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A New Classification of Hemirings through Double-Framed Soft *h*-Ideals

(Pengelasan Baru Hemirings melalui *h*-Ideals Lembut-Dual Kerangka)

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ABSTRACT

Due to lack of parameterization, various ordinary uncertainty theories like theory of fuzzy sets, and theory of probability cannot solve complicated problems of economics and engineering involving uncertainties. The aim of the present paper was to provide an appropriate mathematical tool for solving such type of complicated problems. For the said purpose, the notion of double-framed soft sets in hemirings is introduced. As h*-ideals of hemirings play a central role in the structural theory, therefore, we developed a new type of subsystem of hemirings. Double-framed soft left (right)* h*-ideal, double-framed soft* h*-bi-ideals and double-framed soft* h*-quasi-ideals of hemiring are determined. These concepts are elaborated through suitable examples. Furthermore, we are bridging ordinary* h*-ideals and double-framed soft* h*-ideals of hemirings through double-framed soft including sets and characteristic double-framed soft functions. It is also shown that every double-framed soft* h*-quasi-ideal is double-framed soft* h*-bi-ideal but the converse inclusion does not hold. A well-known class of hemrings i.e.* h*-hemiregular hemirings is characterized by the properties of these newly developed double-framed soft* h*-ideals of*

Keywords: DFS h*-bi-ideal; DFS* h*-hemiregularhemirin; DFS* h*-quasi-idealg; DFS sets;* h*-ideal*

ABSTRAK

Disebabkan oleh kekurangan pemparameteran, pelbagai teori ketidakpastian biasa seperti teori set kabur dan teori kebarangkalian tidak boleh menyelesaikan masalah ekonomi dan kejuruteraan yang rumit yang melibatkan ketidakpastian. Tujuan penulisan kertas ini adalah untuk menyediakan satu alat matematik yang sesuai untuk menyelesaikan masalah rumit yang sedemikian. Untuk tujuan tersebut, satu tanggapan set lembut dual kerangka dalam hemirings diperkenalkan. Oleh kerana h*-ideals hemiring memainkan peranan utama dalam teori struktur, maka kami telah membangunkan satu jenis subsistem hemiring baru.* h*-ideals lembut kiri (kanan) dual kerangka,* h*-dwi-ideal lembut dual kerangka dan* h*-separa-ideal lembut dual kerangka hemirings ditentukan. Konsep ini dihuraikan melalui contoh yang sesuai. Selain itu, kami menghubungkan* h*-ideals biasa dan* h*-ideals lembut dual kerangka hemirings melalui set lembut dual kerangka dan pencirian fungsi lembut dual kerangka. Kajian ini juga menunjukkan bahawa setiap* h*-quasi-ideal lembut dual bingkai adalah* h*-dwi-ideal lembut dual kerangka tetapi rangkuman akas tidak dapat bertahan. Satu kelas hemirings terkenal iaitu* h*-hemisekata hemirings dicirikan oleh sifat* h*-ideals dua bingkai lembut daripada yang baru dibangunkan ini.*

Kata kunci: Set DFS; DFS h*-dwi-ideal; DFS* h*-hemisekata hemiring;* h*-ideal; DFS* h*-separa-ideal*

INTRODUCTION

In modern era, economic and technological advancement plays a remarkable role in the development of any particular country. Due to the high-quality research in advanced fields like control engineering, data analysis, computer science, error correcting codes, economics, decision making, forecasting and robotics, most of the countries are left behind. These advanced countries are spending a huge part of budget on these domains. On the other hand, the aforementioned fields are facing some complicated problems involving uncertainties. These complicated problems cannot be handled through classical methods. There are certain types of theories such as theory of probability, theory of fuzzy sets and theory of rough sets which can be used in aforementioned problems. However, all of these theories have their significance as well as inherent limitations. One major problem faced by these theories is their incompatibility with the parameterization tools. To overcome such type of difficulties, in 1999, Molodtsov initiated the ice breaking concept of soft set theory. The notion of soft sets is a new mathematical approach for dealing with uncertainties. This new approach is free from the difficulties pointed out in the other theories of uncertainties which usually use membership function. Soft sets gain reputation from the last decade due to its parameterization nature and which is free of membership function. Due to its dynamical nature, soft sets are nowadays extensively used in various applied fields. More precisely, soft sets are used in decision making problems (Çagman & Enginoglu 2010a, 2010b; Feng 2011; Feng et al. 2010; Maji et al. 2002; Roya & Maji 2007), soft derivatives, soft integrals and soft numbers along with their applications are thoroughly discussed (Molodtsov et al. 2006), in international trade, soft sets are used for forecasting the export and import volumes (Sezer 2014). Simultaneously, this theory is very much useful due to its applications in information sciences with intelligent systems, approximate reasoning, expert and decision support systems and decision making (Acar et al. 2010; Atagun & Sezgin 2011; Cagman & Enginoglu 2011; Feng et al. 2008, 2011; Jun et al. 2010, 2009a, 2009b, 2008a, 2008b; Majumdar & Samanta 2008; Sezgin & Atagun 2011; Xiao et al. 2010; Yin Li; 2008; Zhan et al. 2010; Zou & Xiao 2008).

It is also important to note that, soft set are used in algebraic framework which successfully leads to the applications of algebraic structures in aforementioned advanced applied fields. Keeping this motivation in view, Maji et al. (2003) presented several operations of algebraic structures in terms of soft sets which is further extended (Ali et al. 2011, 2009).

Presently, among other algebraic structures, semirings (Vandiver 1934), are also used in diverse fields like computer programming, coding theory, fuzzy automata, optimization, formal languages, graph theory and much more (Aho & Ullman 1976; Benson 1989; Conway 1971; Golan 1998; Hebisch & Weinert 1998; Henriksen 1958; Iizuka 1959; Kuich & Salomma 1986; Torre 1965). Among these, several fields such as theory of automata, formal languages and computer sciences used special type of semirings known as hemirings (Benson 1989; Golan 1999; Hebisch & Weinert 1998). Hemirings are those semirings which are commutative with addition and having zero element. Further, ideals of hemirings play a key role in structure theory for many purposes. In 1965, Torre determined *h*-ideals and *h*-ideals in hemirings with several classification of hemirings are discussed in terms of these ideals. The *h*-hemiregularity are investigated (Yin & Li 2008). They also determined *h*-intra hemiregular hemirings and presented various characterization theorems of hemirings in terms of these notions. In 2013, Droste and Kuich discuss hemrings in automata domain. Moreover, Ma and Zhan (2014) characterized hemiregular hemirings by the properties of new type of soft union sets. For other applications of soft union sets in hemirings, the readers refer to (Ma et al. 2016; Zhan & Maji 2014). The concept of cubic *h*-ideals along with several characterization theorems in hemirings is presented (Khan et al. 2015).

Recently, the notion of union and intersectional soft sets is further extended (Jun et al. 2012) to double-framed soft sets and defined double-framed soft subalgebra of a BCK/BCI-algebra. Beside this, Jun et al. (2013) also determined double-framed soft ideals of BCK/BCI-algebra. In 2017a, Khan et al. applied the notion of double-framed soft sets to AG-groupoids and investigated various results. Moreover, double-framed soft sets are further elaborated in LA-semigroups (Khan et al. 2017b).

The aim of the present paper was to apply the idea of double-framed soft sets to hemirings and to investigate double-framed soft *h*-ideals of a hamiring We define double-framed soft left (right) *h*-ideals, double-framed soft *h*-bi-ideals and double-framed soft *h*-quasi-ideals of hemiring *R*. Further, these notions are elaborated through suitable examples. DFS Soft *h*-sum and *h*-product are developed and several results are determined by these notions. On the other hand, we are also bridging ordinary *h*-ideals and double-framed soft *h*-ideals of hemirings through double-framed soft including sets and characteristic double-framed soft functions which is the key milestone of the present paper. It is also shown that every double-framed soft *h*-quasi-ideal is double-framed soft *h*-bi-ideal but for the converse inclusion, the counter example is provided that it does not hold in general. Lastly, *h*-hemiregular hemirings are characterized by the properties of these newly developed double-framed soft *h*-ideals of *R*.

PRELIMINARIES

This section presents the fundamental concepts of hemirings which will be used throughout this paper.

An algebraic system $(R, +, .)$ consists of a non-empty set R with two binary operations addition and multiplication is known as a semiring, if $(R, +)$ and $(R, .)$ are semigroups with the following distributive laws are satisfied $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a,b,c \in R$.

An element $0 \in R$ is called zero of a semiring $(R, +, \cdot)$, if $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all A unit of a semiring is an element $1 \in R$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in R$. A semiring R with zero element and in which $(R, +)$ is a commutative semigroup is known as hemiring. Throughout the paper, ab will be used instead of $a \cdot b$ such that $a, b \in R$ for the sake of simplicity.

Since the objectives of the present research was to discussed several classifications of hemirings by the properties of various types of ideals, therefore, the basic types of ideals in hemirings are necessary to be coated over here. A subhemiring of R is a subset A of R which is both closed under addition and multiplication. A subset A of R is called a left (right) ideal of R if A is closed under addition and $RA \subseteq A$ (resp. $AR \subseteq A$). A subset A of R is called an ideal of R if it is both left and right ideal of R A subset B of R is called a bi-ideal of R if B is closed under addition and multiplication such that $BRB \subseteq B$. A subset Q of R is called a quasi-ideal of R if Q is closed under addition and $RQ \cap QR \subseteq Q$. A subhemiring (left ideal, right ideal, ideal, bi-ideal) A of R is called an h-subhemiring (left h-ideal, right h-ideal, h-ideal, h-bi-ideal), respectively, if for any $x, z \in R$, $a, b \in A$, $x + a + z = b + z \rightarrow x \in A$.

The h-closure $\overline{A}$ of a subset of is defined as

$$\overline{A} = \{x \in R \mid x + a + z = b + z \text{ for some } a, b \in A, z \in R\}.$$

A quasi-ideal Q in a hemiring R is called a h-quasi-ideal of R if $\overline{RQ} \cap \overline{QR} \subseteq Q$ and for any $x, z \in R$, $a, b \in Q$, $x + a + z = b + z \rightarrow x \in Q$.

Note that, for subsets A, B and C of a hemiring R, $A \subseteq \overline{A}$, $\overline{AB} = \overline{\overline{A}\,\overline{B}}$, $\overline{AB} \subseteq A \cap B$ and $\overline{\overline{A}} = \overline{A}$. A subset I in a hemiring R is called an h-idempotent if $I = \overline{I^2}$. Clearly, every left h-ideal of a hemiring is a left ideal of R (the similar case is hold for right, bi- and quasi ideal as well). Every left (resp. right) h-ideal of is an h-quasi-ideal and every h-quasi-ideal is an h-bi-ideal of R but the converses of the aforementioned statements are not true in general (Yin & Li 2008).

SOFT SETS IN HEMIRINGS (BASIC OPERATIONS)

In the last two decades, the uses of soft set theory are achieving another milestone in contemporary mathematics where several mathematical problems involving uncertainties in various field like decision making, automata theory, coding theory, economics and much others which cannot be handle through ordinary mathematical tools (like fuzzy set theory and theory of probability) due to the lack of parameterization. The latest research in this direction and the new investigations of soft set theory is much productive due to the diverse applications of soft sets in the aforementioned fields. It is important to note that Sezgin and Atagun (2011) introduced some new operations on soft set theory and defined soft sets in the following way:

Suppose U be universal set, E be the set of parameters, $P(U)$ be the power set of U and A be a subset of E. Then a soft set f_A over U is an approximate function defined by:

$$f_A : E \rightarrow P(U) \text{ such that } f_A(x) = \emptyset \text{ if } x \notin A.$$

Symbolically a soft set over U is the set of ordered pairs

$$f_A = \{(x, f_A(x)) \colon x \in E, f_A(x) \in P(U)\}.$$

A soft set is a parameterized family of subsets of U, where $S(U)$ denotes the set of all soft sets.

Example 1 Suppose Mr. Lee want to buy various business corners in newly developed supermarket having hundred business corners $\{c_1, c_2, \ldots, c_{100}\} = U$. For the said purpose, Mr. Lee has three different parameters in mind that are "beautiful (e_1)", "cheap (e_2)" and "good location (e_3)". These parameters are represented by the set $E = \{e_1, e_2, e_3\}$. Now for few corners he only consider $\{e_1, e_3\} = A$. Therefore, an approximate function $f_A : E \rightarrow P(U)$ will image $f_A(e_2) = \emptyset$ as $e_2 \notin A$ and ultimately he will have only those choices from P(U) which depend on e_1, e_3. Similarly, for any other subset of parameters, Mr. Lee can select a better corner for his business.

Definition 2 Suppose $f_A, f_B \in S(U)$. Then f_A is said to be subset of f_B denoted by $f_A \tilde{\subseteq} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in E$. Also, two soft sets f_A, f_B are said to be equal denoted by $f_A \cong f_B$ if $f_A \tilde{\subseteq} f_B$ and $f_A \tilde{\supseteq} f_B$ holds.

Definition 3 Let $f_A, f_B \in S(U)$, then the union of f_A and f_B, denoted by $f_A \tilde{\cup} f_B$ is defined by $f_A \tilde{\cup} f_B = f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, where for all $x \in E$.

Definition 4. If $f_A, f_B \in S(U)$, then the intersection of f_A and f_B, denoted by $f_A \tilde{\cap} f_B$ is defined by $f_A \tilde{\cap} f_B = f_{A\cap B}$, where $f_{A\cap B}(x) = f_A(x) \cap f_B(x)$ for all $x \in E$.

Throughout this paper, R will denote a hemiring unless otherwise stated.

Definition 5 A double-framed soft set of A over U is a pair $\left\langle \left(f_A^+, f_A^-\right); A\right\rangle$, where f_A^+ and f_A^- both are mappings from A to P(U). It is denoted by DFS-set of A.

The set of all DFS-set of A over U is denoted by DFS(U).

γ-inclusive set: If $\left\langle \left(f_A^+, f_A^-\right); A\right\rangle$ be a DFS-set of A and γ be a subset of U, then the γ-inclusive set is denoted by $i_A(f_A^+; \gamma)$ and defined as

$$i_A(f_A^+; \gamma) = \{x \in A \mid f_A^+(x) \supseteq \gamma\}$$

δ-exclusive set: If $\left\langle \left(f_A^+, f_A^-\right); A\right\rangle$ be a DFS-set of A and δ be a subset of U, then the δ-exclusive set is denoted by $e_A(f_A^-; \delta)$ and defined as

$$e_A(f_A^-; \delta) = \{x \in A \mid f_A^-(x) \subseteq \delta\}.$$

A double-framed soft including set is of the form

$$DF_A(f_A^+, f_A^-)_{(\gamma,\delta)} = \{x \in A \mid f_A^+(x) \supseteq \gamma, f_A^-(x) \subseteq \delta\}$$

clearly, $DF_A(f_A^+, f_A^-)_{(\gamma,\delta)} = i_A(f_A^+; \gamma) \cap e_A(f_A^-; \delta)$.

In the following, the double-framed soft sum briefly h-sum and int-uni soft product (h-product) for two double-framed soft sets of hemirings are introduced.

Definition 6 Let $f_A = \left\langle \left(f_A^+, f_A^-\right); A\right\rangle$ and $g_A = \left\langle \left(g_A^+, g_A^-\right); A\right\rangle$ be two double-framed soft sets of a hemiring R over U. Then the h-sum is denoted by $f_A \tilde{\oplus} g_A^+ = \left\langle \left(f_A^+ \tilde{\oplus} g_A^+, f_A^- \boxplus g_A^-\right); A\right\rangle$ is defined to be a double-framed soft set of R over U, in which $f_A^+ \tilde{\oplus} g_A^+$ and $f_A^- \boxplus g_A^-$ are soft mappings from R to P(U) given as:

$$f_A^+ \tilde{\oplus} g_A^+ : x \mapsto \begin{cases} \displaystyle\bigcup_{x+a_1+b_1+z=a_2+b_2+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap g_A^+(b_1) \cap g_A^+(b_2) \\ \quad \text{if x can be expressed as } x + a_1 + b_1 + z = a_2 + b_2 + z \\ \emptyset \quad \text{if x can be expressed as } x + a_1 + b_1 + z = a_2 + b_2 + z. \end{cases}$$

$$f_A^- \boxplus g_A^- : x \mapsto \begin{cases} \displaystyle\bigcup_{x+a_1+b_1+z=a_2+b_2+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup g_A^-(b_1) \cup g_A^-(b_2) \\ \quad \text{if x can be expressed as } x + a_1 + b_1 + z = a_2 + b_2 + z \\ U \quad \text{if x can be expressed as } x + a_1 + b_1 + z = a_2 + b_2 + z. \end{cases}$$

Definition 7 Let $f_A = \left\langle \left(f_A^+, f_A^-\right); A \right\rangle$ and $g_A = \left\langle \left(g_A^+, g_A^-\right); A \right\rangle$ be two double-framed soft sets of a hemiring R over U. Then the *h*-product is denoted by $f_A \,\tilde{\Diamond}\, g_A^+ = \left\langle \left(f_A \otimes g_A^+, f_A^- \boxtimes g_A^-\right); A \right\rangle$ is defined to be a double-framed soft set of R over U, in which $f_A^+ \otimes g_A^+$ and $f_A^- \boxtimes g_A^-$ are soft mappings from R to P(U(g) given as

$$f_A^+ \otimes g_A^+ : x \mapsto \begin{cases} \bigcup\limits_{x+a_1b_1+z=a_2b_2+z} \left\{ f_A^+(a_1) \cap f_A^+(a_2) \cap g_A^+(b_1) \cap g_A^+(b_2) \right\} & \text{if x cannot be expressed as } x + a_1b_1 + z = a_2b_2 + z \\ \varnothing & \text{if x cannot be expressed as } x + a_1b_1 + z = a_2b_2 + z. \end{cases}$$

$$f_A^- \boxtimes g_A^- : x \mapsto \begin{cases} \bigcap\limits_{x+a_1b_1+z=a_2b_2+z} \left\{ f_A^-(a_1) \cup f_A^-(a_2) \cup g_A^-(b_1) \cup g_A^-(b_2) \right\} & \text{if x can be expressed as } x + a_1b_1 + z = a_2b_2 + z \\ U & \text{if x cannot be expressed as } x + a_1b_1 + z = a_2b_2 + z. \end{cases}$$

Definition 8 Let $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$ and $g_B = \left\langle (g_B^+, g_B^-); B \right\rangle$ be two double-framed soft sets over U. Then $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$ is called a double-framed soft subset of $g_B = \left\langle (g_B^+, g_B^-); B \right\rangle$ denoted by $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle \widetilde{\subseteq} g_A = \left\langle (g_A^+, g_A^-); A \right\rangle$ if A is the subset of B, $f_A^+(x) \subseteq g_B^+(x)$ and $f_A^-(x) \supseteq g_B^-(x)$ for all $x \in A$. Also two double-framed soft sets $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$ and $g_B = \left\langle (g_B^+, g_B^-); B \right\rangle$ are equal denoted by $\left\langle (f_A^+, f_A^-); A \right\rangle = \left\langle (g_B^+, g_B^-); B \right\rangle$, if $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle \widetilde{\subseteq} g_A = \left\langle (g_A^+, g_A^-); A \right\rangle$ and $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle \widetilde{\supseteq} g_A = \left\langle (g_A^+, g_A^-); A \right\rangle$ both hold.

Definition 9 Let $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$ and $g_A = \left\langle (g_A^+, g_A^-); A \right\rangle$ be two double-framed soft sets of a hemiring R over U. Then the DFS int-uni set of $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$ and $g_A = \left\langle (g_A^+, g_A^-); A \right\rangle$ is to be defined as a DFS set $\left\langle \left(f_A^+ \widetilde{\cap}\, g_A^+, f_A^- \widetilde{\cup}\, g_A^-\right); A \right\rangle$ where $f_A^+ \widetilde{\cap}\, g_A^+$ and $f_A^- \widetilde{\cup}\, g_A^-$ are mappings from A to P(U) such that $(f_A^+ \widetilde{\cap}\, g_A^+)(x) = f_A^+(x) \cap g_A^+(x)$ and $(f_A^- \widetilde{\cup}\, g_A^-)(x) = f_A^+(x) \cup g_A^-(x)$. It is denoted by $\left\langle (f_A^+, f_A^-); A \right\rangle \,\widetilde{\cap}\, \left\langle (g_A^+, g_A^-); A \right\rangle = \left\langle \left(f_A^+ \widetilde{\cap}\, g_A^+, f_A^- \widetilde{\cup}\, g_A^-\right); A \right\rangle$.

Lemma 10 Suppose $f_A = \left\langle (f_A^+, f_A^-); A \right\rangle$, $g_B = \left\langle (g_B^+, g_B^-); B \right\rangle$ and $h_c = \left\langle \left(h_C^+, h_C^-\right); C \right\rangle$ be double-framed soft sets in a hemiring R, then the following hold.

(1). $f_A \,\widetilde{\oplus}\, (g_B \,\widetilde{\cap}\, h_C) = (f_A \,\widetilde{\oplus}\, g_B) \,\widetilde{\cap}\, (f_A \,\widetilde{\oplus}\, h_C)$

(2). $f_A \,\tilde{\Diamond}\, (g_B \,\widetilde{\cap}\, h_C) = (f_A \,\tilde{\Diamond}\, g_B) \,\widetilde{\cap}\, (f_A \,\tilde{\Diamond}\, h_C)$

Proof Let x be an arbitrary element of a hemiring R which cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$. Then, $f_A^+ \oplus (g_B^+ \widetilde{\cap}\, h_C^+)(x) = \varnothing = (f_A^+ \oplus g_B^+) \,\widetilde{\cap}\, (f_A^+ \oplus h_C^+)(x)$ and $(f_A^- \boxplus (g_B^- \widetilde{\cap}\, h_C^-)(x) = U = (f_A^- \boxplus g_B^-) \,\widetilde{\cap}\, (f_A^- \boxplus h_C^-)(x)$. Therefore, $f_A \,\widetilde{\oplus}\, (g_B \,\widetilde{\cap}\, h_C) = (f_A \,\widetilde{\oplus}\, g_B) \,\widetilde{\cap}\, (f_A \,\widetilde{\oplus}\, h_C)$. Now let us suppose that x can be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$, then

$$\begin{aligned}
&f_A^+ \oplus (g_B^+ \widetilde{\cap}\, h_C^+)(x) \\
&= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \left\{ f_A^+(a_1) \cap f_A^+(a_2) \cap (g_B^+ \widetilde{\cap}\, h_C^+)(b_1) \cap (g_B^+ \widetilde{\cap}\, h_C^+)(b_2) \right\} \\
&= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \left\{ f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap h_C^+(b_1) \cap g_B^+(b_2) \cap h_C^+(b_2) \right\} \\
&= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \left\{ \left[f_A^+(a_1) \cap f_A^+(a_2) \right] \cap \left[g_B^+(b_1) \cap g_B^+(b_2) \cap h_C^+(b_1) \cap h_C^+(b_2) \right] \right\} \\
&= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \left\{ \left(f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap g_B^+(b_2) \right) \cap \left(f_A^+(a_1) \cap f_A^+(a_2) \cap h_C^+(b_1) \cap h_C^+(b_2) \right) \right\} \\
&= \left\{ \bigcup_{x+a_1+b_1+z=a_2+b_2+z} f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap g_B^+(b_2) \right\} \cap \left\{ \bigcup_{x+a_1+b_1+z+a_2+b_2+z} f_A^+(a_1) \cap f_A^+(a_2) \cap h_C^+(b_1) \cap h_C^+(b_2) \right\} \\
&= (f_A^+ \oplus g_B^+)(x) \cap (f_A^+ \oplus h_C^+)(x) \\
&= (f_A^+ \oplus g_B^+) \widetilde{\cap} (f_A^+ \oplus h_C^+))(x).
\end{aligned}$$

Also,

$$\begin{aligned}
&f_A^- \boxplus (g_B^- \widetilde{\cap}\, h_C^-)(x) \\
&= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \left\{ f_A^-(a_1) \cap f_A^-(a_2) \cup (g_B^- \widetilde{\cap}\, h_C^-)(b_1) \cup (g_B^- \widetilde{\cap}\, h_C^-)(b_2) \right\} \\
&= \bigcup_{x+a_1\ b_1+z=a_2+b_2+z} \left\{ f_A^-(a_1) \cup f_A^-(a_2) \cup g_B^-(b_1) \cap h_C^-(b_1) \cup g_B^-(b_2) \cap h_C^-(b_2) \right\} \\
&= \bigcup_{x+a_1\ b_1+z=a_2+b_2+z} \left\{ \left[f_A^-(a_1) \cup f_A^-(a_2) \cup g_B^-(b_1) \cup g_B^-(b_2) \right] \cap \left[f_A^-(a_1) \cup f_A^-(a_2) \cup h_C^-(b_1) \cup h_C^-(b_2) \right] \right\} \\
&= \left\{ \bigcup_{x+a_1\ b_1+z=a_2+b_2+z} f_A^-(a_1) \cup f_A^-(a_2) \cup g_B^-(b_1) \cup g_B^-(b_2) \right\} \cap \left\{ \bigcup_{x+a_1+b_1+z=a_2+b_2+z} f_A^-(a_1) \cup f_A^-(a_2) \cup h_C^-(b_1) \cup h_C^-(b_2) \right\} \\
&= (f_A^- \boxplus g_B^-)(x) \cap (f_A^- \boxplus h_C^-)(x) \\
&= (f_A^- \boxplus g_B^-) \,\widetilde{\cap}\, (f_A^- \boxplus h_C^-))(x).
\end{aligned}$$

Thus, $f_A \,\widetilde{\oplus}\, (g_B \,\widetilde{\cap}\, h_C) = (f_A \,\widetilde{\oplus}\, g_B) \,\widetilde{\cap}\, (f_A \,\widetilde{\oplus}\, h_C)$ and so (1) hold.

For (2) Let $x \in R$ which cannot be expressed as $x + a_1b_1 + z = a_2b_2 + z$. Then

$f_A^+ \otimes (g_B^+ \widetilde{\cap}\, h_C^+(x) = \varnothing = (f_A^+ \otimes g_B^+) \,\widetilde{\cap}\, (f_A^+ \otimes h_C^+)(x)$ and $(f_A^- \boxtimes (g_B^- \widetilde{\cap}\, h_C^+)(x) = U = (f_A^- \boxtimes g_B^-) \,\widetilde{\cap}\, (f_A^- \boxtimes h_C^-)(x)$. Therefore, $f_A \,\tilde{\Diamond}\, (g_B \,\widetilde{\cap}\, h_C) = (f_A \,\tilde{\Diamond}\, h_C)$. Now let x can be expressed as $x + a_1b_1 + z = a_2b_2 + z$, then,

$$\begin{aligned}
&f_A^+ \otimes (g_B^+ \widetilde{\cap}\, h_C^+)(x) \\
&= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{ f_A^+(a_1) \cap f_A^+(a_2) \cap (g_B^+ \widetilde{\cap}\, h_C^+)(b_1) \cap (g_B^+ \widetilde{\cap}\, h_C^+)(b_2) \} \\
&= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{ f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap h_C^+(b_1) \cap g_B^+(b_2) \cap h_C^+(b_2) \} \\
&= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{ [f_A^+(a_1) \cap f_A^+(a_2)] \cap [g_B^+(b_1) \cap g_B^+(b_2) \cap h_C^+(b_1) \cap h_C^+(b_2)] \}
\end{aligned}$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{(f_A^+(a_1)\cap f_A^+(a_2)\cap g_B^+(b_1)\cap g_B^+(b_2))\cap(f_A^+(a_1)\cap \cap\{f_A^+(a_2)\cap h_C^+(b_1)\cap h_C^+(b_2))\}$$

$$= \begin{array}{l} \{\bigcup_{x+a_1b_1+z=a_2b_2+z} f_A^+(a_1)\cap f_A^+(a_2)\cap g_B^+(b_1)\cap g_B^+(b_2)\} \\ \cap\{\bigcup_{x+a_1+b_1+z=a_2+b_2+z} f_A^+(a_1)\cap f_A^+(a_2)\cap h_C^+(b_1)\cap h_C^+(b_2)\} \end{array}$$

$$=\left(f_A^+\otimes g_B^+\right)(x)\cap(f_A^+\otimes h_C^+)(x)$$

$$=\left(f_A^+\otimes g_B^+\right)\widetilde{\cap}\,(f_A^+\otimes h_C^+)(x).$$

Also,

$$f_A^-\boxtimes\ (g_B^-\widetilde{\cap}\, h_C^-)(x)$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{f_A^-(a_1)\cap(a_2)\cup(g_B^-\widetilde{\cap}\, h_C^-)(b_1)\cup(g_B^-\widetilde{\cap}\, h_C^-)(b_2)\}$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{f_A^-(a_1)\cup f_A^-(a_2)\cup g_B^-(b_1)\cap h_C^-(b_1)\cup g_B^-(b_2)\cap h_C^-(b_2)\}$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{[f_A^-(a_1)\cup f_A^-(a_2)\cup g_B^-(b_1)\cup g_B^-(b_2)] \cap[f_A^-(a_1)\cup f_A^-(a_2)\cup h_C^-(b_1)\cup h_C^-(b_2)]\}$$

$$= \begin{array}{l} \{\bigcup_{x+a_1b_1+z=a_2b_2+z} f_A^-(a_1)\cup f_A^-(a_2)\cup g_B^-(b_1)\cup g_B^-(b_2)\} \\ \cap\{\bigcup_{x+a_1+b_1+z=a_2+b_2+z} f_A^-(a_1)\cup f_A^-(a_2)\cup h_C^-(b_1)\cup h_C^-(b_2)\} \end{array}$$

$$= (f_A^-\boxtimes\ g_B^-)(x)\cap(f_A^-\boxtimes h_C^-)(x)$$

$$= (f_A^-\boxtimes g_B^-)\,\widetilde{\cap}\,(f_A^-\boxtimes h_C^-))(x).$$

Thus, $f_A \mathbin{\tilde{\Diamond}} (g_B \mathbin{\widetilde{\cap}}\ h_C) = (f_A \mathbin{\tilde{\Diamond}} g_B) \mathbin{\widetilde{\cap}} (f_A \mathbin{\tilde{\Diamond}} h_C)$.

Definition 11 Suppose be a non-empty subset of a hemiring R, then the characteristic double-framed soft mapping of A is a double-framed soft set denoted by $C_A=\left\langle\left(C_A^+,C_A^-\right);A\right\rangle$ where C_A^+,C_A^- are soft mappings from R to P(U) and defined as

$$C_A^+ : x\mapsto\begin{cases} U & \text{if } x\in A,\\ \varnothing & \text{if } x\notin A.\end{cases}$$

and

$$C_A^- : x\mapsto\begin{cases} \varnothing & \text{if } x\in A,\\ U & \text{if } x\notin A.\end{cases}$$

It is important to note that the identity double-framed soft mapping is denoted by $C_R=\left\langle\left(C_R^+,C_R^-\right);R\right\rangle$ where C_R^+ : $x\mapsto U$ and $C_R^-: x\mapsto\varnothing$ for all $x\in R$.

Theorem 12 Suppose A and B be two non-empty subsets of a hemiring R, then the following axioms for characteristic double-framed soft mapping are holds:

(1). $A\subseteq B$ if and only if $C_A\,\widetilde{\subseteq}\,C_B$, i.e., $A\subseteq B \Leftrightarrow C_A^+(x)\subseteq C_B^+(x)$ and $C_A^-(x)\supseteq C_B^-(x)$ for all $x\in A$.

(2). $C_A\,\widetilde{\cap}\,C_B=C_{A\cap B}$, i.e., $\left\langle C_A^+\,\widetilde{\cap}\,C_B^+,C_A^-\,\widetilde{\cup}\,C_B^-\right\rangle=\left\langle C_{A\cap B}^+,C_{A\cap B}^-\right\rangle$

(3). $C_A\,\widetilde{\oplus}\,C_B=C_{\overline{A+B}}$, i.e., $\left\langle C_A^+\oplus C_B^+,C_A^-\boxplus C_B^-\right\rangle=\left\langle C_{\overline{A+B}}^+,C_{\overline{A+B}}^-\right\rangle$

(4). $C_A\ \tilde{\Diamond}\ C_B=C_{\overline{AB}}$, i.e., $\left\langle C_A^+\oplus C_B^+,C_A^-\boxtimes C_B^-\right\rangle=\left\langle C_{\overline{AB}}^+,C_{\overline{AB}}^-\right\rangle$.

Proof The proof of (1) and (2) directly follows from Definition 11.

For the proof of (3) suppose and are two subsets of a hemiring R and $x\in R$. If $x\notin\overline{A+B}$, then x can not be expressed as $x+a_1+b_1+z=a_2+b_2+z$ where $a_1,a_2\in A$, $b_1,b_2,\in B$ and $z\in R$. Hence, $(C_A^+\oplus C_B^+)(x)=\varnothing=C_{\overline{A+B}}^+$ and $(C_A^-\boxplus C_B^-)(x)=U=C_{\overline{A+B}}^-$.Therefore, $C_A\,\widetilde{\oplus}\,C_B\ =C_{\overline{A+B}}$. Now, let $x\in\overline{A+B}$, then x can be expressed as $x+a_1+b_1+z=a_2+\ b_2+z$ such that $a_1,a_2\in A$, $b_1,b_2\in B$ and $z\in R$. Thus

$$(C_A^+\oplus C_B^+)(x)$$

$$= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \{C_A^+(a_1)\cap C_A^+(a_2)\cap C_B^+(b_1)\cap C_B^+(b_2)\}$$

$$= \bigcup_{x+a_1+b_1+z=a_2+b_2+z} \{U\cap U\cap U\cap U\}$$

$$= U$$

$$= C_{\overline{A+B}}^+(x),$$

also,

$$(C_A^-\boxplus C_B^-)(x)$$

$$= \bigcap_{x+a_1+b_1+z=a_2+b_2+z} \{C_A^-(a_1)\cup C_A^-(a_2)\cup C_B^-(b_1)\cup C_B^-(b_2)\}$$

$$= \bigcap_{x+a_1+b_1+z=a_2+b_2+z} \{\varnothing\cap\varnothing\cap\varnothing\cap\varnothing\}$$

$$=\varnothing$$

$$= C_{\overline{A+B}}^-(x).$$

Hence, $C_A\,\widetilde{\oplus}\,C_B\ =C_{\overline{A+B}}$.

(4). Let $x\notin\overline{AB}$, then x can not be expressed as $x+a_1b_1+z=a_2b_2+z$ where $a_1,a_2\in A$, $b_1,b_2\in B$ and $z\in R$. Hence $(C_A^+\otimes C_B^+)(x)=\varnothing=C_{\overline{AB}}^+$ and $(C_A^-\boxtimes C_B^-)(x)=U=C_{\overline{AB}}^-$. Therefore, $C_A\ \tilde{\Diamond}\ C_B=C_{\overline{AB}}$. Now let $x\in\overline{AB}$, then x can be expressed as $x+a_1b_1+z=a_2b_2+z$ such that $a_1,a_2\in A$, $b_1,b_2\in B$ and $z\in R$. Thus

$$(C_A^+\otimes C_B^+)(x)$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{C_A^+(a_1)\cap C_A^+(a_2)\cap C_B^+(b_1)\cap C_B^+(b_2)\}$$

$$= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{U\cap U\cap U\cap U\}$$

$$= U$$

$$= C_{\overline{AB}}^+(x),$$

also,

$$(C_A^-\boxtimes C_B^-)(x)$$

$$= \bigcap_{x+a_1b_1+z=a_2b_2+z} \{C^-_A(a_1)\cup C^-_A(a_2)\cup C^-_B(b_1)\cup C^-_B(b_2)\}$$
$$= \bigcap_{x+a_1b_1+z=a_2b_2+z} \{\varnothing\cap\varnothing\cap\varnothing\cap\varnothing\}$$
$$= \varnothing$$
$$= C^-_{\overline{AB}}(x).$$

Hence, $C_A \,\tilde{\Diamond}\, C_B = C^-_{\overline{AB}}$.

DOUBLE-FRAMED SOFT LEFT (RIGHT) *h*-IDEALS

Double-framed soft structures, are newly developed structures. Comparatively to other structures these can comprehensively discuss and characterized hemirings. The contributions of the present research will play a key role in the structure theory. In this section, hemirings are classified by the properties of double-framed soft left (resp. right) *h*-ideals. Several important results are determined by the above said notions. Note that onward, double-framed soft left (resp. right) *h*-ideal will simply be denoted by DFS left (resp. right) *h*-ideal.

Definition 13 A DFS-set $\langle (f^+_A, f^-_A); A\rangle$ of a hemiring R is said to be a double-framed soft left (resp. right) *h*-ideal of if for a, b $\in$ R, the following conditions hold.

(1a). $f^+_A(a+b) \supseteq f^+_A(a) \cap f^+_A(b)$

(1b). $f^-_A(a+b) \subseteq f^-_A(a) \cup f^-_A(b)$

(2a). $f^+_A(ab) \supseteq f^+_A(b)$(resp. $f^+_A(ab) \supseteq f^+_A(a)$)

(2b). $f^-_A(ab) \subseteq f^-_A(b)$(resp. $f^-_A(ab) \subseteq f^-_A(a)$)

(3a). $\forall a,b,x,z \in R)(x + a + z = b + z \to f^+_A(a) \cap f^+_A(b))$

(3b). $(\forall a,b,x,z \in R)(x + a + z = b + z \to f^-_A(x) \subseteq f^-_A(a) \cup f^-_A(b))$.

Note that a Double-framed soft left *h*-ideal $\langle (f^+_A, f^-_A); A\rangle$ of a hemiring R with zero element satisfies the inequalities $f^+_A(0) \supseteq f^+_A(a)$, $f^-_A(0) \subseteq f^-_A(a)$ for all a $\in$ R.

Example 14 Suppose R = {0, a, b, c} be a set with addition and multiplication defined in the following tables:

+	0	*a*	*b*	*c*
0	0	*a*	*b*	*c*
a	0	*a*	*b*	*c*
b	*b*	*b*	*b*	*c*
c	*c*	*c*	*c*	*b*

.	0	*a*	*b*	*c*
0	0	0	0	0
a	0	*a*	*a*	*a*
b	0	*a*	*a*	*a*
c	0	*a*	*a*	*a*

Define a double-framed soft $\langle (f^+_A, f^-_A); A\rangle$ in R over U = Z^- as follows:

R	0	*a*	*b*	*c*
$f^+_A(x)$	{–1,–2,…,–10}	{–1,–3,…,–9}	{–1,–3,–9}	{–1,–3}
$f^-_A(x)$	{–2,–4,–6,–8}	{–2,–4,–8}	{–2,–4}	{–2}

using Definition 13, $\langle (f^+_A, f^-_A); A\rangle$ is a double-framed soft *h*-ideal of R over Z^-.

In the following Lemma, double-framed soft including sets are used to connect ordinary left *h*-ideals with DFS left *h*-ideals of hemiring *R*.

Lemma 15 If $f_A = \langle (f^+_A, f^-_A); A\rangle$ is a double-framed soft set of a hemiring R, then a non-empty double-framed soft including set $DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$ is left *h*-ideal of R if and only if $f_A = \langle (f^+_A, f^-_A); A\rangle$ is DFS left *h*-ideal of R.

Proof Suppose $\varnothing \neq DF_A(f^+_A, f^-_A)_{(\gamma,\delta)} \in R$ (be a left *h*-ideal, if there exist a, b $\in$ R such that $f^+_A(a+b) \subset f^+_A(a) \cap f^+_A(b) = \gamma_1$ and $f^-_A(a+b) \supset f^-_A(a) \cup f^-_A(b) = \delta_1$ for some γ_1, δ_1 are subsets of U. Then a,b $\in DF_A(f^+_A, f^-_A)_{(\gamma_1,\delta_1)}$ but a + b $\notin DF_A(f^+_A, f^-_A)_{(\gamma_1,\delta_1)}$ which is contradiction to the fact that $DF_A(f^+_A, f^-_A)_{(\gamma_1,\delta_1)}$ is left *h*-ideal. Hence $f^+_A(a+b) \supseteq f^+_A(a) \cap f^+_A(b)$ and $f^-_A(a+b) \subseteq f^-_A(a) \cup f^-_A(b)$ hold for all a, b $\in$ R. Let a, r $\in$ R, assume that $f^+_A(ra) \subset f^+_A(a) = \gamma_2$, $f^-_A(ra) \supset f^-_A(a) = \delta_2$, then a $\in DF_A(f^+_A, f^-_A)_{(\gamma_2,\delta_2)}$ but ra $\notin DF_A(f^+_A, f^-_A)_{(\gamma_2,\delta_2)}$ which is contradiction to the hypothesis so, $f^+_A(ra) \supseteq f^+_A(a)$ and $f^-_A(ra) \subseteq f^-_A(a)$ is true for all a, r $\in$ R. Lastly, if there exist x,a,b,z $\in$ R with the expression x + a + z = b + z such that $f^+_A(x) \subset f^+_A(a) \cap f^+_A(b) = \gamma_3$ and $f^-_A(x) \supset f^-_A(a) \cup f^-_A(b) = \delta_3$, then a,b $\in DF_A(f^+_A, f^-_A)_{(\gamma_3,\delta_3)}$ but x $\notin DF_A(f^+_A, f^-_A)_{(\gamma_3,\delta_3)}$ leads to contradiction again. Thus $f^+_A(x) \supseteq f^+_A(a) \cap f^+_A(b)$ and $f^-_A(x) \subseteq f^-_A(a) \cup f^-_A(b)$ hold for all x, a, b, z $\in$ R. Consequently, $f_A = \langle (f^+_A, f^-_A); A\rangle$ is double-framed soft left *h*-ideal of R.

Conversely, suppose that $f_A = \langle (f^+_A, f^-_A); A\rangle$ is double-framed soft left *h*-ideal of R, need to show that $DF_A(f^+_A, f^-_A)_{(\gamma,\delta)} \neq \varnothing$ (is a left *h*-ideal of for this consider a, b $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$, then $f^+_A(a) \supseteq \gamma$, $f^+_A(b) \supseteq \gamma$ and $f^-_A(a) \subseteq \delta$, $f^-_A(b) \subseteq \delta$. Since $\langle (f^+_A, f^-_A); A\rangle$ is DFS left *h*-ideal of R, so $f^+_A(a+b) \supseteq f^+_A(a) \cap f^+_A(b) \supseteq \gamma \cap \gamma = \gamma$ implies that $f^+_A(a+b) \supseteq \gamma$ and $f^-_A(a+b) \subseteq f^-_A(a) \cup f^-_A(b) \subseteq \delta \cup \delta = \delta$, hence $f^-_A(a+b) \subseteq \delta$. Thus a + b $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$. Similarly, for a $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$ and r $\in$ R, $f^+_A(ra) \supseteq f^+_A(a) \supseteq \gamma$, $f^-_A(ra) \subseteq f^-_A(a) \subseteq \delta$ leads to ra $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$. Finally, let a, b $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$, x, z $\in$ R with the expression x + a + z = b + z, then $f^+_A(a) \supseteq \gamma$, $f^+_A(b) \supseteq \gamma$ and $f^-_A(a) \subseteq \delta$, $f^-_A(b) \subseteq \delta$. So $f^+_A(x) \supseteq f^+_A(a) \cap f^+_A(b) \supseteq \gamma$ and $f^-_A(x) \subseteq f^-_A(a) \cup f^-_A(b) \subseteq \delta$ means that $f^+_A(x) \supseteq \gamma$ and $f^-_A(x) \subseteq \delta$, hence x $\in DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$. Therefore, $DF_A(f^+_A, f^-_A)_{(\gamma,\delta)}$ is a left *h*-ideal of a hemiring R.

The case for the right *h*-ideal can be shown accordingly.

Theorem 16 Suppose A is a non-empty subset of a hemiring R and $C_A = \langle C^+_A, C^-_A\rangle$ is a double-framed soft set on R defined by

$$C^+_A(x) = \begin{cases} \gamma_1 & \text{if } x \in A, \\ \delta_1 & \text{if } x \notin A. \end{cases} \qquad C^-_A(x) = \begin{cases} \delta_2 & \text{if } x \in A, \\ \gamma_2 & \text{if } x \notin A. \end{cases}$$

where $\emptyset \subseteq \delta_1 \subset \gamma_1 \subseteq U$ and $\emptyset \subseteq \delta_2 \subset \gamma_2 \subseteq R$. Then show that C_A is a double-framed soft left h-ideal of R.

Proof The proof of the theorem is similar to the proof of Lemma 15.

For a hemiring R, the double-framed soft sets denoted by $R = \langle (R^+_R, R^-_R); R\rangle$ and $\emptyset = \langle (\emptyset^+_R, \emptyset^-_R); R\rangle$ where (R^+_R, R^-_R) and $(\emptyset^+_R, \emptyset^-_R)$ are soft mappings from to defined by

$$R^+_R : x \mapsto R^+_R(x) = U,$$
$$R^-_R : x \mapsto R^-_R(x) = \emptyset,$$
$$\emptyset^+_R : x \mapsto ?^+_R(x) = \emptyset,$$
$$\emptyset^+_R : x \mapsto \emptyset^-_R(x) = U.$$

for all $x \in R$.

In the following theorem, the necessary and sufficient conditions for double-framed soft left h-ideal is provided.

Theorem 17 Suppose $f_A = \langle (f^+_A, f^-_A); A\rangle$ is a double-framed soft left h-ideal of a hemiring T. Then the following are the necessary and sufficient conditions for $f_A = \langle (f^+_A, f^-_A); A\rangle$ to be a DFS left h-ideal of R are:

(1). $(\forall x, y \in R)(f^+_A(a + b) \supseteq f^+_A(a) \cap f^+_A(b)$ and $f^-_A(a + b) \subseteq f^-_A(a) \cup f^-_A(b))$,

(2). $C_R \tilde{\Diamond} f_A \widetilde{\subseteq} f_A$

(3). $(\forall a, b, x, z \in R)$ $(x + a + z = b + z \rightarrow f^+_A(a) \cap f^+_A(b)$ and $f^-_A(a) \cup f^-_A(b))$.

Proof $(\Rightarrow)$ Suppose that $f_A = \langle (f^+_A, f^-_A); A\rangle$ is a double-framed soft left h-ideal of R. Then, conditions (1) and (3) directly follows from the Definition 13. For Condition (2), if $x \in R$ can not be expressed as $x + a_1b_1 + z = a_2b_2 + z$, then $(C^+_R \otimes f^+_A)(x) = \emptyset \subseteq f^+_A(x)$ and $(C^-_R \boxtimes f^-_A)(x) = U \supseteq f^-_A(x)$. Hence, $C_R \tilde{\Diamond} f_A \widetilde{\subseteq} f_A$. Now assume that x can be expressed as $x + a_1b_1 + z = a_2b_2 + z$, since $f_A = \langle (f^+_A, f^-_A); A\rangle$ is a DFS left h-ideal, therefore, we have,

$$\begin{aligned}
&(C^+_R \otimes f^+_A)(x)\\
&= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{C^+_R(a_1) \cap C^+_R(a_2) \cap f^+_A(b_1) \cap f^+_A(b_2)\}\\
&= \bigcup_{x+a_1b_1+z=a_2b_2+z} \{U \cap U \cap f^+_A(b_1) \cap f^-_A(b_2)\}\\
&\subseteq f^+_A(b_1) \cap f^+_A(b_2)\\
&\subseteq f^+_A(a_1b_1) \cap f^+_A(a_1b_1)\\
&\subseteq \bigcup_{x+a_1b_1+z=a_2b_2+z} f^+_A(a_1b_1) \cap f^+_A(a_1b_1)\\
&= f^+_A(x),
\end{aligned}$$

and

$$\begin{aligned}
&(C^-_R \boxtimes f^-_A)(x)\\
&= \bigcap_{x+a_1b_1+z=a_2b_2+z} \{C^-_R(a_1) \cap C^-_R(a_2) \cap f^-_A(b_1) \cap f^-_A(b_2)\}\\
&= \bigcap_{x+a_1b_1+z=a_2b_2+z} \{\emptyset \cap \emptyset \cap f^-_A(b_1) \cap f^-_A(b_2)\}\\
&\supseteq \bigcap_{x+a_1b_1+z=a_2b_2+z} \{f^-_A(a_1b_1) \cap f^-_A(a_1b_1)\}\\
&= f^-_A(x),
\end{aligned}$$

Hence, $C_R \tilde{\Diamond} f_A \widetilde{\subseteq} f_A$.

$(\Leftarrow)$, now assume that Conditions (1)-(3) holds, let $x, z \in R$, such that $x + (a_1b_1) + z = (a_2b_2) + z$, then using Condition 2, we have,

$$\begin{aligned}
&f^+_A(xy) \supseteq (C^+_R \otimes f^+_A)(xy)\\
&= \bigcup_{xy+a_1b_1+z=a_2b_2+z} \{C^+_R(a_1) \cap C^+_R(a_2) \cap f^+_A(b_1) \cap f^+_A(b_2)\}\\
&= \bigcup_{xy+a_1b_1+z=a_2b_2+z} \{U \cap U \cap f^+_A(b_1) \cap f^+_A(b_2)\}\\
&\supseteq f^+_A(b_1) \cap f^+_A(b_2)\\
&\supseteq \bigcup_{y+b_1+z=b_2+z} \{f^+_A(b_1) \cap f^+_A(b_2)\}\\
&= f^+_A(y)
\end{aligned}$$

also,

$$\begin{aligned}
&f^-_A(xy) \subseteq (C^-_R \boxtimes f^-_A)(xy)\\
&= \bigcap_{xy+a_1b_1+z=a_2b_2+z} \{(C^-_R(a_1) \cup C^-_R(a_2) \cup f^-_A(b_1) \cup f^-_A(b_2)\}\\
&= \bigcap_{xy+a_1b_1+z=a_2b_2+z} \{\emptyset \cup \emptyset \cup f^-_A(b_1) \cup f^-_A(b_2)\}\\
&= \bigcap_{xy+a_1b_1+z=a_2b_2+z} \{f^-_A(b_1) \cup f^-_A(b_2)\}\\
&\subseteq \bigcap_{y+a_1+z=b_2+z} \{f^-_A(b_1) \cup f^-_A(b_2)\}\\
&= f^-_A(y)
\end{aligned}$$

Thus, $f_A = \langle (f^+_A, f^-_A); A\rangle$ is double-framed soft left h-ideal of R.

Lemma 18 Suppose $f_A = \langle (f^+_A, f^-_A); A\rangle$ is a double-framed soft right h-ideal of a hemiring T. Then the following are the necessary and sufficient conditions for $f_A = \langle (f^+_A, f^-_A); A\rangle$ to be a DFS right h-ideal of R are:

(1). $(\forall x, y \in R)(f^+_A(a + b) \supseteq f^+_A(a) \cap f^+_A(b)$ and $f^-_A(a + b) \subseteq f^-_A(a) \cup f^-_A(b))$

(2). $f_A \tilde{\Diamond} C_R \widetilde{\subseteq} f_A$

(3). $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f^+_A(x) \supseteq f^+_A(a) \cap f^+_A(b)$ and $f^-_A(x) \subseteq f^-_A(a) \cup f^-_A(b))$.

Proof The proof of the lemma follows from Theorem 17.

DOUBLE-FRAMED SOFT h-BI-IDEALS AND h-QUASI-IDEALS

An important milestone of the present section is to develop the connection between ordinary h-bi-ideals (h-quasi-ideals) with double-framed soft h-bi-ideals (double-framed soft h-quasi-ideals). For this purpose, double-framed soft including set are used. Further, it is shown that every double-framed soft h-quasi-ideal of a hemiring is a double-framed soft h-bi-ideal but the converse is not true in general.

Definition 19 A DFS-set $\langle (f_A^+, f_A^-); A\rangle$ of a hemiring R is said to be a double-framed soft h-bi-ideal of R if the following conditions hold.

(4a). $(\forall a, b \in R)(f_A^+(a+b) \supseteq f_A^+(a) \cap f_A^+(b))$
(4b). $(\forall a, b \in R)(f_A^-(a+b) \subseteq f_A^-(a) \cup f_A^-(b))$
(5a). $(\forall a, b \in R)(f_A^+(ab) \supseteq f_A^+(a) \cap f_A^+(b))$
(5b). $(\forall a, b \in R)(f_A^-(ab) \subseteq f_A^-(a) \cup f_A^-(b))$
(6a). $(\forall a, b, c \in R)(f_A^+(abc) \supseteq f_A^+(a) \cap f_A^+(c))$
(6b). $(\forall a, b, c \in R)(f_A^-(abc) \subseteq f_A^-(a) \cup f_A^-(c))$
(7a). $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f_A^+(x) \supseteq f_A^+(a) \cap f_A^+(b))$
(7b). $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f_A^-(x) \subseteq f_A^-(a) \cup f_A^-(b))$.

Note that for a double-framed soft h-bi (quasi)-ideal $f_A = \langle (f_A^+, f_A^-); A\rangle$ the inequality $f_A^+(0) \supseteq f_A^+(x)$ and $f_A^-(0) \subseteq f_A^-(x)$ are hold for all $x \in R$.

Example 20 Let U = Z^+ (positive integers) be the universal set. Consider a parameter set A = {0,1,2,3}, the set of non-negative integers module 4 is a hemiring.

Define a double-framed soft $\langle (f_A^+, f_A^-); A\rangle$ in R over $U = Z^+$ as follows:

R	0	1	2	3
$f_A^+(x)$	Z^+	{2, 3}	{1, 2, 3, 5}	{2, 3}
$f_A^-(x)$	{3, 4}	Z^+	{1, 3, 4, 6}	Z^+

using Definition 19, $\langle (f_A^+, f_A^-); A\rangle$ is a double-framed soft h-bi-ideal of R over Z^-. If for the same universal set Z^+, we consider another parameter set:

$$B = \left\{ \begin{bmatrix} x & x \\ y & y \end{bmatrix} \middle| x, y \in Z_2 = \{0,1\} \right\}$$

where Z_2 is the set of non-negative integers module 2. Define a double-framed soft $\langle (f_B^+, f_B^-); B\rangle$ over Z^+ as follows:

R	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
$f_B^+(x)$	Z^+	{2, 3, 4, 5}	{1, 2, 3, 4, 5, 7}	{1, 2, 3, 4, 5, 7, 8}
$f_B^-(x)$	{3, 4}	Z^+	{1, 3, 4, 5, 6, 7, 8}	{1, 3, 4, 5, 7}

then by Definition 19, $\langle (f_B^+, f_B^-); B\rangle$ is a double-framed soft h-bi-ideal of Z^+.

Theorem 21 A DFS-set $f_A = \langle (f_A^+, f_A^-); A\rangle$ of a hemiring R is a double-framed soft h-bi-ideal of R if and only if

(1). $f_A \tilde{\oplus} f_A \tilde{\subseteq} f_A$
(2). $f_A \tilde{\Diamond} f_A \tilde{\subseteq} f_A$
(3). $f_A \tilde{\Diamond} C_R \tilde{\Diamond} f_A \tilde{\subseteq} f_A$.

Proof Let $f_A = \langle (f_A^+, f_A^-); A\rangle$ be a double-framed soft h-bi-ideal of R, and $x \in R$ be such that it can not be expressed in the form $x + (a_1 + b_1) + z = (a_2 + b_2) + z$, then $(f_A^+ \oplus f_A^+)(x) = \emptyset \subseteq f_A^+(x)$ and $(f_A^- \boxtimes f_A^-)(x) = U \supseteq f_A^-(x)$. So $f_A^+ \tilde{\oplus} f_A^+ \tilde{\subseteq} f_A$. Now if x can be expressed in the form $x + (a_1 + b_1) + z = (a_2 + b_2) + z$, then

$$
\begin{aligned}
&(f_A^+ \oplus f_A^+)(x) \\
&= \bigcup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\} \\
&\subseteq \bigcup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^+(a_1+a_2) \cap f_A^+(b_1+b_2)\} && \text{by (4a)} \\
&\subseteq \bigcup_{x+(a_1+b_1)+z=(a_2+b_2)+z} f_A^+(x) && \text{by (7a)} \\
&= f_A^+(x)
\end{aligned}
$$

also,

$$
\begin{aligned}
&(f_A^- \boxplus f_A^-)(x) \\
&= \bigcap_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\} \\
&\supseteq \bigcap_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^-(a_1+a_2) \cup f_A^-(b_1+b_2)\} && \text{by (4b)} \\
&\supseteq \bigcap_{x+(a_1+b_1)+z=(a_2+b_2)+z} f_A^-(x) && \text{by (7b)} \\
&= f_A^-(x)
\end{aligned}
$$

Hence, $f_A \tilde{\oplus} f_A \tilde{\subseteq} f_A$.

Now, if $x \in R$ can not be expressed in the form $x + (a_1 b_1) + z = (a_2 b_2) + z$, then $(f_A^+ \otimes f_A^+)(x) = \emptyset \subseteq f_A^+(x)$ and $(f_A^- \boxtimes f_A^-)(x) = U \supseteq f_A^-(x)$. So $f_A \tilde{\Diamond} f_A \tilde{\subseteq} f_A$. If can be written in the form $x + (a_1 b_1) + z = (a_2 b_2) + z$, then

$$
\begin{aligned}
&(f_A^+ \oplus f_A^+)(x) \\
&= \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\} \\
&\subseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1a_2) \cap f_A^+(b_1b_2)\} && \text{by (5a)} \\
&\subseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} f_A^+(x) && \text{by (7a)} \\
&= f_A^+(x).
\end{aligned}
$$

also,

$$
\begin{aligned}
&(f_A^- \boxtimes f_A^-)(x) \\
&= \bigcap_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}
\end{aligned}
$$

$\supseteq \bigcap_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1a_2) \cup f_A^-(b_1b_2)\}$ by (5b)

$\supseteq \bigcap_{x+(a_1b_1)+z=(a_2b_2)+z} f_A^-(x)$ by (7b)

$= f_A^-(x)$

Thus $f_A \tilde{\Diamond} f_A \tilde{\subseteq} f_A$.

The proof of Condition (3) follows from (1) and (2).

Conversely, assume that conditions (1) to (3) hold. First to show that $f_A^+(0) \supseteq f_A^+(x)$ and $f_A^-(0) \subseteq f_A^-(x)$ for all $x \in R$.

$f_A^+(x) \supseteq (f_A^+ \oplus f_A^+)(0)$

$= \bigcup_{0+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}$

$\supseteq \{f_A^+(x) \cap f_A^+(x) \cap f_A^+(x) \cap f_A^+(x)\}$

$= f_A^+(x)$ *as* $0 + x + x + z = x + x + z$

also

$f_A^-(0) \subseteq (f_A^- \boxplus f_A^-)(0)$

$= \bigcap_{0+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}$

$\subseteq \{f_A^-(x) \cup f_A^-(x) \cup f_A^-(x) \cup f_A^-(x)\}$

$= f_A^-(x)$.

Hence, $f_A^+(0) \supseteq f_A^+(x)$ and $f_A^-(0) \subseteq f_A^-(x)$ hold for all $x \in R$. Now,

$f_A^+(x+y) \supseteq (f_A^+ \oplus f_A^+)(x+y)$

$= \bigcup_{x+y+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}$

$\supseteq \{f_A^+)(0) \cap f_A^+(0) \cap f_A^+(x) \cap f_A^+(y)\}$

now as $x + y + 0 + 0 + z = x + y + z$, so

$f_A^+(x+y) \supseteq f_A^+(x) \cap f_A^+(y)$

also,

$f_A^-(x+y) \subseteq (f_A^- \boxplus f_A^-)(x+y)$

$= \bigcap_{x+y+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}$

$\subseteq \{f_A^-(0) \cup f_A^-(0) \cup f_A^-(x) \cup f_A^-(y)\}$

$= f_A^-(x) \cup f_A^-(y)$.

Further,

$f_A^+(x) \supseteq (f_A^+ \oplus f_A^+)(x)$

$= \bigcup_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}$

If $x + a + z = b + z$, then $x + a + 0 + z = b + 0 + z$, therefore, $f_A^+(x) \supseteq \{f_A^+(a) \cap f_A^+(0) \cap f_A^+(b) \cap f_A^+(0)\} = \{f_A^+(a) \cap f_A^+(b)\}$

similarly,

$f_A^-(x) \subseteq (f_A^- \boxplus f_A^-)(x)$

$= \bigcap_{x+(a_1+b_1)+z=(a_2+b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}$

$\subseteq \{f_A^-(a) \cup f_A^-(0) \cup f_A^-(b) \cup f_A^-(0)\}$

$= f_A^-(x) \cup f_A^-(y)$.

The rest of the conditions can be proved in similar manner. Consequently, $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a double-framed soft h-bi-ideal of R.

Definition 22 A DFS-set $\langle (f_A^+, f_A^-); A \rangle$ of a hemiring R is said to be a double-framed soft h-quasi-ideal of R if the following conditions hold.

(8a). $(\forall a, b \in R)(f_A^+(a+b) \supseteq f_A^+(a) \cap f_A^+(b))$

(8b). $(\forall a, b \in R)(f_A^-(a+b) \subseteq f_A^-(a) \cup f_A^-(b))$

(9a). So $(f_A \tilde{\Diamond} C_R) \tilde{\cap} (C_R \tilde{\Diamond} f_A) \tilde{\subseteq} f_A$

(10a). $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f_A^+(x) \supseteq f_A^+(a) \cap f_A^+(b))$

(10b). $(\forall a, b, x, z \in R)(x + a + z = b + z \rightarrow f_A^-(x) \subseteq f_A^-(a) \cup f_A^-(b))$.

Example 23 The set of all non-negative integers $U = N_0$ is a hemiring with respect to usual addition and multiplication. Suppose $u_1, u_2, u_1', u_2' \in P(U)$ be such that $\emptyset \neq u_1 \subset u_2$ and $\emptyset \neq u_2' \subset u_1'$ where is power set of define a double-framed soft set over as follows:

N_0	if $a \in \langle 3 \rangle$	$a \notin \langle 3 \rangle$
$f_A^+(x)$	u_2	u_1
$f_A^-(x)$	u_2	u_1

then by Definition 22, $\langle (f_A^+, f_A^-); A \rangle$ is a double-framed soft h-quasi-ideal of N_0.

Theorem 24 If $f_A = \langle (f_A^+, f_A^-); A \rangle$ is a double-framed soft set of a hemiring R, then a non-empty double-framed soft including set $DF_A(f_A^-, f_A^-)_{(\gamma,\delta)}$ is h-bi-ideal (resp. h-quasi-ideal) of R if $f_A = \langle (f_A^+, f_A^-); A \rangle$ and only if is DFS h-bi-ideal (resp. h-quasi-ideal) of R.

Proof The proof of the theorem follows from Lemma 15. Using characteristic double-framed soft sets, ordinary h-ideals (h-left (right) ideals, h-bi-ideals, h-quasi-ideals) in a hemiring R are linked with DFS h-ideals (DFS left (right)-ideals, DFS h-bi-ideals, DFS h-quasi-ideals) in the following result.

Corollary 25 If A is any non-empty subset of a hemiring R, then, characteristic double-framed soft set $C_A = \left\langle\left(C_A^+, C_A^-\right); A\right\rangle$ is a DFS h-ideal (resp. DFS h-bi-ideal, DFS h-quasi-ideals) of R if and only if A is an h-ideal (resp. h-bi-ideal, h-quasi-ideal) of R.

Proof Follows from Lemma 15 and Theorem 24.

Theorem 26 If $f_A = \left\langle\left(f_A^+, f_A^-\right); A\right\rangle$ is DFS right h-ideal and $g_B = \left\langle\left(g_B^+, g_B^-\right); B\right\rangle$ is DFS left h-ideal of a hemiring R, then $f_A \tilde{\cap} g_B = \left\langle f_A^+ \tilde{\cap}\, g_B^+, f_A^- \tilde{\cup}\, g_B^-\right\rangle$ is a double framed soft h-quasi-ideal of R.

Proof Assume $x, y \in R$, then

$(f_A^+ \tilde{\cap}\, g_B^+)(x + y) = f_A^+(x + y) \cap g_B^+(x + y)$

$\supseteq \{f_A^+(x) \cap f_A^+(y)\} \cap \{g_B^+(x) \cap g_B^+(y)\}$

$= \{f_A^+(x) \cap g_B^+(x)\} \cap \{f_A^+(y) \cap g_B^+(y)\}$

$= (f_A^+ \tilde{\cap}\, g_B^+)(x) \cap (f_A^+ \tilde{\cap}\, g_B^+)(y)$

also

$(f_A^- \tilde{\cup}\, g_B^-)(x + y) = f_A^-(x + y) \cup g_B^-(x + y)$

$\supseteq \{f_A^-(x) \cup f_A^-(y)\} \cup \{g_B^-(x) \cup g_B^-(y)\}$

$= \{f_A^-(x) \cup g_B^-(x)\} \cup \{f_A^-(y) \cup g_B^-(y)\}$

$= (f_A^- \tilde{\cup}\, g_B^-)(x) \cup (f_A^- \tilde{\cup}\, g_B^-)(y)$.

Now let x, a, b, $z \in R$ with the expression x + a + z = b + z, then

$(f_A^+ \tilde{\cap}\, g_B^+)(x) = f_A^+(x) \cap g_B^+(x)$

$\supseteq \{f_A^+(a) \cap f_A^+(b)\} \cap \{g_B^+(a) \cap g_B^+(b)\}$

$= \{f_A^+(a) \cap \{g_B^+(a)\} \cap \{f_A^+(b) \cap g_B^+(b)\}$

$= (f_A^+ \tilde{\cap}\, g_B^+)(a) \cap (f_A^+ \tilde{\cap}\, g_B^+)(b)$

and

$(f_A^- \tilde{\cup}\, g_B^-)(x) = f_A^-(x) \cup g_B^-(x)$

$\supseteq \{f_A^-(a) \cup f_A^-(b)\} \cup \{g_B^-(a) \cup g_B^-(b)\}$

$= \{f_A^-(a) \cup \{g_B^-(a)\} \cup \{f_A^-(b) \cup g_B^-(b)\}$

$= (f_A^- \tilde{\cup}\, g_B^-)(a) \cup (f_A^- \tilde{\cup}\, g_B^-)(b)$.

Also, $\{(f_A \tilde{\cap}\, g_B)\ \tilde{\Diamond}\ C_R\} \tilde{\cap} \{C_R\ \tilde{\Diamond}\ (f_A \tilde{\cap}\, g_B)\} \tilde{\subseteq} (f_A\ \tilde{\Diamond}\ C_R \tilde{\cap}\, C_R\ \tilde{\Diamond}\ g_B)$ (by Theorem 17, Lemma 18). Hence, $f_A \tilde{\cap}\, g_B$ is Double-framed soft h-quasi-ideal of R.

Next, it is shown that every DFS h-quasi-ideal is DFS h-bi-ideal of a hemiring *R* but the converse is not true in general.

Remark 27 Every double-framed soft h-quasi-ideal is double-framed soft h-bi-ideal in a hemiring R.

Proof Let $f_A = \left\langle\left(f_A^+, f_A^-\right); A\right\rangle$ be a double-framed soft h-quasi-ideal of a hemiring R and a, b, $c \in R$, since $f_A \tilde{\subseteq} (f_A\ \tilde{\Diamond}\ C_R \tilde{\cap}\, C_R\ \tilde{\Diamond}\ g_B)$ therefore,

$f_A^+(abc) \supseteq ((f_A^+ \otimes C_R^+) \tilde{\cap} (C_R^+ \otimes f_A^+))(abc)$

$= (f_A^+ \otimes C_R^+)(abc) \cap (C_R^+ \otimes f_A^+)(abc)$

$$= \left\{\bigcup_{abc+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap C_R^+(b_1) \cap C_R^+(b_2)\}\right\}$$

$$= \cap\left\{\bigcup_{abc+(a_1b_1)+z=(a_2b_2)+z} \{C_R^+(a_1) \cap C_R^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}\right\}$$

as abc + 00 + 0 = a(bc) + 0 and abc + 00 + 0 = (ab)c + 0, therefore

$$\left\{\bigcup_{abc+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap C_R^+(b_1) \cap C_R^+(b_2)\}\right\}$$

$\supseteq f_A^+(0) \cap f_A^+(a) \cap C_A^+(0) \cap C_A^+(bc)$

$= f_A^+(0) \cap f_A^+(a)$

and

$$\left\{\bigcup_{abc+(a_1b_1)+z=(a_2b_2)+z} \{C_R^+(a_1) \cap C_R^+(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}\right\} \supseteq f_A^+(a) \cap f_A^+(c),$$

therefore, $f_A^+(abc) \supseteq f_A^+(a) \cap f_A^+(c)$ and

$f_A^-(abc) \subseteq = ((f_A^- \boxtimes C_R^-) \tilde{\cap} (C_R^- \boxtimes f_A^-))(abc)$

$= (f_A^- \boxtimes C_R^-)(abc) \cap (C_R^- \boxtimes f_A^-))(abc)$

$$\left\{\bigcap_{abc+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2) \cup C_R^-(b_1) \cup C_R^-(b_2)\}\right\}$$

$$\cup\left\{\bigcap_{abc+(a_1b_1)+z=(a_2b_2)+z} \{C_R^-(a_1) \cup C_R^-(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}\right\}$$

$\subseteq f_A^-(a) \cup f_A^-(c)$.

By similar way, we can show that $f_A^+(ab) \supseteq f_A^+(a) \cap f_A^+(b)$, $f_A^-(ab) \subseteq f_A^-(a) \cup f_A^-(b)$. Hence, $f_A = \left\langle\left(f_A^+, f_A^-\right); A\right\rangle$ is double-framed soft h-bi-ideal of R.

The converse of the above proposition is not true in general, as shown in the following example.

Example 28 (Yin & Li 2008). Let R be the set of all 2×2 matrices i.e.,

$$R = \left\{\begin{pmatrix} a_{11} \ldots a_{11} \\ a_{12} \ldots a_{22} \end{pmatrix} \mid a_{ij} \in \mathbb{N}_0 (\text{non-negative integers})\right\}$$

then R is a hemiring with usual addition and multiplication of matrices. Let $Q = \left\{\begin{pmatrix} a \ldots 0 \\ 0 \ldots 0 \end{pmatrix} \mid a \in \mathbb{N}_0\right\}$, then Q is an h-quasi-ideal of R but not an h-ideal of R. Hence, Q is not an h-bi-ideal of R. Using Corollary 25, the double-framed soft characteristic function C_Q is a double-framed soft h-quasi-ideal but not a double-framed soft h-ideal of R. Therefore, it is not a double-framed soft h-bi-ideal of R.

DOUBLE-FRAMED SOFT SETS OF h-HEMIREGULARHEMIRINGS

In this section, well known classification of hemirings called h-hemiregular hemirings are studied. The said important notion h-hemiregular hemirings presented by Zhan and Dudek (2007) are further classified through double-framed soft h-ideals, DFS h-bi-ideals and DFS h-quasi-ideals. Several characterization theorems and results are developed by the aforementioned DFS h-ideals.

Definition 29 (Zhan & Dudek 2007). A hemiring R is h-hemiregular if for all $x \in R$, there exist a_1, a_2, $z \in R$ such that $x + xa_1x + z = xa_2x + z$.

Lemma 30 (Zhan & Dudek 2007). If R is a hemiring, then the following conditions are equivalent:

(i) R is h-hemiregular hemiring.

(ii) $\overline{MN} = M \cap N$, where M is right h-ideal and N is left h-ideal of R.

Lemma 31 (Yin & Li 2008). If R is a hemiring, then the following conditions are equivalent:

(i) R is h-hemiregular.

(ii) $B = \overline{BRB}$, where B is h-bi-ideal of R.

(iii) $Q = \overline{QRQ}$, where Q is h-quasi-ideal of R.

Theorem 32 If R is a hemiring, then the following conditions are equivalent:

(1). R is h-hemiregular hemiring.

(2). If $f_A = \left\langle \left(f_A^+, f_A^-\right); A\right\rangle$, is DFS right h-ideal and $g_B = \left\langle \left(g_B^+, g_B^-\right); B\right\rangle$ is DFS left h-ideal of R, then $f_A \tilde{\Diamond} g_B = f_A \tilde{\cap} g_B$.

Proof (1)⇒(2): Let R is h-hemiregular hemiring and $f_A = \left\langle \left(f_A^+, f_A^-\right); A\right\rangle$, is DFS right h-ideal and $g_B = \left\langle \left(g_B^+, g_B^-\right); B\right\rangle$ is DFS left h-ideal of R. Since f_A, $g_B \tilde{\subseteq} C_R$, therefore, using Theorem 17 and Lemma 18, we have, $f_A \tilde{\Diamond} g_B \tilde{\subseteq} f_A \tilde{\Diamond} C_R$ and $f_A \tilde{\Diamond} g_B \tilde{\subseteq} C_R \tilde{\Diamond} g_B$. Hence, $f_A \tilde{\Diamond} g_B \tilde{\subseteq} f_A \tilde{\cap} g_B$. Now assume $x \in R$, so by hypothesis, there exist a_1, a_2, $z \in R$ such that $x + xa_1x + z = za_2x + z$. Thus

$$(f_A^+ \otimes g_B^+)(x) = \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap g_B^+(b_2)\}$$

$$\supseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(xa_1) \cap f_A^+(xa_2) \cap g_B^+(x) \cap g_B^+(x)\}$$

$$\supseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(x) \cap f_A^+(x) \cap g_B^+(x) \cap g_B^+(x)\}$$

$$= f_A^+(x) \cap g_B^+)(x)$$

$$= (f_A^+ \tilde{\cap} g_B^+)(x),$$

also,

$$(f_A^- \boxtimes f_B^-)(x) = \bigcap_{x+a_1b_1+z=a_2b_2+z} \{f_A^-(a_1) \cup (a_2) \cup g_B^-(b_1) \cup g_B^-(b_2)\}$$

$$\subseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(xa_1) \cup f_A^-(xa_2) \cup g_B^-(x) \cup g_B^-(x)\}$$

$$\subseteq \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(x) \cup f_A^-(x) \cup g_B^-(x) \cup g_B^-(x)\}$$

$$= f_A^+(x) \cup g_B^+(x)$$

$$= (f_A^+ \tilde{\cup} g_B^+(x),$$

Hence, $f_A \tilde{\cap} g_B \tilde{\subseteq} f_A \tilde{\Diamond} g_B$, consequently, $f_A \tilde{\Diamond} g_B = f_A \tilde{\cap} g_B$.

(2)⇒(1): Let A and B be any right and left h-ideals of R, respectively, then by Corollary 25, the characteristic function C_A and C_B are DFS right h-ideal and DFS left h-ideal of R. So by hypothesis, $C_A \tilde{\Diamond} C_B = C_A \tilde{\cap} C_B$. Therefore, by Theorem 12, $C_A \tilde{\Diamond} C_B = C_{\overline{AB}}$ and $C_A \tilde{\cap} C_B = C_{A\cap B}$ which implies that $C_{\overline{AB}} = C_{A\cap B}$, hence $\overline{AB} = A \cap B$. Thus, using Lemma 30, R is h-hemiregular hemiring.

Theorem 33 If R is a hemiring, then the following conditions are equivalent:

(1). R is h-hemiregular hemiring.

(2). $f_A \tilde{\subseteq} f_A \tilde{\Diamond} C_R \tilde{\Diamond} f_A$, for every DFS h-bi-ideal $f_A = \left\langle \left(f_A^+, f_A^-\right); A\right\rangle$ of R.

(3). $f_A \tilde{\subseteq} f_A \tilde{\Diamond} C_R \tilde{\Diamond} f_A$, for every DFS h-quasi-ideal f_A of R.

Proof (1)⇒(2). Let R is -hemiregular and $x \in R$. Suppose f_A is DFS h-bi-ideal of R, there exist a_1, a_2, $z \in R$ such that $x + xa_1x + z\ xa_2x + z$ (R being h-hemiregular). Then

$$(f_A^+ \otimes C_R^+ \otimes f_A^+)(x)$$

$$= \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{(f_A^+ \otimes C_R^+)(a_1) \cap (f_A^+ \otimes C_R^+)(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}$$

$$\supseteq (f_A^+ \otimes C_R^+)(xa_1) \cap (f_A^+ \otimes C_R^+)(xa_2) \cap f_A^+(x)$$

$$= \left\{ \begin{array}{l} \left\{ \bigcup_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap C_R^+(b_1) \cap C_R^+(b_2)\} \right\} \cap \\ \left\{ \bigcup_{xa_2+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap C_R^+(b_1) \cap C_R^+(b_2)\} \right\} \cap f_A^+(x) \end{array} \right\}$$

$$= \left\{ \begin{array}{l} \left\{ \bigcup_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2)\} \right\} \cap \\ \left\{ \bigcup_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2)\} \right\} \cap f_A^+(x) \end{array} \right\}$$

$$\supseteq \{f_A^+(xa_1x) \cap f_A^+(xa_2x)\} \cap \{f_A^+(xa_2x)\} \cap f_A^+(x),$$

this is because, $xa_1 + xa_1xa_1 + za_1 = xa_2xa_1 + za_1$ and $xa_2 + xa_1xa_2 + za_2 = xa_2xa_2 + za_2$.

Therefore,

$(f_A^+ \otimes C_R^+ \otimes f_A^+)(x) \supseteq \{f_A^+(x) \cap f_A^+(x) \cap f_A^+(x)\}$: f_A being DFS h-bi-ideal $= f_A^+(x)$, also

$$(f_A^- \boxtimes C_R^- \boxtimes f_A^-)(x)$$
$$= \bigcap_{x+(a_1b_1)+z=(a_2b_2)+z} \{(f_A^- \boxtimes C_R^-)(a_1) \cup (f_A^- \boxtimes C_R^-)(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}$$
$$\subseteq (f_A^- \boxtimes C_R^-)(xa_1) \cup (f_A^- \boxtimes C_R^-)(xa_2) \cup f_A^-(x)$$
$$= \left\{ \begin{array}{l} \left\{ \bigcap_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1) \cup f_A^-(a_2)\} \right\} \cup \\ \left\{ \bigcap_{xa_2+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1) \cup f_A^-(b_2)\} \right\} \cup f_A^-(x) \end{array} \right\}$$
$$\subseteq \{f_A^-(xa_1x) \cup f_A^-(xa_2x)\} \cup \{f_A^-(xa_1x) \cup f_A^-(xa_2x)\} \cup f_A^-(x)$$
$$\subseteq \{f_A^-(x) \cup f_A^-(x) \cup f_A^-(x)\} = f_A^-(x).$$

(2)⇒(3). This implication hold using Remark 27.

(3)⇒(1). If Q is any h-quasi-ideal of R, then by Corollary 25, the characteristic function C_Q is a DFS h-quasi-ideal of R.Therefore by hypothesis, $C_Q \mathrel{\tilde{\subseteq}} C_Q \tilde{\Diamond} C_R \tilde{\Diamond} C_Q$. Now by Theorem 12, $C_Q \tilde{\Diamond} C_R \tilde{\Diamond} C_Q = C_{\overline{QRQ}}$. Thus, $C_Q \mathrel{\tilde{\subseteq}} C_{\overline{QRQ}}$ implies $Q \subseteq \overline{QRQ}$ and reverse inclusion hold because Q is h-quasi-ideal of R i.e., $\overline{QRQ} \subseteq Q$, which implies that $\overline{QRQ} = Q$. Hence by Lemma 31, is h-hemiregular.

Theorem 34 If R is a hemiring, then the following conditions are equivalent:

(1). R is h-hemiregular hemiring.

(2). $f_A \tilde{\cap} g_B \mathrel{\tilde{\subseteq}} f_A \tilde{\Diamond} g_B \tilde{\Diamond} f_A$, for every DFS h-bi-ideal f_A and every DFS h-ideal g_B of R.

(3). $f_A \tilde{\cap} g_B \mathrel{\tilde{\subseteq}} f_A \tilde{\Diamond} g_B \tilde{\Diamond} f_A$, for every DFS h-quasi-ideal f_A and every DFS h-ideal g_B of R.

Proof (1)⇒(2). Consider R is an h-hemiregular. Suppose f_A is DFS h-bi-ideal and g_B is DFS h-ideal of R, Let $x \in R$, then there exist $a_1, a_2, z \in R$ such that $x + xa_1x$. Therefore,

$$(f_A^+ \otimes g_B^+ \otimes f_A^+)(x)$$
$$= \bigcup_{x+(a_1b_1)+z=(a_2b_2)+z} \{(f_A^+ \otimes g_B^+)(a_1) \cap (f_A^+ \otimes g_B^+)(a_2) \cap f_A^+(b_1) \cap f_A^+(b_2)\}$$
$$\supseteq (f_A^+ \otimes g_B^+)(xa_1) \cap (f_A^+ \otimes g_B^+)(xa_2) \cap f_A^+(x)$$
$$= \left\{ \begin{array}{l} \left\{ \bigcup_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap g_B^+(b_2)\} \right\} \cap \\ \left\{ \bigcup_{xa_2+(a_1b_1)+z=(a_2b_2)+z} \{f_A^+(a_1) \cap f_A^+(a_2) \cap g_B^+(b_1) \cap g_B^+(b_2)\} \right\} \cap f_A^+(x) \end{array} \right\}$$
$$\supseteq (f_A^+(x) \cap g_B^+(a_1xa_1) \cap g_B^+(a_1xa_2)\} \cap \{f_A^+(x) \cap g_B^+(a_1xa_2) \cap g_B^+(a_2xa_2)\} \cap f_A^+(x),$$

this is because, $xa_1 + xa_1xa_1 + za_1 = xa_2xa_1 + za_1$ and $za_1xa_2 + za_2 = xa_2xa_2 + za_2$.

Therefore,

$$(f_A^+ \otimes g_B^+ \otimes f_A^+)(x) \supseteq \{f_A^+(x) \cap g_B^+(x)\} \cap \{f_A^+(x) \cap g_B^+(x)\} \cap f_A^+(x)$$
$$= f_A^+(x) \cap g_B^+(x)$$
$$= (f_A^+ \tilde{\cap} g_B^+)(x),$$

also

$$(f_A^- \boxtimes g_B^- \boxtimes f_A^-)(x)$$
$$= \bigcap_{x+(a_1b_1)+z=(a_2b_2)+z} \{(f_A^- \boxtimes g_B^-)(a_1) \cup (f_A^- \boxtimes g_B^-)(a_2) \cup f_A^-(b_1) \cup f_A^-(b_2)\}$$
$$\subseteq (f_A^- \boxtimes g_B^-)(xa_1) \cup (f_A^- \boxtimes g_B^-)(xa_2) \cup f_A^-(x)$$
$$= \left\{ \begin{array}{l} \left\{ \bigcap_{xa_1+(a_1b_1)+z=(a_2b_2)+z} \{f_A^-(a_1) \cup (f_A^-(a_2) \cup g_B^-(b_1)(b_2)\} \right\} \cup \\ \left\{ \bigcap_{xa_2+(a_1b_1)+z=(a_2b_2)+z} \{(f_A^-(a_1) \cup (f_A^-(a_2) \cup g_B^-(b_1) \cup g_B^-(b_2)\} \right\} \cup f_A^-(x) \end{array} \right\}$$
$$\subseteq \{f_A^-(x) \cup g_B^-(a_1xa_1) \cup g_B^-(a_1xa_2)\} \cup \{f_A^-(x) \cup g_B^-(a_1xa_2)$$
$$\cup g_B^-(a_1xa_2) \cup f_A^-(x),$$
$$\subseteq \{f_A^-(x) \cup g_B^-(x) \cup g_B^-(x)\}$$
$$= f_A^-(x) \cup g_B^-(x)$$
$$= (f_A^- \tilde{\cup} g_B^-)(x).$$

Hence (2) hold. (2)⇒(3). This implication can be shown simply.

(3)⇒(1). Let f_A is any DFS h-quasi-ideal of R, since C_R is DFS h-ideal, so $f_A = f_A \tilde{\cap} C_R \mathrel{\tilde{\subseteq}} f_A \tilde{\Diamond} C_R \tilde{\Diamond} f_A$. Therefore, by Theorem 33, R is an h-hemiregular.

Lemma 35 (Yin & Li 2008). If is a hemiring, then the following conditions are equivalent:

(i) R is h-hemiregular hemiring.

(ii) Both right h-ideal M and left h-ideal N of R are idempotent and $\overline{MN}$ is an h-quasi-ideal of R.

Theorem 36 If R is a hemiring, then the following conditions are equivalent:

(i) R is h-hemiregular hemiring.

(ii) If f_A is DFS right h-ideal and g_B is DFS left h-ideal, then both f_A and g_B are idempotent and $f_A \tilde{\Diamond} g_B$ is a DFS h-quasi-ideal of R.

Proof (i)⇒(ii). If R is h-hemiregular hemiring and f_A is DFS right h-ideal of R, then $f_A \tilde{\Diamond} f_A \mathrel{\tilde{\subseteq}} f_A \tilde{\Diamond} C_R \mathrel{\tilde{\subseteq}} f_A$ (f_A being DFS right h-ideal: by Lemma 18). Also, by Theorem 32, $f_A \mathrel{\tilde{\subseteq}} f_A \tilde{\Diamond} g_B$. Thus $f_A = f_A \tilde{\Diamond} f_A$ implies that f_A is an idempotent. Similarly, g_B is an idempotent. Now using Theorem 32, $f_A \tilde{\Diamond} g_B = f_A \tilde{\cap} g_B$. Since by Theorem 26, $f_A \tilde{\cap} g_B$ is a DFS h-quasi-ideal of R, therefore $f_A \tilde{\Diamond} g_B$ is a DFS h-quasi-ideal of R.

(ii)⇒(i). Suppose A is any right h-ideal, then by Corollary 25, C_A is DFS right h-ideal of R. Therefore, by hypothesis $C_A = C_A \tilde{\Diamond} C_A$, so by Theorem 12, $C_A = C_{\overline{A^2}}$, hence A is idempotent. Similarly, B is also idempotent. Also, $C_A \tilde{\Diamond} C_B$

$= C_{\overline{AB}}$ follows that $\overline{AB}$ is a DFS h-quasi-ideal of R. Hence, using Lemma 35, R is h-hemiregular hemiring.

CONCLUSION

Due to the diverse application of both hemirings and soft sets, the new investigations using soft structures in hemirings are becoming the central focus for researchers. The present research achieved another milestone in the hemiring theory by developing double-framed soft h-ideal theory in hemirings. More precisely, this research introduced double-framed soft left h-ideals, DFS right h-ideals, DFS h-bi-ideals and DFS h-quasi-ideals of hemirings. Double-framed soft including sets and characteristic double-framed soft functions are used to provide the bridge between ordinary h-ideals and double-framed soft h-ideals of hemirings. An important class of hemirings i.e. h-hemiregular hemirings are characterized by the properties of the aforementioned double-framed soft h-ideals of hemirings which yields several characterization theorems of hemirings. The research at hand will further motivate the researcher to apply the concept of double-framed soft sets in other algebraic structures which will ultimately be applied in various applied fields of science.

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