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Fixed and common fixed point theorems in frame of quasi metric spaces under contraction condition based on ultra distance functions*

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Abstract. In this paper, we introduce the notion of ultra distance function. Based on the notion of ultra distance function, we introduce the definitions of (k, ψ, L)-quasi contractions of type (I) and type (II) in the frame of quasi metric spaces. We employ our new definitions to construct and prove many fixed and common fixed point results in the frame of quasi metric spaces. Our results extend and improve many exciting results in the literatures. Also, we introduce some examples and some applications in order to support the usability of our work.

Keywords: quasi metric spaces, fixed point theorems, nonlinear contractions, comparison functions, almost perfect mappings.

1 Introduction and preliminary

A point u is called a fixed point of a function f if $fu = u$. The fixed point iteration is used to prove that an equation of the form $gu = 0$ has a solution. Moreover, fixed point theorems are used to prove the existence and uniqueness of such equations in partial differential equations, integral equations, and ordinary differential equations. The most popular tool for solving some problems in nonlinear analysis is the Banach fixed point theorem [9]. Many authors extended the Banach fixed point theorem to many directions, for example, see [4, 5, 20, 21, 23, 24, 26, 28, 31, 34, 35] and all references cited their.

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In 1931, Wilson [36] introduced the notion of quasi metric spaces as follows:

Definition 1. (See [36].) Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a given function, which satisfies:

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leqslant d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a quasi metric on X, and the pair (X, d) is called a quasi metric space.

Note that every metric space is a quasi metric space. The converse is not true in general. For this instance, see the following example:

Example 1. (See [18].) Let $X = \mathbb{N} \cup \{0\}$ and define the function d on X as follows: $d(0, n) = 1/n$ for all $n \in \mathbb{N}$, $d(n, x) = n$ for $n \neq x$; $n \in \mathbb{N}$, and $d(x, x) = 0$ for all $x \in X$. Then (X, d) is a quasi metric space, which is not a metric space.

A quasi metric d induces a metric q as follows:

$$q(x, y) = \max\bigl\{d(x, y), d(y, x)\bigr\}.$$

The notions of convergence and completeness on quasi metric spaces are given as follows:

Definition 2. (See [7, 22].) Let (X, d) be a quasi metric space, (x_n) be a sequence in X, and $x \in X$. Then the sequence (x_n) converges to x if and only if $\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0$.

Definition 3. (See [7, 22].) Let (X, d) be a quasi metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is left-Cauchy if and if for every $\epsilon > 0$, there is positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $n \geqslant m > N$.

Definition 4. (See [7, 22].) Let (X, d) be a quasi metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is right-Cauchy if and if for every $\epsilon > 0$, there is a positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $m \geqslant n > N$.

Definition 5. (See [7, 22].) Let (X, d) be a quasi metric space and (x_n) be a sequence in X. We say that the sequence (x_n) is Cauchy if and if for every $\epsilon > 0$, there is positive integer $N = N(\epsilon)$ such that $d(x_n, x_m) < \epsilon$ for all $m, n > N$.

Definition 6. (See [7, 22].) Let (X, d) be a quasi metric space. We say that

(i) (X, d) is left-complete if and only if every left-Cauchy sequence in X is convergent;
(ii) (X, d) is right-complete if and only if every right-Cauchy sequence in X is convergent;
(iii) (X, d) is complete if and only if every Cauchy sequence in X is convergent.

It is clear that a sequence (x_n) in quasi metric space (X,d) is Cauchy if and only if (x_n) is left- and right-Cauchy. For some fixed point theorems in quasi metric space, we refer the reader to [2, 3, 6, 15, 17, 30, 32, 33].

The notion of altering distance function plays an important role to improve and extend the Banach contraction theorem to many directions. The notion of altering distance function introduced by Khan [25] as follows:

Definition 7. (See [25].) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is nondecreasing and continuous;
(ii) $\psi(t) = 0$ if and only if $t = 0$.

Abodayeh et al. [1] introduced the notion of almost perfect function as a generalization of the notion of altering distance function and studied some fixed and common fixed point theorems of Ω-distance under various contractive conditions.

Berinde [10, 11] introduced the concept of weak contraction mappings, the concept of almost contraction mappings and studied some nice fixed point theorems. Moreover, many authors introduced and studied many fixed and common fixed point theorems in complete metric spaces for weak and almost contraction mappings in sense of Berinde, see [8, 12–14, 16, 19, 27, 29, 31].

In this paper, we introduce the notion of ultra distance function as a generalization of the notion of the almost perfect function. We utilized the notion of almost contraction mapping and the notion of ultra distance function to introduce the notions of (k,ψ,L)-quasi contractions of type (I) and type (II) in the frame of quasi metric spaces. Then after, we construct many fixed and common fixed point results in the frame of quasi metric spaces.

2 Main result

We start our work by introducing the following definition:

Definition 8. The function $\psi : [0,\infty) \to [0,\infty)$ is called an ultra distance function if the following properties are satisfied:

(i) $\psi(t) = 0$ iff $t = 0$.
(ii) If (x_n) is a sequence in $[0,+\infty)$ such that $\lim_{n\to+\infty} \psi(x_n) = 0$, then $\lim_{n\to+\infty} x_n = 0$.

Here are some examples on ultra distance functions.

Example 2. Define the function $\psi : [0,+\infty) \to [0,\infty)$ by $\psi(t) = \sin(t)$ if $0 \leqslant t \leqslant 3\pi/4$ and $\psi(t) = 1$ if $3\pi/4 < t < \infty$. Then it is clear that ψ is an ultra distance function.

Example 3. Define the function $\psi : [0,+\infty) \to [0,\infty)$ by $\psi(t) = 1 - \cos(t)$ if $0 \leqslant t \leqslant \pi/2$ and $\psi(t) = 1/2$ if $\pi/2 < t < \infty$. Then it is clear that ψ is an ultra distance function.

The following lemma will be very helpful in proving our main results.

Lemma 1. *Let (X, d) be a quasi metric space. Let (x_n) be a sequence in X such that*

$$\lim_{n\to+\infty} d(x_n, x_{n+1}) = \lim_{n\to+\infty} d(x_{n+1}, x_n) = 0.$$

For $m, n \in \mathbb{N}$, where n is odd, m is even, and $m > n$, we have

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Then (x_n) is Cauchy.

Proof. Given $n, m \in \mathbb{N}$ with $m > n$. Now we study the following cases:

Case 1: n is even, and m is odd. By triangle inequality, we have:

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m)$$

and

$$d(x_m, x_n) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_{n+1}) + d(x_{n+1}, x_n).$$

Letting $n, m \to +\infty$ in above inequalities, we get that

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Case 2: n, m are both even. Apply the triangle inequality to get

$$d(x_n, x_m) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_m)$$

and

$$d(x_m, x_n) \leqslant d(x_m, x_{n+1}) + d(x_{n+1}, x_n).$$

On letting $n, m \to +\infty$ in above inequalities, we have

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Case 3: n, m are both odd. Triangle inequality implies that

$$d(x_n, x_m) \leqslant d(x_n, x_{m+1}) + d(x_{m+1}, x_m)$$

and

$$d(x_m, x_n) \leqslant d(x_m, x_{m+1}) + d(x_{m+1}, x_n).$$

By letting n and m tend to infinity in above inequalities, we obtain

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

By collecting all cases together, we conclude that (x_n) is a Cauchy sequence. □

Before proving our first main result, we introduce the following definition.

Definition 9. Let (X,d) be a quasi metric space and $f,g: X \to X$ be two mappings. The pair (f,g) is called (k,ψ,L)-quasi contraction of type (I) if there exist an ultra distance function ψ, $k \in [0,1)$ and $L \geqslant 0$ such that for all $x,y \in X$, we have

$$\begin{aligned}\psi\big(d(fx,gy)\big) &\leqslant k\max\big\{\psi\big(d(x,fx)\big),\psi\big(d(y,gy)\big)\big\}\\ &\quad + L\min\big\{q(x,gy),q(y,fx),q(x,fx)\big\}\end{aligned}$$

and

$$\begin{aligned}\psi\big(d(gx,fy)\big) &\leqslant k\max\big\{\psi\big(d(x,gx)\big),\psi\big(d(y,fy)\big)\big\}\\ &\quad + L\min\big\{q(x,fy),q(y,gx),q(x,gx)\big\}.\end{aligned}$$

Now, we prove our first result:

Theorem 1. *Let (X,d) be a complete quasi metric space and $f,g: X \to X$ be two mappings such that the pair (f,g) is (k,ψ,L)-quasi contraction of type* (I). *If f or g is continuous, then f and g have a unique common fixed point in X.*

Proof. By starting with $x_0 \in X$, we define a sequence (x_n) in X inductively via $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$ for all $n \geqslant 0$. Note that if there exists $r \in \mathbb{N}$ such that $x_{2r} = x_{2r+1}$, then x_{2r} is a fixed point of f. Since the pair (f,g) is (k,ψ,L)-quasi contraction of type (I), we have

$$\begin{aligned}&\psi\big(d(x_{2r+1},x_{2r+2})\big)\\ &\quad = \psi\big(d(fx_{2r},gx_{2r+1})\big)\\ &\quad \leqslant k\max\big\{\psi\big(d(x_{2r},fx_{2r})\big),\psi\big(d(x_{2r+1},gx_{2r+1})\big)\big\}\\ &\qquad + L\min\big\{q(x_{2r},gx_{2r+1}),q(x_{2r+1},fx_{2r}),q(x_{2r},fx_{2r})\big\}\\ &\quad = k\max\big\{\psi\big(d(x_{2r},x_{2r+1})\big),\psi\big(d(x_{2r+1},x_{2r+2})\big)\big\}\\ &\qquad + L\min\big\{q(x_{2r},x_{2r+2}),q(x_{2r+1},x_{2r+1}),q(x_{2r},x_{2r+1})\big\}\\ &\quad = k\psi\big(d(x_{2r+1},x_{2r+2})\big).\end{aligned}$$

The last equality is true only if $\psi(d(x_{2r+1},x_{2r+2})) = 0$. So, we conclude that $d(x_{2r+1}, x_{2r+2}) = 0$. Hence, $x_{2r} = x_{2r+1} = x_{2r+2}$. Therefore, x_{2r} is a common fixed point of f and g.

Now, suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since the pair (f,g) is (k,ψ,L)-quasi contraction of type (I), we have

$$\begin{aligned}&\psi\big(d(x_{2n+1},x_{2n+2})\big)\\ &\quad = \psi\big(d(fx_{2n},gx_{2n+1})\big)\\ &\quad \leqslant k\max\big\{\psi\big(d(x_{2n},fx_{2n})\big),\psi\big(d(x_{2n+1},gx_{2n+1})\big)\big\}\\ &\qquad + L\min\big\{q(x_{2n},gx_{2n+1}),q(x_{2n+1},fx_{2n}),q(x_{2n},fx_{2n})\big\}\\ &\quad = k\max\big\{\psi\big(d(x_{2n},x_{2n+1})\big),\psi\big(d(x_{2n+1},x_{2n+2})\big)\big\}\\ &\qquad + L\min\big\{q(x_{2n},x_{2n+2}),q(x_{2n+1},x_{2n+1}),q(x_{2n},x_{2n+1})\big\}\\ &\quad = k\max\big\{\psi(d(x_{2n},x_{2n+1})),\psi(d(x_{2n+1},x_{2n+2}))\big\}.\end{aligned} \tag{1}$$

If
$$\max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2}))\} = \psi(d(x_{2n+1}, x_{2n+2})),$$
then by (1) we conclude that

$$\psi(d(x_{2n+1}, x_{2n+2})) \leqslant k\psi(d(x_{2n+1}, x_{2n+2})),$$

a contradiction. So,

$$\max\{\psi(d(x_{2n}, x_{2n+1})), \psi(d(x_{2n+1}, x_{2n+2}))\} = \psi(d(x_{2n}, x_{2n+1})).$$

Therefore, (1) becomes

$$\psi(d(x_{2n+1}, x_{2n+2})) \leqslant k\psi(d(x_{2n}, x_{2n+1})). \tag{2}$$

Also, we can show that

$$\psi(d(x_{2n}, x_{2n+1})) \leqslant k\psi(d(x_{2n-1}, x_{2n})). \tag{3}$$

Combining (2) and (3), we conclude that

$$\psi(d(x_n, x_{n+1})) \leqslant k\psi(d(x_{n-1}, x_n)) \tag{4}$$

holds for all $n \in \mathbb{N}$.

Repeating the same arguments as above, we conclude that

$$\psi(d(x_{n+1}, x_n)) \leqslant k\psi(d(x_n, x_{n-1})) \tag{5}$$

holds for all $n \in \mathbb{N}$. From (4) and (5) we have

$$\begin{aligned} &\max\{\psi(d(x_n, x_{n+1})), \psi(d(x_{n+1}, x_n))\} \\ &\quad \leqslant k \max\{\psi(d(x_{n-1}, x_n)), \psi(d(x_n, x_{n-1}))\} \end{aligned} \tag{6}$$

holds for all $n \in \mathbb{N}$.

Repeating (6) n times, we get

$$\begin{aligned} &\max\{\psi(d(x_n, x_{n+1})), \psi(d(x_{n+1}, x_n))\} \\ &\quad \leqslant k \max\{\psi(d(x_{n-1}, x_n)), \psi(d(x_n, x_{n-1}))\} \\ &\quad \leqslant k^2 \max\{\psi(d(x_{n-2}, x_{n-1})), \psi(d(x_{n-1}, x_{n-2}))\} \\ &\quad \leqslant k^3 \max\{\psi(d(x_{n-3}, x_{n-2})), \psi(d(x_{n-2}, x_{n-3}))\} \\ &\quad \leqslant \cdots \\ &\quad \leqslant k^n \max\{\psi(d(x_0, x_1)), \psi(d(x_1, x_0))\}. \end{aligned} \tag{7}$$

Letting $n \to +\infty$ in (7), we get

$$\lim_{n\to+\infty} \psi(d(x_n, x_{n+1})) = 0$$

and

$$\lim_{n\to+\infty} \psi\bigl(d(x_{n+1},x_n)\bigr) = 0.$$

Since ψ is an ultra distance function, we conclude that

$$\lim_{n\to+\infty} d(x_n,x_{n+1}) = 0 \tag{8}$$

and

$$\lim_{n\to+\infty} d(x_{n+1},x_n) = 0. \tag{9}$$

From (8), (9) and the definition of q we have

$$\lim_{n\to+\infty} q(x_{n+1},x_n) = \lim_{n\to+\infty} q(x_{n+1},x_n) = 0. \tag{10}$$

Now, we show that (x_n) is a Cauchy sequence.

Let $n,m\in\mathbb{N}$ be such that n is odd, m is even, and $m>n$. Since the pair (f,g) is (k,ψ,L)-quasi contraction of type (I), we have

$$\begin{aligned}
\psi\bigl(d(x_n,x_m)\bigr) &= \psi\bigl(d(fx_{n-1},gx_{m-1})\bigr)\\
&\leqslant k\max\bigl\{\psi\bigl(d(x_{n-1},fx_{n-1})\bigr),\psi\bigl(d(x_{m-1},gx_{m-1})\bigr)\bigr\}\\
&\quad + L\min\bigl\{q(x_{n-1},gx_{m-1}),q(x_{m-1},fx_{n-1}),q(x_{n-1},fx_{n-1})\bigr\}\\
&= k\max\bigl\{\psi(d\bigl(x_{n-1},x_n)\bigr),\psi\bigl(d(x_{m-1},x_m)\bigr)\bigr\}\\
&\quad + L\min\bigl\{q(x_{n-1},x_m),q(x_{m-1},x_n),q(x_{n-1},x_n)\bigr\}\\
&\leqslant k\max\bigl\{\psi\bigl(d(x_{n-1},x_n)\bigr),\psi\bigl(d(x_{m-1},x_m)\bigr)\bigr\} + Lq(x_{n-1},x_n)\\
&\leqslant k\psi\bigl(d(x_{n-1},x_n)\bigr) + Lq(x_{n-1},x_n)\\
&\leqslant k^n\psi\bigl(d(x_0,x_1)\bigr) + Lq(x_{n-1},x_n).
\end{aligned} \tag{11}$$

Letting $n,m\to+\infty$ in (11). Then using (10) to get

$$\lim_{n,m\to+\infty} \psi\bigl(d(x_n,x_m)\bigr) = 0.$$

From the definition of the function ψ we conclude that

$$\lim_{n,m\to+\infty} d(x_n,x_m) = 0.$$

Again, since the pair (f,g) is (k,ψ,L)-quasi contraction of type (I), we have

$$\begin{aligned}
\psi\bigl(d(x_m,x_n)\bigr) &= \psi\bigl(d(gx_{m-1},fx_{n-1})\bigr)\\
&\leqslant k\max\bigl\{\psi\bigl(d(x_{m-1},gx_{m-1})\bigr),\psi\bigl(d(x_{n-1},fx_{n-1})\bigr)\bigr\}\\
&\quad + L\min\bigl\{q(x_{m-1},fx_{n-1}),q(x_{n-1},gx_{m-1}),q(x_{m-1},gx_{m-1})\bigr\}\\
&\leqslant k\max\bigl\{\psi\bigl(d(x_{m-1},x_m)\bigr),\psi\bigl(d(x_{n-1},x_n)\bigr)\bigr\}\\
&\quad + L\min\bigl\{q(x_{m-1},x_n),q(x_{n-1},x_m),q(x_{m-1},x_m)\bigr\}\\
&\leqslant k\psi\bigl(d(x_{n-1},x_n)\bigr) + Lq(x_{m-1},x_m)\\
&\leqslant k^n\psi\bigl(d(x_0,x_1)\bigr) + Lq(x_{m-1},x_m).
\end{aligned} \tag{12}$$

On letting $n, m \to +\infty$ in (12), we get

$$\lim_{n,m\to+\infty} \psi\bigl(d(x_m, x_n)\bigr) = 0.$$

Since ψ is an ultra distance function, we have

$$\lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

By Lemma 1, we conclude that (x_n) is Cauchy. By completeness of the quasi metric space (X, d), there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n, u) = \lim_{n\to+\infty} d(u, x_n) = 0.$$

Hence,

$$\lim_{n\to+\infty} q(x_n, u) = \lim_{n\to+\infty} q(u, x_n) = 0.$$

Without loss of generality, we may assume that f is continuous. So,

$$\lim_{n\to+\infty} d(fx_{2n}, fu) = \lim_{n\to+\infty} d(fu, fx_{2n}) = 0.$$

Now, we show that $u = fu$. By triangle inequality, we have

$$d(u, fu) \leqslant d(u, x_{2n+1}) + d(x_{2n+1}, fu).$$

Letting $n \to +\infty$ and using the fact that $x_{2n+1} = fx_{2n}$, we conclude that $u = fu$. Since the pair (f, g) is (k, ψ, L)-quasi contraction of type (I), we conclude that

$$\begin{aligned}
\psi\bigl(d(u, gu)\bigr) &= \psi\bigl(d(fu, gu)\bigr) \\
&\leqslant k \max\bigl\{\psi\bigl(d(u, fu)\bigr), \psi\bigl(d(u, gu)\bigr)\bigr\} \\
&\quad + L \min\bigl\{q(u, gu), q(u, fu), q(u, fu)\bigr\} \\
&= k\psi\bigl(d(u, gu)\bigr).
\end{aligned}$$

Since $k \in [0, 1)$, we conclude that $\psi(d(u, gu)) = 0$. Hence, $gu = u$. Thus, u is a common fixed point of f and g. To prove the uniqueness of the common fixed point of f and g, we assume that u and v are common fixed points of f and g. Then $fu = gu = u$ and $fv = gv = v$. By the first contractive condition, we have

$$\begin{aligned}
\psi\bigl(d(u, v)\bigr) &= \psi\bigl(d(fu, gv)\bigr) \\
&\leqslant k \max\bigl\{\psi\bigl(d(u, fu)\bigr), \psi\bigl(d(v, gv)\bigr)\bigr\} \\
&\quad + L \min\bigl\{q(u, gv), q(v, fu), q(u, fu)\bigr\} \\
&= 0.
\end{aligned}$$

So, $\psi(d(u, v)) = 0$ and hence $d(u, v) = 0$. Thus, $u = v$. So, f and g have a unique common fixed point. □

By taking $L = 0$ in Theorem 1, we get the following result:

Corollary 1. *Let (X, d) be a complete quasi metric space and $f, g : X \to X$ be two mappings. Assume the following hypotheses*:

(i) *There exist an ultra distance function ψ and a real number k with $k \in [0, 1)$ such that for all $x, y \in X$, we have*

$$\psi\big(d(fx, gy)\big) \leqslant k \max\big\{\psi\big(d(x, fx)\big), \psi\big(d(y, gy)\big)\big\}$$

and

$$\psi\big(d(gx, fy)\big) \leqslant k \max\big\{\psi\big(d(x, gx)\big), \psi\big(d(y, fy)\big)\big\}.$$

(ii) *f or g is continuous.*

Then f and g have a unique common fixed point in X.

Taking $g = f$ in Theorem 1, we have the following result:

Corollary 2. *Let (X, d) be a complete quasi metric space and $f : X \to X$ be a mapping. Suppose the following hypotheses*:

(i) *There exist an ultra distance function ψ, $k \in [0, 1)$ and $L \geqslant 0$ such that for all $x, y \in X$, we have*

$$\begin{aligned}\psi\big(d(fx, fy)\big) \leqslant{}& k \max\big\{\psi\big(d(x, fx)\big), \psi\big(d(y, fy)\big)\big\} \\ &+ L \min\big\{q(x, fy), q(y, fx), q(x, fx)\big\}.\end{aligned}$$

(ii) *f is continuous.*

Then f has a unique fixed point in X.

Corollary 3. *Let (X, d) be a complete quasi metric space and $f, g : X \to X$ be two mappings. Assume the following hypotheses*:

(i) *There exist an ultra distance function ψ, two positive real numbers a and b with $a + b < 1$ and $L > 0$ such that for all $x, y \in X$, we have*

$$\begin{aligned}\psi\big(d(fx, gy)\big) \leqslant{}& a\psi\big(d(x, fx)\big) + b\psi\big(d(y, gy)\big) \\ &+ L \min\big\{q(x, gy), q(y, fx), q(x, fx)\big\}\end{aligned}$$

and

$$\begin{aligned}\psi\big(d(gx, fy)\big) \leqslant{}& a\psi\big(d(x, gx)\big) + b\psi\big(d(y, fy)\big) \\ &+ L \min\big\{q(x, fy), q(y, gx), q(x, gx)\big\}.\end{aligned}$$

(ii) *f or g is continuous.*

Then f and g have a unique common fixed point in X.

Proof. Follows from Theorem 1 by noting that for any $x, y \in X$, we have

$$a\psi\big(d(x, gx)\big) + b\psi\big(d(y, fy)\big) \leqslant (a + b) \max\big\{\psi\big(d(x, gx)\big), \psi\big(d(y, fy)\big)\big\}. \qquad \square$$

Before, we present our next main result, we introduce the following definitions:

Definition 10. Let $\psi : [0, +\infty) \to [0, +\infty)$ be a function and (X, d) be a quasi metric space. We say that (X, d) is bounded with respect to ψ if the exists $M > 0$ such that $\psi(d(x, y)) \leqslant M$ for all $x, y \in X$.

Definition 11. Let (X, d) be a bounded quasi metric space and $f, g : X \to X$ be two mappings. The pair (f, g) is called (k, ψ, L)-quasi contraction of type (II) if there exist an ultra distance function ψ, $k \in [0, 1)$ and $L \geqslant 0$ such that for all $x, y \in X$, we have

$$\psi\big(d(fx, gy)\big) \leqslant k\psi\big(d(x, y)\big) + L \min\big\{\psi\big(d(fx, y)\big), \psi\big(d(x, gy)\big), \psi\big(d(x, fx)\big)\big\}$$

and

$$\psi\big(d(gx, fy)\big) \leqslant k\psi\big(d(x, y)\big) + L \min\big\{\psi\big(d(gx, y)\big), \psi\big(d(x, fy)\big), \psi\big(d(x, gx)\big)\big\}.$$

Theorem 2. *Let (X, d) be a complete quasi metric space and $f, g : X \to X$ be two mappings satisfying the following conditions*:

(i) *f or g is continuous.*
(ii) *The pair (f, g) is (k, ψ, L)-quasi contraction of type (II).*
(iii) *(X, d) is bounded with respect to ψ.*

Then f and g have a unique common fixed point in X.

Proof. As in proof of Theorem 1, we construct a sequence (x_n) in X such that $x_{2n+1} = fx_{2n}$ and $x_{2n+2} = gx_{2n+1}$. Assume there exists $r \in \mathbb{N}$ such that $x_{2r} = x_{2r+1}$. Since the pair (f, g) is (k, ψ, L)-quasi contraction of type (II), we have

$$\begin{aligned}
&d(x_{2r+1}, x_{2r+2}) \\
&\quad = d(fx_{2r}, gx_{2r+1}) \\
&\quad \leqslant k\psi\big(d(x_{2r}, x_{2r+1})\big) \\
&\qquad + L \min\big\{\psi\big(d(fx_{2r}, x_{2r+1})\big), \psi\big(d(x_{2r}, gx_{2r+1})\big), \psi\big(d(x_{2r}, fx_{2r})\big)\big\} \\
&\quad = k\psi\big(d(x_{2r}, x_{2r+1})\big) \\
&\qquad + L \min\big\{\psi\big(d(x_{2r+1}, x_{2r+1})\big), \psi\big(d(x_{2r}, x_{2r+2})\big), \psi\big(d(x_{2r}, x_{2r+1})\big)\big\} \\
&\quad = 0.
\end{aligned}$$

So, we conclude that $d(x_{2r+1}, x_{2r+2}) = 0$. Hence, $x_{2r} = x_{2r+1} = x_{2r+2}$. Therefore, x_{2r} is a common fixed point of f and g.

Now, assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Since (f, g) is (k, ψ, L)-quasi contraction of type (II), we have

$$\begin{aligned}
&\psi\big(d(x_{2n+1}, x_{2n+2})\big) \\
&\quad = \psi\big(d(fx_{2n}, gx_{2n+1})\big) \\
&\quad \leqslant k\psi\big(d(x_{2n}, x_{2n+1})\big) \\
&\qquad + \min\big\{\psi\big(d(fx_{2n}, x_{2n+1})\big), \psi\big(d(x_{2n}, gx_{2n+1})\big), \psi\big(d(x_{2n}, fx_{2n})\big)\big\}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant k\psi\big(d(x_{2n}, x_{2n+1})\big)\\
&\quad + \min\big\{0, \psi\big(d(x_{2n}, x_{2n+2})\big), \psi\big(d(x_{2n}, x_{2n+1})\big)\big\}\\
&= k\psi\big(d(x_{2n}, x_{2n+1})\big). \qquad (13)
\end{aligned}
$$

On other hand, we have

$$
\begin{aligned}
&\psi\big(d(x_{2n}, x_{2n+1})\big)\\
&\quad = \psi\big(d(gx_{2n-1}, fx_{2n})\big)\\
&\quad \leqslant k\psi\big(d(x_{2n-1}, x_{2n})\big)\\
&\qquad + L\min\big\{\psi\big(d(gx_{2n-1}, x_{2n})\big), \psi\big(d(x_{2n-1}, fx_{2n})\big), \psi\big(d(x_{2n-1}, gx_{2n-1})\big)\big\}\\
&\quad = k\psi\big(d(x_{2n-1}, x_{2n})\big)\\
&\qquad + L\min\big\{0, \psi\big(d(x_{2n-1}, x_{2n+1})\big), \psi\big(d(x_{2n-1}, x_{2n})\big)\big\}\\
&\quad = k\psi\big(d(x_{2n-1}, x_{2n})\big). \qquad (14)
\end{aligned}
$$

Inequalities (13) and (14) imply that

$$\psi\big(d(x_n, x_{n+1})\big) \leqslant k\psi\big(d(x_{n-1}, x_n)\big), \quad n \in \mathbb{N}. \qquad (15)$$

Repeating (15) n times, we obtain

$$\psi\big(d(x_n, x_{n+1})\big) \leqslant k^n\psi\big(d(x_0, x_1)\big), \quad n \in \mathbb{N}. \qquad (16)$$

Moreover, one can use the same arguments to prove that

$$\psi\big(d(x_{n+1}, x_n)\big) \leqslant k^n\psi\big(d(x_1, x_0)\big), \quad n \in \mathbb{N}.$$

Now, we prove that (x_n) is a Cauchy sequence in X. Let $n, m \in \mathbb{N}$ be such that n is odd, m is even, and $m > n$. By using (16), we have

$$
\begin{aligned}
&\psi\big(d(x_n, x_m)\big)\\
&\quad = \psi\big(d(fx_{n-1}, gx_{m-1})\big)\\
&\quad \leqslant k\psi\big(d(x_{n-1}, x_{m-1})\big)\\
&\qquad + L\min\big\{\psi\big(d(fx_{n-1}, x_{m-1})\big), \psi\big(d(x_{n-1}, gx_{m-1})\big), \psi\big(d(x_{n-1}, fx_{n-1})\big)\big\}\\
&\quad = k\psi\big(d(x_{n-1}, x_{m-1})\big)\\
&\qquad + L\min\big\{\psi\big(d(x_n, x_{m-1})\big), \psi\big(d(x_{n-1}, x_m)\big), \psi\big(d(x_{n-1}, x_n)\big)\big\}\\
&\quad \leqslant k\psi\big(d(x_{n-1}, x_{m-1})\big) + L\psi\big(d(x_{n-1}, x_n)\big)\\
&\quad \leqslant k\psi\big(d(x_{n-1}, x_{m-1})\big) + Lk^{n-1}\psi\big(d(x_0, x_1)\big). \qquad (17)
\end{aligned}
$$

On other hand, we have

$$
\begin{aligned}
&\psi\big(d(x_{n-1}, x_{m-1})\big)\\
&\quad = \psi\big(d(gx_{n-2}, fx_{m-2})\big)\\
&\quad \leqslant k\psi\big(d(x_{n-2}, x_{m-2})\big)\\
&\qquad + L\min\big\{\psi\big(d(gx_{n-2}, x_{m-2})\big), \psi\big(d(x_{n-2}, fx_{m-2})\big), \psi\big(d(x_{n-2}, gx_{n-2})\big)\big\}
\end{aligned}
$$

$$\begin{aligned}
&= k\psi\big(d(x_{n-2}, x_{m-2})\big) \\
&\quad + L\min\big\{\psi\big(d(x_{n-1}, x_{m-2})\big), \psi\big(d(x_{n-2}, x_{m-1})\big), \psi\big(d(x_{n-2}, x_{n-1})\big)\big\} \\
&\leqslant k\psi\big(d(x_{n-2}, x_{m-2})\big) + L\psi\big(d(x_{n-2}, x_{n-1})\big) \\
&\leqslant k\psi\big(d(x_{n-2}, x_{m-2})\big) + Lk^{n-2}\psi\big(d(x_0, x_1)\big). \qquad (18)
\end{aligned}$$

Repeating (17) and (18) n times and using the assumption that of (X, d) is bounded with respect to ψ, we have

$$\begin{aligned}
\psi\big(d(x_n, x_m)\big) &\leqslant k\psi\big(d(x_{n-1}, x_{m-1})\big) + Lk^{n-1}\psi\big(d(x_0, x_1)\big) \\
&\leqslant k^2\psi\big(d(x_{n-2}, x_{m-2})\big) + 2Lk^{n-1}\psi\big(d(x_0, x_1)\big) \\
&\leqslant k^3\psi\big(d(x_{n-3}, x_{m-3})\big) + 3Lk^{n-1}\psi\big(d(x_0, x_1)\big) \\
&\leqslant \cdots \\
&\leqslant k^n\psi\big(d(x_0, x_{m-n}\big) + Lnk^{n-1}\psi\big(d(x_0, x_1)\big) \\
&\leqslant k^nM + Lnk^{n-1}M. \qquad (19)
\end{aligned}$$

By letting $n, m \to +\infty$ in (19), we deduce that

$$\lim_{n,m\to+\infty} \psi\big(d(x_n, x_m)\big) = 0.$$

Similarly, we can show that

$$\lim_{n,m\to+\infty} \psi\big(d(x_m, x_n)\big) = 0.$$

Since ψ is an ultra distance function, we conclude that

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

By Lemma (1), we conclude that (x_n) is a Cauchy sequence in (X, d). So, there exists $u \in X$ such that

$$\lim_{n\to+\infty} d(x_n, u) = \lim_{n\to+\infty} d(u, x_n) = 0.$$

Without loss of generality, we may assume that f is continuous. So,

$$\lim_{n\to+\infty} d(fx_{2n}, fu) = \lim_{n\to+\infty} d(fu, fx_{2n}) = 0.$$

Hence,

$$\lim_{n\to+\infty} d(x_{2n+1}, fu) = \lim_{n\to+\infty} d(fu, x_{2n+1}) = 0.$$

So, we can conclude that $fu = u$. To show that $gu = u$, we use the triangle inequality,

$$\begin{aligned}
\psi\big(d(u, gu)\big) &= \psi\big(d(fu, gu)\big) \\
&\leqslant k(0) + L\min\big\{0, \psi\big(d(u, gu)\big)\big\} \\
&= 0.
\end{aligned}$$

Hence, $\psi(d(u, gu)) = 0$. Therefore, $d(u, gu) = 0$ and hence $gu = u$. So, u is a common fixed point of f and g. Now, let u and v be two common fixed points of f and g; that is, $fu = gu = u$ and $fv = gv = v$. Since the pair (f, g) is (k, ψ, L)-quasi contraction of type (II), we have

$$\begin{aligned}\psi\bigl(d(u,v)\bigr) &= \psi\bigl(d(fu,gv)\bigr)\\ &\leqslant k\psi\bigl(d(u,v)\bigr) + L\min\bigl\{\psi\bigl(d(fu,v)\bigr), \psi\bigl(d(u,gv)\bigr), \psi\bigl(d(u,fu)\bigr)\bigr\}\\ &= k\psi\bigl(d(u,v)\bigr).\end{aligned}$$

Since $k \in [0, 1)$, we conclude that $\psi(d(u, v)) = 0$. Hence, $d(u, v) = 0$. So, $u = v$. Therefore, the common fixed point of f and g is unique. □

By taking $L = 0$ in Theorem 2, we get the following result:

Corollary 4. *Let (X, d) be a complete quasi metric space and $f, g : X \to X$ be two mappings satisfying the following conditions*:

(i) *There exist an ultra distance function ψ and a real number $k \in [0, 1)$ such that for all $x, y \in X$, we have*

$$\psi\bigl(d(fx,gy)\bigr) \leqslant k\psi\bigl(d(x,y)\bigr) \quad \textit{and} \quad \psi\bigl(d(gx,fy)\bigr) \leqslant k\psi\bigl(d(x,y)\bigr).$$

(ii) *f or g is continuous.*

(iii) *(X, d) is bounded with respect to ψ.*

Then f and g have a unique common fixed point in X.

Taking $g = f$ in Theorem 2, we deduce the following result:

Corollary 5. *Let (X, d) be a complete quasi metric space and $f : X \to X$ be a mapping. Assume the following hypotheses*:

(i) *There exist an ultra distance function ψ, $k \in [0, 1)$ and $L \geqslant 0$ such that for all $x, y \in X$, we have*

$$\psi\bigl(d(fx,fy)\bigr) \leqslant k\psi\bigl(d(x,y)\bigr) + L\min\bigl\{\psi\bigl(d(fx,y)\bigr), \psi\bigl(d(x,fy)\bigr), \psi\bigl(d(x,fx)\bigr)\bigr\}.$$

(ii) *f is continuous.*

(iii) *(X, d) is bounded with respect to ψ.*

Then f has a unique fixed point in X.

3 Examples and applications

In this section, we introduce some examples and some application in order to support the useability of our results.

Example 4. Let $X = \{0, 1, 2, 3, \dots\}$. Define $d : X \times X \to X$ by

$$d(m,n) = \begin{cases} m & \text{if } n = 0, m \neq 0, \\ 1 & \text{if } m = 0, n \neq 0, \\ 0 & \text{if } m = n, \\ m & \text{if } m \neq n \end{cases}$$

and the mappings $f, g : X \to X$ by

$$fx = \begin{cases} 0 & \text{if } x \in \{0,1,2,3,4,5,6,7,8\}, \\ 1 & \text{if } x \in \{9,10,11,\dots\} \end{cases}$$

and

$$gx = \begin{cases} 0 & \text{if } x \in \{0,1,2,3,4\}, \\ 1 & \text{if } x \in \{5,6,7,\dots\}. \end{cases}$$

Also, define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} \mathrm{e}^t - 1 & \text{if } t \in [0,1], \\ \mathrm{e}^t - 2 & \text{if } t \in (1, +\infty). \end{cases}$$

Then:

(i) ψ is an ultra distance function.
(ii) (X, d) is a complete quasi metric space.
(iii) f is continuous.
(iv) For all $x, y \in X$, we have

$$\psi\bigl(d(fx, gy)\bigr) \leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, fx)\bigr), \psi\bigl(d(y, gy)\bigr)\bigr\} + L\min\bigl\{q(x, gy), q(y, fx), q(x, fx)\bigr\}$$

and

$$\psi\bigl(d(gx, fy)\bigr) \leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, gx)\bigr), \psi\bigl(d(y, fy)\bigr)\bigr\} + L\min\bigl\{q(x, fy), q(y, gx), q(x, gx)\bigr\}.$$

Proof. The proof of (i) is clear. Also, it is an easy matter to figure out that d is quasi metric. To show that d is complete, we let (x_n) be a Cauchy sequence in X. Then

$$\lim_{n\to+\infty} d(x_n, x_m) = \lim_{n\to+\infty} d(x_m, x_n) = 0.$$

So, we deduce that $x_n = x_m$ for all $n, m \in \{0, 1, 2, 3, \dots\}$ but possibly at finitely many. So, (x_n) is a convergent sequence in X. Thus, we conclude that (X, d) is a complete quasi metric space.

To prove (iii), let (x_n) be a sequence in X such that $x_n \to x \in X$. So,

$$\lim_{n\to+\infty} d(x_n, x) = \lim_{n\to+\infty} d(x, x_n) = 0.$$

Thus, we conclude that $x_n = x$ for all $n \in \mathbb{N}$ but possible at finitely many. Thus, $f(x_n) = f(x)$ for all but possible at finitely many. So, $f(x_n) \to fx$. Thus, f is continuous.

To prove (iv), given $x, y \in X$. We divide the proof into the following cases:

Case 1: $x \in \{0,1,2,3,4,5,6,7,8\}$ and $y \in \{0,1,2,3,4\}$. Then

$$\begin{aligned}\psi\bigl(d(fx, gy)\bigr) &= 0\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, fx)\bigr), \psi\bigl(d(y, gy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x, gy), q(y, fx), q(x, fx)\bigr\}.\end{aligned}$$

Now, if $x \in \{0,1,2,3,4\}$, then we have

$$\begin{aligned}\psi\bigl(d(gx, fy)\bigr) &= 0\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, gx)\bigr), \psi\bigl(d(y, fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x, fy), q(y, gx), q(x, gx)\bigr\}.\end{aligned}$$

While, if $x \in \{5,6,7,8\}$, we have

$$\begin{aligned}\psi\bigl(d(gx, fy)\bigr) &= \psi\bigl(d(1,0)\bigr) = \psi(1) = \mathrm{e} - 1 \leqslant \frac{1}{2}(\mathrm{e}^5 - 2) \leqslant \frac{1}{2}(\mathrm{e}^x - 2)\\ &= \frac{1}{2}\psi(x) = \frac{1}{2}\psi\bigl(d(x,1)\bigr) = \frac{1}{2}\psi\bigl(d(x, gx)\bigr)\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, gx)\bigr), \psi\bigl(d(y, fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x, fy), q(y, gx), q(x, gx)\bigr\}.\end{aligned}$$

Case 2: $x \in \{0,1,2,3,4,5,6,7,8\}$ and $y \in \{5,6,7,\dots\}$. So,

$$\begin{aligned}\psi\bigl(d(fx, gy)\bigr) &= \psi(1) = \mathrm{e} - 1 \leqslant \frac{1}{2}\bigl(\mathrm{e}^5 - 2\bigr) \leqslant 12\bigl(\mathrm{e}^y - 2\bigr)\frac{1}{2}\psi(y) = \frac{1}{2}\psi\bigl(d(y, gy)\bigr)\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, fx)\bigr), \psi\bigl(d(y, gy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x, gy), q(y, fx), q(x, fx)\bigr\}.\end{aligned}$$

Now, if $x \in \{0,1,2,3,4\}$ and $y \in \{5,6,7,8\}$, then

$$\begin{aligned}\psi\bigl(d(gx, fy)\bigr) &= 0\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x, gx)\bigr), \psi\bigl(d(y, fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x, fy), q(y, gx), q(x, gx)\bigr\}.\end{aligned}$$

If $x \in \{0,1,2,3,4\}$ and $y \in \{9,10,11,\ldots\}$, then

$$\begin{aligned}\psi\bigl(d(gx,fy)\bigr) &= \psi\bigl(d(0,1)\bigr) = \psi(1) = \mathrm{e}-1 \leqslant \frac{1}{2}\bigl(\mathrm{e}^9-2\bigr) \leqslant \frac{1}{2}\bigl(\mathrm{e}^y-2\bigr)\\ &= \frac{1}{2}\psi(y) = \frac{1}{2}\psi\bigl(d(y,1)\bigr) = \frac{1}{2}\psi\bigl(d(y,fy)\bigr)\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,gx)\bigr),\psi\bigl(d(y,fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,fy),q(y,gx),q(x,gx)\bigr\}.\end{aligned}$$

If $x \in \{5,6,7,8\}$ and $y \in \{5,6,7,8\}$, then

$$\begin{aligned}\psi\bigl(d(gx,fy)\bigr) &= \psi(1) = \mathrm{e}-1 \leqslant \frac{1}{2}\bigl(\mathrm{e}^5-2\bigr)\\ &\leqslant \frac{1}{2}\bigl(\mathrm{e}^y-2\bigr) = \frac{1}{2}\psi(y) = \frac{1}{2}\psi\bigl(d(y,1)\bigr) = \frac{1}{2}\psi\bigl(d(y,fy)\bigr)\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,gx)\bigr),\psi\bigl(d(y,fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,fy),q(y,gx),q(x,gx)\bigr\}.\end{aligned}$$

If $x \in \{5,6,7,8\}$ and $y \in \{9,10,11,\ldots\}$, then

$$\begin{aligned}\psi\bigl(d(gx,fy)\bigr) &= \psi\bigl(d(1,1)\bigr) = 0\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,gx)\bigr),\psi\bigl(d(y,fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,fy),q(y,gx),q(x,gx)\bigr\}.\end{aligned}$$

Case 3: $x \in \{9,10,11,\ldots\}$ and $y \in \{0,1,2,3,4\}$. Then

$$\begin{aligned}\psi\bigl(d(fx,gy)\bigr) &= \psi\bigl(d(fx,gy)\bigr) = \psi(1) = \mathrm{e}-1\\ &\leqslant \frac{1}{2}\bigl(\mathrm{e}^9-1\bigr) \leqslant \frac{1}{2}\bigl(\mathrm{e}^x-2\bigr) = \frac{1}{2}\psi(x) =\leqslant \frac{1}{2}\psi\bigl(d(x,fx)\bigr)\\ &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,fx)\bigr),\psi\bigl(d(y,gy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,gy),q(y,fx),q(x,fx)\bigr\}.\end{aligned}$$

Similarly, one can show that

$$\begin{aligned}\psi\bigl(gx,fy)\bigr) &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,gx)\bigr),\psi\bigl(d(y,fy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,fy),q(y,gx),q(x,gx)\bigr\}.\end{aligned}$$

Case 4: $x \in \{9,10,11,\ldots\}$ and $y \in \{5,6,7,\ldots\}$. Then

$$\begin{aligned}\psi\bigl(d(fx,gy)\bigr) = 0 &\leqslant \frac{1}{2}\max\bigl\{\psi\bigl(d(x,fx)\bigr),\psi\bigl(d(y,gy)\bigr)\bigr\}\\ &\quad + L\min\bigl\{q(x,gy),q(y,fx),q(x,fx)\bigr\}.\end{aligned}$$

Also, if $x \in \{9, 10, 11, \dots\}$ and $y \in \{5, 6, 7, 8\}$, then

$$\begin{aligned}\psi(gx, fy) &= \psi\big(d(1,0)\big) = \psi(1) = \mathrm{e} - 1 \leqslant \frac{1}{2}\big(\mathrm{e}^9 - 2\big) \\ &\leqslant \frac{1}{2}\big(\mathrm{e}^x - 2\big) = \frac{1}{2}\psi(x) = \frac{1}{2}\psi\big(d(x, gx)\big) \\ &\leqslant \frac{1}{2}\max\big\{\psi\big(d(x, gx)\big), \psi\big(d(y, fy)\big)\big\} \\ &\quad + L\min\big\{q(x, fy), q(y, gx), q(x, gx)\big\}.\end{aligned}$$

If $x \in \{9, 10, 11, \dots\}$ and $y \in \{9, 10, 11, \dots\}$, then

$$\begin{aligned}\psi(gx, fy) = \psi\big(d(1,1)\big) = 0 &\leqslant \frac{1}{2}\max\big\{\psi\big(d(x, gx)\big), \psi\big(d(y, fy)\big)\big\} \\ &\quad + L\min\big\{q(x, fy), q(y, gx), q(x, gx)\big\}.\end{aligned}$$

Note that f and g satisfy all the conditions of Theorem 1. So, f and g have a unique common fixed point. Here 0 is the unique common fixed point of f and g. □

Example 5. Let $X = [0, 1/2]$. Define $d : X \times X \to [0, +\infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ x + y & \text{if } x \neq y \end{cases}$$

and the mappings $f, g : X \to X$ by $fx = x^2/2$ and $gx = x^2/4$. Also, define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} t^2 & \text{if } t \in [0, 2], \\ \frac{t}{1+t} & \text{if } t \in (2, +\infty). \end{cases}$$

Then:

(i) ψ is an ultra distance function.
(ii) (X, d) is a complete quasi metric space.
(iii) (X, d) is bounded with respect to ψ.
(iv) f is continuous.
(v) The pair (f, g) is (k, ψ, L)-quasi contraction of type (II).

Proof. To show that d is complete, we let (x_n) be a Cauchy sequence in X. Then

$$\lim_{n,m\to+\infty} d(x_n, x_m) = \lim_{n,m\to+\infty} d(x_m, x_n) = 0.$$

Given $\varepsilon > 0$, then there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n \geqslant k$. So, we have two cases:

Case 1: $x_n = x_m = x$ for all but finitely many. In this case, we have $x_n \to x$.

Case 2: $x_n \neq x_m$ for all but finitely many. In this case, we have $x_n + x_m < \varepsilon$ for all $m > n \geqslant k$. So, $x_n < \varepsilon$ for all $n \geqslant k$. So, $x_n \to 0$. From both cases, we conclude that X is complete.

To prove that (X, d) is bounded with respect to ψ, Given $x, y \in X$. If $x = y$, then $d(x, y) = 0$ and hence $\psi(d(x, y)) = 0 < 1$. If $x \neq y$, then $d(x, y) = x + y < 1$. So, $\psi(d(x, y)) = (x + y)^2 < 1$. So, (X, d) is bounded with respect to ψ. To prove the continuity of f, let (x_n) be a sequence in X such that $x_n \to x \in X$. So,

$$\lim_{n\to+\infty} d(x_n, x) = \lim_{n\to+\infty} d(x, x_n) = 0.$$

If $x_n = x$ for all but finitely many, then $fx_n = fx$ for all but finitely many. So, $fx_n \to fx$. If $x_n \neq x$ for all but finitely many. Here we conclude that $x = 0$. So, $f(x) = 0$. Hence, $d(fx_n, fx) = d(fx, fx_n) = x_n^2/2 \to 0$. So, $fx_n \to fx$. So, we conclude that f is continuous.

To prove that the pair (f, g) is (k, ψ, L)-quasi contraction of type (II), given $x, y \in X$. Then $fx = x^2/2$ and $gy = y^2/4$.

Case 1: $y^2 = 2x^2$. Here we have

$$\begin{aligned}\psi\big(d(fx, gy)\big) &= 0 \\ &\leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(fx, y)\big), \psi\big(d(x, gy)\big), \psi\big(d(x, fx)\big)\big\}.\end{aligned}$$

Also,

$$\begin{aligned}\psi\big(d(gx, fy)\big) &= \psi\left(d\left(\frac{1}{4}x^2, \frac{1}{2}y^2\right)\right) = \left(\frac{1}{4}x^2 + x^2\right)^2 \\ &= \frac{25}{16}x^4 \leqslant \frac{25}{64}x^2 \leqslant \frac{1}{4}(1+\sqrt{2})^2x^2 \\ &= \frac{1}{4}\big(d(x, \sqrt{2}x)\big)^2 = \frac{1}{4}\psi\big(d(x, y)\big) \\ &\leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(gx, y)\big), \psi\big(d(x, fy)\big), \psi\big(d(x, gx)\big)\big\}.\end{aligned}$$

Case 2: $y^2 \neq 2x^2$. We divide this case into the following subcases:

Subcase 1: $y^2 \neq 2x^2$ and $x \neq y$. Here we have

$$\begin{aligned}\psi\big(d(fx, gy)\big) &= \psi\left(d\left(\frac{1}{2}x^2, \frac{1}{4}y^2\right)\right) = \psi\left(\frac{1}{2}x^2 + \frac{1}{4}y^2\right) \\ &= \left(\frac{1}{2}x^2 + \frac{1}{4}y^2\right)^2 \leqslant \frac{1}{4}(x+y)^2 = \frac{1}{4}\psi\big(d(x, y)\big) \\ &\leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(fx, y)\big), \psi\big(d(x, gy)\big), \psi\big(d(x, fx)\big)\big\}.\end{aligned}$$

Subcase 2: $y^2 \neq 2x^2$ and $x = y$. Here we have

$$\psi\big(d(fx, gy)\big) = \psi\left(d\left(\frac{1}{2}x^2, \frac{1}{4}y^2\right)\right) = \left(\frac{1}{2}x^2 + \frac{1}{4}y^2\right)^2 = \left(\frac{3}{4}x^2\right)^2,$$

$$\psi\big(d(fx, y)\big) = \psi\left(d\left(\frac{1}{2}x^2, y\right)\right) = \left(\frac{1}{2}x^2 + y\right)^2 = \left(\frac{1}{2}x^2 + x\right)^2,$$

$$\psi\big(d(x, gy)\big) = \psi\left(d\left(x, \frac{1}{4}y^2\right)\right) = \left(x + \frac{1}{4}y^2\right)^2 = \left(x + \frac{1}{4}x^2\right)^2,$$

and

$$\psi\big(d(x, fx)\big) = \psi\left(d\left(x, \frac{1}{2}x^2\right)\right) = \left(x + \frac{1}{2}x^2\right)^2.$$

So, we have

$$\psi\big(d(fx, gy)\big) \leqslant \min\big\{\psi\big(d(fx, y)\big), \psi\big(d(x, gy)\big), \psi\big(d(x, fx)\big)\big\}.$$

Therefore,

$$\psi\big(d(fx, gy)\big) \leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(fx, y)\big), \psi\big(d(x, gy)\big), \psi\big(d(x, fx)\big)\big\}.$$

Subcase 3: $y^2 \neq 2x^2$ and $x^2 = 2y^2$. Here,

$$\begin{aligned}\psi\big(d(gx, fy)\big) &= \psi\left(d\left(\frac{1}{2}y^2, \frac{1}{2}y^2\right)\right) = 0 \\ &\leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(gx, y)\big), \psi\big(d(x, fy)\big), \psi\big(d(x, gx)\big)\big\}.\end{aligned}$$

Subcase 4: $y^2 \neq 2x^2$, $x^2 \neq 2y^2$, and $x \neq y$. Here we have

$$\begin{aligned}\psi\big(d(gx, fy)\big) &= \psi\left(d\left(\frac{1}{4}x^2, \frac{1}{2}y^2\right)\right) = \psi\left(\frac{1}{4}x^2 + \frac{1}{2}y^2\right) \\ &= \left(\frac{1}{4}x^2 + \frac{1}{2}y^2\right)^2 \leqslant \frac{1}{4}(x + y)^2 = \frac{1}{4}\psi\big(d(x, y)\big) \\ &\leqslant \frac{1}{4}\psi\big(d(x, y)\big) + \min\big\{\psi\big(d(gx, y)\big), \psi\big(d(x, fy)\big), \psi\big(d(x, gx)\big)\big\}.\end{aligned}$$

Subcase 5: $y^2 \neq 2x^2$, $x^2 \neq 2y^2$, and $x = y$. Here we have

$$\psi\big(d(gx, fy)\big) = \psi\left(d\left(\frac{1}{4}x^2, \frac{1}{2}y^2\right)\right) = \left(\frac{1}{4}x^2 + \frac{1}{2}y^2\right)^2 = \left(\frac{3}{4}x^2\right)^2,$$

$$\psi\big(d(gx, y)\big) = \psi\left(d\left(\frac{1}{4}x^2, y\right)\right) = \left(\frac{1}{4}x^2 + y\right)^2 = \left(\frac{1}{4}x^2 + x\right)^2,$$

$$\psi\big(d(x, fy)\big) = \psi\left(d\left(x, \frac{1}{2}y^2\right)\right) = \left(x + \frac{1}{2}y^2\right)^2 = \left(x + \frac{1}{2}x^2\right)^2,$$

and

$$\psi\bigl(d(x,gx)\bigr) = \psi\left(d\left(x, \frac{1}{4}x^2\right)\right) = \left(x + \frac{1}{4}x^2\right)^2.$$

So, we have

$$\psi\bigl(d(gx,fy)\bigr) \leqslant \min\bigl\{\psi\bigl(d(gx,y)\bigr), \psi\bigl(d(x,fy)\bigr), \psi\bigl(d(x,gx)\bigr)\bigr\}.$$

Therefore,

$$\psi\bigl(d(gx,fy)\bigr) \leqslant \frac{1}{4}\psi\bigl(d(x,y)\bigr) + \min\bigl\{\psi\bigl(d(gx,y)\bigr), \psi\bigl(d(x,fy)\bigr), \psi\bigl(d(x,gx)\bigr)\bigr\}.$$

By combining all cases together, we conclude that for all $x, y \in X$, we have

$$\psi\bigl(d(fx,gy)\bigr) \leqslant \psi\bigl(d(x,y)\bigr) + \min\bigl\{\psi\bigl(d(fx,y)\bigr), \psi\bigl(d(x,gy)\bigr), \psi\bigl(d(x,fx)\bigr)\bigr\}$$

and

$$\psi\bigl(d(gx,fy)\bigr) \leqslant \psi\bigl(d(x,y)\bigr) + \min\bigl\{\psi\bigl(d(gx,y)\bigr), \psi\bigl(d(x,fy)\bigr), \psi\bigl(d(x,gx)\bigr)\bigr\}.$$

Thus, the pair (f,g) is $(1/4,\psi,1)$-quasi contraction of type (II).

So, all conditions of Theorem 2 are satisfied. Therefore, f and g have a unique common fixed point, here 0 is the common fixed point of f and g. □

Now, we will introduce an example to show that our results can be used to prove the existence and uniqueness of solution of such nontrivial equations.

Example 6. The equation

$$x^2 + 2 = 4x^3 + 16x \tag{20}$$

has a unique real solution.

Proof. Let $X = [0,1]$. Define $d : X \times X \to [0,+\infty)$ by $d(x,y) = |x-y|$. Then (X,d) is a complete quasi metric space. Also, define the mapping $f : X \to X$ by $fx = (x^2+2)/(4x^2+16)$. Note that f is continuous. Moreover, define $\psi : [0,+\infty) \to [0,+\infty)$ by

$$\psi(t) = \begin{cases} t^2 & \text{if } t \in [0,1], \\ t - \frac{1}{2} & \text{if } t \in (1,+\infty). \end{cases}$$

Note that ψ is an ultra distance function and (X,d) is bounded with respect to ψ. Now, we will prove that the pair (f,f) is $(1/16,\psi,0)$-quasi contraction of type (II). Given $x, y \in X$. Then

$$\begin{aligned}\psi\bigl(d(fx,fy)\bigr) &= \left|\frac{x^2+2}{4x^2+16} - \frac{y^2+2}{4y^2+16}\right|^2 = \frac{(8|x^2-y^2|)^2}{(4x^2+16)^2(4y^2+16)^2} \\ &\leqslant \frac{(16|x-y|)^2}{(4x^2+16)^2(4y^2+16)^2} \leqslant \frac{1}{16}\psi\bigl(d(x,y)\bigr).\end{aligned}$$

Thus, f satisfies all conditions of Theorem 2. Therefore, f has a unique fixed point. Note that the unique fixed point of f is the unique solution of equation (20). □

As an application of our results, we construct some fixed point theorems of integral types.

Let Γ denoted to the set of functions $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) λ is Lebesgue-integrable on each compact of $\mathbb{R}^+$.

(ii) For each $\epsilon > 0$, we have

$$\int_0^{\epsilon} \lambda(z)\,\mathrm{d}z > 0.$$

Theorem 3. *Let (X, d) be a complete quasi metric space and f, g be two self mappings on X. Assume that f and g satisfy the following hypothesis*:

(i) *There exist $\lambda \in \Gamma$, $L > 0$, and two positive numbers a and b, $a + b < 1$, such that*

$$\int_0^{d(fx,gy)} \lambda(z)\,\mathrm{d}z \leqslant a \int_0^{d(x,fx)} \lambda(z)\,\mathrm{d}z + b \int_0^{d(y,gy)} \lambda(z)\,\mathrm{d}z + L \int_0^{d_1} \mathrm{d}z,$$

where $d_1 = \min\{q(x, gy), q(y, fx), q(x, fx)\}$, and

$$\int_0^{d(gx,fy)} \lambda(z)\,\mathrm{d}z \leqslant a \int_0^{d(x,gx)} \lambda(z)\,\mathrm{d}z + b \int_0^{d(y,fy)} \lambda(z)\,\mathrm{d}z + L \int_0^{d_2} \mathrm{d}z,$$

where $d_2 = \min\{q(x, fy), q(y, gx), q(x, gx)\}$.

(ii) *f or g is continuous.*

Then f and T have a unique common fixed point.

Proof. Define the function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ via $\psi(t) = \int_0^t \lambda(z)\,\mathrm{d}z$. Note that ψ is an ultra distance function. The result follows from Corollary 3 by noting that the two functions f and g satisfy all the conditions of Corollary 3. □

Theorem 4. *Let (X, d) be a complete quasi metric space and f, g be two self mappings on X. Assume the following hypothesis*:

(i) *There exist $\lambda \in \Gamma$, $L > 0$, and $k \in [0, 1)$ such that*

$$\int_0^{d(fx,gy)} \lambda(z)\,\mathrm{d}z \leqslant k \int_0^{d(x,y)} \lambda(z)\,\mathrm{d}z + L \int_0^{d_3} \mathrm{d}z,$$

where $d_3 = \min\{\psi(d(x, gy)), \psi(d(y, fx)), \psi(d(x, fx))\}$, and

$$\int_0^{d(gx,fy)} \lambda(z)\,\mathrm{d}z \leqslant k \int_0^{d(x,y)} \lambda(z)\,\mathrm{d}z + L \int_0^{d_4} \mathrm{d}z,$$

where $d_4 = \min\{\psi(d(x, fy)), \psi(d(y, gx)), \psi(d(x, gx))\}$.

(ii) *There exists* $M > 0$ *such that for all* $x, y \in X$, *we have* $\int_0^{d(x,y)} \lambda(z)\,\mathrm{d}z \leqslant M$.
(iii) f *or* g *is continuous.*

Then f *and* T *have a unique common fixed point.*

Proof. Define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \int_0^t \lambda(z)\,\mathrm{d}z$. Note that ψ is an ultra distance function. From condition (ii), we deduce that (X, d) is bounded with respect to ψ. Also, from condition (i), we conclude that pair (f, g) is (k, ψ, L)-quasi contraction of type (II). Thus, the functions f and g satisfy all the conditions of Theorem 2. Thus, f and g have a unique common fixed point. □

Now, we furnish an application to show that our work can be used to prove the existence and uniqueness solution of some integral equations.

Let $X = C([0, 1], \mathbb{R})$ be the set of all continuous functions defined on $[0, 1]$. Define $d : X \times X \to [0, +\infty)$ by

$$d(x, y) = \|x - y\|_\infty =: \sup\bigl\{\bigl|x(t) - y(t)\bigr|\colon t \in [0, 1]\bigr\}.$$

Note that (X, d) is a complete quasi metric space. Consider the following integral equation:

$$x(t) = \int_0^1 s(t, u) f\bigl(u, x(u)\bigr)\,\mathrm{d}u, \tag{21}$$

where $f : [0, 1] \times X \to \mathbb{R}$ is a continuous function, and $s : [0, 1] \times [0, 1] \to [0, +\infty)$ is a function such that $s(t, .)$ is integrable for all $t \in [0, 1]$.

Now, we introduce and prove the following theorem, which ensure that the integral equation (21) has a unique solution.

Theorem 5. *Suppose the following conditions are satisfied*:

(i) *There exists an ultra distance function* ψ *such that for all* $u \in [0, 1]$ *and* $x, y \in X$, *we have*
$$\bigl|f\bigl(u, x(u)\bigr) - f\bigl(u, y(u)\bigr)\bigr| \leqslant \psi\bigl(\bigl|x(u) - y(u)\bigr|\bigr).$$
(ii) *There exist* $M_1, M_2 > 0$ *such that for any* $x \in X$ *and* $t \in [0, 1]$, *we have*
$$\psi\bigl(\bigl|x(t)\bigr|\bigr) \leqslant M_1 \psi\bigl(\|x\|_\infty\bigr) \leqslant M_2 \|x\|_\infty.$$
(iii) *There exists* $k \in [0, 1/M_2)$ *such that for any* $t \in [0, 1]$, *we have* $\int_0^1 s(t, u)\,\mathrm{d}u < k$.
(iv) *There exists* $M > 0$ *such that for any* $x, y \in X$, *we have* $\psi(d(x, y)) \leqslant M$.

Then the integral equation (21) *has a unique solution.*

Proof. Define an operator $T : X \times X \to [0, +\infty)$ by

$$Tx(t) = \int_0^1 s(t, u) f\bigl(u, x(u)\bigr)\,\mathrm{d}u.$$

Note that any fixed point of T is a solution for the integral equation (21). So, we shall show that T has a unique fixed point. Note that T is continuous. Now, given $x, y \in X$.

Then for any $t \in [0,1]$, we have

$$\begin{aligned}\left|Tx(t) - Ty(t)\right| &= \left| \int_0^1 s(t,u) f\big(u, x(u)\big) - s(t,u) f\big(u, y(u)\big)\,\mathrm{d}u \right| \\ &\leqslant \int_0^1 s(t,u) \big| f\big(u, x(u)\big) - f\big(u, y(u)\big) \big|\,\mathrm{d}u \\ &\leqslant \int_0^1 s(t,u)\psi\big(\big|x(u) - y(u)\big|\big)\,\mathrm{d}u \\ &\leqslant \int_0^1 M_1 s(t,u)\psi\big(\|x - y\|_\infty\big)\,\mathrm{d}u \leqslant kM_1\psi\big(\|x - y\|_\infty\big).\end{aligned}$$

Thus, $\|Tx - Ty\|_\infty \leqslant kM_1\psi(\|x - y\|_\infty)$. From condition (ii) we have

$$\begin{aligned}\psi\big(d(Tx, Ty)\big) = \psi\big(\|Tx - Ty\|_\infty\big) &\leqslant \frac{M_2}{M_1}\|Tx - Ty\|_\infty \\ &\leqslant kM_2\psi\big(\|x - y\|_\infty\big) = kM_2\psi\big(d(x,y)\big).\end{aligned}$$

So, the pair (T, T) is $(kM_2, \psi, 0)$-quasi contraction of type (II). From condition (iv) we conclude that (X, d) is bounded with respect to ψ. Thus, all conditions of Theorem 2 are satisfied. So, T has a unique solution. Therefore, the integral equation (21) has a unique solution. □

We introduce the following example to illustrate Theorem 5.

Example 7. The following integral equation:

$$x(t) = \int_0^1 \frac{u|x(u)|\cos u}{(2+t)(1+|x(u)|)}\,\mathrm{d}u \tag{22}$$

has a unique solution in $C([0,1], \mathbb{R})$.

Proof. Let $X = C([0,1], \mathbb{R})$ be the set of all continuous function on $[0,1]$. Define $f : [0,1] \times X \to \mathbb{R}$ and $s : [0,1] \times [0,1] \to [0, +\infty)$ by

$$f\big(t, y(t)\big) = \frac{\cos t|y(t)|}{1 + |y(t)|},$$

and $s(t,u) = u/(2+t)$. Moreover, define $\psi : [0, +\infty) \to [0, +\infty)$ by

$$\psi(t) = \begin{cases} \frac{t}{1+t} & \text{if } t \in [0,1], \\ 1 & \text{if } t \in (1, +\infty). \end{cases}$$

Then f, s, and ψ satisfy all conditions of Theorem 5 with $M_1 = M_2 = 1$ and $k = 1/2$. Thus, the integral equation (22) has a unique solution in X. □

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