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Positive solutions of higher order fractional integral boundary value problem with a parameter*

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Abstract. In this paper, we study a higher order fractional differential equation with integral boundary conditions and a parameter. Under different conditions of nonlinearity, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter. Our approach relies on the Guo–Krasnoselskii fixed point theorem on cones.

Keywords: positive solution, fractional differential equation, integral boundary condition, fixed point theorem, cone.

1 Introduction

In this paper, we investigate the following fractional differential equation with integral boundary conditions and a parameter:

$$\begin{aligned} &-D_{0+}^{\eta-2}\big(u''(t)\big)+\lambda f\big(t,u(t)\big)=0, \quad t\in(0,1),\\ &u''(0)=u'''(0)=\cdots=u^{(n-2)}(0)=0, \qquad D_{0+}^{\kappa-2}\big(u''(t)\big)\big|_{t=1}=0,\\ &\alpha u(0)-\beta u'(0)=\int_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1)+\delta u'(1)=\int_0^1 u(s)\,\mathrm{d}B(s), \end{aligned} \tag{1}$$

where $D_{0+}^{\eta-2}$, $D_{0+}^{\kappa-2}$ are the standard Riemann–Liouville fractional derivative of orders $\eta-2$ and $\kappa-2$, respectively. $n-1<\eta\leqslant n$, $\eta\geqslant 4$, $2\leqslant\kappa\leqslant n-2$, $\alpha,\beta,\gamma,\delta>0$,

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$\int_0^1 u(s)\,\mathrm{d}A(s)$ and $\int_0^1 u(s)\,\mathrm{d}B(s)$ denote the Riemann–Stieltjes integrals of u with respect to A and B, respectively. $A(t)$, $B(t)$ are nondecreasing on $[0,1]$, $f\colon [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous, $\lambda > 0$ is a parameter.

Fractional differential equations describe many phenomena in various fields of scientific and engineering disciplines such as physics, aerodynamics, viscoelasticity, electromagnetics, control theory, chemistry, biology, economics etc.; see, for example, [32, 34, 36, 55]. For the latest development direction of the fractional differential equations, see the references [1–4, 7, 9–11, 16, 19, 25, 28, 39, 40, 45, 46, 49, 53, 54, 56].

Boundary value problems (BVPs for short) with integral boundary conditions for ordinary differential equations represent a very interesting and important class of problems and arise in the study of various biological, physical and chemical processes [5, 6, 31, 44] such as heat conduction, thermo-elasticity, chemical engineering, underground water flow and plasma physics. The existence of solutions or positive solutions for such class of problems has attracted much attention; see, for example, [8, 12–15, 20–24, 26, 27, 29, 30, 35, 37, 41–43, 47, 48, 50–52] and the references therein.

Recently, Gunendi and Yaslan [17] considered the multi-point BVP for higher order fractional differential equation

$$
\begin{aligned}
&-D_{0+}^{\eta-2}\bigl(u''(t)\bigr) + f\bigl(t,u(t)\bigr) = 0, \quad t \in [0,1],\\
&u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u'''(1) = 0,\\
&\alpha u(0) - \beta u'(0) = \sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)\,\mathrm{d}s, \qquad \gamma u(1) + \delta u'(1) = \sum_{p=1}^{m-2} b_p \int_0^{\xi_p} u(s)\,\mathrm{d}s,
\end{aligned}
$$

where $D_{0+}^{\eta-2}$ denotes the Riemann–Liouville fractional derivative of order $\eta-2$, $n-1 < \eta \leqslant n$, $m,n \geqslant 3$, $\alpha,\beta,\gamma,\delta > 0$, $a_p, b_p \geqslant 0$ are given constants, $0 < \xi_1 < \cdots < \xi_{m-2} < 1$, $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. The existence results of at least one, two and three positive solutions are obtained by the four functionals fixed point theorem, the Avery–Henderson fixed point theorem and the Legget–Williams fixed point theorem, respectively.

In the present paper, we consider the more general fractional differential equation integral BVP (1). Under different conditions of the function f, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter λ. Our approach relies on the Guo–Krasnoselskii fixed point theorem on cones.

We express the fixed point operator with a Green's function, which is a convolution. The idea constructing Green's functions as convolutions of Green's functions for lower order BVPs is from the work of Eloe and Neugebauer [10]. The paper [10] contains some interesting ideas and develops the convolution method to several families of BVPs.

This paper is arranged as follows. In Section 2, we present some definitions and preliminary lemmas. In Section 3, we establish the existence and nonexistence of positive solutions for BVP (1) by using the fixed point theorem on cones. An example is also given to illustrate the main results in Section 4.

2 Preliminaries

We present the definitions of fractional calculus and some auxiliary results that are useful to the proof of our main results.

Definition 1. (See [32, 34, 36, 55].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $h\colon (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)\,\mathrm{d}s, \quad t > 0,$$

provided the right-hand side is pointwise defined on $(0, +\infty)$.

Definition 2. (See [32, 34, 36, 55].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h\colon (0, +\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} h(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{h(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where n is the smallest integer not less than α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Lemma 1. (*See* [32, 34, 36].)

(i) *If* $u \in L^1[0,1]$, $\rho > \sigma > 0$ *and* $n \in \mathbb{N}$, *then*

$$D_{0+}^{\sigma} I_{0+}^{\rho} u(t) = I_{0+}^{\rho-\sigma} u(t), \qquad D_{0+}^{\sigma} I_{0+}^{\sigma} u(t) = u(t),$$
$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\bigl(D_{0+}^{\sigma} u(t)\bigr) = D_{0+}^{n+\sigma} u(t).$$

(ii) *If* $\nu \geqslant 0$, $\sigma > 0$, *then*

$$D_{0+}^{\nu} t^{\sigma} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\nu+1)} t^{\sigma-\nu}.$$

(iii) *Let* $\alpha > 0$. *Then the following equality holds for* $u \in L^1[0,1]$ *and* $D_{0+}^{\alpha} u \in L^1[0,1]$:

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, 3, \ldots, n$, $n - 1 < \alpha \leqslant n$.

Let $-u''(t) = y(t)$, then the BVP

$$-D_{0+}^{\eta-2}\bigl(u''(t)\bigr) + \lambda f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,1),$$
$$u''(0) = u'''(0) = \cdots = u^{(n-2)}(0) = 0, \qquad D_{0+}^{\kappa-2}\bigl(u''(t)\bigr)\big|_{t=1} = 0$$

becomes

$$\begin{aligned}
&D_{0+}^{\eta-2}y(t)+\lambda f\big(t,u(t)\big)=0, \quad t\in(0,1),\\
&y(0)=y'(0)=\cdots=y^{(n-4)}(0)=0, \qquad D_{0+}^{\kappa-2}y(1)=0.
\end{aligned}$$

Using Lemma 1, by arguments similar to Lemma 2.4 in [17], we have the following result.

Lemma 2. *Let* $h\in C[0,1]$. *Then the BVP*

$$\begin{aligned}
&D_{0+}^{\eta-2}y(t)+h(t)=0, \quad t\in(0,1),\\
&y(0)=y'(0)=\cdots=y^{(n-4)}(0)=0, \qquad D_{0+}^{\kappa-2}y(1)=0
\end{aligned}$$

has a unique solution

$$y(t)=\int\limits_0^1 H(\kappa;t,s)h(s)\,\mathrm{d}s, \quad t\in[0,1],$$

where

$$H(\kappa;t,s)=\begin{cases}(1-s)^{\eta-\kappa-1}t^{\eta-3}/\Gamma(\eta-2), & 0\leqslant t\leqslant s\leqslant 1,\\ (1-s)^{\eta-\kappa-1}t^{\eta-3}-(t-s)^{\eta-3}/\Gamma(\eta-2), & 0\leqslant s\leqslant t\leqslant 1.\end{cases}$$

By direction computations we obtain the properties of $H(\kappa;t,s)$.

Lemma 3.

(i) $$0\leqslant H(\kappa;t,s)\leqslant\frac{(1-s)^{\eta-\kappa-1}t^{\eta-3}}{\Gamma(\eta-2)}\leqslant\frac{1}{\Gamma(\eta-2)}, \quad t,s\in[0,1].$$

(ii) *If* $2\leqslant\kappa_1<\kappa_2\leqslant n-2$, *then*

$$0<H(\kappa_1;t,s)<H(\kappa_2;t,s), \quad (t,s)\in(0,1)\times(0,1).$$

(iii) *If* $3\leqslant\kappa\leqslant n-2$, *then* $H_t(\kappa;t,s)\geqslant 0$ *for* $(t,s)\in[0,1]\times[0,1]$; *if* $2\leqslant\kappa<3$, *then* $H_t(\kappa;t,s)$ *changes sign on* $[0,1]\times[0,1]$.

Now we consider the following integral BVP:

$$\begin{aligned}
&-u''(t)=y(t), \quad t\in(0,1),\\
&\alpha u(0)-\beta u'(0)=\int\limits_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1)+\delta u'(1)=\int\limits_0^1 u(s)\,\mathrm{d}B(s).
\end{aligned}$$

Let

$$\phi(t)=\alpha t+\beta, \qquad \psi(t)=\gamma+\delta-\gamma t, \qquad w=\alpha\gamma+\alpha\delta+\beta\gamma,$$

$$G_0(t,s)=\begin{cases}\phi(t)\psi(s)/w, & 0\leqslant t\leqslant s\leqslant 1,\\ \phi(s)\psi(t)/w, & 0\leqslant s\leqslant t\leqslant 1,\end{cases}$$

then $G_0(t,s)$ is the Green's function of the following homogeneous differential equation BVP:

$$-u''(t) = 0, \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = 0, \qquad \gamma u(1) + \delta u'(1) = 0.$$

Define

$$a(t) = \frac{\psi(t)}{w}, \qquad b(t) = \frac{\phi(t)}{w},$$

then $a(t)$ and $b(t)$ are the solutions of

$$-a''(t) = 0, \quad t \in (0,1),$$
$$\alpha a(0) - \beta a'(0) = 1, \qquad \gamma a(1) + \delta a'(1) = 0$$

and

$$-b''(t) = 0, \quad t \in (0,1),$$
$$\alpha b(0) - \beta b'(0) = 0, \qquad \gamma b(1) + \delta b'(1) = 1,$$

respectively.

Denote

$$v_1 = 1 - \int_0^1 a(t)\,\mathrm{d}A(t), \qquad v_2 = 1 - \int_0^1 b(t)\,\mathrm{d}B(t),$$
$$v_3 = \int_0^1 a(t)\,\mathrm{d}B(t), \qquad v_4 = \int_0^1 b(t)\,\mathrm{d}A(t),$$

$$V(s) = \frac{v_2 \int_0^1 G_0(t,s)\,\mathrm{d}A(t) + v_4 \int_0^1 G_0(t,s)\,\mathrm{d}B(t)}{v_1 v_2 - v_3 v_4},$$
$$W(s) = \frac{v_1 \int_0^1 G_0(t,s)\,\mathrm{d}B(t) + v_3 \int_0^1 G_0(t,s)\,\mathrm{d}A(t)}{v_1 v_2 - v_3 v_4}.$$

We will use the following assumption:

(H) $v_1 > 0,\ v_1 v_2 - v_3 v_4 > 0.$

Lemma 4. (*See* [33].) *Assume that* (H) *holds. For any* $y \in C[0,1]$, u *is the solution of the BVP*

$$-u''(t) = y(t), \quad t \in (0,1),$$
$$\alpha u(0) - \beta u'(0) = \int_0^1 u(s)\,\mathrm{d}A(s), \qquad \gamma u(1) + \delta u'(1) = \int_0^1 u(s)\,\mathrm{d}B(s)$$

if and only if u *can be expressed by*

$$u(t) = \int_0^1 G(t,s)y(s)\,\mathrm{d}s, \quad t \in [0,1],$$

where

$$G(t,s) = G_0(t,s) + a(t)V(s) + b(t)W(s), \quad t,s \in [0,1].$$

Lemma 5. (*See* [33].)

$$0 < \gamma_0 G_0(s,s) \leqslant G_0(t,s) \leqslant G_0(s,s) \leqslant \frac{M^2}{w}, \quad t,s \in [0,1],$$

where

$$M = \max\{\alpha+\beta,\, \gamma+\delta\}, \quad \gamma_0 = \frac{1}{M}\min\{\beta,\delta\}.$$

Lemma 6. *Assume that* (H) *holds, then*

$$\gamma_0 \Phi(s) \leqslant G(t,s) \leqslant \Phi(s), \quad t,s \in [0,1],$$

where

$$\Phi(s) = G_0(s,s) + \frac{\gamma+\delta}{w}V(s) + \frac{\alpha+\beta}{w}W(s).$$

Proof. By using Lemma 5, for any $t, s \in [0,1]$, we obtain

$$G(t,s) \leqslant G_0(s,s) + \frac{\gamma+\delta}{w}V(s) + \frac{\alpha+\beta}{w}W(s) = \Phi(s).$$

On the other hand, by Lemma 5, we deduce

$$\begin{aligned} G(t,s) &\geqslant \gamma_0 G_0(s,s) + \frac{\delta}{w}V(s) + \frac{\beta}{w}W(s) \\ &\geqslant \gamma_0\left[G_0(s,s) + \frac{\gamma+\delta}{w}V(s) + \frac{\alpha+\beta}{w}W(s)\right] \\ &= \gamma_0\Phi(s), \quad t,s \in [0,1]. \end{aligned}$$ □

By using Lemmas 2 and 4 a solution of integral equation

$$u(t) = \lambda\int_0^1 G(t,s)\int_0^1 H(\kappa;s,\tau)f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0,1],$$

is a solution for BVP (1). As in [10], the integral equation can be rewritten in terms of a Green's function, which is a convolution of G and H. In fact,

$$u(t) = \lambda\int_0^1 \mathcal{G}(\kappa;t,s)f\big(s,u(s)\big)\,\mathrm{d}s,$$

where $\mathcal{G}(\kappa; t, s)$ is the Green's function for BVP (1); in particular,

$$\mathcal{G}(\kappa; t, s) = \int_0^1 G(t, \tau) H(\kappa; \tau, s) \, \mathrm{d}\tau, \quad (t, s) \in [0, 1] \times [0, 1].$$

Lemma 7. *Assume that* (H) *holds. Then the function* $\mathcal{G}(\kappa; t, s)$ *has the properties*:

(i) *If* $2 \leqslant \kappa_1 < \kappa_2 \leqslant n - 2$*, then*

$$0 < \mathcal{G}(\kappa_1; t, s) < \mathcal{G}(\kappa_2; t, s), \quad (t, s) \in (0, 1) \times (0, 1).$$

(ii) $$\gamma_0 \overline{\mathcal{G}}(s) \leqslant \mathcal{G}(\kappa; t, s) \leqslant \overline{\mathcal{G}}(s), \quad t, s \in [0, 1],$$

where

$$\overline{\mathcal{G}}(s) = \int_0^1 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau, \quad s \in [0, 1].$$

Proof. By Lemma 3 and expression of $\mathcal{G}(\kappa; t, s)$ it is easy to see that (i) holds. In the following, we will prove (ii). By using Lemma 6, for any $t, s \in [0, 1]$, we deduce

$$\mathcal{G}(\kappa; t, s) \leqslant \int_0^1 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau = \overline{\mathcal{G}}(s),$$

and

$$\mathcal{G}(\kappa; t, s) \geqslant \int_0^1 \gamma_0 \Phi(\tau) H(\kappa; \tau, s) \, \mathrm{d}\tau = \gamma_0 \overline{\mathcal{G}}(s). \qquad \square$$

Set $E = C[0, 1]$, then E is a Banach space with the norm $\|u\| = \sup_{t \in [0,1]} |u(t)|$. Let

$$P = \Big\{ u \in E\colon\ u(t) \geqslant 0,\ \min_{0 \leqslant t \leqslant 1} u(t) \geqslant \gamma_0 \|u\| \Big\}.$$

It is easy to see that P is a cone in E. We define the operator $T\colon E \to E$ as

$$Tu(t) = \lambda \int_0^1 \mathcal{G}(\kappa; t, s) f\big(s, u(s)\big) \, \mathrm{d}s, \quad t \in [0, 1].$$

It is clear that if $u \in P$ is a fixed point of T, then u is a positive solution of BVP (1). By using standard arguments we obtain the following lemma with respect to completely continuous operator.

Lemma 8. *Assume that* (H) *holds, then* $T\colon P \to P$ *is a completely continuous operator.*

The main tool in the paper is the following Guo–Krasnoselskii fixed point theorem on cones.

Lemma 9. (*See* [18].) *Let E be a Banach space and P be a cone in E. Assume Ω_1 and Ω_2 are bounded open subsets of E with $\theta \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let $A\colon P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator. If the following conditions are satisfied*:

(i) $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$, $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Ax\| \geqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_1$, $\|Ax\| \leqslant \|x\|$ *for all* $x \in P \cap \partial\Omega_2$,

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3 Main results

Denote

$$f_0^s = \limsup_{x\to 0^+} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_0^i = \liminf_{x\to 0^+} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$f_\infty^s = \limsup_{x\to \infty} \max_{t\in[0,1]} \frac{f(t,x)}{x}, \qquad f_\infty^i = \liminf_{x\to \infty} \min_{t\in[0,1]} \frac{f(t,x)}{x},$$
$$L = \int_0^1 \overline{\mathcal{G}}(s)\,\mathrm{d}s, \qquad K_1 = \frac{1}{\gamma_0^2 L f_\infty^i}, \qquad K_2 = \frac{1}{L f_0^s}, \qquad K_3 = \frac{1}{\gamma_0^2 L f_0^i}, \qquad K_4 = \frac{1}{L f_\infty^s}.$$

By expressions of $\Phi(\tau)$ and $H(\kappa;\tau,s)$ we obtain $0 < L < +\infty$.

Theorem 1. *Assume that* (H) *holds. If $f_0^s, f_\infty^i \in (0,\infty)$ and $K_1 < K_2$, then for any $\lambda \in (K_1, K_2)$, BVP* (1) *has at least one positive solution.*

Proof. For any $\lambda \in (K_1,K_2)$, there exists $0 < \varepsilon < f_\infty^i$ such that

$$\frac{1}{\gamma_0^2 L(f_\infty^i - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{L(f_0^s + \varepsilon)}.$$

By definition of f_0^s there exists $R_1 > 0$ such that

$$f(t,x) \leqslant \big(f_0^s + \varepsilon\big)x, \quad t \in [0,1],\ 0 \leqslant x \leqslant R_1.$$

We define $\Omega_1 = \{u \in E\colon \|u\| < R_1\}$. For any $u \in P \cap \partial\Omega_1$ and $t \in [0,1]$, we have

$$Tu(t) \leqslant \lambda \int_0^1 \overline{\mathcal{G}}(s)\big(f_0^s + \varepsilon\big)u(s)\,\mathrm{d}s \leqslant \lambda L\big(f_0^s + \varepsilon\big)\|u\| \leqslant \|u\|.$$

Therefore, we obtain

$$\|Tu\| \leqslant \|u\|, \quad u \in P \cap \partial\Omega_1. \tag{2}$$

On the other hand, by definition of f_∞^i, there exists $\overline{R}_2 > 0$ such that

$$f(t,x) \geqslant \big(f_\infty^i - \varepsilon\big)x, \quad t \in [0,1],\ x \geqslant \overline{R}_2.$$

We choose $R_2 = \max\{2R_1, \overline{R}_2/\gamma_0\}$ and define $\Omega_2 = \{u \in E\colon \|u\| < R_2\}$. Then for any $u \in P \cap \partial\Omega_2$ and $t \in [0,1]$, we obtain $u(t) \geqslant \gamma_0\|u\| \geqslant \overline{R}_2$ and

$$Tu(t) \geqslant \lambda \int_0^1 \gamma_0 \overline{\mathcal{G}}(s)\bigl(f^i_\infty - \varepsilon\bigr)u(s)\,\mathrm{d}s \geqslant \lambda\gamma_0^2 L\bigl(f^i_\infty - \varepsilon\bigr)\|u\| \geqslant \|u\|.$$

Then

$$\|Tu\| \geqslant \|u\|, \quad u \in P \cap \partial\Omega_2. \tag{3}$$

By (2), (3) and Lemma 9 we conclude that T has a fixed point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 2. *Assume that* (H) *holds. If f^i_0, $f^s_\infty \in (0,\infty)$ and $K_3 < K_4$, then for any $\lambda \in (K_3, K_4)$, BVP* (1) *has at least one positive solution.*

Proof. For any $\lambda \in (K_3, K_4)$, there exists $0 < \varepsilon < f^i_0$ such that

$$\frac{1}{\gamma_0^2 L(f^i_0 - \varepsilon)} \leqslant \lambda \leqslant \frac{1}{L(f^s_\infty + \varepsilon)}.$$

By definition of f^i_0 there exists $R_3 > 0$ such that

$$f(t,x) \geqslant \bigl(f^i_0 - \varepsilon\bigr)x, \quad t \in [0,1],\ 0 \leqslant x \leqslant R_3.$$

Let $\Omega_3 = \{u \in E\colon \|u\| < R_3\}$. For any $u \in P \cap \partial\Omega_3$, we obtain

$$Tu(t) \geqslant \lambda \int_0^1 \gamma_0 \overline{\mathcal{G}}(s)\bigl(f^i_0 - \varepsilon\bigr)u(s)\,\mathrm{d}s \geqslant \lambda\gamma_0^2 L\bigl(f^i_0 - \varepsilon\bigr)\|u\| \geqslant \|u\|, \quad t \in [0,1].$$

Therefore,

$$\|Tu\| \geqslant \|u\|, \quad u \in P \cap \partial\Omega_3. \tag{4}$$

We define $f^*\colon [0,1] \times [0,+\infty) \to [0,+\infty)$ as follows:

$$f^*(t,x) = \max_{u \in [0,x]} f(t,u), \quad t \in [0,1],\ x \geqslant 0,$$

then for any $t \in [0,1]$ and $u \in [0,x]$, we have $f(t,u) \leqslant f^*(t,x)$. Clearly, $f^*(t,x)$ is nondecreasing on x. By the proof of [38] we have

$$\limsup_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x} \leqslant f^s_\infty.$$

From the above inequality there exists $\overline{R}_4 > 0$ such that

$$\frac{f^*(t,x)}{x} \leqslant \limsup_{u\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x} + \varepsilon \leqslant f^s_\infty + \varepsilon, \quad x \geqslant \overline{R}_4,\ t \in [0,1],$$

then $f^*(t,x) \leqslant (f_\infty^s+\varepsilon)x$ for $x \geqslant \overline{R}_4$, $t \in [0,1]$. We define now $R_4 = \max\{2R_3, \overline{R}_4/\gamma_0\}$ and $\Omega_4 = \{u \in E\colon \|u\| < R_4\}$. Then for any $u \in P \cap \partial\Omega_4$ and $t \in [0,1]$, we have $u(t) \geqslant \gamma_0\|u\| \geqslant \overline{R}_4$, thus

$$Tu(t) \leqslant \lambda \int_0^1 \overline{\mathcal{G}}(s) f^*\bigl(s, \|u\|\bigr)\,\mathrm{d}s \leqslant \lambda L\bigl(f_\infty^s + \varepsilon\bigr)\|u\| \leqslant \|u\|.$$

Therefore, we obtain

$$\|Tu\| \leqslant \|u\|, \quad u \in P \cap \partial\Omega_4. \tag{5}$$

By (4), (5) and Lemma 9 we conclude that T has a fixed point $u \in P \cap (\overline{\Omega}_4 \setminus \Omega_3)$. □

Theorem 3. *Assume that* (H) *holds. If $f_0^s, f_\infty^s < \infty$, then there exists $\lambda^* > 0$ such that BVP* (1) *has no positive solution for $\lambda \in (0, \lambda^*)$.*

Proof. By definitions of f_0^s and f_∞^s, there exists $M_1 > 0$ such that

$$f(t,x) \leqslant M_1 x, \quad t \in [0,1],\ x \geqslant 0.$$

Set $\lambda^* = 1/(LM_1)$. Then for any $\lambda \in (0, \lambda^*)$, BVP (1) has no positive solution. Otherwise, we suppose that BVP (1) has a positive solution u, then

$$u(t) = Tu(t) \leqslant \lambda \int_0^1 \overline{\mathcal{G}}(s) M_1 u(s)\,\mathrm{d}s \leqslant \lambda L M_1\|u\|, \quad t \in [0,1],$$

and

$$\|u\| \leqslant \lambda L M_1 \|u\| < \lambda^* L M_1 \|u\| = \|u\|.$$

This contradiction shows that BVP (1) has no positive solution. □

Theorem 4. *Assume that* (H) *holds. If $f_0^i, f_\infty^i > 0$, then there exists $\tilde\lambda > 0$ such that BVP* (1) *has no positive solution for $\lambda > \tilde\lambda$.*

Proof. By definitions of f_0^i and f_∞^i there exists $m > 0$ such that

$$f(t,x) \geqslant mx, \quad t \in [0,1],\ x \geqslant 0.$$

Set $\tilde\lambda = 1/(\gamma_0^2 Lm)$. Then for any $\lambda > \tilde\lambda$, BVP (1) has no positive solution. Otherwise, we suppose that BVP (1) has a positive solution u, then

$$u(t) = Tu(t) \geqslant \lambda \int_0^1 \gamma_0 \overline{\mathcal{G}}(s) m u(s)\,\mathrm{d}s \geqslant \lambda\gamma_0^2 Lm\|u\|, \quad t \in [0,1],$$

thus

$$\|u\| \geqslant \lambda\gamma_0^2 Lm\|u\| > \tilde\lambda\gamma_0^2 Lm\|u\| = \|u\|.$$

This contradiction shows that BVP (1) has no positive solution. □

Remark 1. In this paper, compare with paper [17], we study the fractional differential equation with more general boundary conditions and a parameter. Motivated by the paper [10], we consider a family of BVPs with the family ranging over the higher order boundary condition at 1, which is different from [17]. We express the fixed point operator with a Green's function, which is a convolution. Some properties of the associated Green's function are obtained. Under different conditions of the function f, existence and nonexistence results for positive solutions are derived in terms of different intervals of parameter λ.

4 An example

Let $\alpha = \beta = \delta = 1$, $\gamma = 2$, $\eta = 9/2$, $\kappa = 3$, $A(s) = B(s) = s$. We consider the following fractional integral BVP:

$$\begin{aligned}
&-D_{0+}^{5/2}\big(u''(t)\big) + \lambda f\big(t, u(t)\big) = 0, \quad t \in (0,1),\\
&u''(0) = u'''(0) = 0, \qquad u'''(1) = 0,\\
&u(0) - u'(0) = \int_0^1 u(s)\,\mathrm{d}s, \qquad 2u(1) + u'(1) = \int_0^1 u(s)\,\mathrm{d}s.
\end{aligned} \tag{6}$$

Direct computation shows that

$$\upsilon_1 = \frac{3}{5}, \quad \upsilon_2 = \frac{7}{10}, \quad \upsilon_3 = \frac{2}{5}, \quad \upsilon_4 = \frac{3}{10}, \quad \gamma_0 = \frac{1}{3}, \quad L = \frac{36736}{2079\sqrt{3}\pi}.$$

So assumption (H) is satisfied.

1. We choose $f(t,x) = (\sin(\pi t/2) + 1)(2x+1)x/(5x+90)$, then $f_0^s = 1/45$, $f_\infty^i = 2/5$, $K_1 \approx 6.927$, $K_1 \approx 13.854$. By Theorem 1 we conclude that BVP (6) has at least one positive solution for any $\lambda \in (K_1, K_2)$.
2. We choose $f(t,x) = (t+2)(x+9)x/(18(2x+1))$, then $f_0^i = 1$, $f_\infty^s = 1/12$, $K_3 \approx 2.771$, $K_4 \approx 3.694$. By Theorem 2 we conclude that BVP (6) has at least one positive solution for any $\lambda \in (K_3, K_4)$.
3. We choose $f(t,x) = ((t+1)/2000)\ln(x+1)$, then $f_0^s = 1/1000$, $f_\infty^s = 0$. For any $t \in [0,1]$, $x \in [0,+\infty)$, we have $f(t,x) \leqslant x/1000$. Let $\lambda^* = 1/(LM_1) \approx 307.866$. By Theorem 3 we conclude that BVP (6) has no positive solution for any $\lambda \in (0, \lambda^*)$.
4. We choose $f(t,x) = \sqrt{t+1}(\mathrm{e}^x - 1)/2$, then $f_0^i = 1/2$, $f_\infty^i = +\infty$. For any $t \in [0,1]$, $x \in [0,+\infty)$, we have $f(t,x) \geqslant x/2$. Let $\tilde{\lambda} = 1/(\gamma_0^2 Lm) \approx 5.542$. By Theorem 4 we conclude that BVP (6) has no positive solution for any $\lambda > \tilde{\lambda}$.

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