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Stability analysis of fractional differential equations with unknown parameters

Mehmet Emir Koksal

Department of Mathematics, Ondokuz Mayis University,
55139 Atakum, Samsun, Turkey
mekoksal@omu.edu.tr

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Abstract. In this paper, the stability of fractional differential equations (FDEs) with unknown parameters is studied. Using the graphical based D-decomposition method, the parametric stability analysis of FDEs is investigated without complicated mathematical analysis. To achieve this, stability boundaries are obtained firstly by a conformal mapping from s-plane to parameter space composed by unknown parameters of FDEs, and then the stability region set depending on the unknown parameters is found. The applicability of the presented method is shown considering some benchmark equations, which are often used to verify the results of a new method. Simulation examples show that the method is simple and give reliable stability results.

Keywords: stability, fractional differential equations, fractional derivative, unknown parameters, parametric analysis.

1 Introduction

Fractional differential equations [18, 28] are generalization of classical integer-order differential equations through the application of fractional calculus [30], which has been developed by pure mathematicians firstly since after half of the 19th century, though engineers and physicists found applications of fractional calculus for various concepts 100 years later [50]. As a field of mathematical analysis, fractional calculus studies the possibility of taking real or complex number powers of differential operators. It may be considered an old branch of mathematical analysis, but it is a novel topic yet [20]. Especially, fractional calculus has gained a great deal of popularity in modelling some physical and engineering systems as well as fractal phenomena in the last few decades [3, 12, 35, 36]. In fact, many systems in the real world are now better characterized by FDEs and analysed by numerical techniques developed for solving differential equations involving noninteger-order derivatives [16]. FDEs are also known as extraordinary differential equations.

Continuing technological developments have required new methods in basic sciences, especially in mathematics for analysis and design of physical systems and their control tools. These methods which are easily implemented with the advancement of high speed computers aimed to better and better characterization, design tools and control performance of modern technological products of engineering systems of developing civilization. These developments, which had covered only static system models involving geometry and algebra until 1965, had started using dynamical models involving differential and integral calculus since 1965; and now have been accelerating since the 1960's with the fractional-order modelling involving FDEs, which have gained force and dare with high speed computers. Hence, FDEs have become a powerful tool in studying, designing and control of physical systems and engineering products of the present world and it still constitutes a popular research area resulting with new definitions of fractional derivative and its applications in need.

Atangana and Baleauno have proposed a new fractional derivative with nonlocal and no singular kernel and applied it to solve fractional heat transfer model [2]; Wang and Liu have used a different solution procedure for nonlinear fractional porous media equation based on a new fractional derivative [34]; Sayevand and Pichaghci have analyzed nonlinear fractional KdV equation based on He's fractional derivative [31]; Yang et al. have proposed a new numerical technique for solving the local fractional diffusion equation by using two-dimensional extended differential transform approach [48]; Yang with some other co-authors has also done a comprehensive study of the methods, which have been used for the solutions of the problems containing fractional derivatives and integral operators [39]. In a very recent paper by Ortigueira and Machado, a framework for compatible integer and fractional calculus is described and how suitable fractional formulations are really extensions of integer-order definitions; the particular case of fractional linear systems is considered and the problem of initial conditions are tackled [27].

Continuing to review the recent literature about fractional derivatives, Yang et al. have addressed in 2017 new general fractional derivatives (GFDs) involving the kernels of the extended Mittag–Leffler-type functions; with the aid of the GFDs in the mentioned kernels, they analysed and discussed the mathematical models for the anomalous diffusion of fractional order; their formulation were also used to describe complex phenomena occurring in heat transfer [44]. The same year, fractional-order relaxation equations of constants and variable orders in the sense of Caputo type are modelled from mathematical view of point for the first time [37]; the comparative results of the anomalous relaxation among the various fractional derivatives are also given, they are very efficient in description of the complex phenomenon arising in heat transfer.

Further contributions of X.-J. Yang alone and/or with his colleges appear in [40–43]: in [41], a new fractional derivative without singular kernel is defined and its potential application for modelling the steady heat-flow conduction problem is shown; in [42], a new fractional operator of variable order is proposed in sense of Caputo type, and the results for the anomalous diffusion equations of variable order are discussed; in [43], the growths of populations by means of local fractional calculus is modelled; in [40], a family of the special functions by Mittag–Leffler function defined on the Cantor sets is investigated, and the nonlinear local fractional-order differential equations are presented by following the rules of local fractional derivative.

Yet, there have been many other recent publications about the local fractional derivatives and fractional derivatives with special functions, some of them are summarized as follows: in [49], general fractional derivatives with a nonsingular power-law kernel are investigated, and the anomalous diffusion models with nonsingular power-law kernel are discussed; in [47], a new family of the local fractional PDEs is investigated, and the linear, quasilinear, semilinear and nonlinear local fractional PDEs are presented; in [46], a nondifferentiable model of the LC-electric circuit described by a local fractional differential equation of fractal dimensional order is addressed; in [38], the local fractional Klein–Gordon equation and Helmholtz equation in fractal $(1+1)$-dimensional space are solved by the local fractional Laplace series expansion method; in [45], based on the local fractional derivative, a new Boussinesq-type model in a fractional domain is derived; in [10], problem similar to the anomalous diffusion phenomena in heat transfer [23, 44] is considered, and the general fractional calculus of Liouville–Weyl- and Liouville–Caputo-type with the nonsingular power-low kernel is suggested to model the problem; finally, in [11], which is stated as giving an accurate and efficient technique for solving the local PDEs, a coupling of the variational iteration method with the Sumudu transform by the local fractional calculus operator is proposed for the first time, and the exact solution for the local fractional diffusion equation in fractal one-dimensional space is obtained.

Stability is one of the most important objects in the analysis and design of dynamical systems. If the differential equation of a system has not a stable property, the system may burn out, disintegrate or saturate when a signal is applied [7]. Therefore an unstable system is useless in practice and needs a stabilization process via an additional control element mostly [14]. If a system has unknown parameters, the stability analysis is called as parametric stability analysis. It is more difficult than the classical stability analysis, which has simple analysis methods such as Routh–Hurwitz method, Nyquist stability theorem, etc. In the literature, there are some methods on the stability of FDEs with uncertain parameters [8]. The uncertainty in these studies is considered by a certain interval of the unknown parameters. However, to consider the unknown parameters' values in a whole parametric interval from zero to infinity is more useful than a certain little interval, which is the subject of this paper.

Motivated by the need of stability analysis for FDEs with unknown parameters, we suggest in this paper an efficient graphical based stability analysis using the D-decomposition method [21]. The D-decomposition method provides a powerful and simple stability work environment to the analyst. This method is based on a conformal mapping from frequency domain to parametric domain of unknown parameters. With this mapping, the imaginary axis, which is the stability bound of complex s-plane, converts to three types of stability boundaries, which are named as real, complex and infinite root boundaries, in the parametric space. These boundaries give us the stability regions, which are important tools including useful stability knowledge. The algorithm presented in this paper has a reliable result, which is illustrated by several examples, and is practically useful in the computerized analysis of FDEs having unknown parameters.

The paper is organized as follows. The basic principles of FDEs are revisited in the next section. The stability concept for FDEs is explained in Section 3. A derivation of the stability boundary formulae for the stability regions of fractional differential equations with unknown parameters is given in Section 4. The next section illustrates the effectiveness of the stability analysis proposed with three simulation examples. Finally, Section 6 gives some concluding remarks.

2 Principles of fractional differential equations

In the most general case, FDEs are expressed by the following form [19]:

$$F\big(t, y(t), {}_\eta Dt^{\alpha_1} y(t), \dots, {}_\eta Dt^{\alpha_n} y(t)\big) = G\big(t, u(t), {}_\eta Dt^{\beta_1} u(t), \dots, {}_\eta Dt^{\beta_m} u(t)\big), \quad (1)$$

where F and G are fractional differential functions, α_i ($i = 1 \sim n$) and β_k ($k = 1 \sim m$) are positive real numbers such that $0 < \alpha_1 < \alpha_2 < \dots < \alpha_n$, $0 < \beta_1 < \beta_2 < \dots < \beta_m$ and $m < n$. ${}_\eta D_t^\gamma$ is fractional-order derivative and integral operator, and it is defined as follows [8]:

$${}_\eta Dt^\gamma = \begin{cases} \mathrm{d}^\gamma/\mathrm{d}t^\gamma, & \Re(\gamma) > 0, \\ 1, & \Re(\gamma) = 0, \\ \int_\eta^t (\mathrm{d}\tau)^{-\gamma}, & \Re(\gamma) < 0. \end{cases}$$

Here γ is fractional order, $\Re(\gamma)$ is the real part of fractional order and η is a constant coefficient related with initial conditions. Commonly, t is an independent variable representing time, $u(t)$ is input exiting function and $y(t)$ is the output response function of a dynamical system. There are various definitions for fractional derivative. Riemann–Liouville, Grünwald–Letnikov, Caputo and Mittag–Leffler are the well-known and common definitions among them. (For a more detail of these definitions, the reader can see [4, 28].)

One of the most common types of FDEs is linear time-invariant fractional differential equation

$$\sum_{i=0}^{n} a_{i0} D_t^{\alpha_i} y(t) = \sum_{k=0}^{m} b_{k0} D_t^{\beta_k} u(t), \quad \alpha_n \neq 0,\ \alpha_0 = 0, \quad (2)$$

where a_i and b_k are real numbers, α_i and β_k are as defined for Eq. (1), $\beta_m \leqslant \alpha_n$ for stability reason [27]. Equation (2) is also called as noncommensurate-order FDE. As a special case, fractional orders a_i and b_k may be multiple of same real number α like $a_i = i\alpha$ and $b_k = k\alpha$. In this case, Eq. (2) is named by commensurate-order FDE [17,28].

The Laplace transform method is commonly used in engineering systems and their analysis. According to Grünwald–Letnikov definition, Laplace transform of differential operator ${}_0D_t^\gamma$ is given by

$$L\big\{{}_0D^\gamma f(t)\big\} = s^\gamma F(s),$$

where s is the Laplace operator. Hence, for the differential equation in Eq. (2), the transfer function giving the input-output expression of a system with zero initial conditions is given by

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\sum_{i=0}^{m} b_i s^{\beta_i}}{a_o + \sum_{i=1}^{n} a_i s^{\alpha_i}} = \frac{N(s)}{D(s)}. \quad (3)$$

In this equation, $U(s)$ is the Laplace transform of the exciting function $u(t)$; similarly, $Y(s)$ is that of the response $y(t)$. $N(s)$ and $D(s)$ are the numerator and denominator polynomials of the transfer function, respectively. Being the stability as a first, the function given in Eq. (3) contains many important system information and concepts.

3 Stability analysis of fractional differential equations

There are many ways of testing the stability of a linear time-invariant differential equation. Stability of the differential equation can be examined by applying Routh–Hurwitz test on its denominator polynomial, checking the locations of the poles of its transfer function whether they are on the left half s-plane or not and exploring whether the output remains bounded with an impulse or step input excitation [26]. If the differential equation involves time-delay or fractional-order terms, in this case, Routh–Hurwitz' criteria cannot be applied.

The denominator seen in Eq. (3) of a FDE is in the form of a quasi-polynomial, and it is expressed by

$$D(s) = a_n s^{a_n} + a_{n-1} s^{a_{n-1}} + \cdots + a_1 s^{a_1} + a_0. \tag{4}$$

For the stability analysis, the quasi-polynomial in Eq. (4) is transformed to the following commensurate-order quasi-polynomial:

$$D_c(s) = a_n \prod_{i=0}^{n} \left(s^\alpha + \lambda_i\right) = \prod_{i=0}^{n} P\left(s^\alpha\right),$$

where α is the least common multiple of $\alpha_1, \alpha_2, \ldots, \alpha_n$. Stability condition of this fractional-order polynomial was given by Matignon [23] in 1996 as follows:

$$\left|\arg(-\lambda_i)\right| > \alpha\frac{\pi}{2} \quad \forall i = 1, 2, \ldots, n,$$

where $-\lambda_i$ $(i = 1, 2, \ldots, n)$ are the zeros of the pseudopolynomial $P(s^\alpha)$ (pseudopoles of the transfer function in Eq. (3)) [29]. Matignon's stability analysis is applicable for only linear FDEs whose coefficients are known and invariant. For the FDEs changing their coefficients/parameters in an interval, Chen et al. [8] proposed a very effective method for the stability analysis. However, to investigate the stability, it may be more useful in the case when the parameters of a FDE change between minus and plus infinity. In this paper, a method is presented for this type of stability analysis. The method is based on obtaining stability boundaries and it contains a graphical presentation. The most important property of this method is to construct a conformal mapping from s-plane to parameter space composed by unknown parameters of the FDE. Therefore, the method is called as parametric stability analysis and based on the D-decomposition method (see [25] for more detail).

In the D-decomposition method, there are three stability boundaries of a polynomial [15, 25]. The first boundary belongs to a real pole, which changes its stability property when passing through origin and crossing the opposite half of s-plane with the parameter changes. Therefore, this boundary is called real root boundary. It is obtained by putting zero instead of s in Eq. (4) with

$$D(s)|_{s=0} = 0 \quad \Longrightarrow \quad a_0 = 0. \tag{5}$$

Second boundary is named by infinite root boundary because of the fact that the boundary belongs to a pole, which changes stability property at infinity with the parameter changes. Infinite root boundary is determined by equalizing the coefficient of the greatest-order term to zero:

$$a_n = 0. \tag{6}$$

In order to obtain the boundaries from Eqs. (5) and (6), the coefficients a_0 and a_n should contain unknown parameters. Otherwise, these boundaries do not exist for the considered FDE.

The last boundary results a couple of complex poles passing to one half-plane from the other half-plane over the imaginary axis of the s-plane with the parameter changes. This is the main boundary, which determines the stability region of the FDE and is named by complex root boundary. To obtain this boundary, s in Eq. (4) is replaced by $j\omega$ as follows:

$$D(jw) = a_n(j\omega)^{a_n} + a_{n-1}(j\omega)^{a_{n-1}} + \cdots + a_1(j\omega)^{a_1} + a_0. \tag{7}$$

Using the expansion $j^x = \cos(0.5\pi x) + j\sin(0.5\pi x)$ in Eq. (7) for the fractional-order powers of the complex number j, we get

$$D(jw) = \sum_{i=1}^{n} \bigl\{ a_i \bigl[\cos(0.5\pi\alpha_i) + j\sin(0.5\pi\alpha_i)\bigr]\omega^{\alpha_i} \bigr\} + a_0.$$

By decomposing $D(jw)$ into real and imaginary parts, we obtain

$$D_r(w) = \sum_{i=1}^{n} a_i \cos(0.5\pi\alpha_i)\omega^{\alpha_i} + a_0, \qquad D_i(w) = \sum_{i=1}^{n} a_i \sin(0.5\pi\alpha_i)\omega^{\alpha_i}. \tag{8}$$

By equalizing $D_r(\omega)$ and $D_i(\omega)$ to zero separately, two variable equations system whose variables are the unknown parameters of the FDE are obtained. The complex root boundary is found by solving of this system with respect to ω for $0 < \omega < \infty$. The stability region of the FDE is enclosed by these three boundaries in the parameter space. The main property of this region is that all parameters in this area make the FDE stable.

4 Parametric stability analysis of fractional differential equations

In this section, we consider the FDEs, which are often encountered in the engineering systems and have the following form:

$$a_0 D_t^{\alpha_2} y(t) + b_0 D_t^{\alpha_1} y(t) + cy(t) = ku(t), \tag{9}$$

where $u(t)$ and $y(t)$ are forcing and response functions, respectively; a, b, c and k are real coefficients, α_1 and α_2 are the fractional-order powers such that $0 < \alpha_1 < \alpha_2 < 2$. In particular, when $\alpha_1 = 1$, $\alpha_2 = 2$, and $u = 0$, this equation becomes the describing equation of, for example, velocity of a mass sliding on a frictional surface and attached to a wall through a spring and stimulated by its initial kinetic energy (initial velocity), or describing equation of capacitor voltage variation in a series RLC resonator circuit stimulated by the initial condition voltage on the capacitor [26]. For $\alpha_1 = 0$, $\alpha_2 = 1$, and $u = 0$, it reduces simply to a first-order linear time-invariant differential equation describing, for example, discharging process of a capacitor through a resistor in an RC electrical circuit.

As it is pointed out in the previous section, the transfer function of a FDE encloses important stability information. By taking the Laplace transform of the FDE in Eq. (9), the transfer function giving input-output relation is obtained as

$$G(s) = \frac{Y(s)}{U(s)} = \frac{k}{as^{\alpha_2} + bs^{\alpha_1} + c}. \tag{10}$$

If the coefficients a, b, c and the fractional orders α_1 and α_2 are constant or they change in a specific interval, the stability of Eq. (10) can be determined by Matignon's method [23] and Chen et al.'s method [8] easily as mentioned in Section 3. Here we investigate the parametric stability for the full variation range of these coefficients using the D-decomposition method.

For obtaining the stability boundaries, we consider the denominator of Eq. (10)

$$D(s) = as^{\alpha_2} + bs^{\alpha_1} + c. \tag{11}$$

The real root boundary for Eq. (11) is determined as

$$c = 0. \tag{12}$$

Since $\alpha_2 > \alpha_1$, the infinite root boundary is found by applying Eq. (6) to Eq. (11) as follows:

$$a = 0. \tag{13}$$

For the complex root boundary, the real and imaginary parts of Eq. (8) are equalized to zero and the following system of equations is obtained:

$$a\omega^{\alpha_2}\cos(0.5\pi\alpha_2) + b\omega^{\alpha_1}\cos(0.5\pi\alpha_1) + c = 0,$$
$$a\omega^{\alpha_2}\sin(0.5\pi\alpha_2) + b\omega^{\alpha_1}\sin(0.5\pi\alpha_1) = 0.$$

By solving this system of equations with respect to b, α_1, α_2 for the coefficients a and c, we get the following formulas:

$$a = -b\omega^{\alpha_1-\alpha_2}\frac{\sin(0.5\pi\alpha_1)}{\sin(0.5\pi\alpha_2)}, \tag{14}$$

$$c = -b\omega^{\alpha_1}\frac{\sin[0.5\pi(\alpha_2-\alpha_1)]}{\sin(0.5\pi\alpha_2)}. \tag{15}$$

By changing ω from zero to infinity for different values of b, α_1 and α_2, the complex root boundaries are obtained.

Definition 1. When three boundary conditions defined in Eqs. (12), (13), (14) and (15) are drawn in (a, c)-plane together, the plane separates to many regions. The most basic property of these regions is that all points in any region have the same number of stable or unstable roots [25]. With this reference, the region whose all poles are stable are called as stability region. Every point in this region makes the FDE in Eq. (9) stable. The stability of each region can be determined by selecting a test point in the region and checking the stability of Eq. (11) in every region [23]. Plot of stability boundaries and stability regions on (a, c)-plane will yield the parametric investigation of the stability of the given fractional-order system. This provides for example to design a more robust system.

5 Simulation examples

In this section, stability analysis of some fractional differential equations commonly used in the literature are given for illustration of the validity of the presented method. In the first example, fractional Basset equation defining the dynamical motion of an object, which submerged into a fluid, is considered. In the next example, as a more general case, stability analysis is investigated for a commensurate-order FDE. In the last example, the stability ranges of the parameters a, b and c for an industrial heating furnace is investigated.

Example 1. The motion dynamic of an object, which is submerged into an incompressible fluid, is one of the frequently studied topics in engineering literature. Basset [6] proposed an equation and its solution for a sphere moving in a viscous liquid when the sphere is moving in a straight line under the action of a constant force, such as gravity, and also when the sphere is surrounded by viscous liquid and is set in rotation about a fixed diameter and then left to itself [1]. This equation is named Basset equation, and it is expressed by the following fractional-order differential equation:

$$a_0 D_t^1 y(t) + b_0 D_t^\alpha y(t) + c y(t) = u(t), \tag{16}$$

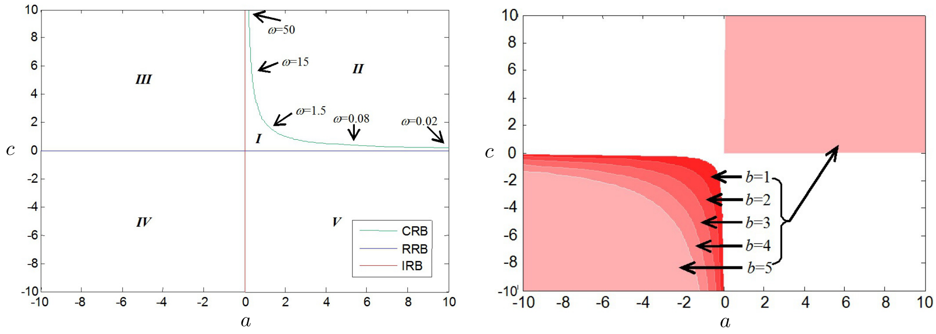
where $\alpha \in (0, 1)$, and $a \neq 0, b, c$ are arbitrary real coefficients [5]. While Eq. (16) is called classical Basset equation for $\alpha = 0.5$, it is named generalized Basset equation for $0 < \alpha < 1$ [22].

For the stability analysis of Basset equation, Govindaraj and Balachandran proposed an analytical method in [13]. Even though they found solutions for the changes in the coefficients a, b and c separately, they have not presented a general solution with respect to changing in the coefficients. The goal in this example is to obtain a simple graphical result giving the stability or instability of the equation according to the changing of the parameters for the system given in Eq. (16).

Real and infinite root boundaries are the same with in Eqs. (12) and (13), respectively. For the complex root boundary, putting $\alpha_2 = 1$ and $0 < \alpha_1 = \alpha < 1$ in Eqs. (14) and (15), the expressions giving complex root boundary are found by

$$a = -b\omega^{\alpha-1} \sin \alpha \frac{\pi}{2}, \qquad c = -b\omega^{\alpha} \cos \alpha \frac{\pi}{2}.$$

For the stability analysis of Eq. (16), we consider the classical Basset equation firstly. For $\alpha = 0.5$, the stability boundaries are



(a) (b)

Figure 1. Stability boundaries (a) and region (b) of the classical Basset equation for $b = -2$.

(i) real root boundary: $c = 0$,
(ii) infinite root boundary: $a = 0$,
(iii) complex root boundary: $a = -(\sqrt{2}/2)b\omega^{-0.5}$ and $c = -(\sqrt{2}/2)b\omega^{0.5}$.

For changing ω from 0 to ∞, these boundaries decompose the (a, c)-plane into many number of regions for various values of b. For example, the stability boundaries constitute five regions in the (a, c)-plane for the value of $b = -2$ as shown in Fig. 1(a), where complex root boundary is $ac = b^2/2 = 2$, $a, c > 0$ from (iii). Since the regions are unlimited throughout the axes of a and c, the figure is limited for good visibility in the interval of $[-10, 10]$ for these axes. The most important characteristic of these regions is that all points in every region have the same number of stable and unstable roots. Because of this reason, to determine, which regions are stable or not among these five areas, it is sufficient to select only one testing point from every region and checking the stability of Eq. (16) according to these points. As shown in Fig. 1(b), it is found that the second and fourth regions are the stability regions. For verification of these regions, it can be seen that the results are suitable with the following results:

(i) asymptotically stable for $a = -3$, $b = -2$, $c = -4$,
(ii) periodically stable for $a = 1$, $b = -2$, $c = 2$,
(iii) unstable for $a = 1$, $b = -2$, $c = -1$,

which are given by Govindaraj and Balachandran [13] for the classical Basset equation. By varying b and repeating the above procedure, different stability regions are obtained for each b value as shown in Fig. 2. It is seen from this figure that smaller values of $|b|$ provide bigger stability regions. To illustrate the graphical results more clearly, the overall stability region can then be visualized in a 3D plot as shown in Fig. 3.

In order to make a more general stability analysis, we consider the generalized Basset equation for $0 < \alpha < 1$ lastly. Figure 4 shows the stability regions of the generalized Basset equation for different values of the parameter α for $b = -2$ and $b = 4$. Notice that the change of α affects only the curve of complex root boundary but does not influence the other boundaries. However, the variation of the complex root boundary remains only

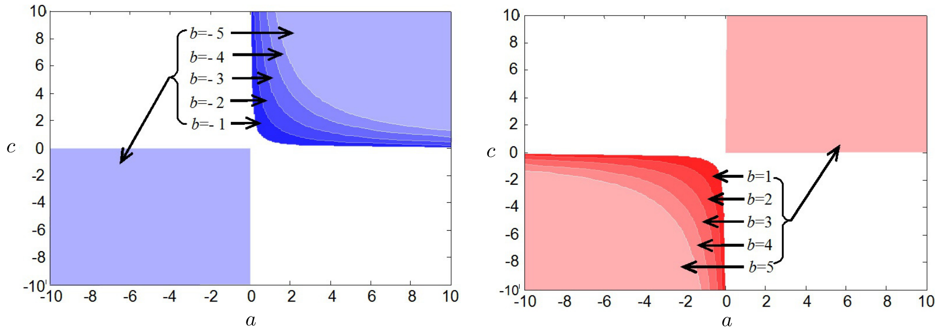


Figure 2. Changing the stability regions of the classical Basset equation for various values of b.

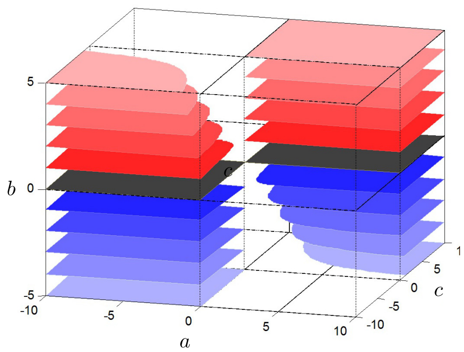


Figure 3. The overall 3D stability region for the classical Basset equation.

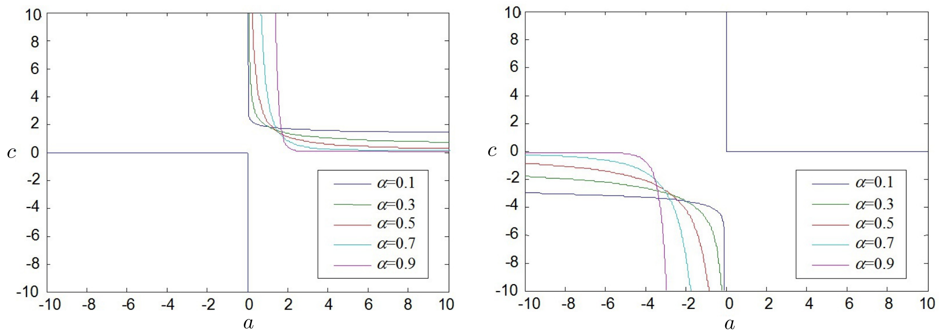


Figure 4. Stability regions of the generalized Basset equation for different values of α: (a) $b = -2$, (b) $b = 4$.

in a bounded area with the variation of α. To generalize this fact clearly, the change of complex root boundary is plotted for 100 different values of α in the interval of $(0, 1)$ for $b = -2$ in Fig. 5. Robust stability region can be defined for the area taking the shape of

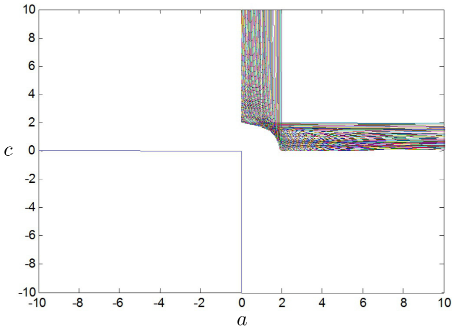


Figure 5. The variation of stability regions of generalized Basset equation for $b = -2$ as α changes in $(0, 1)$ with the increments of 0.01.

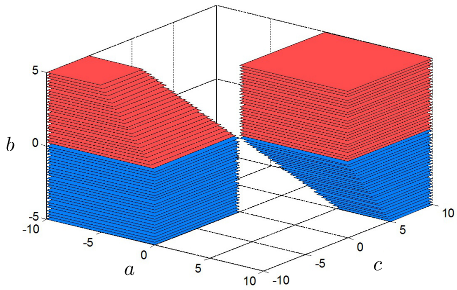


Figure 6. 3D robust stability region independent of α parameter of the generalized Basset equation.

a rectangle $-b = 2 < a, c \leqslant 10$, which remains inside the set of complex root boundaries and also the rectangular area $-10 \leqslant a, c < 0$ at the left bottom. This robust region always gives the guaranteed stable results for all values of a, b, c without depending on the values of α in $(0, 1)$. With reference to this result, three-dimensional robust stability region, which does not depend on the parameter α of generalized Basset equation, is shown in Fig. 6. Any (a, b, c) point selected in this region makes the fractional Basset equation in Eq. (16) absolutely stable for any value of α in $(0, 1)$. One of the other advantages of obtaining three-dimensional robust stability region is that it gives the limit of how much any parameter can be changed without affecting the stability of the Basset equation for any point in the stability region. For example, from Fig. 6 it is clear that the Basset equation is stable for $a = -5$, $b = 1$ and $c = -10$. As reference to this result, it is possible to say that the stability of Basset equation is not affected by the change in the values of a and c parameters in the range of $(-\infty, 5)$ and b parameter in the range of $(-\infty, 5)$.

Remark 1. It can be observed from Fig. 1(a) that if the complex root boundary curve, which is derived for any b value of commensurate-order equation obtained for $\alpha = 0.5$, passes through any (a, c) point, it also passes from the point (c, a) symmetrically. This result originates from the powers of Eq. (9) to be commensurate orders of 1 and 0.5. However, this characteristic is invalid for the general incommensurate orders, i.e. when the value of α is different from 0.5.

Example 2. Commensurate-order FDEs are commonly used for modelling of physical systems and industrial processes [33]. In this example, the following equation containing two fractional terms is considered:

$$a_0 D_t^{2\alpha} y(t) + b_0 D_t^{\alpha} y(t) + c y(t) = u(t). \tag{17}$$

This equation represents a FDEs family for different values of α and it also contains classical Basset equation for $\alpha = 0.5$. Members of this family are named multi-term differential equations if the power of the greatest derivative term is greater than 1 (or

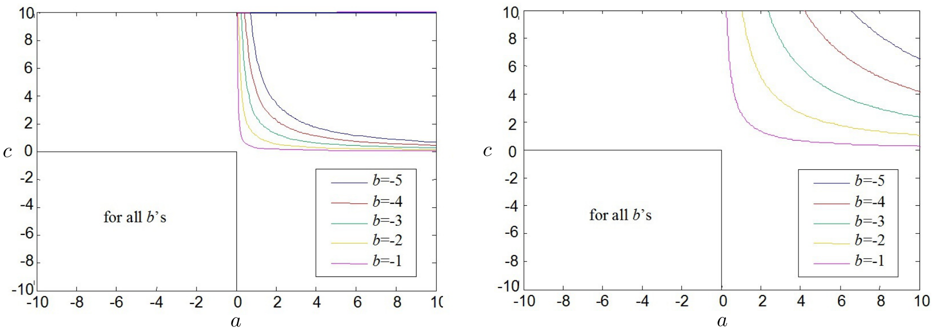


Figure 7. Stability regions for the values of the parameter b in the interval of $[-5, -1]$ as α changes: (a) $\alpha = 0,2$, (b) $\alpha = 0.8$.

$\alpha > 0.5$) and single-term differential equations if the power of the greatest derivative term is less than 1 (or $\alpha < 0.5$) [9].

Expressions, which belong to the complex root boundary of commensurate-order differential equation in Eq. (17) whose real and infinite root boundaries are given by Eqs. (12) and (13), are obtained for $\alpha_2 = 2\alpha$ and $\alpha_1 = \alpha$ in Eqs. (14) and (15) as follows:

$$a = -b\omega^{-\alpha}\frac{\sin(0.5\pi\alpha)}{\sin(\pi\alpha)}, \qquad c = -a\omega^{2\alpha}\cos(\pi\alpha) - b\omega^{\alpha}\cos(0.5\pi\alpha). \tag{18}$$

For the various values of α, the stability regions can be easily obtained according to the values of b. For example, the stability regions for $\alpha = 0.2$ and $\alpha = 0.8$ are plotted for $b \in [-5, -1]$ as shown in Fig. 7, where the complex root boundary calculated from (18) is

$$ac = \frac{b^2}{2(1 + \cos\alpha\pi)}, \quad a, c > 0.$$

Stability regions for the values of b changing in the interval $[1, 5]$ are replacements of the same regions with respect to origin symmetrically. In Fig. 8, the stability regions for five different values of α are seen when $b = -2$ and $b = 3$. For $b = -2$, when the values of α are changed more often for instance, 100 times in the range of $(0, 1)$, the change of stability region is appeared from Fig. 9. It is seen from this figure that the stability region fills right upper part of the (a, c)-plane if the value of α approaches to 0 and the stability region is getting smaller if the value of α approaches to 1. As a result, the stability of Eq. (17) for any value of α depending on parameter varying can be analyzed from the figures easily. However, it is seen that obtaining a robust stability region is very difficult on the contrary of Example 1.

Example 3. In this example, the incommensurate-order FDE

$$a_0 D_t^{1.31} y(t) + b_0 D_t^{0.97} y(t) + c y(t) = u(t) \tag{19}$$

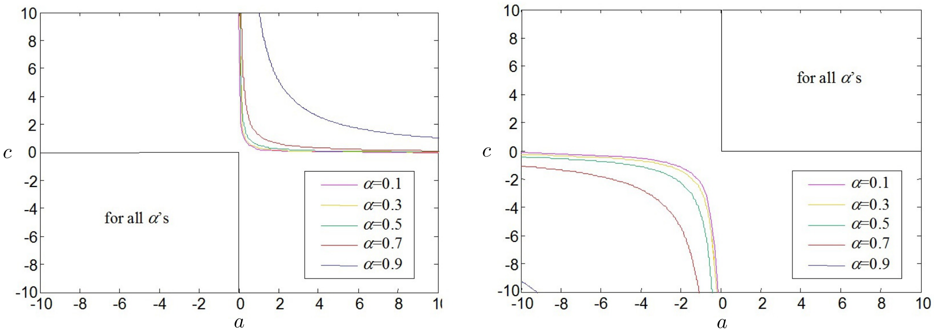


Figure 8. Stability regions for various values of α: (a) $b = -2$, (b) $b = 3$.

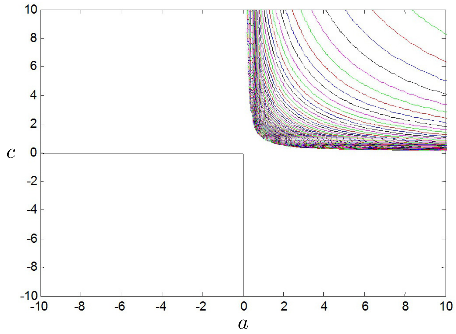


Figure 9. The variation of stability regions for 100 values of α in $(0, 1)$ when $b = -2$.

is considered for modelling an industrial heating furnace. In this equation, nominal values of a, b and c parameters are given as $a = 14994$, $b = 6009.5$ and $c = 1.69$ in [24]. Sondhi and Hote [32] have shown the stability of the equation for these nominal values. The goal in this example is to verify this stability result and to determine the stability intervals by assuming these parameters are varying.

Stability boundaries for Eq. (19) are as follows:

(i) real root boundary: $c = 0$,
(ii) infinite root boundary: $a = 0$,
(iii) complex root boundary: $a = -1.1303b\omega^{-0.34}$ and $c = -0.576b\omega^{0.97}$.

The stability region of the system for $b = 6009.5$ is shown in Fig. 10, where from (iii) the lower left stability region is found bounded by the curve

$$(-a)(-c)^{0.350515464} = 118190.408. \tag{20}$$

It is seen from this figure, the nominal values of a and c stay in the stability region. So, the stability result of Sondhi and Hote's has been verified easily without any complicated calculations. For a more general case, three-dimensional stability region derived for

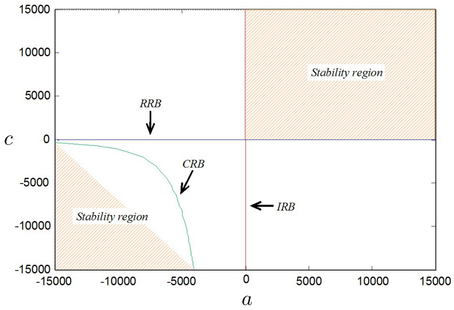


Figure 10. Stability region for industrial heating furnace.

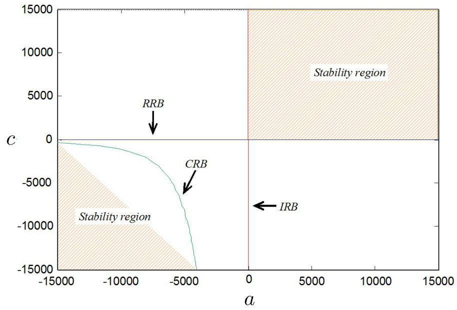


Figure 11. Three-dimensional overall stability region as b changes in $[-10000, 10000]$.

various values of b is seen in Fig. 11. From this figure evolution of the stability as contrast to system parameter changes can be investigated. The stability is conserved at all positive values of a and c, and at all negative values of a and c staying in the oval part when b is positive; at all negative values of a and c and at all positive values of a and c staying in the oval part when b is negative.

6 Conclusions

In this paper, a graphical based stability analysis method is presented for FDEs. The analysis concept is based on the derivation of stability boundaries and then the determination of the stability region including the parameter set, which makes the FDE stable. One of the most important advantages of the method is that the stability analysis is done in a visual environment without considering complex analytical solutions. This method can be used not only for the investigation of the stability of a differential equation whose parameters are not changed but also for observation of the parametric robust stability of the equation whose parameters are varying in a large interval. From this aspect, having a large usage perspective of the method in comparison with the other methods is one of the other advantages of the method. This provides opportunity to engineers in their analyses about discussion more detail. Simulation examples have been selected from the benchmark problems encountered in engineering systems. As evidenced by the results given in these examples, it can be concluded that the proposed graphical based method is a method reliable not only for stability analysis but also parametric robust stability analysis according to the parameter changes.

The presented method can be generalized for stability analyses of fractional differential equations having time delay, which is a very popular subject in the last decade. Moreover, the differential equations having more fractional terms can be also studied. Here, when the number of unknown parameters increases, the three-dimensional graphs will be insufficient. In this case, more than one graphs or four-dimensional graphs with the fourth dimension assuming by colour can be used.

References

1. A. Armand, T. Allahviranloo, Z. Gouyandeh, General solution of Basset equation with Caputo generalized Hukuhara derivative, *J. Appl. Anal. Comput.*, **6**(1):119–130, 2016.
2. A. Atangana, D. Baleanu, New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20**(2):763–769, 2016.
3. Z.B. Bai, Y.Q. Chen, H.R. Lian, S.J. Sun, On the existence of blow up solutions for a class of fractional differential equations, *Fract. Calc. Appl. Anal.*, **17**(4):1175–1187, 2014.
4. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, 2012.
5. D. Baleanu, R. Garra, I. Petras, A fractional variational approach to the fractional Basset-type equation, *Rep. Math. Phys.*, **72**(1):57–64, 2013.
6. A.B. Basset, On the motion of a sphere in a viscous liquid, *Philos. Trans. R. Soc. Lond., A*, **179**:43–63, 1888.
7. C.T. Chen, *Linear System Theory and Design*, Oxford Univ. Press, New York, 1999.
8. Y.Q. Chen, H.-S. Ahn, I. Podlubny, Robust stability check of fractional order linear time invariant systems with interval uncertainties, *Signal Process.*, **86**(19):2611–2618, 2006.
9. J.T. Edwards, J.F. Neville, A.C. Simpson, The numerical solution of linear multi-term fractional differential equations: Systems of equations, *J. Comput. Appl. Math.*, **148**(2):401–418, 2002.
10. F. Gao, General fractional calculus in non-singular power-law kernel applied to model anomalous diffusion phenomena in heat transfer problems, *Therm. Sci.*, **21**(1):11–18, 2017.
11. F. Gao, H.M. Srivastava, Y.N. Gao, X.J. Yang, Coupling method involving the Sumudu transform and the variational iteration method for a class of local fractional diffusion equations, *J. Nonlinear Sci. Appl.*, **9**(11):5830–5835, 2016.
12. F. Ge, Y.Q. Chen, C. Kou, I. Podlubny, On the regional controllability of the sub-diffusion process with Caputo fractional derivative, *Fract. Calc. Appl. Anal.*, **19**(5):1262–1281, 2016.
13. V. Govindaraj, K. Balachandran, Stability of Basset equation, *J. Fract. Calc. Appl.*, **5**(20):1–15, 2014.
14. S.E. Hamamci, M. Koksal, Calculation of all stabilizing fractional-order PD controllers, *Comput. Math. Appl.*, **59**:1621–1629, 2010.
15. N. Hohenbichler, J. Ackermann, Synthesis of robust PID controllers for time delay systems, in *Proceedings of the 2003 European Control Conference, UK, September 1–4, 2003*, IEEE, Cambridge, 2003, pp. 1169–1174.
16. C. Hwang, Y.-C Cheng, A numerical algorithm for stability testing of fractional delay systems, *Automatica*, **42**(5):825–831, 2006.
17. D. Idiou, A. Charef, A. Djouambi, Linear fractional order system identification using adjustable fractional order differentiator, *IET Signal Process.*, **8**(4):398–409, 2014.
18. A.A. Kilbas, H.M. Srivastava, J.J. Trujiilo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.
19. A.A. Kilbas, J.J. Trujillo, Differential equations of fractional order: Methods, results and problem—I, *Appl. Anal.*, **78**(1–2):153–192, 2001.

20. V. Lakshmikanthama, A.S. Vatsala, Basic theory of fractional differential equations, *Nonlinear Anal., Theory Methodas Appl.*, **69**:2677–2682, 2008.
21. B.N. Le, Q.G. Wang, T.H. Lee, Development of D-decomposition method for computing stabilizing gain ranges for general delay systems, *J. Process Control*, **25**:94–104, 2015.
22. F. Mainardi, Some basic problems in continuum and statistical mechanics, in A. Carpinteri, F. Mainardi (Eds.), *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Wien, 1997, pp. 291–348.
23. D. Matignon, Stability results for fractional differential equations with applications to control processing, in *Proceedings of the International Conference on Computational Engineering in Systems Applications, France, July, 1996*, IEEE-SMC, Lille, 1996, pp. 963–968.
24. F. Merrikh-Bayat, M. Karimi-Ghartemani, Method for designing $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ stabilisers for minimum-phase fractional-order systems, *IET Control Theory Appl.*, **4**(1):61–70, 2010.
25. J.I. Neimark, D-decomposition of the space of quasi-polynomials (on the stability of linearized distributive systems, *Transl., Ser. 2, Am. Math. Soc.*, **102**:95–131, 1973.
26. K. Ogata, *Modern Control Engineering*, Prentice Hall, Englewood Cliffs, NJ, 2009.
27. M. Ortigueria, J. Machado, Which derivative?, *Fractal and Fractional*, **1**(3):1–13, 2017.
28. I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, CA, 1999.
29. M. Rivero, S.V. Rogosin, J.A.T. Machado, J.J. Trujillo, Stability of fractional order systems, *Math. Probl. Eng.*, **2013**:356215, 2013.
30. J. Sabatier, O. Agrawal, J.T. Machado, *Advances in Fractional Calculus. Theoretical Developments and Applications in Physics and Engineering*, Springer, Berlin, 2007.
31. K. Sayevand, K. Pichaghchi, Analysis of nonlinear fractional KdV equation based on He's fractional derivative, *Nonlinear Sci. Lett.*, **7**(3):77–85, 2016.
32. S. Sondhi, Y.V. Hote, Stability testing and IMC based fractional order PID controller design for heating furnace system, in *Proceedings of the 2014 Annual IEEE India Conference, India, December 11–13, 2014*, IEEE, Pune, 2014, pp. 1–5.
33. B.M. Vinagre, V. Feliu, Optimal fractional controllers for commensurate order systems: A special case of the Wiener–Hopf method, in *Proceedings of the 39th IEEE Conference on Decision and Control, Australia, December 12–15, 2000*, IEEE, Sydney, 2000, pp. 97–102.
34. K.L. Wang, S.Y. Liu, A new solution procedure for nonlinear fractional porous media equation based on a new fractional derivative, *Nonlinear Sci. Lett.*, **7**(4):135–140, 2016.
35. A. Yakar, M.E. Koksal, Existence results for solutions of nonlinear fractional differential equations, *Abstr. Appl. Anal.*, **2012**:267108, 2012.
36. Q. Yang, D.L. Chen, T.B. Zhao, Y.Q. Chen, Fractional calculus in image processing, *Fract. Calc. Appl. Anal.*, **19**(5):1222–1249, 2016.
37. X.J. Yang, Fractional derivatives of constant and variable orders applied to anomalous relaxation models in heat transfer problems, *Therm. Sci.*, **21**(3):1161–1171, 2017.
38. X.J. Yang, D. Baleanu, F. Gao, New analytical solutions for Klein–Gordon and Helmholtz equations in fractal dimensional space, *Proc. Rom. Acad., Ser. A, Math. Phys. Tech. Sci. Inf. Sci.*, **18**(3):231–238, 2017.
39. X.J. Yang, D. Baleanu, H.M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, Amsterdam, 2016.

40. X.J. Yang, F. Gao, H.M. Srivastava, Non-differentiable exact solutions for the nonlinear odes defined on fractal sets, *Fractals*, **25**(4):1–9, 2017.

41. X.J. Yang, J.A.T. Machado, A new fractional derivative without singular kernel: Application to themodelling of the steady heat flow, *Therm. Sci.*, **20**(2):753–756, 2016.

42. X.J. Yang, J.A.T. Machado, A new fractional operator of variable order: Application in the description of anomalous diffusion, *Physica A*, **481**:276–283, 2017.

43. X.J. Yang, J.A.T. Machado, A new insight into complexity from the local fractional calculus view point: Modelling growths of populations, *Math. Method. Appl. Sci.*, **40**(17):6070–6075, 2017.

44. X.J. Yang, J.A.T. Machado, D. Baleanu, Anomalous diffusion models with general fractional derivatives within the kernels of the extended Mittag–Leffler type functions, *Rom. Rep. Phys.*, **69**(4):1–19, 2017.

45. X.J. Yang, J.A.T. Machado, D. Baleanu, On exact traveling-wave solution for local fractional Boussinesq equation in fractal domain, *Fractals*, **25**(4):1–7, 2017.

46. X.J. Yang, J.A.T. Machado, C. Cattani, F. Gao, On a fractal LC-electric circuit modeled by local fractional calculus, *Commun. Nonlinear Sci.*, **47**:200–206, 2017.

47. X.J. Yang, J.A.T. Machado, J.J. Nieto, A new family of the local fractional PDEs, *Fundam. Inform.*, **151**(1–4):63–75, 2017.

48. X.J. Yang, J.A.T. Machado, H.M. Srivastava, A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach, *Appl. Math. Comput.*, **274**:143–151, 2016.

49. X.J. Yang, H.M. Srivastavac, D.F.M. Torres, A. Debbouche, General fractional-order anomalous diffusion with non-singular power-law kernel, *Therm. Sci.*, **21**(1):1–9, 2017.

50. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.