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Application of fractional subequation method to nonlinear evolution equations

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Abstract. In this paper, we constructed a traveling wave solutions expressed by three types of functions: hyperbolic, trigonometric, and rational. We used a fractional subequation method for some space-time fractional nonlinear partial differential equations (FNPDE), which are considered as models for different phenomena in natural and social sciences fields like engineering, physics, geology, etc. This method is very effective and easy to investigate exact traveling wave solutions to FNPDE with the aid of the modified Riemann–Liouville derivative.

Keywords: fractional Cahn–Hilliard equation, modified Riemann–Liouville derivative, fractional subequation method, spinodal decomposition, phase ordering dynamics.

1 Introduction

Over the last decades, the field of fractional calculus has thrived in pure mathematics as well as in scientific applications, and its utility has become more and more conspicuous. Rating the fractional calculus as a young science would be completely wrong. In fact, the ancestry of fractional calculus can be outputted from Leibniz's letter to l'Hôpital, see [25] in which the meaning of fractional derivatives, especially the case of the one-half order, was first discussed with some remarks about its possibility. The last reference of fractional calculus during the lifetime of Leibniz can be found in [26]. With the death of Leibniz in 1716, the topic of fractional derivatives did not end, and many famous mathematicians have worked on this topic and related questions. A list of mathematicians, who have provided important contributions up to the middle of last century, includes Laplace, Fourier, Abel, Liouville, Riemann, Grünwald, Letnikov, Lévy, Marchaud, and Riesz [21, 22].

The fractional differential equation may be considered as a straightforward development of the classical differential equation. Thus, the studies related to them have received considerable attention in more recently years. Fractional differential equations [2] have been considered as powerful mathematical tools for factual and more accurate description of different phenomena. They appear in various areas, including mathematical chemistry [9, 18], viscoelasticity [27], biology [22], electrochemistry, physics [12], semiconductors, seismology, scattering theory, heat conduction, fluid flow, metallurgy, population dynamics, optimal control theory, mathematical economics, and chemical reaction. As the employment of fractional partial differential equations (FPDEs) is increasing in many social and scientific fields [23, 38], the main challenge we confront is to obtain solutions for them. Unfortunately, for most of these FPDEs, no one able to achieve analytic solutions for such problems. There are an extraordinary number of demonstrating and fractional-order differential equations, which have been illuminated numerically utilizing different methods, see [1–4, 7, 8].

Moreover, several analytical techniques are presented to solve the fractional differential equation such as an iterative Laplace transform method [32, 33], adaptive observer [39], and a new analytical technique (NAT) [35]. On the other hand, very few techniques for the analytical solution of FNPDEs have been presented. For example, the fractional variational iteration method [10, 37], the Adomian decomposition method [8, 28], the homotopy perturbation method [11, 15], the finite element method [14], the (G'/G)-expansion method [31, 36]. Fractional differential-algebraic equations is solved in [6] using waveform relaxation method. A spectral decomposition [3] based on Fourier and Laplace transforms is introduced to solve time-fractional diffusion equation. Analytical soliton solutions are listed by Navickas et al. [24] for solving nonlinear fractional-order differential equations. Time-fractional diffusion equation is solved using spectral decomposition method with Fourier and Laplace transforms [3]. Fourier series expansion [29] is used to construct semianalytical solutions for time-fractional telegraph equations. For Riesz fractional advection-dispersion equation, the authors in [34] used the Laplace and Fourier transforms to treat the time and space variables. While, nonlinear time-fractional differential equations are solves using the three-time-splitting scheme [5].

Here, we applied a fractional subequation method [40, 41] for finding new solutions of some FPDEs like the space-time fractional Cahn–Hilliard equation [19], the space-time fractional fifth-order Sawda–Kotera equation [20, 30], and the space-time fractional modified equal-width (EW) equation [13]. We use a fractional subequation method to introduce another solutions for the mentioned problems in the sense of the modified Riemann–Liouville derivative defined by Jumarie [16, 17], which is a fractional version of the known (G'/G) method. This method is based on the fractional ODE

$$D_{\xi}^{2\alpha}G(\xi) + \lambda D_{\xi}^{\alpha}G(\xi) + \mu G(\xi) = 0, \tag{1}$$

where $D_\xi^\alpha G(\xi)$ is the modified Riemann–Liouville derivative of order α for $G(\xi)$ with respect to ξ.

This paper is organized as follows. After this introduction, some basic properties of Jumarie's modified Riemann–Liouville derivative are given in Section 2. In Section 3, the main steps of a fractional subequation method are given. In Section 4, we construct the solutions of some FNPDEs by a fractional subequation method. Finally, In Section 5, the conclusions are illustrated.

2 Jumarie's modified Riemann–Liouville derivative and general expression for $(D_\xi^\alpha G/G)$

Jumarie's modified Riemann–Liouville derivative of order α is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (t-\xi)^{-\alpha}(f(\xi)-f(0))\,\mathrm{d}\xi, & 70<\alpha<1, \\ (f^{(n)}(t))^{(\alpha-n)}, & n \leqslant \alpha < n+1,\ n \geqslant 1. \end{cases}$$

We list some important properties for the modified Riemann–Liouville derivative as follows:

$$\begin{gathered} D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha}, \\ D_t^\alpha \big(f(t)g(t)\big) = g(t)D_t^\alpha f(t) + f(t) D_t^\alpha g(t), \\ D_t^\alpha f\big[g(t)\big] = f'_g\big[g(t)\big] D_t^\alpha g(t) = D_g^\alpha f\big[g(t)\big]\big(g'(t)\big)^\alpha. \end{gathered} \tag{2}$$

The general solution of Eq. (2) is given as

$$\frac{H'(\eta)}{H(\eta)} = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\varDelta}}{2}\frac{C_1\sinh\frac{\sqrt{\varDelta}}{2}\eta + C_2\cosh\frac{\sqrt{\varDelta}}{2}\eta}{C_1\cosh\frac{\sqrt{\varDelta}}{2}\eta + C_2\sinh\frac{\sqrt{\varDelta}}{2}\eta}, & \varDelta > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{-\varDelta}}{2}\frac{-C_1\sin\frac{\sqrt{-\varDelta}}{2}\eta + C_2\cos\frac{\sqrt{-\varDelta}}{2}\eta}{C_1\cos\frac{\sqrt{-\varDelta}}{2}\eta + C_2\sin\frac{\sqrt{-\varDelta}}{2}\eta}, & \varDelta < 0, \\ -\frac{\lambda}{2} + \frac{C_2}{C_1+C_2\eta}, & \varDelta = 0, \end{cases}$$

where $\varDelta = \lambda^2 - 4\mu$, C_1, C_2 are arbitrary constants. Since $D_\xi^\alpha G(\xi) = D_\xi^\alpha H(\eta) = H'(\eta) D_\xi^\alpha \eta = H'(\eta)$, we obtain

$$\frac{D_\xi^\alpha G(\xi)}{G(\xi)} = \begin{cases} -\frac{\lambda}{2} + \frac{\sqrt{\varDelta}}{2}\frac{C_1\sinh\frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cosh\frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\sinh\frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}, & \varDelta > 0, \\ -\frac{\lambda}{2} + \frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\frac{-C_1\sin\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cos\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\eta + C_2\sin\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}, & \varDelta < 0, \\ -\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}, & \varDelta = 0. \end{cases} \tag{3}$$

3 Description of a fractional subequation method

In this section, we list the main steps of the fractional subequation method for finding the exact solutions of FNPDEs. Suppose that a FNPDE, say in the independent variables $t, x_1, x_2, \dots, x_n$, is given by

$$\begin{aligned} &P\big(u_1, \dots, u_k, D_t^\alpha u_1, \dots, D_t^\alpha u_k, D_{x_1}^\alpha u_1, \dots, D_{x_1}^\alpha u_k, D_{x_n}^\alpha u_1, \dots, \\ &\quad D_{x_n}^2 \alpha u_k, D_t^{2\alpha} u_1, \dots, D_t^{2\alpha} u_k, D_{x_1}^{2\alpha} u_1, \dots\big) = 0, \end{aligned} \tag{4}$$

where $u_i = u_i(t, x_1, x_2, \dots, x_n)$, $i = 1, \dots, k$, are unknown functions, P is a polynomial in u_i, and their various partial derivatives include fractional derivatives.

Step 1. Suppose that $U_i(\xi) = u_i(t, x_1, x_2, \dots, x_n)$, $\xi = ct + k_1 x_1 + k_2 x_2 + \dots + k_n x_n + \xi_0$. Use the previous transformation, then Eq. (4) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} &Q\big(U_1, \dots, U_k, c^\alpha D_t^\alpha U_1, \dots, c^\alpha D_t^\alpha U_k, k_1^\alpha D_\xi^\alpha U_1, \dots, k_1^\alpha D_\xi^\alpha U_k, k_n^\alpha D_\xi^\alpha U_1, \dots, \\ &\quad k_n^\alpha D_\xi^2 \alpha U_k, c^{2\alpha} D_\xi^{2\alpha} U_1, \dots, c^{2\alpha} D_\xi^{2\alpha} U_k, k_1^{2\alpha} D_\xi^{2\alpha} U_1, \dots\big) = 0. \end{aligned} \tag{5}$$

Step 2. Suppose that the solution of Eq. (5) can be expressed by a polynomial in $D_\xi^\alpha G(\xi)/G(\xi)$ as follows:

$$U_i(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right)^i, \quad j = 1, 2, \dots, k, \tag{6}$$

where $G = G(\xi)$ satisfies Eq. (1), and $a_{j,i}$, $i = 0, 1, \dots, m$, $j = 1, 2, \dots, k$, are constants to be determined later with $a_{j,m} \neq 0$. The positive integer m can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms appearing in Eq. (5).

Step 3. Substituting Eq. (6) into Eq. (5) and using Eq. (1), collecting all terms with the same order of $D_\xi^\alpha G(\xi)/G(\xi)$ together, the left-hand side of Eq. (5) is converted into another polynomial in $D_\xi^\alpha G(\xi)/G(\xi)$. Equating each coefficient of this polynomial to zero yields a set of algebraic equations for $a_{j,i}$, $i = 0, 1, \dots, m$, $j = 1, 2, \dots, k$. Solving the equation system in Step 3 and using Eq. (3), we can construct a variety of exact solutions for Eq. (4).

4 Applications

In this section, we will construct solutions for some nonlinear FNPDEs, namely the space-time fractional Cahn–Hilliard equation, the space-time fractional fifth-order Sawda–Kotera equation, and the space-time fractional modified equal-width equation by applying the fractional subequation method in which FNPDEs are very important in mathematical physics and have been paid attention by many researchers.

4.1 The space-time fractional Cahn–Hilliard equation

Consider the space-time fractional Cahn–Hilliard equation of the form

$$D_t^\alpha u - \gamma D_x^\alpha u - 6u\big(D_x^\alpha\big)^2 - \big(3u^2-1\big)D_x^{2\alpha}u + D_x^{4\alpha}u = 0, \tag{7}$$

where $0 < \alpha \leqslant 1$, γ is a real constant that represent the different diffusion power, and u is a function of (x,t). For the case corresponding to $\alpha = 1$, this equation is related with a number of interesting physical phenomena like the spinodal decomposition, phase separation, and phase ordering dynamics. On the other hand, it becomes important in material sciences. However, we notice that this equation is very difficult to be solved and several articles investigated it [19]. Now we will apply the described method above to Eq. (7). Let

$$u(x,t) = U_i(\xi), \quad \xi = ct + kx + \xi_0,$$

where c, k, ξ_0 are all constants with $k \neq 0$. Then using the second equality in Eq. (2), Eq. (7) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$c^\alpha D_\xi^\alpha U - \gamma k^\alpha D_\xi^\alpha U - 6U\big(k^\alpha D_\xi^\alpha\big)^2 - k^{2\alpha}\big(3U^2-1\big)D_\xi^{2\alpha}U + k^{4\alpha}D_\xi^{4\alpha}U = 0. \tag{8}$$

Suppose that the solution of Eq. (8) can be expressed by

$$U_i(\xi) = \sum_{i=0}^{m_j} a_{j,i}\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)^i, \quad j = 1,2,\dots,k, \tag{9}$$

where $G = G(\xi)$ satisfies Eq. (1). By balancing the order between the highest-order derivative term and nonlinear term in Eq. (8) we can obtain $m = 1$. So, we have

$$U(\xi) = a_0 + a_1\frac{D_\xi^\alpha G(\xi)}{G(\xi)}. \tag{10}$$

Substituting Eq. (10) into Eq. (8) and collecting all the terms with the same power of $D_\xi^\alpha G(\xi)/G(\xi)$ together, equating each coefficient to zero, yield a set of algebraic equations. Solving these equations by mathematica programm yields

Case 1. $a_0 = 0, \quad a_1 = -\sqrt{2}\,k^\alpha, \quad c = \left(\frac{k^{-\alpha}}{\gamma}\right)^{-1/\alpha};$

Case 2. $a_0 = 0, \quad a_1 = \sqrt{2}\,k^\alpha, \quad c = \left(\frac{k^{-\alpha}}{\gamma}\right)^{-1/\alpha}.$

Substituting the result above into Eq. (9) and combining with Eq. (3), we can obtain the following solutions to Eq. (7).

When $\Delta > 0$,

$$u_1(x,t) = \mp\sqrt{2}\,k^\alpha\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\,\frac{C_1\sinh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cosh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\sinh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right). \tag{11}$$

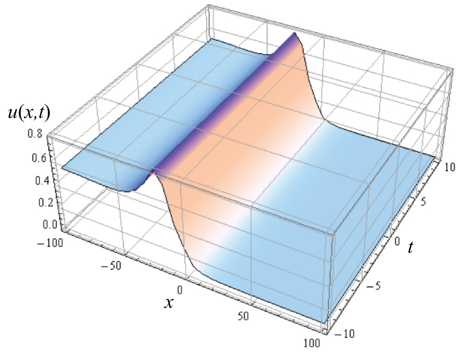

Figure 1. The variation of the solitary wave profile u_1 defined by (11) of Eq. (7) corresponding to the values $\lambda = 2$, $\alpha = 0.7$, $\gamma = 0.02$, $\mu = 0.2$.

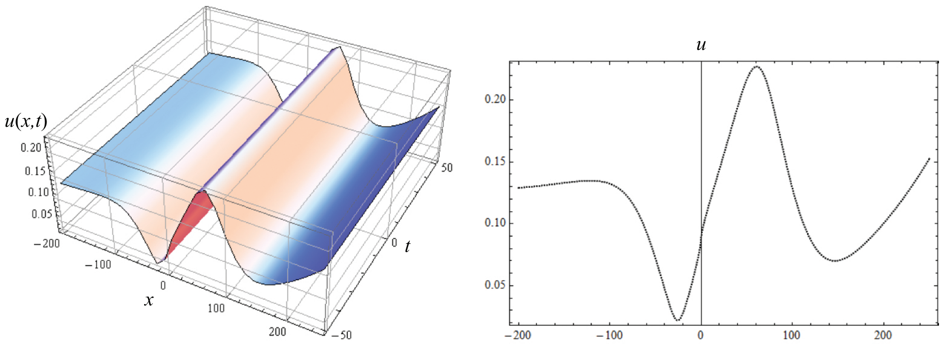

Figure 2. The solitary wave solution u_2 defined by (12) of Eq. (7) for the parameters $\lambda = 0.5$, $\alpha = 0.7$, $\gamma = 0.02$, $\mu = 0.2$.

When $\varDelta < 0$,

$$u_2(x,t) = \mp\sqrt{2}\,k^\alpha\left(-\frac{\lambda}{2} + \frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)} \times \frac{-C_1 \sin\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\eta + C_2 \sin\frac{\sqrt{-\varDelta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right). \tag{12}$$

When $\varDelta = 0$,

$$u_3(x,t) = \mp\sqrt{2}\,k^\alpha\left(-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}\right),$$

where $\xi = ct + kx + \xi_0$.

These solutions were represented graphically in Figs. 1, 2.

4.2 The space-time fractional fifth-order Sawda–Kotera equation

Consider the space-time fractional fifth-order Sawda–Kotera equation [20,30] of the form

$$D_t^\alpha u + D_x^{5\alpha} u + 45u^2\, D_x^\alpha + 15 D_x^\alpha D_x^{2\alpha} u + 15u D_x^{3\alpha} u = 0, \tag{13}$$

where $0 < \alpha \leqslant 1$, and u are the functions of (x,t). As example (1), we will apply the above described method to Eq. (13). Let

$$u(x,t) = U_i(\xi), \quad \xi = ct + kx + \xi_0,$$

where c, k, ξ_0 are all constants with $k \neq 0$. Then by means of the second equality in Eq. (2), Eq. (13) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$\begin{aligned} &c^\alpha D_\xi^\alpha U + k^{5\alpha} D_\xi^{5\alpha} U + 45k^\alpha\, U^2 D_\xi^\alpha U \\ &\quad + 15k^{3\alpha} D_\xi^\alpha D_\xi^{2\alpha} U + 15k^{3\alpha} U D_\xi^{3\alpha} U = 0. \end{aligned} \tag{14}$$

Suppose that the solution of Eq. (14) can be expressed by

$$U_i(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right)^i, \quad j = 1, 2, \ldots, k,$$

where $G = G(\xi)$ satisfies Eq. (1). By balancing the order between the highest-order derivative term and nonlinear term in Eq. (14) we can obtain $m = 2$. So, we have

$$U(\xi) = a_0 + a_1 \frac{D_\xi^\alpha G(\xi)}{G(\xi)} + a_2 \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right)^2. \tag{15}$$

Substituting Eq. (15) into Eq. (14) and collecting all the terms with the same power of $D_\xi^\alpha G(\xi)/G(\xi)$ together, equating each coefficient to zero, yield a set of algebraic equations. Solving these equations by mathematica programm yields

$$a_0 = \pm \frac{1}{30\lambda^2} \sqrt{\pm 20 c^\alpha \lambda^{9/2} + \frac{5k^{6\alpha}\lambda^5 \Delta^2}{k^{2\alpha}\lambda}} - \frac{1}{6} k^{2\alpha} \left(\lambda^2 + 8\mu \right),$$
$$a_1 = -2k^{2\alpha}\lambda, \qquad a_2 = -2\, k^{2\alpha}.$$

Substituting the result above into Eq. (15) and combining with Eq. (3), we can obtain the following solutions to Eq. (13).

When $\Delta > 0$,

$$u_1(x,t) = \pm \frac{1}{30\lambda^2} \sqrt{\pm 20 c^\alpha \lambda^{9/2} + \frac{5k^{6\alpha}\lambda^5 \Delta^2}{k^{2\alpha}\lambda}} - \frac{1}{6} k^{2\alpha} \left(\lambda^2 + 8\mu \right)$$

$$- 2k^{2\alpha}\lambda\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\,\frac{C_1 \sinh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)$$
$$- 2k^{2\alpha}\left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\,\frac{C_1 \sinh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cosh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cosh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \sinh \frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)^2. \tag{16}$$

When $\Delta < 0$,

$$\begin{aligned} u_2(x,t) &= \pm\frac{1}{30\lambda^2}\sqrt{\pm 20c^\alpha\lambda^{9/2} + \frac{5k^{6\alpha}\lambda^5\Delta^2}{k^{2\alpha}\lambda}} - \frac{1}{6}k^{2\alpha}\left(\lambda^2 + 8\mu\right) \\ &\quad - 2k^{2\alpha}\lambda\left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\,\frac{-C_1 \sin \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\eta + C_2 \sin \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right) \\ &\quad - 2k^{2\alpha}\left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\,\frac{-C_1 \sin \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2 \cos \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1 \cos \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\eta + C_2 \sin \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right)^2. \end{aligned} \tag{17}$$

When $\Delta = 0$,

$$\begin{aligned} u_3(x,t) &= \pm\frac{1}{30\lambda^2}\sqrt{\pm 20c^\alpha\lambda^{9/2} + \frac{5k^{6\alpha}\lambda^5\Delta^2}{k^{2\alpha}\lambda}} - \frac{1}{6}k^{2\alpha}\left(\lambda^2 + 8\mu\right) \\ &\quad - 2k^{2\alpha}\lambda\left(-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}\right) \\ &\quad - 2k^{2\alpha}\left(-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}\right)^2, \end{aligned} \tag{18}$$

where $\xi = ct + kx + \xi_0$.

These solutions were represented by the graphs, which shown in Figs. 3–5.

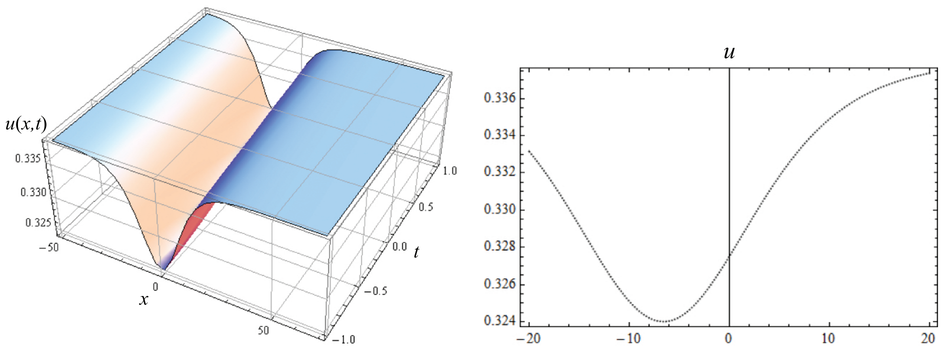

Figure 3. The solitary wave solution u_1 defined by (16) of Eq. (13) with singularity corresponding to the values $\lambda = 2$, $\alpha = 1$, $\mu = 0.2$.

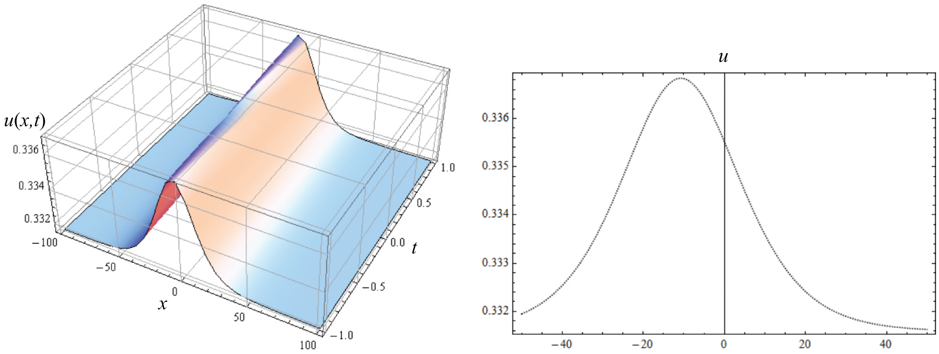

Figure 4. The solitary wave solution u_2 defined by (17) of Eq. (13) with singularity corresponding to the values $\lambda = 1.5$, $\alpha = 1$, $\mu = 0.3$.

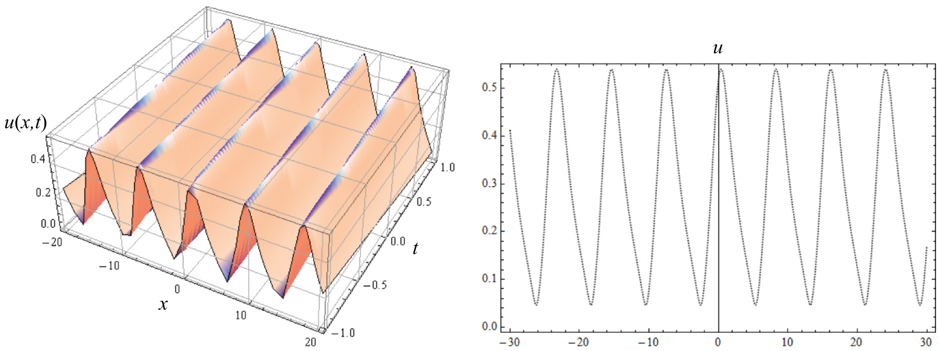

Figure 5. The periodic travelling wave solution u_3 defined by (18) of Eq. (13) for a set of parameters $\lambda = 1.2$, $\alpha = 1$, $\mu = 0.8$.

4.3 The space-time fractional modified equal-width (EW) equation

Consider the space-time fractional modified EW equation [13] of the form

$$D_t^\alpha u + \epsilon D_x^\alpha u^3 - \delta D_{xxt}^{3\alpha} u = 0, \tag{19}$$

where $0 < \alpha \leqslant 1$, $t > 0$, and u is a function of (x, t). For the case corresponding to $\alpha = 1$, this equation becomes the modified equal-width wave equation based upon the equal-width wave equation, which was suggested by Morrison et al. and used as a model partial differential equation for the simulation of one-dimensional wave propagation in nonlinear media with dispersion processes. This equation is related to the modified regularized long wave MRLW equation and modified Korteweg–de Vries (MKdV) equation to govern a large number of important physical phenomena such as the nonlinear transverse waves in shallow water, ion-acoustic and magnetohydrodynamic waves in plasma, and phonon packets in nonlinear crystals. Now, also we will apply the above described method to Eq. (19). Let

$$u(x,t) = U_i(\xi), \quad \xi = ct + kx + \xi_0,$$

where c, k, ξ_0 are all constants with $k \neq 0$. Then by use of the second equality in Eq. (2), Eq. (19) can be turned into the following fractional ordinary differential equation with respect to the variable ξ:

$$c^\alpha D_\xi^\alpha U + 3\epsilon k^\alpha U^2 D_\xi^\alpha U + -\delta c^\alpha k^{2\alpha} U D_\xi^{3\alpha} U = 0. \tag{20}$$

Suppose that the solution of Eq. (20) can be expressed by

$$U_i(\xi) = \sum_{i=0}^{m_j} a_{j,i} \left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)} \right)^i, \quad j = 1, 2, \ldots, k,$$

where $G = G(\xi)$ satisfies Eq. (1). By balancing the order between the highest-order derivative term and nonlinear term in Eq. (20), we can obtain $m = 1$. So, we have

$$U(\xi) = a_0 + a_1 \frac{D_\xi^\alpha G(\xi)}{G(\xi)}. \tag{21}$$

Substituting Eq. (21) into Eq. (20) and collecting all the terms with the same power of $D_\xi^\alpha G(\xi)/G(\xi)$ together, equating each coefficient to zero, yield a set of algebraic equations. Solving these equations by mathematica programm yields.

Case 1. $$a_0 = \pm \frac{(-1)^{1/4}\sqrt{3} c^{\alpha/2} \delta^{1/4} \lambda}{(2)^{3/4} (\varepsilon^2 \varDelta)^{1/4}}, \qquad a_1 = \pm \frac{(-2)^{1/4}\sqrt{3} c^{\alpha/2} \delta^{1/4}}{(\varepsilon^2 \varDelta)^{1/4}},$$
$$k = 2^{1/2\alpha} \left(\frac{\mathrm{i}\varepsilon}{\varepsilon\sqrt{\delta\varDelta}} \right)^{1/\alpha};$$

Case 2. $$a_0 = \pm \frac{(-1)^{3/4}\sqrt{3} c^{\alpha/2} \delta^{1/4} \lambda}{(2)^{3/4} (\varepsilon^2 \varDelta)^{1/4}}, \qquad a_1 = \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3} c^{\alpha/2} \delta^{1/4}}{(\varepsilon^2 \varDelta)^{1/4}},$$
$$k = 2^{1/2\alpha} \left(\frac{-\mathrm{i}\varepsilon}{\varepsilon\sqrt{\delta\varDelta}} \right)^{1/\alpha}.$$

Substituting the result above into Eq. (20) and combining with Eq. (3), we can obtain the following solutions to Eq. (19):

Case 1. When $\varDelta > 0$,

$$\begin{aligned} u_1(x,t) = {} & \pm \frac{(-1)^{1/4}\sqrt{3} c^{\alpha/2} \delta^{1/4} \lambda}{(2)^{3/4} (\varepsilon^2 \varDelta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3} c^{\alpha/2} \delta^{1/4}}{(\varepsilon^2 \varDelta)^{1/4}} \\ & \times \left(-\frac{\lambda}{2} + \frac{\sqrt{\varDelta}}{2} \frac{C_1 \sinh \frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \cosh \frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)} \xi^\alpha}{C_1 \cosh \frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)} \xi^\alpha + C_2 \sinh \frac{\sqrt{\varDelta}}{2\Gamma(1+\alpha)} \xi^\alpha} \right). \end{aligned} \tag{22}$$

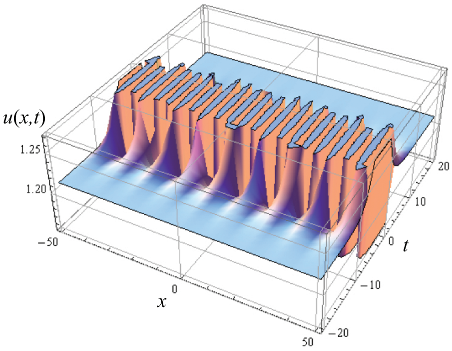

Figure 6. The periodic travelling wave solution u_1 defined by (22) of Eq. (19) for a different set of parameters $\lambda = 1.2$, $\alpha = 1$, $\mu = 0.8$.

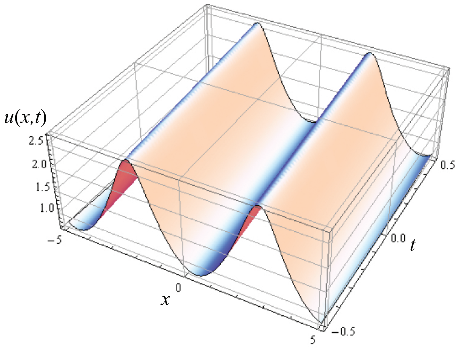

Figure 7. The travelling wave solution u_2 defined by (23) of Eq. (19) for a different set of parameters $\lambda = 0.8$, $\alpha = 1$, $\gamma = 0.2$, $\mu = 0.1$.

When $\Delta < 0$,

$$u_2(x,t) = \pm\frac{(-1)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}\lambda}{(2)^{3/4}(\varepsilon^2\Delta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}}{(\varepsilon^2\Delta)^{1/4}} \times \left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\,\frac{-C_1\sin\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cos\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\eta + C_2\sin\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right). \tag{23}$$

When $\Delta = 0$,

$$u_3(x,t) = \pm\frac{(-1)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}\lambda}{(2)^{3/4}(\varepsilon^2\Delta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}}{(\varepsilon^2\Delta)^{1/4}} \times \left(-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}\right),$$

where $\xi = c\,t + kx + \xi_0$, and $k = 2^{1/2\alpha}(\mathrm{i}\varepsilon/(\varepsilon\sqrt{\delta\Delta}))^{1/\alpha}$.

Solutions u_1, u_2 were represented by the graphs, which shown in Figs. 6, 7.

Case 2. When $\Delta > 0$, we have

$$u_4(x,t) = \pm\frac{(-1)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}\lambda}{(2)^{3/4}(\varepsilon^2\Delta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}}{(\varepsilon^2\Delta)^{1/4}} \times \left(-\frac{\lambda}{2} + \frac{\sqrt{\Delta}}{2}\,\frac{C_1\sinh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cosh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cosh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\sinh\frac{\sqrt{\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right). \tag{24}$$

When $\Delta < 0$,

$$u_5(x,t) = \pm\frac{(-1)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}\lambda}{(2)^{3/4}(\varepsilon^2\Delta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}}{(\varepsilon^2\Delta)^{1/4}} \times \left(-\frac{\lambda}{2} + \frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\,\frac{-C_1\sin\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha + C_2\cos\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}{C_1\cos\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\eta + C_2\sin\frac{\sqrt{-\Delta}}{2\Gamma(1+\alpha)}\xi^\alpha}\right). \tag{25}$$

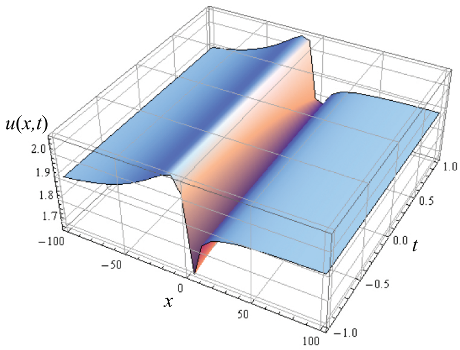

Figure 8. The solitary wave solution u_4 defined by (24) of Eq. (19) for a different set of parameters $\lambda = 1.2$, $\alpha = 1$, $\mu = 0.8$.

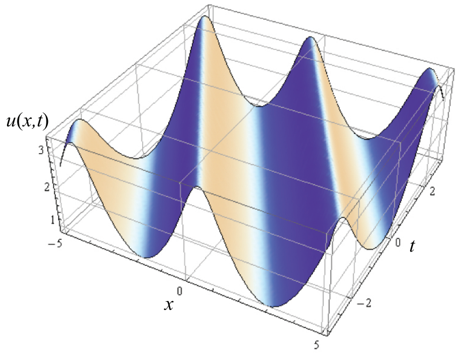

Figure 9. The periodic travelling wave solution u_5 defined by (25) of Eq. (19) for a different set of parameters $\lambda = 0.8$, $\alpha = 1$, $\gamma = 0.2$, $\mu = 0.1$.

When $\varDelta = 0$,

$$u_6(x,t) = \pm \frac{(-1)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}\lambda}{(2)^{3/4}(\varepsilon^2\varDelta)^{1/4}} \pm \frac{\mathrm{i}(-2)^{1/4}\sqrt{3}c^{\alpha/2}\delta^{1/4}}{(\varepsilon^2\varDelta)^{1/4}} \times \left(-\frac{\lambda}{2} + \frac{C_2\Gamma(1+\alpha)}{C_1 2\Gamma(1+\alpha) + C_2\xi^\alpha}\right),$$

where $\xi = c\,t + kx + \xi_0$, and $k = 2^{1/2\alpha}(-\mathrm{i}\varepsilon/(\varepsilon\sqrt{\delta\varDelta}))^{1/\alpha}$.

Solutions u_4, u_5 were represented by the graphs, which shown in Figs. 8, 9.

5 Conclusion

In this paper, we have focused on some of the most important space-time FPDEs by a fractional subequation method, we used it to construct new solutions to the space-time fractional Cahn–Hilliard equation, the space-time fractional fifth-order Sawda–Kotera equation, and the space-time fractional modified equal-width equation. After applying this method we notice that it is reliable and effective method since, it introduce several new solutions. We think that this method can also be applied to other generalized FNPDEs. In our future studies, we are going to solve FNPDEs by this approach and develop it to introduce new and different solutions for FNPDEs.

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