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Group analysis and conservation laws of an integrable Kadomtsev–Petviashvili equation*

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Abstract. In this paper, an integrable KP equation is studied using symmetry and conservation laws. First, on the basis of various cases of coefficients, we construct the infinitesimal generators. For the special case, we get the corresponding geometry vector fields, and then from known soliton solutions we derive new soliton solutions. In addition, the explicit power series solutions are derived. Lastly, nonlinear self-adjointness and conservation laws are constructed with symmetries.

Keywords: integrable KP equation, symmetry analysis, soliton solutions, nonlinear self-adjointness, conservation laws.

1 Introduction

It is well known that Kadomtsev–Petviashvili (KP) equation is mainly used to describe the nonlinear wave phenomenon. It is first derived by physicists Boris B. Kadomtsev and Vladimir I. Petviashvili in 1970 [5]. This equation play a very key role in the field of mathematical physics. There are many authors studied various versions of KP equation with different method. In [12], the authors studied $(3+1)$-dimensional generalized KP and BKP (Bogoyavlenskii–Kadomtsev–Petviashvili) equations using the multiple exp-function algorithm. The authors [7] investigated extended KP-like equation. The authors [22, 23] considered mixed lump-kink solutions to the KP, BKP equation. In [10], the authors studied diversity of interaction solutions to the $(2+1)$-dimensional Itô equation. The authors [6], with the use of the normal form, derived an extended KP equation with higher-order correction.

Recently, the authors [20] derived a new integrable KP equation from pseudo-differential formalism perspective. Motivated by the above papers, we study the more general

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case of KP equation with arbitrary coefficients

$$m_t + am_{xxy} + b\partial_x^{-2} m_{yyy} + cm_x \partial_x^{-1} m_y + emm_y = 0.$$

For the special case, $a = b = -1/2$, $c = -2$, $d = -4$, this equation reduce the case in paper [20]. In [21], the authors considered the multiple solitons of the special coefficients of the equation. Using the same transformation [21]

$$m(x, y, t) = u_{xx}(x, y, t),$$

one can get

$$u_{xxt} + au_{xxxxy} + bu_{yyy} + cu_{xxx}u_{xy} + du_{xxy}u_{xx} = 0, \tag{1}$$

here a, b, c and d are constants.

In this paper, we try to use the symmetry and conservation laws to study this equation. Symmetry [1–3, 13–19] and conservation laws play a key roles in the fields of applied mathematics and physics. Ibragimov [4] give a new theorem to derive the conservation laws. Recently, Ma [8, 9] studied the conservation laws by using symmetries and adjoint symmetries in details. The authors [11] investigated a few generalized KP and BKP equations via Hirota bilinear forms. The paper is divided as follows. In Section 2, we deal with differen cases for different coefficients, and then, the corresponding infinitesimal generators are derived. In Section 3, we consider the symmetry reductions and explicit solutions. In Section 4, first, the nonlinear self-adjointness are considered, and then conservation laws are derived with symmetries.

2 Symmetry analysis

Based on the symmetry analysis [1–3, 13–19], for the vector fields,

$$V = \xi_t(x, y, t, u)\frac{\partial}{\partial t} + \xi_x(x, y, t, u)\frac{\partial}{\partial x} + \xi_y(x, y, t, u)\frac{\partial}{\partial y} + \eta(x, y, t, u)\frac{\partial}{\partial u},$$

we directly get the following results for various cases of coefficients.

Case 1. $a = 0,\ b = 0,\ c = 0,\ d = 0$:

$$\begin{aligned}
&\eta = \mathcal{F}_3[y, t] + x\mathcal{F}_4[y, t] + \mathcal{F}_5[x, y] + u\big(\mathcal{F}_8[y] + x\mathcal{F}_9[y] + \mathcal{F}_{10}[y] + x\mathcal{F}_{11}[y]\big),\\
&\xi_x = \mathcal{F}_6[y] + x\big(\mathcal{F}_7[y] + x\big(\mathcal{F}_9[y] + \mathcal{F}_{11}[y]\big)\big), \qquad \xi_y = \mathcal{F}_2[y], \qquad \xi_t = \mathcal{F}_1[y, t].
\end{aligned}$$

Case 2. $a \neq 0,\ b = 0,\ c = 0,\ d = 0$:

$$\begin{aligned}
&\eta = u\mathbf{c}_2 + \mathcal{F}_1[x, y, t],\\
&\xi_x = \mathbf{c}_1 + \frac{1}{2}x(\mathbf{c}_4 - \mathbf{c}_6), \qquad \xi_y = \mathbf{c}_5 + y\mathbf{c}_6, \qquad \xi_t = \mathbf{c}_3 + t\mathbf{c}_4.
\end{aligned}$$

Case 3. $a = 0,\ b = 0,\ c \neq 0,\ d = -c$:

$$\begin{aligned}
\eta &= -6dtu\mathbf{c}_2 + x^2(\mathbf{c}_1 + y\mathbf{c}_2) + u\mathbf{c}_3 + \mathcal{F}_1[y,t] + x\mathcal{F}_2[y,t],\\
\xi_x &= \frac{1}{2}\biggl(x(-4dt\mathbf{c}_2 + \mathbf{c}_3 - \mathbf{c}_5 + \mathbf{c}_7) - 2d\int \mathcal{F}_{2y}\,dt + 2\mathcal{F}_3[y]\biggr),\\
\xi_y &= 2dt(\mathbf{c}_1 + y\mathbf{c}_2) + \mathbf{c}_4 + y\mathbf{c}_5, \qquad \xi_t = 2dt^2\mathbf{c}_2 + \mathbf{c}_6 + t\mathbf{c}_7.
\end{aligned}$$

Case 4. $a = 0,\ b = 0,\ c \neq 0,\ d = 0$:

$$\begin{aligned}
\eta &= u(4ct\mathbf{c}_2 + \mathbf{c}_3) + \mathcal{F}_1[y,t] + x\biggl(x(\mathbf{c}_1 + y\mathbf{c}_2) + \mathcal{F}_3[t] + \frac{y\mathcal{F}_2'}{c}\biggr),\\
\xi_x &= \frac{1}{2}\bigl(x(4ct\mathbf{c}_2 + \mathbf{c}_3 - \mathbf{c}_5 + \mathbf{c}_7) + 2\mathcal{F}_2[t]\bigr),\\
\xi_y &= \mathbf{c}_4 + y\mathbf{c}_5, \qquad \xi_t = \mathbf{c}_6 + t\mathbf{c}_7.
\end{aligned}$$

Case 5. $a \neq 0,\ b = 0,\ c = 0,\ d \neq 0$:

$$\begin{aligned}
\eta &= \frac{x^2\mathbf{c}_2}{2d} + \mathcal{F}_1[y,t] + x\mathcal{F}_2[y,t],\\
\xi_x &= \mathbf{c}_1 + \frac{1}{2}x(-\mathbf{c}_4 + \mathbf{c}_6), \qquad \xi_y = t\mathbf{c}_2 + \mathbf{c}_3 + y\mathbf{c}_4, \qquad \xi_t = \mathbf{c}_5 + t\mathbf{c}_6.
\end{aligned}$$

Case 6. $a = 0,\ b = 0,\ cd(c + d) \neq 0$:

$$\begin{aligned}
\eta &= 4ctu\mathbf{c}_2 - 2dtu\mathbf{c}_2 + x^2(\mathbf{c}_1 + y\mathbf{c}_2) + u\mathbf{c}_3 + \mathcal{F}_1[y,t]\\
&\quad + x\biggl(\mathcal{F}_3[t] + \frac{y\mathcal{F}_2'}{c}\biggr),\\
\xi_x &= \frac{1}{2}\bigl(x(4ct\mathbf{c}_2 + \mathbf{c}_3 - \mathbf{c}_5 + \mathbf{c}_7) + 2\mathcal{F}_2[t]\bigr),\\
\xi_y &= 2dt(\mathbf{c}_1 + y\mathbf{c}_2) + \mathbf{c}_4 + y\mathbf{c}_5, \qquad \xi_t = 2dt^2\mathbf{c}_2 + \mathbf{c}_6 + t\mathbf{c}_7.
\end{aligned}$$

Case 7. $a = 0,\ b = 0,\ c = 0,\ d \neq 0$:

$$\begin{aligned}
\eta &= \mathbf{c}_1 + x\mathbf{c}_2 + x^2\mathbf{c}_3 + x^2\mathbf{c}_5 - 2dtu\mathbf{c}_6 + x^2y\mathbf{c}_6 + u\mathbf{c}_7\\
&\quad + \mathcal{F}_1[y,t] + x\mathcal{F}_2[y,t] + \mathcal{F}_3[y] + x\mathcal{F}_4[y],\\
\xi_x &= \mathbf{c}_4 + \frac{1}{2}x(\mathbf{c}_7 - \mathbf{c}_9 + \mathbf{c}_{11}), \qquad \xi_y = 2dt(\mathbf{c}_3 + \mathbf{c}_5 + y\mathbf{c}_6) + \mathbf{c}_8 + y\mathbf{c}_9,\\
\xi_t &= 2dt^2\mathbf{c}_6 + \mathbf{c}_{10} + t\mathbf{c}_{11}.
\end{aligned}$$

Case 8. $a = 0,\ b \neq 0,\ c = 0,\ d = 0$:

$$\begin{aligned}
\eta &= u\mathbf{c}_1 + \mathcal{F}_1[x,y,t],\\
\xi_x &= \mathbf{c}_2 + x\mathbf{c}_3, \qquad \xi_y = \mathbf{c}_4 + y\mathbf{c}_5, \qquad \xi_t = -2t\mathbf{c}_3 + 3t\mathbf{c}_5 + \mathbf{c}_6,
\end{aligned}$$

in addition,

$$-b\mathcal{F}_{1yyy} - \mathcal{F}_{1xxt} = 0.$$

Case 9. $a \neq 0,\ b = 0,\ c \neq 0,\ d = 2c$:

$$\begin{aligned}
\eta &= x^2(\mathbf{c}_1 + y\mathbf{c}_2) + \mathcal{F}_1[y,t] + x\left(\mathcal{F}_3[t] + \frac{2y\mathcal{F}_2'}{d}\right),\\
\xi_x &= \frac{1}{2}x(2dt\mathbf{c}_2 - \mathbf{c}_4 + \mathbf{c}_6) + \mathcal{F}_2[t], \qquad \xi_y = 2dt(\mathbf{c}_1 + y\mathbf{c}_2) + \mathbf{c}_3 + y\mathbf{c}_4,\\
\xi_t &= 2dt^2\mathbf{c}_2 + \mathbf{c}_5 + t\mathbf{c}_6.
\end{aligned}$$

Case 0. $a \neq 0,\ b = 0,\ c(2c-d)d \neq 0$:

$$\begin{aligned}
\eta &= \mathcal{F}_1[y,t] + x\left(x\mathbf{c}_1 + \mathcal{F}_3[t] + \frac{y\mathcal{F}_2'}{c}\right),\\
\xi_x &= \frac{1}{2}x(-\mathbf{c}_3 + \mathbf{c}_5) + \mathcal{F}_2[t], \qquad \xi_y = 2dt\mathbf{c}_1 + \mathbf{c}_2 + y\mathbf{c}_3, \qquad \xi_t = \mathbf{c}_4 + t\mathbf{c}_5.
\end{aligned}$$

Case 11. $a \neq 0,\ b = 0,\ c \neq 0,\ d = 0$:

$$\begin{aligned}
\eta &= \mathcal{F}_1[y,t] + x\left(x\mathbf{c}_1 + \mathcal{F}_3[t] + \frac{y\mathcal{F}_2'}{c}\right),\\
\xi_x &= \frac{1}{2}x(-\mathbf{c}_3 + \mathbf{c}_5) + \mathcal{F}_2[t], \qquad \xi_y = \mathbf{c}_2 + y\mathbf{c}_3, \qquad \xi_t = \mathbf{c}_4 + t\mathbf{c}_5.
\end{aligned}$$

Case 12. $a \neq 0,\ b \neq 0,\ c = 0,\ d = 0$:

$$\begin{aligned}
\eta &= u\mathbf{c}_1 + \mathcal{F}_1[x,y,t],\\
\xi_x &= x\mathbf{c}_3 + \mathbf{c}_4, \qquad \xi_y = \mathbf{c}_2 + 2y\mathbf{c}_3, \qquad \xi_t = 4t\mathbf{c}_3 + \mathbf{c}_5,
\end{aligned}$$

and

$$-b\mathcal{F}_{1yyy} - \mathcal{F}_{1xxt} - a\mathcal{F}_{1xxxxy} = 0.$$

Case 13. $a = 0,\ b \neq 0,\ c(2c-d)d \neq 0$:

$$\begin{aligned}
\eta &= 2u\mathbf{c}_1 + 2u\mathbf{c}_2 + x\mathcal{F}_3[t] + \mathcal{F}_4[t] + y\mathcal{F}_5[t] + y^2\mathcal{F}_6[t]\\
&\quad + \frac{x^2\mathcal{F}_1'}{2d} + \frac{xy\mathcal{F}_2'}{c} - \frac{y^3\mathcal{F}_1''}{6bd},\\
\xi_x &= \frac{cx(2\mathbf{c}_1 + \mathbf{c}_2) + (2c-d)\mathcal{F}_2[t]}{2c-d},\\
\xi_y &= \frac{y(2c\mathbf{c}_1 + d(\mathbf{c}_1 + \mathbf{c}_2)) + (2c-d)\mathcal{F}_1[t]}{2c-d},\\
\xi_t &= \frac{t(2c(\mathbf{c}_1 - \mathbf{c}_2) + 3d(\mathbf{c}_1 + \mathbf{c}_2))}{2c-d} + \mathbf{c}_3.
\end{aligned}$$

Case 14. $a = 0,\ b \neq 0,\ c \neq 0,\ d = 0$:

$$\eta = 2u\mathbf{c}_2 + 2u\mathbf{c}_3 + x^2\mathcal{F}_2[t] + x\mathcal{F}_3[t] + \mathcal{F}_4[t] + y\mathcal{F}_5[t] + y^2\mathcal{F}_6[t] + \frac{xy\mathcal{F}_1'}{c} - \frac{y^3\mathcal{F}_2'}{3b},$$

$$\xi_x = x\mathbf{c}_2 + \frac{x\mathbf{c}_3}{2} + \mathcal{F}_1[t], \qquad \xi_y = \mathbf{c}_1 + y\mathbf{c}_2, \qquad \xi_t = t(\mathbf{c}_2 - \mathbf{c}_3) + \mathbf{c}_4.$$

Case 15. $a \neq 0,\ b \neq 0,\ c(2c-d)d \neq 0$:

$$\eta = x\mathcal{F}_3[t] + \mathcal{F}_4[t] + y\mathcal{F}_5[t] + y^2\mathcal{F}_6[t] + \frac{x^2\mathcal{F}_1'}{2d} + \frac{xy\mathcal{F}_2'}{c} - \frac{y^3\mathcal{F}_1''}{6bd},$$

$$\xi_x = \frac{-cx\mathbf{c}_1 + (2c-d)\mathcal{F}_2[t]}{2c-d}, \qquad \xi_y = \frac{-2cy\mathbf{c}_1 + (2c-d)\mathcal{F}_1[t]}{2c-d},$$

$$\xi_t = -\frac{4ct\mathbf{c}_1}{2c-d} + \mathbf{c}_2.$$

Case 16. $a = 0,\ b \neq 0,\ c \neq 0,\ d = 2c$:

$$\eta = u\mathbf{c}_1 + \mathcal{F}_5[t] + y\mathcal{F}_6[t] + y^2\mathcal{F}_7[t] + \frac{x(4d\mathcal{F}_4[t] + 2x\mathcal{F}_2' + 8y\mathcal{F}_3' + xy\mathcal{F}_1'')}{4d} - \frac{y^3\mathcal{F}_2''}{6bd} - \frac{y^4\mathcal{F}_1'''}{48bd},$$

$$\xi_x = \mathcal{F}_3[t] + \frac{1}{8}x(3\mathbf{c}_1 + 2\mathcal{F}_1'), \qquad \xi_y = \mathcal{F}_2[t] + \frac{1}{4}y(\mathbf{c}_1 + 2\mathcal{F}_1'),$$

$$\xi_t = \mathcal{F}_1[t].$$

Case 17. $a = 0,\ b \neq 0,\ c = 0,\ d \neq 0$:

$$\eta = -4u\mathbf{c}_1 + 2u\mathbf{c}_3 + x\mathcal{F}_2[t] + xy\mathcal{F}_3[t] + xy^2\mathcal{F}_4[t] + \mathcal{F}_5[t] + y\mathcal{F}_6[t] + y^2\mathcal{F}_7[t] + \frac{x^2\mathcal{F}_1'}{2d} - \frac{y^3\mathcal{F}_1''}{6bd},$$

$$\xi_x = x\mathbf{c}_1 + \mathbf{c}_2, \qquad \xi_y = 4y\mathbf{c}_1 - y\mathbf{c}_3 + \mathcal{F}_1[t], \qquad \xi_t = 10t\mathbf{c}_1 - 3t\mathbf{c}_3 + \mathbf{c}_4.$$

Case 18. $a \neq 0,\ b \neq 0,\ c \neq 0,\ d = 0$:

$$\eta = \mathcal{F}_4[t] + y\mathcal{F}_5[t] + y^2\mathcal{F}_6[t] + x\left(x\mathcal{F}_2[t] + \mathcal{F}_3[t] + \frac{y\mathcal{F}_1'}{c}\right) - \frac{y^3\mathcal{F}_2'}{3b},$$

$$\xi_x = x\mathbf{c}_2 + \mathcal{F}_1[t], \quad \xi_y = \mathbf{c}_1 + 2y\mathbf{c}_2, \qquad \xi_t = 4t\mathbf{c}_2 + \mathbf{c}_3.$$

Case 19. $a \neq 0,\ b \neq 0,\ c \neq 0,\ d = 2c$:

$$\eta = \mathcal{F}_5[t] + y\mathcal{F}_6[t] + y^2\mathcal{F}_7[t] + \frac{x(4d\mathcal{F}_4[t] + 2x\mathcal{F}_2' + 8y\mathcal{F}_3' + xy\mathcal{F}_1'')}{4d} - \frac{y^3\mathcal{F}_2''}{6bd} - \frac{y^4\mathcal{F}_1'''}{48bd},$$

$$\xi_x = \mathcal{F}_3[t] + \frac{x\mathcal{F}_1'}{4}, \qquad \xi_y = \mathcal{F}_2[t] + \frac{y\mathcal{F}_1'}{2}, \qquad \xi_t = \mathcal{F}_1[t].$$

Case 20. $a \neq 0,\ b \neq 0,\ c = 0,\ d \neq 0$:

$$\begin{aligned} \eta &= \mathbf{c}_4 + x\mathbf{c}_5 + \mathcal{F}_2[t] + x\mathcal{F}_3[t] + x\mathcal{F}_4[t] + xy\mathcal{F}_5[t] \\ &\quad + xy^2\mathcal{F}_6[t] + \mathcal{F}_7[t] + y\mathcal{F}_8[t] + y^2\mathcal{F}_9[t] + \frac{x^2\mathcal{F}_1'}{2d} - \frac{y^3\mathcal{F}_1''}{6bd}, \\ \xi_x &= x\mathbf{c}_1 + \mathbf{c}_2, \qquad \xi_y = 2y\mathbf{c}_1 + \mathcal{F}_1[t], \qquad \xi_t = 4t\mathbf{c}_1 + \mathbf{c}_3. \end{aligned}$$

Case 21. $a = b = 1/2,\ c = -2,\ d = -4$:

$$\begin{aligned} \eta &= \mathcal{F}_5[t] + x\mathcal{F}_6[t] - \frac{1}{8}x^2\mathcal{F}_3' + y\left(\mathcal{F}_7[t] - \frac{1}{16}x(8\mathcal{F}_1' + x\mathcal{F}_2'')\right) \\ &\quad - \frac{1}{96}y^2\bigl(-96\mathcal{F}_4[t] + y(8\mathcal{F}_3'' + y\mathcal{F}_2'''))\bigr), \\ \xi_x &= \mathcal{F}_1[t] + \frac{x\mathcal{F}_2'}{4}, \qquad \xi_y = \mathcal{F}_3[t] + \frac{y\mathcal{F}_2'}{2}, \qquad \xi_t = \mathcal{F}_2[t]. \end{aligned}$$

It is clear that, for the case $\mathcal{F}_2[t] = 4c_1t + c_2, \mathcal{F}_3[t] = c_3, \mathcal{F}_1[t] = c_4, \mathcal{F}_5[t] = c_5, \mathcal{F}_4[t] = \mathcal{F}_6[t] = \mathcal{F}_7[t] = 0$, we have the following results:

$$\eta = c_5, \qquad \xi_x = c_1x + c_4, \qquad \xi_y = 2c_1y + c_3, \qquad \xi_t = 4c_1t + c_2.$$

It is easily get the following vector fields

$$\begin{gathered} v_1 = \partial_u, \qquad v_2 = \partial_x, \qquad v_3 = \partial_y, \qquad v_4 = \partial_t, \\ v_5 = x\partial_x + 2y\partial_y + 4t\partial_t. \end{gathered}$$

Consequently, we get the following invariant group:

$$\begin{aligned} g(x,y,t,u) &\mapsto g(x+\varepsilon,\ y,\ t,\ u), \\ g(x,y,t,u) &\mapsto g(x,\ y+\varepsilon,\ t,\ u), \\ g(x,y,t,u) &\mapsto g(x,\ y,\ t+\varepsilon,\ u), \\ g(x,y,t,u) &\mapsto g(x,\ y,\ t,\ u+\varepsilon), \\ g(x,y,t,u) &\mapsto g\bigl(\mathrm{e}^{-\varepsilon}x,\ \mathrm{e}^{-2\varepsilon}y,\ \mathrm{e}^{-4\varepsilon}t, u\bigr). \end{aligned}$$

Therefore, based the obtained results, we can construct new exact solutions of KP equation for known exact solutions. We will show these results in the next section.

For various cases of coefficients, we get different vector fields. In the following, we try to derive symmetry reductions and exact solutions.

3 Symmetry reductions and exact solutions

3.1 Symmetry reduction

Here, we just consider the following case, as other cases can derived in a similar way.

Case A. Travelling wave reduction

For this case, we let $\xi = x + y - vt$, where v is the speed of wave. We can get the reduced equation is

$$(b-v)f''' + af^{(5)} + (c+d)f'''f'' = 0.$$

Integral once, and let the integral constant equal to zero, one can arrive at

$$(b-v)f'' + af^{(4)} + \frac{(c+d)}{2}(f'')^2 = 0. \tag{2}$$

In order to further simplify the equation, let $h = f''$, which leads to the following results:

$$(b-v)h + ah'' + \frac{(c+d)}{2}h^2 = 0.$$

So, now, if we get the h, we can get the exact solutions of the original equation.

3.1.1 Case B. Scalar reduction

For the case v_5, we can get the invariant solutions and invariants are

$$u = f(\xi,\eta), \quad \xi = xt^{-1/4}, \quad \eta = yt^{-1/2}.$$

In this way, we get the reduced equation is

$$-\frac{1}{4}\xi f_{\xi\xi\xi} - \frac{1}{2}\eta f_{\xi\xi\eta} - \frac{1}{2}f_{\xi\xi} + af_{\xi\xi\xi\xi\eta} + bf_{\eta\eta\eta} + cf_{\xi\xi\xi}f_{\xi\eta} + df_{\xi\xi\eta}f_{\xi\xi} = 0.$$

In fact, we can further reduce this equation based on the symmetry analysis. We, however, for brevity, do not list all of them.

3.2 Soliton solutions via the known soliton solutions

For one-parameter groups, that is space-invariance, $g(x,y,t,u) \mapsto g(x+\varepsilon,y,t,u)$, we can construct new exact solutions via the known soliton solutions. For example, for the single soliton solution [21], we have the following new exact solutions

$$u(x,y,t) = \ln\left(1 + \exp\left\{k_1(x+\varepsilon) + r_1 y + \frac{k_1^4 r_1 + r_1^3}{2k_1^2 t}\right\}\right),$$

so, the final soliton solutions of original equation is

$$v(x,y,t) = k_1^2 \exp\left\{k_1(x+\varepsilon) + r_1 y + \frac{k_1^4 r_1 + r_1^3}{2k_1^2}t\right\} \times \left(1 + \exp\left\{k_1(x+\varepsilon) + r_1 y + \frac{k_1^4 r_1 + r_1^3}{2k_1^2}t\right\}\right)^{-2}.$$

For two soliton solutions [21],

$$\begin{aligned}u(x,y,t) = \ln\Biggl(1 &+ \exp\biggl\{k_1(x+\varepsilon) + r_1 y + \frac{k_1^4 r_1 + r_1^3}{2k_1^2}t\biggr\} \\ &+ \exp\biggl\{k_2(x+\varepsilon) + r_2 y + \frac{k_2^4 r_2 + r_2^3}{2k_2^2}t\biggr\} \\ &+ \frac{k_1^2 k_2^2 (k_1-k_2)^2 - (k_1 r_2 - k_2 r_1)^2}{k_1^2 k_2^2 (k_1+k_2)^2 - (k_1 r_2 - k_2 r_1)^2} \\ &\times \exp\biggl\{(k_1+k_2)(x+\varepsilon) + (r_1+r_2)y + \biggl(\frac{k_1^4 r_1 + r_1^3}{2k_1^2} + \frac{k_2^4 r_2 + r_2^3}{2k_2^2}\biggr)t\biggr\}\Biggr).\end{aligned}$$

For the invariant group $g(x,y,t,u) \mapsto g(\mathrm{e}^{-\varepsilon}x, \mathrm{e}^{-2\varepsilon}y, \mathrm{e}^{-4\varepsilon}t, u)$, we can get new two soliton solutions are

$$\begin{aligned}&u(x,y,t) \\ &\quad = \ln\Biggl(1 + \exp\biggl\{k_1\bigl(\mathrm{e}^{-\varepsilon}x\bigr) + r_1\mathrm{e}^{-2\varepsilon}y + \frac{k_1^4 r_1 + r_1^3}{2k_1^2}\mathrm{e}^{-4\varepsilon}t\biggr\} \\ &\qquad + \exp\biggl\{k_2\bigl(\mathrm{e}^{-\varepsilon}x\bigr) + r_2\mathrm{e}^{-2\varepsilon}y + \frac{k_2^4 r_2 + r_2^3}{2k_2^2}\mathrm{e}^{-4\varepsilon}t\biggr\} \\ &\qquad + \frac{k_1^2 k_2^2 (k_1-k_2)^2 - (k_1 r_2 - k_2 r_1)^2}{k_1^2 k_2^2 (k_1+k_2)^2 - (k_1 r_2 - k_2 r_1)^2} \\ &\qquad \times \exp\biggl\{(k_1+k_2)\bigl(\mathrm{e}^{-\varepsilon}x\bigr) + (r_1+r_2)\mathrm{e}^{-2\varepsilon}y + \biggl(\frac{k_1^4 r_1 + r_1^3}{2k_1^2} + \frac{k_2^4 r_2 + r_2^3}{2k_2^2}\biggr)\mathrm{e}^{-4\varepsilon}t\biggr\}\Biggr).\end{aligned}$$

We can also construct other new explicit soliton solutions via other invariant group. Here, we do not list all of them.

3.3 The explicit power series solutions

Now, we deal with (2). Assume that (2) has the following solution:

$$f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n. \tag{3}$$

Putting (3) into (2), one has

$$\begin{aligned}24ac_4 &+ a\sum_{n=1}^{\infty}(n+1)(n+2)(n+3)(n+4)c_{n+4}\xi^n \\ &+ 2(b-v)c_2 + (b-v)\sum_{n=1}^{\infty}(n+1)(n+2)c_{n+2}\xi^n + 4\frac{c+d}{2}c_2^2 \\ &+ \frac{c+d}{2}\sum_{n=1}^{\infty}\sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\xi^n = 0.\end{aligned}$$

Consider the case when $n = 0$, one leads to

$$c_4 = \frac{(v-b)c_2 - (c+d)c_2^2}{12a}.$$

For this case, it requires that $a \neq 0$. Consider the general case $n \geqslant 1$, one gets

$$c_{n+4} = \frac{1}{a(n+1)(n+2)(n+3)(n+4)}\biggl((v-b)(n+1)(n+2)c_{n+2} - \frac{c+d}{2}\sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\biggr).$$

Therefore, we have the following results:

$$\begin{aligned}
f(\xi) &= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + c_4\xi^4 + \sum_{n=1}^{\infty} c_{n+4}\xi^{n+4} \\
&= c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3 + \frac{(v-b)c_2 - (c+d)c_2^2}{12a}\xi^4 \\
&\quad + \sum_{n=1}^{\infty}\biggl(\frac{1}{a(n+1)(n+2)(n+3)(n+4)}\biggl((v-b)(n+1)(n+2)c_{n+2} \\
&\quad - \frac{c+d}{2}\sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\biggr)\biggr)\xi^{n+4}.
\end{aligned}$$

At last, we get the explicit solutions of (1)

$$\begin{aligned}
u(x,t) = \biggl[&c_0 + c_1(x+y-vt) + c_2(x+y-vt)^2 + c_3(x+y-vt)^3 \\
&+ \frac{(v-b)c_2 - (c+d)c_2^2}{12a}(x+y-vt)^4 \\
&+ \sum_{n=1}^{\infty}\biggl(\frac{1}{a(n+1)(n+2)(n+3)(n+4)}\biggl((v-b)(n+1)(n+2)c_{n+2} \\
&- \frac{c+d}{2}\sum_{k=1}^{n}(k+1)(n+2-k)(n+3-k)c_{k+1}c_{n+3-k}\biggr)\biggr)(x+y-vt)^{n+4}\biggr].
\end{aligned}$$

Here c_i $(i = 0,1,2,3)$ are arbitrary constants, one can get the other coefficients c_n $(n \geqslant 4)$ from the similar way.

In order to provide the help for numerical results, we rewrite it in approximate form

$$\begin{aligned}
u(x,y,t) = c_0 &+ c_1(x+y-vt) + c_2(x+y-vt)^2 + c_3(x+y-vt)^3 \\
&+ \frac{(v-b)c_2 - (c+d)c_2^2}{12a}(x+y-vt)^4 + \cdots.
\end{aligned}$$

Remark: It is easily to verify the convergence, we do not give the proof for simplicity.

4 Nonlinear self-adjointness and conservation laws

In this section, we consider the nonlinear self-adjointness and conservation laws of (1). We need to use the following results [4]

Theorem 1. *Every Lie point, Lie–Bäcklund and nonlocal symmetry provides a conservation law for* (1) *and the adjoint equation. Then the elements of conservation vector* (C^1, C^2, C^3) *are defined by the following expression*:

$$\begin{aligned} C^i = \xi^i L + W^\alpha \left[\frac{\partial L}{\partial u_i^\alpha} - D_j \frac{\partial L}{\partial u_{ij}^\alpha} + D_j D_k \frac{\partial L}{\partial u_{ijk}^\alpha} \right] \\ + D_j W^\alpha \left[\frac{\partial L}{\partial u_{ij}^\alpha} - D_k \frac{\partial L}{\partial u_{ijk}^\alpha} + \cdots \right], \end{aligned}$$

where $W^\alpha = \eta^\alpha - \xi^j u_j^\alpha$.

Based on the definition in [4], we get the adjoint equation of (1) as follows:

$$\begin{aligned} &-b\upsilon_{yyy} - \upsilon_{xxt} - 3c\upsilon_{xx}u_{xxy} - du_{xx}\upsilon_{xxy} + c\upsilon_{xy}u_{xxx} - 2d\upsilon_{xy}u_{xxx} \\ &\quad - cu_{xy}\upsilon_{xxx} - 2c\upsilon_x u_{xxxy} + c\upsilon_y u_{xxxx} - d\upsilon_y u_{xxxx} - a\upsilon_{xxxxy} = 0. \end{aligned}$$

It is easily found that this equation is not self-adjointness. In order to get the conditions, we let $\upsilon = F(u)$,

$$\begin{aligned} &- b\big(F''' u_y^3 + 3F'' u_y u_{yy} + F' u_{yyy}\big) - F''' u_t u_x^2 - 2F'' u_x u_{xt} - F'' u_t u_{xx} \\ &\quad - 3c\big(F'' u_x^2 + F' u_{xx}\big)u_{xxy} - du_{xx}\big(F''' u_y u_x^2 + 2F'' u_x u_{xy} + F'' u_y u_{xx} + F' u_{xxy}\big) \\ &\quad + c(F'' u_y u_x + F' u_{xy})u_{xxx} - 2d(F'' u_y u_x + F' u_{xy})u_{xxx} \\ &\quad - cu_{xy}\big(F''' u_x^3 + 3F'' u_x u_{xx} + F' u_{xxx}\big) - 2cF' u_x u_{xxxy} + cF' u_y u_{xxxx} \\ &\quad - dF' u_y u_{xxxx} - F'(-bu_{yyy} - du_{xx}u_{xxy} - cu_{xy}u_{xxx} - au_{xxxxy}) \\ &\quad - a\big(F^{(5)} u_y u_x^4 + 4F'''' u_x^3 u_{xy} + 3F'''' u_y u_x^2 u_{xx} + 6F''' u_x u_{xy} u_{xx}\big) \\ &\quad + a\big(3u_x(F'''' u_y u_x + F''' u_{xy})u_{xx} + 3F''' u_x^2 u_{xxy} + 3F'' u_{xx} u_{xxy}\big) \\ &\quad + a\big(u_{xx}(F''' u_y u_{xx} + F'' u_{xxy}) + u_{xx}(2F''' u_y u_{xx} + 2F'' u_{xxy})\big) \\ &\quad + a\big(F''' u_x(u_{xy}u_{xx} + u_x u_{xxy}) + F''' u_x(2u_{xy}u_{xx} + 2u_x u_{xxy})\big) \\ &\quad + a\big(F''' u_y u_x u_{xxx} + F'' u_{xy} u_{xxx} + (F''' u_y u_x + F'' u_{xy})u_{xxx}\big) \\ &\quad + (2F''' u_y u_x + 2F'' u_{xy})u_{xxx} + 4F'' u_x u_{xxxy} + F'' u_y u_{xxxx} + F' u_{xxxxy} = 0. \end{aligned}$$

It is clear that for this case $F = c_1 y + c_2$, this equation is strictly self adjoint for all parameters.

Based on Theorem 1, we get

$$C^t = \xi^t L + W\left(D_{xx}\frac{\partial L}{\partial u_{xxt}}\right) + D_x W\left(-D_x\frac{\partial L}{\partial u_{xxt}}\right) + D_{xx} W\left(\frac{\partial L}{\partial u_{xxt}}\right),$$

$$
\begin{aligned}
C^x &= \xi^x L + W\biggl(-D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}} + D_{xx}\frac{\partial L}{\partial u_{xxx}} + D_{xy}\frac{\partial L}{\partial u_{xxy}} \\
&\quad + D_{xt}\frac{\partial L}{\partial u_{xxt}} + D_{xxxy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_x W\biggl(\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} - D_y\frac{\partial L}{\partial u_{xxy}} - D_t\frac{\partial L}{\partial u_{xxt}} - D_{xxy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_y W\biggl(\frac{\partial L}{\partial u_{xy}} - D_x\frac{\partial L}{\partial u_{xxy}} - D_{xx}\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_t W\biggl(-D_x\frac{\partial L}{\partial u_{xxt}}\biggr) \\
&\quad + D_{xx} W\biggl(\frac{\partial L}{\partial u_{xxx}} + D_{xy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_{xxx} W\biggl(-D_y\frac{\partial L}{\partial u_{xxxxy}}\biggr),
\end{aligned}
$$

$$
\begin{aligned}
C^y &= \xi^y L + W\biggl(-D_x\frac{\partial L}{\partial u_{xy}} + D_{xx}\frac{\partial L}{\partial u_{xxy}} + D_{yy}\frac{\partial L}{\partial u_{yyy}} + D_{xxxx}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_x W\biggl(\frac{\partial L}{\partial u_{xy}} - D_x\frac{\partial L}{\partial u_{xxy}} - D_{xxx}\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_y W\biggl(-D_y\frac{\partial L}{\partial u_{yyy}}\biggr) \\
&\quad + D_{yy} W\biggl(\frac{\partial L}{\partial u_{yyy}}\biggr) + D_{xx} W\biggl(\frac{\partial L}{\partial u_{xxy}} + D_{xx}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_{xxx} W\biggl(-D_x\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_{xxxx} W\biggl(\frac{\partial L}{\partial u_{xxxxy}}\biggr).
\end{aligned}
$$

Now, for the special case of $w = -u_t$, we derive the following results:

$$
\begin{aligned}
C^t &= \xi^t L + (-u_t)\biggl(D_{xx}\frac{\partial L}{\partial u_{xxt}}\biggr) + D_x(-u_t)\biggl(-D_x\frac{\partial L}{\partial u_{xxt}}\biggr) + D_{xx}(-u_t)\biggl(\frac{\partial L}{\partial u_{xxt}}\biggr) \\
&= -u_t v_{xx} + u_{xt} v_x - u_{xxt} v,
\end{aligned}
$$

$$
\begin{aligned}
C^x &= \xi^x L + (-u_t)\biggl(-D_x\frac{\partial L}{\partial u_{xx}} - D_y\frac{\partial L}{\partial u_{xy}} + D_{xx}\frac{\partial L}{\partial u_{xxx}} + D_{xy}\frac{\partial L}{\partial u_{xxy}} \\
&\quad + D_{xt}\frac{\partial L}{\partial u_{xxt}} + D_{xxxy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_x(-u_t)\biggl(\frac{\partial L}{\partial u_{xx}} - D_x\frac{\partial L}{\partial u_{xxx}} - D_y\frac{\partial L}{\partial u_{xxy}} - D_t\frac{\partial L}{\partial u_{xxt}} - D_{xxy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&\quad + D_y(-u_t)\biggl(\frac{\partial L}{\partial u_{xy}} - D_x\frac{\partial L}{\partial u_{xxy}} - D_{xx}\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_t(-u_t)\biggl(-D_x\frac{\partial L}{\partial u_{xxt}}\biggr) \\
&\quad + D_{xx}(-u_t)\biggl(\frac{\partial L}{\partial u_{xxx}} + D_{xy}\frac{\partial L}{\partial u_{xxxxy}}\biggr) + D_{xxx}(-u_t)\biggl(-D_y\frac{\partial L}{\partial u_{xxxxy}}\biggr) \\
&= (-u_t)(v_{xt} + av_{xxxy} - cv_y u_{xxx} + cv_{xx} u_{xy} + 2cv_x u_{xxy} + dv_{xy} u_{xx} + dv_y u_{xxx}) \\
&\quad - u_{xt}(-cv_x u_{xy} - cvu_{xxy} - dv_y u_{xx} - v_t - av_{xxy}) \\
&\quad - u_{yt}(cvu_{xxx} - dv_x u_{xx} - dvu_{xxx} - av_{xx}) + u_{tt} v_x \\
&\quad - u_{txx}(cvu_{xy} + avv_{xy}) + u_{txxx} av_y,
\end{aligned}
$$

$$
\begin{aligned}
C^y &= \xi^y L + (-u_t)\left(-D_x\frac{\partial L}{\partial u_{xy}} + D_{xx}\frac{\partial L}{\partial u_{xxy}} + D_{yy}\frac{\partial L}{\partial u_{yyy}} + D_{xxxx}\frac{\partial L}{\partial u_{xxxxy}}\right) \\
&\quad + D_x(-u_t)\left(\frac{\partial L}{\partial u_{xy}} - D_x\frac{\partial L}{\partial u_{xxy}} - D_{xxx}\frac{\partial L}{\partial u_{xxxxy}}\right) + D_y(-u_t)\left(-D_y\frac{\partial L}{\partial u_{yyy}}\right) \\
&\quad + D_{yy}(-u_t)\left(\frac{\partial L}{\partial u_{yyy}}\right) + D_{xx}(-u_t)\left(\frac{\partial L}{\partial u_{xxy}} + D_{xx}\frac{\partial L}{\partial u_{xxxxy}}\right) \\
&\quad + D_{xxx}(-u_t)\left(-D_x\frac{\partial L}{\partial u_{xxxxy}}\right) + D_{xxxx}(-u_t)\left(\frac{\partial L}{\partial u_{xxxxy}}\right) \\
&= (-u_t)(-cv_x u_{xxx} - cvu_{xxxx} + dvu_{xxxx} + 2dv_x u_{xxx} + dv_{xx}u_{xx} + bv_{yy} + av_{xxxx}) \\
&\quad - u_{tx}(avu_{xxx} - dvu_{xxx} - dv_x u_{xx} - av_{xxx}) + u_{ty}bv_{yyy} \\
&\quad - u_{tyy}bv - u_{txx}(dvu_{xx} + av_{xx}) + u_{txxx}av_x - u_{txxxx}av.
\end{aligned}
$$

5 Conclusions

In this paper, based on symmetries and conservation laws, we studied a new integrable KP equation. First, we considered the corresponding infinitesimal generators for different coefficients. In particular, for the special case, we get the geometric vector fields and get the corresponding invariant group. Then, based on the invariant group, some new soliton solutions are presented. In addition, the explicit power series solutions are derived. Meanwhile, the recursive relationship between the coefficients is found. Subsequently, nonlinear self-adjointness of this equation are presented. Particulary, strictly self-adjointness conditions is explained. Lastly, conservation laws are obtained. In future works, we will study the nonlocal symmetries, inverse scattering and other properties, also including other solutions using various method.

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