,\mathrm{d}x+\beta\int\limits_{\mathbb{R}^N}k(x)|u_n|^q\phi_\rho\,\mathrm{d}x+o_n(1), \qquad (10)
\end{aligned}$$

Since ϕ_ρ has compact support, letting $n \to \infty$ in (10), we can deduce from (7)–(9) that

$$m_0\mu\big(\{x_j\}\big) \leqslant \alpha\nu_j.$$

Combining this fact with Proposition 2, we obtain

(i) $\nu_j = 0$, or
(ii) $\nu_j \geqslant (m_0\alpha^{-1}S)^{N/(ps)}$,

which implies that I is finite. The claim is thereby proved. □

To obtain the possible concentration of mass at infinity, we similarly define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. We can verify that $\{u_n\phi_R\}_n$ is bounded in $D^{s,p}_A(\mathbb{R}^N,\mathbb{C})$, hence $\langle J'(u_n), u_n\phi_R\rangle \to 0$

as $n \to \infty$, which implies

$$\begin{aligned} & M\big([u_n]_{s,A}^p\big) \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - \mathrm{e}^{\mathrm{i}(x-y)A((x+y)/p)}u_n(y)|^p \phi_R(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \mathrm{Re}\big\{M\big([u_n]_{s,A}^p\big)L(u_n, u_n\phi_R)\big\} \\ &\quad = \alpha \int\limits_{\mathbb{R}^N} |u_n|^{p_s^*}\varphi_R\,\mathrm{d}x + \beta \int\limits_{\mathbb{R}^N} k(x)|u_n|^q \varphi_R(x)\,\mathrm{d}x. \end{aligned} \tag{11}$$

It is easy to verify that

$$\limsup_{R\to\infty}\limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{||u_n(x)| - |u_n(y)||^p \phi_R(y)}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y = \mu_\infty$$

and

$$\big|\big\{M\big([u_n]_{s,A}^p\big)L(u_n, u_n\phi_R)\big\}\big| \leqslant C\bigg(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_R(x) - \phi_R(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y\bigg)^{1/p}.$$

Note that

$$\begin{aligned} & \limsup_{R\to\infty}\limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|\phi_R(x) - \phi_R(y)|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y \\ &\quad = \limsup_{R\to\infty}\limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|(1-\phi_R(x)) - (1-\phi_R(y))|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

In a way similar to the proof of Lemma 3.4 in [42], we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^p|(1-\phi_R(x)) - (1-\phi_R(y))|^p}{|x-y|^{N+ps}}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

By the Hölder inequality and the definition of S,

$$\begin{aligned} & \int\limits_{\mathbb{R}^N} k(x)|u_n|^q\varphi_R\,\mathrm{d}x \\ &\quad \leqslant \bigg(\int\limits_{\{|x|>2R\}} |u_n|^{p_s^*}\,\mathrm{d}x\bigg)^{q/p_s^*}\bigg(\int\limits_{\{|x|>2R\}} \big|k(x)\big|^{p_s^*/(p_s^*-q)}\,\mathrm{d}x\bigg)^{(p_s^*-q)/p_s^*} \\ &\quad \leqslant S^{-q/p_s^*}[u_n]_{s,p}^q\bigg(\int\limits_{\{|x|>2R\}} \big|k(x)\big|^{p_s^*/(p_s^*-q)}\,\mathrm{d}x\bigg)^{(p_s^*-q/p_s^*} \\ &\quad \leqslant S^{-q/p_s^*}\|u_n\|_W^q\bigg(\int\limits_{\{|x|>2R\}} \big|k(x)\big|^{p_s^*/(p_s^*-q)}\,\mathrm{d}x\bigg)^{(p_s^*-q)/p_s^*}, \end{aligned}$$

which implies

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} k(x)|u_n|^q\varphi_R\,\mathrm{d}x \\
&\quad\leqslant C\lim_{R\to\infty}\biggl(\int\limits_{\{|x|>2R\}} \bigl|k(x)\bigr|^{p_s^*/(p_s^*-q)}\,\mathrm{d}x\biggr)^{(p_s^*-q)/p_s^*} = 0.
\end{aligned}$$

Therefore, by letting $R\to\infty$ and $n\to\infty$ in (11), we have

$$m_0\mu_\infty \leqslant \alpha\nu_\infty. \tag{12}$$

By Proposition 2, and (12), we conclude that either

(iii) $\nu_\infty = 0$ or
(iv) $\nu_\infty \geqslant (m_0\alpha^{-1}S)^{N/(ps)}$.

Next, we claim that (ii) and (iv) cannot occur if α and β are chosen properly. To this end, from the Hölder inequality we have

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\biggl[J(u_n) - \frac{1}{p_s^*}\bigl\langle J'(u_n), u_n\bigr\rangle\biggr] \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)M\bigl([u_0]_{s,A}^p\bigr)[u_0]_{s,A}^p - \beta\biggl(\frac{1}{q} - \frac{1}{p_s^*}\biggr)\bigl\|k(x)\bigr\|_r\biggl(\int\limits_{\mathbb{R}^N}|u_0|^{p_s^*}\,\mathrm{d}x\biggr)^{q/p_s^*} \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)m_0[u_0]_{s,A}^p - \beta\biggl(\frac{1}{q} - \frac{1}{p_s^*}\biggr)\bigl\|k(x)\bigr\|_r S^{-q/p}[u_0]_{s,A}^{q/p} \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)m_0 S\|u_0\|_{p_s^*}^p - \beta\biggl(\frac{1}{q} - \frac{1}{p_s^*}\biggr)\bigl\|k(x)\bigr\|_r\|u_0\|_{p_s^*}^q.
\end{aligned}$$

Thus, it follows that

$$\|u_0\|_{p_s^*} \leqslant C\beta^{1/(p-q)}. \tag{13}$$

If (iv) occurs, we obtain by (13) that

$$\begin{aligned}
0 > c &= \lim_{R\to\infty}\lim_{n\to\infty}\biggl[J(u_n) - \frac{1}{p_s^*}\bigl\langle J'(u_n), \varphi_R\bigr\rangle\biggr] \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)m_0\mu_\infty - \beta\biggl(\frac{1}{q} - \frac{1}{p_s^*}\biggr)\bigl\|k(x)\bigr\|_r\|u_0\|_{p_s^*}^q \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)m_0\mu_\infty - \beta\biggl(\frac{1}{q} - \frac{1}{p_s^*}\biggr)\bigl\|k(x)\bigr\|_r C\beta^{q/(p-q)} \\
&\geqslant \biggl(\frac{1}{p\theta} - \frac{1}{p_s^*}\biggr)m_0\alpha^{-N/(ps)}S^{N/(ps)} - C\beta^{p/(p-q)}.
\end{aligned}$$

However, since $\theta \in [1, N/(N-ps))$, $q < p$, if $\alpha > 0$ is given, we can take small β_* such that for every $0 < \beta < \beta_*$, the term on the right-hand side above is greater than zero, which is a contradiction. Similarly, if $\beta > 0$ is given, we can choose small α_* such that for every $0 < \alpha < \alpha_*$, the term on the right-hand side above is greater than zero. Similarly, we can prove that (ii) cannot occur. Hence,

$$\int\limits_{\mathbb{R}^N} |u_n|^{p_s^*}\,\mathrm{d}x \to \int\limits_{\mathbb{R}^N} |u_0|^{p_s^*}\,\mathrm{d}x \quad \text{as } n \to \infty.$$

On the other hand, since $k \in L^r(\mathbb{R}^N)$, we have

$$\int\limits_{\mathbb{R}^N} k(x)\bigl(|u_n|^q - |u|^q\bigr)\,\mathrm{d}x \leqslant \bigl\|k(x)\bigr\|_r \bigl\||u_n|^q - |u|^q\bigr\|_{p^*/q} \to 0, \quad n \to +\infty.$$

By the weak lower semicontinuity of the norm, conditon (A1), and the Brézis–Lieb lemma, we have

$$\begin{aligned}
o(1)\|u_n\| &= \bigl\langle J'(u_n), u_n\bigr\rangle \\
&= M\bigl([u_n]_{s,A}^p\bigr)[u_n]_{s,A}^p - \alpha \int\limits_{\mathbb{R}^N} |u_n|^{p_s^*}\,\mathrm{d}x - \beta \int\limits_{\mathbb{R}^N} k(x)|u_n|^q\,\mathrm{d}x \\
&\geqslant m_0\bigl([u_n]_{s,A}^p - [u_0]_{s,A}^p\bigr) + M\bigl([u_0]_{s,A}^p\bigr)[u_0]_{s,A}^p \\
&\quad - \alpha \int\limits_{\mathbb{R}^N} |u_0|^{p_s^*}\,\mathrm{d}x - \beta \int\limits_{\mathbb{R}^N} k(x)|u_0|^q\,\mathrm{d}x \\
&\geqslant m_0\|u_n - u_0\|^p + o(1)\|u_0\|.
\end{aligned}$$

Here we use the fact that $J'(u_0) = 0$. Thus we have proved that $\{u_n\}_n$ strongly converges to u_0 in $D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$. Hence, the proof is complete. □

4 Main results

To prove the multiplicity result stated in Theorem 1, we will use some topological results introduced by Krasnoselskii in [20]. For the sake of completeness and for reader's convenience, we recall here some basic notions on the Krasnoselskii's genus. Let X be a Banach space, and let us denote by Σ the class of all closed subsets $A \subset X \setminus \{0\}$ that are symmetric with respect to the origin, that is, $u \in A$ implies $-u \in A$.

Definition 3. Let $A \in \Sigma$. The Krasnoselskii's genus $\gamma(A)$ of A is defined as being the least positive integer n such that there is an odd mapping $\phi \in C(A, \mathbb{R}^N)$ such that $\phi(x) \neq 0$ for any $x \in A$. If n does not exist, we set $\gamma(A) = \infty$. Furthermore, we set $\gamma(\emptyset) = 0$.

In the sequel, we will recall only the properties of the genus that will be used throughout this work. More information on this subject may be found in the references [19,20,29].

Proposition 4. *Let A and B be closed symmetric subsets of X which do not contain the origin. Then the following hold*:

(i) *If there exists an odd continuous mapping from A to B, then $\gamma(A) \leqslant \gamma(B)$*;
(ii) *If there is an odd homeomorphism from A to B, then $\gamma(A) = \gamma(B)$*;
(iii) *If $\gamma(B) < \infty$, then $\gamma\overline{(A \setminus B)} \geqslant \gamma(A) - \gamma(B)$*;
(iv) *n-dimensional sphere S^n has a genus of $n+1$ by the Borsuk–Ulam theorem*;
(v) *If A is compact, then $\gamma(A) < +\infty$ and there exists $\delta > 0$ such that $N_\delta(A) \subset \Sigma$ and $\gamma(N_\delta(A)) = \gamma(A)$ with $N_\delta(A) = \{x \in X\colon \operatorname{dist}(x, A) \leqslant \delta\}$.*

We conclude this section recalling the symmetric mountain pass lemma introduced by Kajikiya in [19]. The proof of Theorem 1 is based on the application of the following result.

Lemma 3. *Let E be an infinite-dimensional space and $J \in C^1(E, \mathbb{R})$ and suppose the following conditions hold*:

(J1) *$J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais–Smale condition, i.e., for some $\bar{c} > 0$, in the case when every sequence $\{u_n\}_n$ in E satisfying $\lim_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence*;
(J2) *For each $n \in \mathbb{N}$, there exists an $A_n \in \Sigma_n$ such that $\sup_{u\in A_n} J(u) < 0$.*

Then either (i) *or* (ii) *below holds*:

(i) *There exists a sequence $\{u_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) < 0$ and $\{u_n\}$ converges to zero.*
(ii) *There exist two sequences $\{u_n\}_n$ and $\{v_n\}_n$ such that $J'(u_n) = 0$, $J(u_n) < 0$, $u_n \neq 0$, $\lim_{n\to\infty} u_n = 0$, $J'(v_n) = 0$, $J(v_n) < 0$, $\lim_{n\to\infty} J(v_n) = 0$, and $\{v_n\}_n$ converges to a nonzero limit.*

To obtain infinitely many solutions, we need some technical lemmas. Let $J(u)$ be the functional defined as above, $1 < q < 2$, $\alpha > 0$, and $\beta > 0$. Then

$$\begin{aligned}
J(u) &= \frac{1}{p}\widetilde{M}\big([u]_{s,A}^p\big) - \frac{\alpha}{p_s^*}\int\limits_{\mathbb{R}^N} |u|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\int\limits_{\mathbb{R}^N} k(x)|u|^q\,\mathrm{d}x \\
&\geqslant \frac{1}{p\theta}M\big([u]_{s,A}^p\big)[u]_{s,A}^p - \frac{\alpha}{p_s^*}\int\limits_{\mathbb{R}^N} |u|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\int\limits_{\mathbb{R}^N} k(x)|u|^q\,\mathrm{d}x \\
&\geqslant \frac{1}{p\theta}m_0[u]_{s,A}^p - \frac{\alpha}{p_s^*}\int\limits_{\mathbb{R}^N} |u|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\big\|k(x)\big\|_r\|u\|_{p_s^*}^q \\
&\geqslant \frac{1}{p\theta}m_0[u]_{s,A}^p - \frac{\alpha}{p_s^*}\big(S^{-1}[u]_{s,A}^p\big)^{(p_s^*)/p} - \frac{\beta}{q}\|k(x)\|_r\big(S^{-1}[u]_{s,A}^p\big)^{q/p} \\
&\geqslant C_1[u]_{s,A}^p - \alpha C_2[u]_{s,A}^{p_s^*} - \beta C_3[u]_{s,A}^q.
\end{aligned}$$

Define

$$h(t) = C_1 t^p - \alpha C_2 t^{p_s^*} - \beta C_3 t^q.$$

Then it is easy to see that, for the given $\alpha > 0$, we can choose $\beta^* > 0$ so small that if $0 < \beta < \beta^*$, there exists $0 < t_0 < t_1$ such that $h(t) < 0$ for $0 < t < t_0$; $h(t) > 0$ for $t_0 < t < t_1$; $h(t) < 0$ for $t > t_1$.

Similarly, for the given $\beta > 0$, we can choose $\alpha^* > 0$ so small that if $0 < \alpha < \alpha^*$, there exists $0 < t_0^* < t_1^*$ such that $h(t) < 0$ for $0 < t < t_0^*$; $h(t) > 0$ for $t_0^* < t < t_1^*$; $h(t) < 0$ for $t > t_1^*$.

Clearly, $h(t_0) = 0 = h(t_1)$. Following the same idea as in [5], we consider the truncated functional

$$\tilde{J}(u) = \frac{1}{p}\widetilde{M}\big([u]_{s,A}^p\big) - \frac{\alpha}{p_s^*}\psi(u)\int_{\mathbb{R}^N} |u|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\int_{\mathbb{R}^N} k(x)|u|^q\,\mathrm{d}x,$$

where $\psi(u) = \tau(\|u\|)$, and $\tau : \mathbb{R}^+ \to [0,1]$ is a nonincreasing C^∞ function such that $\tau(t) = 1$ if $t \leqslant t_0$ and $\tau(t) = 0$ if $t \geqslant t_1$. Obviously, $\tilde{J}(u)$ is even. Thus, from Lemma 2 we obtain the following lemma.

Lemma 4. *Let $c < 0$ and $1 < q < p$. Then*:

(i) *$\tilde{J} \in C^1$ and $\tilde{J}$ is bounded from below.*
(ii) *If $\tilde{J}(u) < 0$, then $\|u\| < t_0$ and $\tilde{J}(u) = J(u)$.*
(iii) *For each $\alpha > 0$, there exists $\tilde{\beta}^* = \min\{\beta_*, \beta^*\} > 0$ such that if $0 < \beta < \tilde{\beta}^*$, then $\tilde{J}$ satisfies* $(\mathrm{PS})_\mathrm{c}$.
(iv) *For each $\beta > 0$, there exists $\tilde{\alpha}^* = \min\{\alpha_*, \alpha^*\} > 0$ such that if $0 < \alpha < \tilde{\alpha}^*$, then $\tilde{J}$ satisfies* $(\mathrm{PS})_\mathrm{c}$.

Proof. Obviously, (i) and (ii) are immediate. To prove (iii) and (iv), observe that all $(\mathrm{PS})_\mathrm{c}$-sequences for $\tilde{J}$ with $c < 0$ must be bounded, similar to the proof of Lemma 2, there exists a strong convergent subsequence in $D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$. □

Remark 4. Denote $K_c = \{u \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})\colon\ \tilde{J}'(u) = 0,\ \tilde{J}(u) = c\}$. If α, β are as in (iii) or (iv) above, then it follows from $(\mathrm{PS})_\mathrm{c}$ that K_c $(c < 0)$ is compact.

Lemma 5. *Denote $\tilde{J}_c := \{u \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})\colon\ \tilde{J}'(u) = 0,\ \tilde{J}(u) \leqslant c\}$. Given $n \in \mathbb{N}$, there exists $\epsilon_n < 0$ such that*

$$\gamma(\tilde{J}^{\epsilon_n}) := \gamma\big(\big\{u \in D_A^{s,p}\big(\mathbb{R}^N, \mathbb{C}\big)\colon\ \tilde{J}(u) \leqslant \epsilon_n\big\}\big) \geqslant n.$$

Proof. Let X_n be a n-dimensional subspace of $D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$. For any $u \in X_n$, $u \neq 0$, write $u = r_n w$ with $w \in X_n$, $\|w\| = 1$, and then $r_n = \|u\|$. From condition (A3) it is easy to see that, for every $w \in X_n$ with $\|w\| = 1$, there exists $d_n > 0$ such that $\int_{\mathbb{R}^N} k(x)|w|^q\,\mathrm{d}x \geqslant d_n$. Thus, for $0 < r_n < t_0$, by the continuity of M, we have

$$\tilde{J}(u) = \frac{1}{p}\widetilde{M}\big([u]_{s,A}^p\big) - \frac{\alpha}{p_s^*}\psi(u)\int_{\mathbb{R}^N} |u|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q}\int_{\mathbb{R}^N} k(x)|u|^q\,\mathrm{d}x$$

$$\begin{aligned}
&\leqslant \frac{1}{p} r_n^p \mathcal{M}\big([w]_{s,A}^p\big) - \frac{\alpha}{p_s^*} r_n^{p_s^*} \int\limits_{\mathbb{R}^N} |w|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{q} r_n^q \int\limits_{\mathbb{R}^N} k(x)|w|^q\,\mathrm{d}x \\
&\leqslant \frac{C_1}{p} r_n^p - \frac{\alpha}{p_s^*} r_n^{p_s^*} \int\limits_{\mathbb{R}^N} |w|^{p_s^*}\,\mathrm{d}x - \frac{\beta}{p} d_n r_n^q \\
&= \epsilon_n.
\end{aligned}$$

Therefore, we can choose $r_n \in (0, t_0)$ so small that $\tilde{J}(u) \leqslant \epsilon_n < 0$. Let

$$S_{r_n} = \big\{u \in X_n\colon\ \|u\| = r_n\big\}.$$

Then $S_{r_n} \cap X_n \subset \tilde{J}^{\epsilon_n}$. Hence, by Proposition 4,

$$\gamma\big(\tilde{J}^{\epsilon_n}\big) \geqslant \gamma(S_{r_n} \cap X_n) = n.$$

As desired. □

According to Lemma 4, we denote $\Sigma_n = \{A \in \Sigma\colon\ \gamma(A) \geqslant n\}$, and let

$$c_n = \inf_{A\in\Sigma_n} \sup_{u\in A} \tilde{J}(u). \tag{14}$$

Then

$$-\infty < c_n \leqslant \epsilon_n < 0 \tag{15}$$

because $\tilde{J}^{\epsilon_n} \in \Sigma_n$ and $\tilde{I}$ is bounded from below.

Lemma 6. *Let α, β be as in* (iii) *or* (iv) *of Lemma* 4. *Then all c_n* (*given by* (14)) *are critical values of $\tilde{J}$, and $c_n \to 0$.*

Proof. Since $\Sigma_{n+1} \subset \Sigma_n$, it is clear that $c_n \leqslant c_{n+1}$. By (15), we have $c_n < 0$. Hence, there is a $\bar{c} \leqslant 0$ such that $c_n \to \bar{c} \leqslant 0$. Moreover, since that all c_n are critical values of $\tilde{J}$ (see [29]), we claim that $\bar{c} = 0$. If $\bar{c} < 0$, then by Remark 4, $K_{\bar{c}} = \{u \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})\colon\ \tilde{J}'(u) = 0,\ \tilde{J}(u) = \bar{c}\}$ is compact, and $K_{\bar{c}} \in \Sigma$, then $\gamma(K_{\bar{c}}) = n_0 < +\infty$, and there exists $\delta > 0$ such that $\gamma(K_{\bar{c}}) = \gamma(N_\delta(K_{\bar{c}})) = n_0$, here $N_\delta(K_{\bar{c}}) = \{x \in D_A^{s,p}(\mathbb{R}^N, \mathbb{C})\colon\ \|x - K_{\bar{c}}\| \leqslant \delta\}$. By the deformation lemma (see [36]), there exist $\epsilon > 0$ ($\bar{c} + \epsilon < 0$) and an odd homeomorphism $\eta : D_A^{s,p}(\mathbb{R}^N, \mathbb{C}) \to D_A^{s,p}(\mathbb{R}^N, \mathbb{C})$ such that

$$\eta(\tilde{J}^{\bar{c}+\epsilon} \setminus N_\delta(K_{\bar{c}})) \subset \tilde{J}^{\bar{c}-\epsilon}.$$

Since c_n is increasing and converges to $\bar{c}$, there exists $n \in \mathbb{N}$ such that $c_n > \bar{c} - \epsilon$ and $c_{n+n_0} \leqslant \bar{c}$. Choose $A \in \Sigma_{n+n_0}$ such that $\sup_{u\in A} \tilde{J}(u) < \bar{c} + \epsilon$, that is $A \subset \tilde{J}^{\bar{c}+\epsilon}$. By the properties of γ, we have

$$\gamma\big(\overline{A \setminus N_\delta(K_{\bar{c}})}\big) \geqslant \gamma(A) - \gamma\big(N_\delta(K_{\bar{c}})\big) \geqslant n, \qquad \gamma\big(\overline{\eta\big(A \setminus N_\delta(K_{\bar{c}})\big)}\big) \geqslant n.$$

Hence, we have $\overline{\eta(A \setminus N_\delta(K_{\bar{c}}))} \in \Sigma_n$. Consequently,

$$\sup_{u\in\overline{\eta(A\setminus N_\delta(K_{\bar{c}}))}} \tilde{J}(u) \geqslant c_n > \bar{c} - \epsilon,$$

a contradiction, hence $c_n \to 0$. □

Proof of Theorem 1. By Lemma 4(ii), $\tilde{J}(u) = J(u)$ if $\tilde{J}(u) < 0$. By Lemmas 4–6, one can see that all the assumptions of Lemma 3 are satisfied. This completes the proof. □

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