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$\overline{\mathbf{R}}$- Fuzzy Sets in Factor Space

(Set Kabur $\overline{\mathbf{R}}$ dalam Ruang Faktor)

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ABSTRACT

Interval-valued fuzzy sets, both-branch fuzzy sets and $\overline{\mathbf{R}}$*-Fuzzy sets are three extended forms of ordinary fuzzy sets. Compared to traditional fuzzy sets, interval-valued fuzzy sets have a stronger ability to express uncertainty because they provide more choices for the descriptions of attributes. Using both-branch fuzzy sets has solved some problems in engineering decision and engineering control. For* $\overline{\mathbf{R}}$*-Fuzzy sets, a new fuzzy system is constructed, with both-branch fuzzy sets as a special case. The resulting theorems improve the flexibility of uncertainty system models and expand the range of fuzzy system applications as well.*

Keywords: Envelope of feedback extension; factor space; feedback extension; representation extension; $\overline{\mathbf{R}}$*-Fuzzy set*

ABSTRAK

Set kabur nilai selang, kedua-dua cabang set kabur dan set kabur $\overline{\mathbf{R}}$ *adalah tiga bentuk lanjutan daripada set kabur biasa. Berbanding dengan set kabur tradisi, set kabur nilai selang mempunyai lebih keupayaan untuk menunjukkan ketidakpastian kerana ia menyediakan lebih banyak pilihan untuk gambaran atribut. Penggunaan kedua-dua cabang set kabur telah menyelesaikan beberapa masalah dalam keputusan kejuruteraan dan kawalan kejuruteraan. Untuk set kabur* $\overline{\mathbf{R}}$*, satu sistem kabur baru dibina, dengan kedua-dua cabang set kabur dijadikan sebagai kes khas. Teorem yang terhasil meningkatkan kelenturan model sistem yang tidak menentu dan mengembangkan pelbagai aplikasi sistem kabur juga.*

Kata kunci: Perluasan maklum balas; perwakilan maklum balas; ruang faktor; sampul perluasan maklum balas; set kabur $\overline{\mathbf{R}}$

INTRODUCTION

In the early 1980s, Professor Peizhuang Wang introduced the concept of factor spaces. In 1982, he published the first article on factor spaces (Garg & Arora 2018; Van Hoof et al. 2018; Zhou et al. 2017), and subsequently this concept was further developed and applied. However, factor spaces were introduced based on Zadeh fuzzy sets, whereas the fuzzy decision and control of $(-\infty, 0)$ and $(1, +\infty)$ are also important. Professor Kaiquan Shi proposed the concept of both-branch fuzzy sets (Verma & Parihar 2017) in 1997, and the proposal of the $\overline{\mathbf{R}}$-Fuzzy set in 2008 by Professor Hongxing Li accommodated for the shortcomings of Zadeh fuzzy sets.

First, several concepts are defined:

Definition 1.1 (Konieczny et al. 2017): For a given left matching $(U, V]$, if a factor family $F \subset V$, then the set family $\{X(f)\}_{f \subset T}$ is a factor space in U if it satisfies the axioms:

$F = F(\wedge, \vee, c, 1, 0)$ is a complete Boolean algebra; $X(0) = \{0\}$; For any $T \subset F$, if factor family T are pairwise independent, then $\underset{f \in T}{\vee} f = \prod_{f \in T} f \cdot \prod_{f \in T} f$ is herein referred to as the direct product of mappings (factors).

F is the factor family, $f \subset F$ are the factors, and $X(f)$ is the state space of f.

Definition 1.2 (Rikka et al. 2018): Zadeh fuzzy set μ_A: $X \to [0,1]$, $x \to \mu_A(x)$. μ_A is the membership function, and $\mu_A(x)$ is the membership degree of x to A.

Definition 1.3 (Neill et al. 2017): Both-branch fuzzy set: $X \to [-1,1]$, $x \to S(x)$. $S(x)$ is the fuzzy kiss function of x to S. For the given $x_0 \subset X$, $S(x_0)$ is the fuzzy kiss degree of x_0 to S.

Professor Hongxing Li expanded the Zadeh fuzzy set and, with both-branch fuzzy sets as a special case, put forward the concept of $\overline{\mathbf{R}}$-Fuzzy set.

Definition 1.4 (Story et al. 2018): $\overline{\mathbf{R}}$-Fuzzy set: $A : X \to °$ and ° represents the generalized set of real numbers, i.e. $\bar{°} = ° \cup \{-\infty, +\infty\}$.

All $\overline{\mathbf{R}}$-Fuzzy sets on X are written as $\bar{°}^X$. When A is a bounded function, it is called aset. All bounded Fuzzy sets on X are denoted $BF(X)$. Assuming $A \in BF(X)$, then there are $c, d \in °$, $c \leq d$, making $A : X \to °$. It can be represented $A : X \to [c,d]$. IF $c \geq 0$, $d \leq 1$, bounded $\overline{\mathbf{R}}$-Fuzzy sets reduce to Zadeh fuzzy sets.

In fact, mappings can be used to connect Zadeh fuzzy sets with $\overline{\mathbf{R}}$ - Fuzzy sets, i.e.

$$A(x) \triangleq \begin{cases} -\infty & x = 0; \\ \tan[\pi(x-0.5)] & x \in (0,1); \\ +\infty & x = 1; \end{cases} \quad (1)$$

In Zadeh fuzzy sets, the [0, 1] measure is used to represent degree of membership. Likewise, for a bounded Fuzzy set A, $A(x) \in [0,d]$ shows the belonging degree of x to A. When $A(x) = d$, x can be supposed to totally belong to A; when $A(x) = c$, x does not belong to A; when $c < A(x) < d$, $A(x) \in [0,d]$ means the degree of membership is some degree between c and d. Specifically, when $c < 0$ and $d \geq 0$, $A(x) \in [0,d]$ means the degree of x belonging to A; $A(x) \in [c,0)$ refers to the degree of x not belonging to A.

The explanation for $\overline{\mathbf{R}}$ - Fuzzy sets A : when A is unbounded, if $\sup\{A(x) | x \in X\} = +\infty$, i.e., $\forall d > 0$, $\exists x \in X$, making $A(x) > d$, this represents that there are no elements totally belonging to A in X, but there are elements that are an arbitrarily close approximation to 'totally belonging'. If a concept of totally belonging is needed, it can be achieved by a supplementary provision. For example, by the inverse transformation of (1), a $\overline{\mathbf{R}}$ - Fuzzy set can be turned into an ordinary fuzzy.

OPERATION AND EXTENSION OF FUZZY SETS

$F = F(\vee, \wedge, c, 1, 0)$ in Definition 1.1 1) is a complete Boolean algebra. Now we are going to study the algebraic structure of F when 0 and 1 are substituted with $-\infty$ and $-\infty$.

Definition 2.1 Two operations "∨" and "∧" are defined in a set $(-\infty, +\infty)$: $\forall \alpha, \beta, \gamma \in (-\infty, +\infty)$, ∨ @sup, ∧ @inf .

Idempotency: $\alpha \vee \alpha = \alpha, \alpha \wedge \alpha = \alpha$; Commutativity: $\alpha \vee \beta = \beta \vee \alpha$, $\alpha \wedge \beta = \beta \wedge \alpha$; Associativity: $(\alpha \vee \beta) \vee \gamma = \alpha \vee (\beta \vee \gamma)$, $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$; Absorption law: $(\alpha \vee \beta) \wedge \alpha = \alpha$, $(\alpha \wedge \beta) \vee \alpha = \alpha$, so that $(\vee, \wedge, -\infty, +\infty)$ is a grid.

Definition 2.2 If the grid $(\vee, \wedge, -\infty, +\infty)$ satisfies the distributive law, $(\alpha \vee \beta) \wedge \gamma = (\alpha \wedge \gamma) \vee (\beta \wedge \gamma)$, $(\alpha \wedge \beta) \vee \gamma = (\alpha \vee \gamma) \wedge (\beta \vee \gamma)$, then $(\vee, \wedge, -\infty, +\infty)$ is a distribution grid. In $(\vee, \wedge, c, 1, 0)$, there is the maximum element 1 and the minimum element 0; likewise, in $(\vee, \wedge, c, +\infty, -\infty)$, the 'maximal element' and the 'minimal element' are defined.

Definition 2.3 In the grid $(\vee, \wedge, -\infty, +\infty)$, $+\infty$ is defined as the maximal element and $-\infty$ is the minimal element. If they satisfy:

$$\forall x \in (-\infty, +\infty), x \vee (-\infty) = x, x \wedge (-\infty; x \vee (+\infty, x \wedge (+\infty) = x;$$

then there are maximal element $+\infty$ and minimal element $-\infty$ in $(\vee, \wedge, -\infty, +\infty)$.

Definition 2.4 If a complement c is defined in a distribution grid $(\vee, \wedge, -\infty, +\infty)$ with maximal element and minimal element, i.e. $\forall x \in (-\infty, +\infty)$, then $x^c = -x$. If it satisfies both resiliency law: $c(c\alpha) = c(-\alpha) = \alpha$; and complementarity law: $\alpha \vee (c\alpha) = +\infty$, $\alpha \wedge (c\alpha) = -\infty$, then $(\vee, \wedge, -\infty, +\infty)$ is a Boolean algebra.

Definition 2.5 If the Boolean algebra $(\vee, \wedge, -\infty, +\infty)$ satisfies:

$$\vee\{\alpha | \alpha \in (-\infty, +\infty)\} \ @\sup\{\alpha | \alpha \in (-\infty, +\infty)\};$$

$$\wedge\{\alpha | \alpha \in (-\infty, +\infty)\} \ @\inf\{\alpha | \alpha \in (-\infty, +\infty)\};$$

then $(\vee, \wedge, -\infty, +\infty)$ is a complete Boolean algebra. In this way, $(\vee, \wedge, c, 1, 0)$ is extended to $(\vee, \wedge, c, +\infty, -\infty)$.

EXTENDED DEFINITION OF FACTOR SPACE

Definition 3.1 For a given left matching $(U, V]$, if factor family $F \subset V$, then $\{X(f)\}_{(f \in F)}$ is a factor space in U if it satisfies the axioms:

$F = F(\vee, \wedge, c, +\infty, -\infty)$ is a complete Boolean algebra; $X(0) = \{0\}$; For any $T \subset F$, if factor family T is pairwise independent, then $\vee_{f \in T} f = \prod_{f \in T} f$. $\prod_{f \in T} f$ is herein referred to as the direct product of mappings (regarding the factors as mappings). F is the factor set, $f \in F$ is the factor and $X(f)$ is the state space of f.

BASIC PROPERTIES AND THEOREMS OF $\overline{\mathbf{R}}$ - Fuzzy SETS

In 2008, Professor Hongxing Li introduced the concept of $\overline{\mathbf{R}}$ - Fuzzy sets, which have a wider range of applications compared to the classic fuzzy sets, interval value fuzzy sets and both-branch fuzzy sets. For specific applications, please refer to reference (Baggen et al. 2017; Jaffe et al. 2018; Zvika & Guy 2017).

Definition 4.1 For a given domain, $A: X \to P((-\infty, +\infty))$, $X^+(A) = \{x \in X | A(x) \text{I} [0, +\infty) \neq \varnothing\}$ is the upper domain of A; $X^-(A) = \{x \in X | A(x) \text{I} [-\infty, 0) \neq \varnothing\}$ is the lower domain of A; and $X^*(A) = X^+(A) \text{I} X^-(\text{A})$ is the domain of A.

A^+ is the $\overline{\mathbf{R}}$ - Fuzzy set in the upper domain X^+, which is called the upper $\overline{\mathbf{R}}$ - Fuzzy set; A^- is the $\overline{\mathbf{R}}$ - Fuzzy set in the upper domain X^-, which is called lower $\overline{\mathbf{R}}$ - Fuzzy set; A^* is the $\overline{\mathbf{R}}$ - Fuzzy set in the upper domain X^*, which is called the domain $\overline{\mathbf{R}}$ - Fuzzy set.

Definition 4.2 Assuming, $\lambda \in (-\infty, +\infty)$, $A_\lambda, A_{\underset{\cdot}{\lambda}} \in P(X)$ are λ- cut set and λ-strong cut set of $A \in \overset{-X}{\circ}$, respectively, if they satisfy: $A_\lambda = \{x \| A(x) |\geq| \lambda |\}$, $A_{\underset{\cdot}{\lambda}} = \{x \| A(x) |>| \lambda |\}$. Hence, the λ- cut set A_λ^+ and λ-strong cut set $A_{\underset{\cdot}{\lambda}}^+$ of A in upper domain X^+ are: $A_\lambda^+ = \{x | A^+(x) \geq \lambda\}$, $A_{\underset{\cdot}{\lambda}}^+ = \{x | A(x) > \lambda\}$.

The λ- cut set A_λ^- and λ-strong cut set λ of A in upper domain X^- are: $A^{-} = \{x | A^-(x) \leq \lambda\}$. Obviously: $A_\lambda^+, A_{\underset{\cdot}{\lambda}}^+ \subset X^+ \subset X$; $A_\lambda^-, A_{\underset{\cdot}{\lambda}}^- \subset X^- \subset X$.

Definition 4.3 Assuming $A^{+}_{\lambda}\in P(X)$ is the λ- cut set of $A\in\bar{\circ}^{X}$ in $X^{+}\cup X^{*}\subset X$, then $A^{+}_{\lambda}(x)$ is the feature function of x with respect to $A^{+}_{\lambda}\subset X+\cup X^{*}$, and

$$A^{+}_{\lambda}(x)=\begin{cases}+\infty & x\in A^{+}_{\lambda}\\ 0 & x\notin A^{+}_{\lambda}\end{cases}.$$

Definition 4.4 Assuming is the λ - cut set of $A\in\bar{\circ}^{X}$ in $X^{-}\cup X^{*}\subset X$, then $A^{-}_{\lambda}(x)$ is the feature function of x with respect to $A^{-}_{\lambda}\subset X-\cup X^{*}$, and

$$A^{-}_{\lambda}(x)=\begin{cases}-\infty & x\in A^{-}_{\lambda}\\ 0 & x\notin A^{-}_{\lambda}\end{cases}.$$

Definition 4.5 Assume $A\in\bar{\circ}^{X}$, $\lambda\in(-\infty,+\infty)$. Let $\lambda A\in\bar{\circ}^{X}$, with the membership function of λA defined as: $\lambda A(x)=\lambda\wedge A(x)$.

At this point, the preparation has been finished. Next, the decomposition theorems of $\bar{\mathbf{R}}$-Fuzzy sets are introduced.

Theorem 4.1 ($\bar{\mathbf{R}}$-Fuzzy set-decomposition theorem I) Assuming $A\in\bar{\circ}^{X}$ is an $\bar{\mathbf{R}}$-Fuzzy set in X and $A_{\lambda}\in P(X)$ is the λ-cut set of $A\in\bar{\circ}^{X}$, then $A=\bigcup\limits_{\lambda\in(-\infty,+\infty)}\lambda A_{\lambda}$.

Proof

If $\lambda\in(-\infty,0]$, then $\forall x\in X^{-}\cup X^{*}$, i.e.

$$\begin{aligned}\bigcup_{\lambda\in(-\infty,+\infty)}\lambda A_{\lambda}(x)&=\bigvee_{\lambda\in(-\infty,0]}(\lambda\wedge A_{\lambda}(x));\\
&=(\bigvee_{\lambda\in(-\infty,A(x)]}(\lambda\wedge A_{\lambda}(x)))\vee(\bigvee_{\lambda\in(A(x),0]}(\lambda\wedge A_{\lambda}(x)))\\
&=\bigvee_{\lambda\in(-\infty,A(x)]}(\lambda\wedge A_{\lambda}(x))\\
&=\bigvee_{\lambda\in(-\infty,A(x)]}\lambda\\
&=A(x)\end{aligned}$$

If $\lambda\in[0,-\infty)$, it can be proven by the same argument. Therefore, $A=\bigcup\limits_{\lambda\in[0,+\infty)}\lambda A_{\lambda}$.

Theorem 4.2 ($\bar{\mathbf{R}}$-Fuzzy set-decomposition theorem II) Assuming $A\in\bar{\circ}^{X}$ is an $\bar{\mathbf{R}}$-Fuzzy set in X and $A_{\underset{\cdot}{\lambda}}\in P(X)$ is the λ-strong cut set of $A\in\bar{\circ}^{X}$, then $A=\bigcup\limits_{\lambda\in(-\infty,+\infty)}\lambda A_{\underset{\cdot}{\lambda}}$.

Proof The same as that of Theorem 4.1.

Theorem 4.3 ($\bar{\mathbf{R}}$-Fuzzy set- decomposition theorem III) Assume $A\in\bar{\circ}^{X}$ is an $\bar{\mathbf{R}}$-Fuzzy set in X, while mapping $H:(-\infty,+\infty)\to P(X)$, $\lambda\to H(\lambda)$ satisfies that if $\lambda\in(-\infty,+\infty)$, then $A_{\underset{\cdot}{\lambda}}\subseteq H(\lambda)\subseteq A_{\lambda}$. Therefore, $A=\bigcup\limits_{\lambda\in(-\infty,+\infty)}\lambda H(\lambda)$.

Proof 1) If $\lambda\in(-\infty,0]$, since $A_{\underset{\cdot}{\lambda}}\subseteq H(\lambda)\subseteq A_{\lambda}$, hence $\lambda A_{\underset{\cdot}{\lambda}}\subseteq\lambda H(\lambda)\subseteq\lambda A_{\lambda}$.

From Theorem 4.1 it is known that $A=\bigcup\limits_{\lambda\in(-\infty,0]}\lambda A_{\lambda}$, and $A=\bigcup\limits_{\lambda\in(-\infty,0]}\lambda A_{\underset{\cdot}{\lambda}}$ from Theorem 4.2. So $A=\bigcup\limits_{\lambda\in(-\infty,0]}\lambda A_{\underset{\cdot}{\lambda}}\subseteq\bigcup\limits_{\lambda\in(-\infty,0]}\lambda H(\lambda)\subseteq\bigcup\limits_{\lambda\in(-\infty,0]}\lambda A_{\lambda}=A$, and from the squeeze theorem we can know $A=\bigcup\limits_{\lambda\in(-\infty,0]}\lambda H(\lambda)$.

If $\lambda\in[0,+\infty)$, the proof is the same as 1).

Therefore, $A=\bigcup\limits_{\lambda\in(-\infty,+\infty)}\lambda H(\lambda)$.

The representation theorem of $\bar{\mathbf{R}}$-Fuzzy sets is introduced herewith. First, several relevant concepts and lemmas are introduced.

Definition 4.6 Assume the mapping $H:(-\infty,+\infty)\to P(X)$ satisfies 1) $\forall\lambda_1,\lambda_2\in(-\infty,0]$, and $\lambda_1\le\lambda_2$, then $H(\lambda_1)\supseteq H(\lambda_2)$; 2) $\forall\lambda_1,\lambda_2\in[0,+\infty)$, and $\lambda_1\le\lambda_2$, then $H(\lambda_1)\subseteq H(\lambda_2)$. H is then called the nested set in X. All the nested sets in X is marked as $R(X)$. (H is an ordinary set rather than a fuzzy set).

Definition 4.7 Assuming $H(\lambda)\in R(X)$, if $\lambda\in(-\infty,0]$, $H(\lambda)$ is the feature function of x with respect to $H(\lambda)$, and

$$H(\lambda)(x)=\begin{cases}-\infty & x\in H(\lambda)\\ 0 & x\notin H(\lambda)\end{cases}.$$

Definition 4.8 Assuming $H(\lambda)\in R(X)$, if $\lambda\in[0,+\infty)$, $H(\lambda)$ is the feature function of x with respect to $H(\lambda)$, and

$$H(\lambda)(x)=\begin{cases}+\infty & x\in H(\lambda)\\ 0 & x\notin H(\lambda)\end{cases}.$$

Definition 4.9 Assuming $H(\lambda)\in R(X)$ and $\lambda\in(-\infty,+\infty)$, define $(\lambda H(\lambda))(x)=\lambda\wedge H(\lambda)(x)$. Obviously, if $\lambda\in(-\infty,0]$, then

$$(\lambda H(\lambda))(x)=\begin{cases}-\infty & x\in H(\lambda)\\ \lambda & x\notin H(\lambda)\end{cases};$$ if $\lambda\in[0,+\infty)$, then

$$(\lambda H(\lambda))(x)=\begin{cases}\lambda & x\in H(\lambda)\\ 0 & x\notin H(\lambda)\end{cases}.$$

Lemma 4.1 Let $A\in\bar{\circ}^{X}$ be an $\bar{\mathbf{R}}$-Fuzzy set on X.

If $\lambda_1,\lambda_2\in(-\infty,0]$ and $\lambda_1<\lambda_2$, then $A_{\lambda_1}\subseteq A_{\lambda_2}$, $A_{\underset{\cdot}{\lambda_1}}\subseteq A_{\underset{\cdot}{\lambda_2}}$, $A_{\lambda_1}\subseteq A_{\underset{\cdot}{\lambda_2}}$; 2) If $\lambda_1,\lambda_2\in[0,+\infty)$ and $\lambda_1<\lambda_2$, then $A_{\lambda_1}\supseteq A_{\lambda_2}$, $A_{\underset{\cdot}{\lambda_1}}\supseteq A_{\underset{\cdot}{\lambda_2}}$, $A_{\underset{\cdot}{\lambda_1}}\supseteq A_{\lambda_2}$.

Proof It obviously holds.

Lemma 4.2 Assume $A\in\bar{\circ}^{X}$, $t\in T$ $\lambda_t\in(-\infty,+\infty)$. Then:

If $\lambda_t\in(-\infty,0]$, then $\bigcap\limits_{t\in T}A_{\lambda_t}=A_{(\bigwedge\limits_{t\in T}\lambda_t)}$, $\bigcup\limits_{t\in T}A_{\lambda_t}=A_{(\bigvee\limits_{t\in T}\lambda_t)}$; 2) If $\lambda_t\in[0,+\infty)$, then $\bigcap\limits_{t\in T}A_{\lambda_t}=A_{(\bigvee\limits_{t\in T}\lambda_t)}$, $\bigcup\limits_{t\in T}A_{\underset{\cdot}{\lambda_t}}=A_{\underset{\cdot}{(\bigwedge\limits_{t\in T}\lambda_t)}}$.

Proof

1) Assuming $\lambda_t\in(-\infty,0]$, then $u\in A_{(\bigwedge\limits_{t\in T}\lambda_t)}\Leftrightarrow A(u)\le\bigwedge\limits_{t\in T}\lambda_t$
$\Leftrightarrow\forall t\in T$, then $A(u)\le\lambda_t$
$\Leftrightarrow\forall t\in T$, $U\in A_{\lambda_t}$,
$\Leftrightarrow u\in\bigcap\limits_{t\in T}A_{\lambda_T}$

i.e. $\bigcap_{t\in T} A_{\lambda_t} = A_{(\underset{t\in T}{\wedge}\lambda_t)}$

$u \in A_{\underset{\cdot}{(\underset{t\in T}{\vee}\lambda_t)}} \Leftrightarrow A(u) < \underset{t\in T}{\vee}\lambda_t$

$\Leftrightarrow \exists t_0 \in T$, making $A(u) < \lambda_{t_0}$

$\Leftrightarrow \exists t_0 \in T, u \in A_{\underset{\cdot}{\lambda_{t_0}}}$

$\Leftrightarrow u \in \bigcup_{t\in T} A_{\underset{\cdot}{\lambda_t}}$,

i.e. $\bigcup_{t\in T} A_{\underset{\cdot}{\lambda_t}} = A_{\underset{\cdot}{(\underset{t\in T}{\vee}\lambda_t)}}$

2) The same argument is used.

Theorem 4.4 (Representation theorem of $\overline{\mathbf{R}}$ - Fuzzy set) Assuming $H(\lambda) \in R(X)$, then $\bigcup_{\lambda\in(-\infty,+\infty)} \lambda H(\lambda)$ is a $\overline{\mathbf{R}}$ - Fuzzy set on X and is marked as A; $\bigcup_{\lambda\in(-\infty,0]} \lambda H(\lambda)$ is a lower $\overline{\mathbf{R}}$ - Fuzzy set on X and $\bigcup_{\lambda\in[0,+\infty)} \lambda H(\lambda)$ is an upper $\overline{\mathbf{R}}$ - Fuzzy set on X.

In addition, 1) If $\lambda \in (-\infty,0]$, then

a) $A_\lambda = \bigcap_{k>\lambda} H(k) \quad \lambda \neq 0$ (1)

b) $A_{\underset{\cdot}{\lambda}} = \bigcup_{k>\lambda} H(k) \quad \lambda \neq +\infty$ (2)

2) If $\lambda \in [0, +\infty)$, then

a) $A_\lambda = \bigcap_{k<\lambda} H(k) \quad \lambda \neq 0$ (3)

b) $A_{\underset{\cdot}{\lambda}} = \bigcup_{k>\lambda} H(k) \quad \lambda \neq +\infty$ (4)

Proof $\overline{\mathbf{R}}$ - Fuzzy set: $\forall\lambda \in (-\infty,+\infty)$, that is $\forall\lambda \in (-\infty,+\infty)$ and $\bigcup_{\lambda\in(-\infty,+\infty)} \lambda H(\lambda) \in \bar{\circ}^{X}$, let $A = \bigcup_{\lambda\in(-\infty,+\infty)} \lambda H(\lambda)$. From Definition 4.9 it follows that $\bigcup_{\lambda\in(-\infty,0]} \lambda H(\lambda) \in (-\infty,0]$, so $\bigcup_{\lambda\in(-\infty,0]} \lambda H(\lambda)$ is a lower $\overline{\mathbf{R}}$ - Fuzzy set on X. With the same argument, from Definition 4.9 it follows that $\bigcup_{\lambda\in[0,+\infty)} \lambda H(\lambda) \in [0,+\infty)$, so $\bigcup_{\lambda\in[0,+\infty)} \lambda H(\lambda)$ is a upper $\overline{\mathbf{R}}$ - Fuzzy set on X.

The four formulas are proven as follows:

To prove these four formulas, $\forall\lambda \in (-\infty,+\infty)$, $A_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda$ has to be proven first.

Proof If $\lambda \in (-\infty,0]$, assume $x \notin H(\lambda)$, i.e. $H(\lambda) = 0$. Since

$$A(x) = \bigcup_{\alpha\in(-\infty,0]} \alpha H(\alpha)(x) = \underset{\alpha\in(-\infty,0]}{\vee} (\alpha \wedge H(\alpha))(x) \geq \lambda \wedge H(\lambda)(x) = \lambda,$$

i.e. $A(x) \geq \lambda$, then $x \notin A_{\underset{\cdot}{\lambda}}$ and $A_{\underset{\cdot}{\lambda}} \subseteq H(\lambda)$.

Assume $x \notin A_\lambda$, i.e. $A(x) > \lambda$. Since $A(x) = \bigcup_{\alpha\in(-\infty,0]} \alpha H(\alpha)(x)$, then $\bigcup_{\alpha\in(-\infty,0]} \alpha H(\alpha)(x) > \lambda$, $\underset{\alpha\in(-\infty,0]}{\vee} (\alpha \wedge H(\alpha))(x) > \lambda$.

Therefore, $\exists\alpha_0 \in (-\infty,0]$, making $\alpha_0 \wedge H(\alpha_0)(x) > \lambda$. So $\exists\alpha_0 \in (-\infty,0]$, making $\alpha_0 > \lambda$ and $H(\alpha_0)(x) > \lambda$.

Since $H(\alpha_0)(x)$ can only be $-\infty$ or 0, it follows that $H(\alpha_0)(x) = 0$, so it follows that $x \notin H(\alpha_0)$ from Definition 4.7.

Since $\alpha_0 > \lambda$, it can be known that $H(\alpha_0) \subseteq H(\lambda)$ from Definition 4.6, hence $x \notin H(\lambda)$ and $H(\lambda) \subseteq A_\lambda$.

Therefore, if $\forall\lambda \in (-\infty,0]$, $A_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda$.

If $\lambda \in [0,+\infty)$, it can be proven by the same argument. Therefore, $\forall\lambda \in (-\infty,+\infty)$, $A_{\underset{\cdot}{\lambda}} \subseteq H(\lambda) \subseteq A_\lambda$ holds.

The proof of the four formulas:

Formula (1): For $\forall\lambda \in (-\infty,0]$, $k > \lambda$, it can be known that $A_{\underset{\cdot}{\lambda}} \subseteq A_k$ from Lemma 4.1.

Since $A_{\underset{\cdot}{k}} \subseteq H(k) \subseteq A_k$ holds, it follows that $A_\lambda \subseteq A_{\underset{\cdot}{k}} \subseteq H(k) \subseteq A_k$ and $A_\lambda \subseteq \bigcap_{k>\lambda} H(k), \lambda \neq 0$; since $H(k) \subseteq A_k$, it follows that $H(k) \subseteq A_k$. From Lemma 4.2 it is known that $\bigcap_{k>\lambda} A_k = A_{(\underset{k>\lambda}{\wedge} k)}$ and $A_{(\underset{k>\lambda}{\wedge} k)} = A_\lambda$, so $A_\lambda \supseteq \bigcap_{k>\lambda} H(k), \lambda \neq 0$. Therefore, $A_\lambda = \bigcap_{k>\lambda} H(k), \lambda \neq 0$, that is, Formula (1) holds.

Formula (2): For $\forall\lambda \in (-\infty,0]$, $k < \lambda$, it can be known that $A_k \subseteq A_{\underset{\cdot}{\lambda}}$ from Lemma 4.1. Since $A_{\underset{\cdot}{k}} \subseteq H(k) \subseteq A_k$ holds, it follows $A_{\underset{\cdot}{k}} \subseteq H(k) \subseteq A_k \subseteq A_{\underset{\cdot}{k}}$ and; since $H(k) \supseteq A_{\underset{\cdot}{k}}$, it follows that $\bigcup_{k<\lambda} H(k) \supseteq \bigcup_{k<\lambda} A_{\underset{\cdot}{k}}$. From Lemma 4.2 it can be known that $\bigcup_{k<\lambda} A_k = A_{\underset{\cdot}{(\underset{k>\lambda}{\vee} k)}}$ and $A_{\underset{\cdot}{(\underset{k<\lambda}{\vee} k)}} = A_{\underset{\cdot}{\lambda}}$, so $A_{\underset{\cdot}{\lambda}} \subseteq \bigcup_{k<\lambda} H(k), \lambda \neq -\infty$. Therefore, $A_{\underset{\cdot}{\lambda}} = \bigcup_{k<\lambda} H(k), \lambda \neq -\infty$, that is, Formula (1) holds. Formula (3) and Formula (4) can be proven by the same arguments. In summary, the proof of Theorem 4.4 is completed.

EXTENSION PRINCIPLE OF SET

Let $\bar{\circ}^{X}$ be a set of $\overline{\mathbf{R}}$ - Fuzzy sets on X, and $\bar{\circ}^{Y}$ be a set of $\overline{\mathbf{R}}$ - Fuzzy sets on Y.

Definition 5.1 (Max-max extension principle) Assume mapping $f: X \to Y, x \mapsto f(x)$. Then f can induce a mapping from to and a mapping from $\bar{\circ}^{X}$ to $\bar{\circ}^{Y}$

$$f; \bar{\circ}^{X} \to \bar{\circ}^{Y}, A \mapsto f(A)$$

$$f^{-1}; \bar{\circ}^{Y} \to \bar{\circ}^{X}, B \mapsto f^{-1}(B).$$

The fuzzy kiss functions of $f(A)$ and $f^{-1}(B)$ are defined as:

$f(A)^-(y) = \underset{f(x)=y}{\vee} A^-(x)$, $f(A)^+(y) = \underset{f(x)=y}{\vee} A^+(x)$, respectively, (when $\{x \in X \mid f(x) = y\} = \varnothing$, $f(A)(y) = 0$). $f(A)$ is the image of A and $f^{-1}(B)$ is the inverse image of B.

Definition 5.2 (Min-min extension principle) Assume mapping $f: X \to Y, x \mapsto f(x)$. Then f can induce a mapping from $\bar{\circ}^{X}$ to $\bar{\circ}^{Y}$ and a mapping from $\bar{\circ}^{Y}$ to $\bar{\circ}^{X}$

$$f: \bar{\circ}^{X} \to \bar{\circ}^{Y}, A \mapsto f(A)$$

$$f^{-1}: \bar{\circ}^{Y} \to \bar{\circ}^{X}, B \mapsto f^{-1}(B).$$

The fuzzy kiss functions of $f(A)$ and $f^{-1}(B)$ are defined as:

$f(A)^{-}(y) = \underset{f(x)=y}{\wedge} A^{-}(x), f(A)^{+}(y) = \underset{f(x)=y}{\wedge} A^{+}(x)$, respectively, (when $\{x \in X \mid f(x) = y\} = \varnothing, f(A)(y) = 0$). $f(A)$ is the image of A and $f^{-1}(B)$ is the inverse image of B.

Definition 5.3 (Min-max extension principle) Assume mapping $f: X \to Y, x \mapsto f(x)$. Then f can induce a mapping from $\bar{\circ}^{X}$ to $\bar{\circ}^{Y}$ and a mapping from $\bar{\circ}^{Y}$ to $\bar{\circ}^{X}$.

$$f: \bar{\circ}^{X} \to \bar{\circ}^{Y}, A \mapsto f(A)$$

$$f^{-1}: \bar{\circ}^{Y} \to \bar{\circ}^{X}, B \mapsto f^{-1}(B).$$

The fuzzy kiss functions of $f(A)$ and $f^{-1}(B)$ are defined as:

$f(A)^{-}(y) = \underset{f(x)=y}{\wedge} A^{-}(x), f(A)^{+}(y) = \underset{f(x)=y}{\vee} A^{+}(x)$, respectively, (when $\{x \in X \mid f(x) = y\} = \varnothing, f(A)(y) = 0$). is the image of A and $f^{-1}(B)$ is the inverse image of B.

Definition 5.4 (Max-min extension principle) Assume mapping $f: X \to Y, x \mapsto f(x)$. Then f can induce a mapping from $\bar{\circ}^{X}$ to $\bar{\circ}^{Y}$ and a mapping from $\bar{\circ}^{Y}$ to $\bar{\circ}^{X}$.

$$f: \bar{\circ}^{X} \to \bar{\circ}^{Y}, A \mapsto f(A)$$

$$f^{-1}: \bar{\circ}^{Y} \to \bar{\circ}^{X}, B \mapsto f^{-1}(B)$$

The fuzzy kiss functions of $f(A)$ and $f^{-1}(B)$ are defined as:

$f(A)^{-}(y) = \underset{f(x)=y}{\wedge} A^{-}(x), f(A)^{+}(y) = \underset{f(x)=y}{\vee} A^{+}(x)$, respectively, (when $\{x \in X \mid f(x) = y\} = \varnothing, f(A)(y) = 0$). $f(A)$ is the image of A and $f^{-1}(B)$ is the inverse image of B.

To distinguish them, the various $f(A)$ obtained by the four extension principles are referred to as $f_{\text{max-max}}(A)$, $f_{\text{min-min}}(A)$, $f_{\text{min-max}}(A)$, $f_{\text{max-min}}(A)$.

The rationality of the discussed definitions can be proven by the representation theorem of $\overline{\mathbf{R}}$ - Fuzzy sets. The rationality of the max-max extension principle is proven herewith.

Theorem 5.1 Assume Mapping $f: X \to Y, x \mapsto f(x)$

1) If $A \in F(X)$, then

$f_{\text{max-max}}(A)^{-}(y) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f(A_{\lambda}))(y), f_{\text{max-max}}(A)^{+}(y) = (\underset{\lambda \in [0,+\infty)}{\cup} \lambda f(A_{\lambda}))(y);$

If $B \in F(Y)$, then

$$(f^{-2}(B))^{-}(x) = B^{-}(f(x)) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f^{-1}(B_{\lambda}))(x);$$

$$(f^{-1}(B))^{+}(x) = B^{+}(f(x)) = (\underset{\lambda \in [0,+\infty)}{\cup} \lambda f^{-1}(B_{\lambda}))(x)$$

Proof 1) If $\{x \in X \mid f(x) = y\} = \varnothing$, i.e. $f(A)(y) = 0$, then the theorem holds obviously. For $\{x \in X \mid f(x) = y\} \neq \varnothing$, if $\lambda \in (-\infty, 0]$, then

$$\begin{aligned}(\underset{\lambda \in (-\infty,0]}{\cup} \lambda f(A_{\lambda}))(y) &= \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge f(A_{\lambda})(y)) \\ &= \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge (\underset{f(x)=y}{\vee} A_{\lambda}(x))) \\ &= \underset{f(x)=y}{\vee} (\lambda \wedge (\underset{\lambda \in (-\infty,0]}{\vee} A_{\lambda}(x))) \\ &= \underset{f(x)=y}{\vee} A^{-}(x) \\ &= f_{\text{max-max}}(A)^{-}(y)\end{aligned}$$

It can also be proven by the same argument if $\lambda \in [0, +\infty)$. Therefore,

$$f_{\text{max-max}}(A)^{-}(y) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f(A_{\lambda}))(y),$$

$$f_{\text{max-max}}(A)^{+}(y) = (\underset{\lambda \in [0,+\infty)}{\cup} \lambda f(A_{\lambda}))(y).$$

2) If $\lambda \in (-\infty, 0]$, then

$$\begin{aligned}(\underset{\lambda \in (-\infty,0]}{\cup} \lambda f^{-1}(B_{\lambda}))(x) &= \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge f^{-1}(B_{\lambda})(x)) \\ &= \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge B_{\lambda}(f(x))) \\ &= B^{-}(f(x)) \\ &= (f^{-}(B))^{-}(x)\end{aligned}$$

It can also be proven by the same argument if $\lambda \in [0, +\infty)$. Therefore,

$$(f^{-1}(B))^{-}(x) = B^{-}(f(x)) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f^{-1}(B_{\lambda}))(x).$$

The rationality of the min-min extension principle is proven herewith.

Theorem 5.2 Assume mapping $f: X \to Y, x \mapsto f(x)$

1) If $A \in F(X)$, then

$$f_{\text{min-min}}(A)^{-}(y) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f(A_{\lambda}))(y),$$

$$f_{\text{min-min}}(A)^{+}(y) = (\underset{\lambda \in [0,+\infty)}{\cup} \lambda f(A_{\lambda}))(y);$$

2) If $B \in F(Y)$, then

$$(f^{-}(B))^{-}(x) = B^{-}(f(x)) = (\underset{\lambda \in (-\infty,0]}{\cup} \lambda f^{-1}(B_{\lambda}))(x);$$

$$(f^{-1}(B))^{+}(x) = B^{+}(f(x)) = (\underset{\lambda \in [0,+\infty)}{\cup} \lambda f^{-1}(B_{\lambda}))(x).$$

Proof 1) If $\{x \in X \mid f(x) = y\} = \varnothing$, i.e. $f(A)(y) = 0$, then the theorem holds obviously.

For $\{x \in X \mid f(x) = y\} \neq \varnothing$, if $\lambda \in (-\infty, 0]$, then

$$(\underset{\lambda \in (-\infty,0]}{\cup} \lambda f(A_{\lambda}))(y) = \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge (\underset{f(x)=y}{\wedge} A_{\lambda}(x))).$$

Obviously $\lambda \wedge (\underset{f(x)=y}{\wedge} A_{\lambda}(x)) \leq \lambda \wedge A_{\lambda}(x)$ s o $\underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge A_{\lambda}(x))) \leq \underset{\lambda \in (-\infty,0]}{\vee} (\lambda \wedge A_{\lambda}(x)))$.

Therefore, $\vee_{\lambda\in(-\infty,0]}(\lambda\wedge(\wedge_{f(x)=y} A_\lambda(x))) \le \wedge_{f(x)=y}(\vee_{\lambda\in(-\infty,0]}(\lambda\wedge A_\lambda(x)))$.

The proof of the establishment of the 'equal sign' is presented herewith. A proof by contradiction is used: if there is $\alpha\in(-\infty, 0]$, then $\vee_{\lambda\in(-\infty,0]}(\lambda \wedge (\wedge_{f(x)=y} A_\lambda(x))) \le \alpha < \wedge_{f(x)=y}(\vee_{\lambda\in(-\infty,0]}(\lambda \wedge A_\lambda(x)))$, but $A_\alpha(x)$ can only be $-\infty$ or 0.

a) If $\wedge_{f(x)=y} A_\alpha(x) = 0$, then

$$\alpha > \vee_{\lambda\in(-\infty,0]}(\lambda\wedge(\wedge_{f(x)=y} A_\lambda(x))) \ge \alpha\wedge(\wedge_{f(x)=y} A_\alpha(x)) = \alpha\wedge 0 = \alpha.$$

Therefore, it is a contradiction.

b) If $\wedge_{f(x)=y} A_\alpha(x) = -\infty$, then $x_0 \in (-\infty,0]$, and then $f(x_0) = y$ and $A_\alpha(x_0) = -1$. So, for any $\lambda \ge \alpha$, $A_\lambda(x_0) = -1$.

$$\begin{aligned}\alpha &< \wedge_{f(x)=y}(\vee_{\lambda\in(-\infty,0]}(\lambda \wedge A_\lambda(x))) \le \vee_{\lambda\in(-\infty,0]}(\lambda \wedge A_\lambda(x_0))\\ &= (\vee_{\lambda\in(-\infty,\alpha]}(\lambda \wedge A_\lambda(x_0))) \vee (\vee_{\lambda\in[\alpha,0]}(\lambda \wedge A_\lambda(x_0)))\\ &= \vee_{\lambda\in(-\infty,\alpha]}(\lambda \wedge A_\lambda(x_0)) \le \vee_{\lambda\in(-\infty,\alpha]} \lambda = \alpha\end{aligned}$$

Therefore, it is a contradiction.

Therefore, the 'equal sign' holds. That is, $\vee_{\lambda\in(-\infty,0]}(\lambda\wedge(\wedge_{f(x)=y} A_\lambda(x))) = \wedge_{f(x)=y}(\vee_{\lambda\in(-\infty,0]}(\lambda \wedge A_\lambda(x)))$.

Therefore, $(\bigcup_{\lambda\in(-\infty,0]} \lambda f(A_\lambda))(y) = \wedge_{f(x)=y}(\vee_{\lambda\in(-\infty,0]}(\lambda\wedge A_\lambda(x))) = \wedge_{f(x)=y} A^-(x) = f_{\text{min-min}}(A)^-(y)$. That is, if, $\lambda \in(-\infty,0]$, $(\bigcup_{\lambda\in(-\infty,0]} \lambda f(A_\lambda))(y) = f_{\text{min-min}}(A)^-(y)$, and it can also be proven by the same argument if $\lambda \in [0,+\infty)$.

Therefore, $f_{\text{min-min}}(A)^-(y) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f(A_\lambda))(y)$, $f_{\text{min-min}}(A)^+(y) = (\bigcup_{\lambda\in[0,+\infty)} \lambda f(A_\lambda))(y)$.

2) If $\lambda \in(-\infty, 0]$, then

$$\begin{aligned}(\bigcup_{\lambda\in(-\infty,0]} \lambda f^{-1}(B_\lambda))(x) &= \vee_{\lambda\in(-\infty,0]}(\lambda \wedge f^{-1}(B_\lambda)(x))\\ &= \vee_{\lambda\in(-\infty,0]}(\lambda \wedge B_\lambda(f(x)))\\ &= B^-(f(x))\\ &= (f^-(B))^-(x)\end{aligned}.$$

It can also be proven by the same argument if $\lambda\in[0,+\infty)$. So

$$(f^{-1}(B))^-(x) = B^-(f(x)) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f^{-1}(B_\lambda))(x);$$

$$(f^{-1}(B))^+(x) = B^+(f(x)) = (\bigcup_{\lambda\in[0,+\infty)} \lambda f^{-1}(B_\lambda))(x).$$

The rationality of the min-max extension principle is stated herewith.

Theorem 5.3 Assume mapping $f: X \to Y, x \mapsto f(x)$

1) If $A \in F(X)$, then

$$f_{\text{min-max}}(A)^-(y) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f(A_\lambda))(y),$$

$$f_{\text{min-max}}(A)^+(y) = (\bigcup_{\lambda\in[0,+\infty)} \lambda f(A_\lambda))(y)$$

2) If $B \in F(Y)$, then

$$(f^{-1}(B))^-(x) = B^-(f(x)) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f^{-1}(B_\lambda))(x)$$

The rationality of the max-min extension principle is stated herewith.

Theorem 5.4 Assume mapping $f: X \to Y, x \mapsto f(x)$

1) If $A \in F(Y)$, then

$$f_{\text{max-min}}(A)^-(y) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f(A_\lambda))(y),$$

$$f_{\text{max-min}}(A)^+(y) = (\bigcup_{\lambda\in[0,+\infty)} \lambda f(A_\lambda))(y)\cdot$$

2) If, then

$$(f^{-1}(B))^-(x) = B^-(f(x)) = (\bigcup_{\lambda\in(-\infty,0]} \lambda f^{-1}(B_\lambda))(x);$$

$$(f^{-1}(B))^+(x) = B^+(f(x)) = (\bigcup_{\lambda\in[0,+\infty)} \lambda f^{-1}(B_\lambda))(x).$$

The proofs of the two theorems are the same as those of Theorem 5.1 and 5.2, therefore, they are omitted.

Although we can study some properties of these extension principles, they are not further described here since they are not the focus of this article.

FOUR TYPES OF REPRESENTATION EXTENSIONS AND FEEDBACK EXTENSIONS OF THE $\overline{\mathbf{R}}$ - Fuzzy CONCEPT

With the extension principle mentioned before, every type of representation extension of $\overline{\mathbf{R}}$ - Fuzzy can be defined. With representation extensions, the feedback extension of a concept can be constructed and the feedback extension envelope can be defined as an approximation of the concept (Sefusatti et al. 2018; Siwak et al. 2017; Zhang et al. 2017).

Definition 6.1 For a given description frame (U, C, F), if $\alpha \in C$, assume the extension of concept α in U is the $\overline{\mathbf{R}}$ - Fuzzy set A. For all $f \in F$, mark

$$f(A): X(f) \to [0,1], x \mapsto f(A)(x).$$

$f(A)$ is a $\overline{\mathbf{R}}$ - Fuzzy set of representation domain $X(f)$ i.e. $f(A) \in \bar{\circ}^X$.

1) If $f(A) = f_{\text{max-max}}(A)$, and $\bar{\circ}^X$, then $f(A)$ is the max-max type representation extension of concept α in representation domain $X(f)$;
2) If $f(A) = f_{\text{min-min}}(A)$, and $f_{\text{min-min}}(A)^-(y) = \wedge_{f(x)=y} A^-(x)$, $f_{\text{min-min}}(A)^+(y) = \wedge_{f(x)=y} A^+(x)$, then $f(A)$ is the min-min type representation extension of concept α in representation domain $X(f)$;
3) If $f(A) = f_{\text{min-max}}(A)$, and $f_{\text{min-max}}(A)^-(y) = \wedge_{f(x)=y} A^-(x)$, $f_{\text{min-max}}(A)^+(y)\ \wedge_{f(x)=y} A^+(x)$, then $f(A)$ is the min-max type representation extension of concept α in representation domain $X(f)$;
4) If $f(A) = f_{\text{max-min}}(A)$, and $f_{\text{max-min}}(A)^-(y) = \wedge_{f(x)=y} A^-(x)$, $f_{\text{max-min}}(A)^+(y) = \wedge_{f(x)=y} A^+(x)$, then $f(A)$ is the max-

min type representation extension of concept α in representation domain $X(f)$;

Definition 6.2 For a given description frame $(U, C, F]$, if, assuming the representation extension of concept α in representation domain $X(f)$ is $B(f) \in \bar{\circ}^{X}$, then $f^{-1}(B(f))$ is a max-max (min-min, min-max, max-min) type feedback extension if $B(f)$ is a max-max (min-min, min-max, max-min) type representation extension (Anderson et al. 2018; Burkert et al. 2018; Francoenzástiga et al. 2017).

Theorem 6.1 For a given description frame $(U, C, F]$, if $\alpha \in C$, assuming the extension of concept α in U is an $\overline{\mathbf{R}}$ - Fuzzy set A, then the following conclusions hold:

$$\forall f, g \in F, f \geq g \Rightarrow \uparrow_g^f g_{\max-\max}(A) \supseteq f_{\max-\max}(A);$$

2) $\forall f \in F, f^{-1}(f_{\text{max-max}}(A)) \supseteq A$, the equation holds if f is an injective function; $\forall f; g \in F; f \geq g \Rightarrow f^{-1}(f_{max\text{-}max}(A)) \subseteq g^{-1}$; $\infty^{-1}(\infty_{\text{max-max}}(A)) = A$; $\forall u \in U, 0^{-1}(0_{\text{max-max}}(A)^{-}((u) = \vee_{u' \in U} A^{-}(u'), 0^{-1}(0_{\text{max-max}}(A)+)u = \vee_{u' \in U} A+(u')$; Assuming $G \subseteq F$, $h = \vee_{f \in G} f$, if the elements in G are independent, then $\bigcap_{f \in G} f^{-1}(f_{\text{max-max}}(A)) = h^{-1}(\bigcap_{f \in G}(\uparrow_f^h f_{\text{max-max}}(A)))$.

Proof

1) If $(x, y) \in X(f) = X(g) \times X(f - g)$, then

$$\uparrow_g^f g_{\max-\max}(A)(x,y) = g_{\max-\max}(A)(x) = \vee_{g(u)=x} A(u)$$

$$f_{\max-\max}(A)(x,y) = \vee_{f(u)=(x,y)} A(u)$$

It follows from $f \geq g$ that for $\forall u \in U$, $f(u) = (x, y) \Rightarrow g(u) = x$. Therefore, $\vee_{g(u)=x} A(u) \geq \vee_{f(u)=(x,y)} A(u)$, i.e. $\uparrow_g^f g_{\text{max-max}}(A) \supseteq f_{\text{max-max}}(A)$.

2) For any $u \in U$, there is

$$f^{-1}(f_{\max-\max}(A))(u) = f_{\max-\max}(A)(f(u)) = \vee_{f(u')=f(u)} A(u') \geq A(u),$$

therefore, $f^{-1}(f_{\text{max-max}}(A)) \supseteq A$ holds. Obviously, the equation holds when f is an injective function (Bernardo et al. 2017; Hayatsu et al. 2017; Huang et al. 2017).

3) For any $u \in U$, there are

$$f^{-1}(f_{\max-\max}(A))(u) = f_{\max-\max}(A)(f(u)) = \vee_{f(u')=f(u)} A(u')$$

$$g^{-1}(g_{\max-\max}(A))(u) = g_{\max-\max}(A)(g(u)) = \vee_{g(u')=g(u)} A(u').$$

From $f \geq g$ it follows that

$\forall u' \in U, f(u') = f(u) \Rightarrow g(u)$. Therefore, $\forall u \in U$, there is $f^{-1}(f_{\text{max-max}}(A))(u) \leq g^{-1}(g_{\text{max-max}}(A))(u)$, so

$f^{-1}(f_{\text{max-max}}(A)) \subseteq g^{-1}(g_{\text{max-max}}(A))$.

4) Since $f: ([0,2], \vee, \wedge) \rightarrow (\overline{\mathbf{R}}, \vee, \wedge)$, x a $f(x)$

$$@\begin{cases} -\infty, & x = 0; \\ \tan[\pi(x - 0.5)], & x \in (0,1); \\ +\infty, & x = 1, \end{cases} \text{ is an}$$

isomorphic mapping and 1 is an injective function, so ∞ is an injective function, and therefore it can be proven by 2) (Hidaka et al. 2017; Li et al. 2017; Milione et al. 2017; Ross et al. 2017; Tatler et al. 2017).

5) For any $u \in U$, there is

$$0^{-1}(0_{\max-\max}(A))(u) = 0_{\max-\max}(A)(0(u)) = \vee_{0(u')=0(u)} A(u') = \vee_{u' \in U} A(u').$$

6) To prove the conclusion when there are only two factors f, g in G:

For any $u \in U$, there is

$$\begin{aligned} & [h^{-1}((\uparrow_f^h f_{\max-\max}(A)) \cap (\uparrow_g^h g_{\max-\max}(A)))](u) \\ &= [(\uparrow_f^h f_{\max-\max}(A)) \cap (\uparrow_g^h g_{\max-\max}(A))](h(u)) \\ &= (\uparrow_f^h f_{\max-\max}(A))(f(u), g(u)) \wedge (\uparrow_g^h g_{\max-\max}(A))(f(u), g(u)) \\ &= f_{\max-\max}(A)(f(u)) \wedge g_{\max-\max}(A)(g(u)) \\ &= f^{-1}(f_{\max-\max}(A))(u) \wedge g^{-1}(g_{\max-\max}(A))(u) \\ &= [f^{-1}(f_{\max-\max}(A)) \cap g^{-1}(g_{\max-\max}(A))](u) \end{aligned}$$

therefore

$$\begin{aligned} & h^{-1}((\uparrow_f^h f_{\max-\max}(A)) \cap (\uparrow_g^h g_{\max-\max}(A))) \\ &= f^{-1}(f_{\max-\max}(A)) \cap g^{-1}(g_{\max-\max}(A)) \end{aligned}$$

To generalize the results, we can get

$$\bigcap_{f \in G} f^{-1}(f_{\max-\max}(A)) = h^{-1}(\bigcap_{f \in G}(\uparrow_f^h f_{\max-\max}(A))).$$

The conclusions for min-min type representation extension and feedback extension are presented herewith (Datta 2018; Jog 2018; Martin et al. 2017; Shaffrey et al. 2017):

Theorem 6.2 For a given description frame $(U, C, F]$, if $\alpha \in C$, assuming the extension of concept α in U is an $\overline{\mathbf{R}}$ - Fuzzy set A, then the following conclusions hold:

1) $\forall f, g \in F, f \geq g \Rightarrow \uparrow_g^f g_{\min-\min}(A) \subseteq f_{\min-\min}(A)$;
2) $\forall f \in F, f^{-1}(f_{\max-\max}(A)) \supseteq A$, the equation holds if f is an injective function;
3) $\forall f, g \in F, f \geq g \Rightarrow f^{-1}(f_{\min-\min}(A)) \supseteq g^{-1}(g_{\min-\min}(A))$;
4) $\infty^{-1}(\infty_{\min-\min}(A)) = A$;
5) $\forall u \in U, 0^{-1}(0_{\min}(A)^{-})(u) = \wedge_{u' \in U} A^{-}(u')$,

$0^{-1}(0_{\min}(A)^{+})(u) = \wedge_{u' \in U} A^{+}(u')$;

6) Assuming $G \subseteq F, h = \vee_{f \in G} f$, if the elements in G are independent, then

$$\bigcup_{f \in G} f^{-1}(f_{\min}(A)) = h^{-1}(\bigcup_{f \in G}(\uparrow_f^h f_{\min}(A))).$$

The proof is omitted since it is the same as that of Theorem 6.1.

The conclusions for max-min type representation extension and feedback extension are presented herewith:

Theorem 6.3 For a given description frame (*U, C, F*], if $\alpha \in C$, assuming the extension of concept α in *U* is an $\overline{\mathbf{R}}$-Fuzzy set *A*, then the following conclusions hold:

1) $\forall f, g \in F, f \geq g \Rightarrow \uparrow_g^f g_{\max-\min}(A)^- \supseteq f_{\max-\min}(A)^-$, $\uparrow_g^f g_{\max-\min}(A)^+ \subseteq f_{\max-\min}(A)^+$;
2) $\forall f \in F, f^{-1}(f_{\max-\min}(A)^-) \supseteq A^-, f^{-1}(f_{\max-\min}(A)^+) \subseteq A^+$, the equation holds if *f* is an injective function;
3) $\forall f, g \in F, f \geq g \Rightarrow f^{-1}(f_{\max-\min}(A)^-) \subseteq g^{-1}(g_{\max-\min}(A)^-)$ $f^{-1}(f_{\max-\min}(A)^+) \supseteq g^{-1}(g_{\max-\min}(A)^+)$;
4) $\infty^{-1}(\infty_{\max-\min}(A)) = A$;
5) $\forall u \in U, 0^{-1}(0_{\max-\min}(A)^-)(u) = \vee_{u' \in U} A^-(u')$, $0^{-1}(0_{\max-\min}(A)^+)(u) = \wedge_{u' \in U} A^+(u')$;
6) Assuming $G \subseteq F, h = \vee_{f \in G} f$, if the elements in G are independent, then

$$\bigcap_{f \in G} f^{-1}(f_{\max-\min}(A)^-) = h^{-1}(\bigcap_{f \in G}(\uparrow_f^h f_{\max-\min}(A)^-)),$$

$$\bigcup_{f \in G} f^{-1}(f_{\max-\min}(A)^+) = h^{-1}(\bigcup_{f \in G}(\uparrow_f^h f_{\max-\min}(A)^+)).$$

The proof is omitted since it is the same as that of Theorem 6.1.

The conclusions for min-max type representation extension and feedback extension are presented herewith:

Theorem 6.4 For a given description frame (*U, C, F*], if $\alpha \in C$, assuming the extension of concept *α* in *U* is an $\overline{\mathbf{R}}$-Fuzzy set *A*, then the following conclusions hold:

1) $\forall f, g \in F, f \geq g \Rightarrow \uparrow_g^f g_{\min-\max}(A)^- \subseteq f_{\min-\max}(A)^-$, $\uparrow_g^f g_{\min-\max}(A)^+ \supseteq f_{\min-\max}(A)^+$;
2) $\forall f \in F, f^{-1}(f_{\min-\max}(A)^-) \subseteq A^-, f^{-1}(f_{\min-\max}(A)^+) \supseteq A^+$, the equation holds if *f* is an injective function;
3) $\forall f, g \in F, f \geq g \Rightarrow f^{-1}(f_{\min-\max}(A)^-) \supseteq g^{-1}(g_{\min-\max}(A)^-)$ $f^{-1}(f_{\min-\max}(A)^+) \subseteq g^{-1}(g_{\min-\max}(A)^+)$;
4) $\infty^{-1}(\infty_{\min-\max}(A)) = A$;
5) $\forall u \in U, 0^{-1}(0_{\min-\max}(A)^-)(u) = \wedge_{u' \in U} A^-(u')$, $0^{-1}(0_{\min-\max}(A)^+)(u) = \vee_{u' \in U} A^+(u')$;
6) Assuming $G \subseteq F, h = \vee_{f \in G} f$, if the elements in G are independent, then

$$\bigcap_{f \in G} f^{-1}(f_{\min-\max}(A)^-) = h^{-1}(\bigcap_{f \in G}(\uparrow_f^h f_{\min-\max}(A)^-)),$$

$$\bigcup_{f \in G} f^{-1}(f_{\min-\max}(A)^+) = h^{-1}(\bigcup_{f \in G}(\uparrow_f^h f_{\min-\max}(A)^+)).$$

The proof is omitted since it is the same as that of Theorem 6.1.

Additionally, collective forces of simple factors can be utilized to approximate the extension of an $\overline{\mathbf{R}}$-Fuzzy concept. Here are four types of feedback extension envelopes.

Definition 6.3 For a given description frame (*U, C, F*], let concept $\alpha \in C$, and let the extension of concept α in *U* be an $\overline{\mathbf{R}}$-Fuzzy set *A*. If $G \subseteq F$, the elements in *G* are independent, thus

$$A^-_{\max-\max}[G] = \bigcap_{f \in G} f^-(f_{\max-\max}(A)^-), A^+_{\max-\max}[G] = \bigcap_{f \in G} f^-(f_{\max-\max}(A)^+).$$

$A^-_{\max-\max}[G]$ and $A^+_{\max-\max}[G]$ are the max type feedback extension lower envelope and max type feedback extension upper envelope of *A*, respectively (Huang et al. 2017; Sefusatti et al. 2018; Travin et al. 2017; Yan et al. 2018).

Definition 6.4 For a given description frame (*U, C, F*], let concept $\alpha \in C$, and let the extension of concept α in *U* be an $\overline{\mathbf{R}}$-Fuzzy set *A*. If $G \subseteq F$, the elements in *G* are independent. Therefore:

$$A^-_{\min-\min}[G] = \bigcup_{f \in G} f^-(f_{\min-\min}(A)^-), A^+_{\min-\min}[G] = \bigcup_{f \in G} f^-(f_{\min-\min}(A)^+).$$

$A^-_{\min-\min}[G]$ and $A^+_{\min-\min}[G]$ are the min type feedback extension lower envelope and min type feedback extension upper envelope of *A*, respectively.

Definition 6.5 For a given description frame (*U, C, F*], let concept $\alpha \in C$, and let the extension of concept α in *U* be an $\overline{\mathbf{R}}$-Fuzzy set *A*. If $G \subseteq F$, the elements in *G* are independent. Therefore:

$$A^-_{\min-\min}[G] = \bigcup_{f \in G} f^-(f_{\min-\min}(A)^-), A^+_{\min-\min}[G] = \bigcup_{f \in G} f^-(f_{\min-\min}(A)^+).$$

$A^-_{\min-\min}[G]$ and $A^+_{\min-\min}[G]$ are the max type feedback extension lower envelope and min type feedback extension upper envelope of *A*, respectively.

Definition 6.6 For a given description frame (*U, C, F*], let concept $\alpha \in C$, and let the extension of concept α in *U* be an $\overline{\mathbf{R}}$-Fuzzy set *A*. Take $G \subseteq F$, the elements in *G* are independent. Therefore:

$$A^-_{\max-\min}[G] = \bigcap_{f \in G} f^-(f_{\max-\min}(A)^-), A^+_{\max-\min}[G] = \bigcup_{f \in G} f^-(f_{\max-\min}(A)^+).$$

$A^{-}_{\max-\min}[G]$ and $A^{+}_{\min-\max}[G]$ are the min type feedback extension lower envelope and max type feedback extension upper envelope of A, respectively.

In summary, four approximation approaches for $\overline{\mathbf{R}}$ - Fuzzy concepts extension have been presented.

CONCLUSION

The four approaches discussed are suitable for different situations: For the max type feedback extension lower envelope and max type feedback extension upper envelope, it tends to be more helpful when approximating the extension of concept. However, the result will be very conservative when the min type feedback extension lower envelope and min type feedback extension upper envelope are applied to approximate the concept extension. In contrast, the max type feedback extension lower envelope with min type feedback extension upper envelope, and the min type feedback extension lower envelope with max type feedback extension upper envelope, are both closer to the extension of concept when approximating, but their approximation processes are more complicated. Therefore, these envelopes should be applied appropriately considering specific conditions when approximating the extension of concepts.

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