Planar Graphs Have Bounded Nonrepetitive Chromatic Number

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Abstract: A colouring of a graph is *nonrepetitive* if for every path of even order, the sequence of colours on the first half of the path is different from the sequence of colours on the second half. We show that planar graphs have nonrepetitive colourings with a bounded number of colours, thus proving a conjecture of Alon, Grytczuk, Hałuszczak and Riordan (2002). We also generalise this result for graphs of bounded Euler genus, graphs excluding a fixed minor, and graphs excluding a fixed topological minor.

1   Introduction

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a *k-colouring* of a graph G is a function ϕ that assigns one of k colours to each vertex of G. A path $(v_1, v_2, \ldots, v_{2t})$ of even order in G is *repetitively* coloured by ϕ if $\phi(v_i) = \phi(v_{t+i})$ for $i \in \{1, \ldots, t\}$. A colouring ϕ of G is *nonrepetitive* if no path of G is repetitively coloured by ϕ. Observe that a nonrepetitive colouring is *proper*, in the sense that adjacent vertices are coloured differently. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer k such that G admits a nonrepetitive k-colouring.

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The classical result in this area is by Thue [51], who proved in 1906 that every path is nonrepetitively 3-colourable. Starting with the seminal work of Alon, Grytczuk, Hałuszczak, and Riordan [3], nonrepetitive colourings of general graphs have recently been widely studied; see the surveys [11, 26–28, 50] and other references [1–14, 16, 17, 20–23, 25–38, 40–45, 48–54].

Several graph classes are known to have bounded nonrepetitive chromatic number. In particular, cycles are nonrepetitively 3-colourable (except for a finite number of exceptions) [12], trees are nonrepetitively 4-colourable [9, 36], outerplanar graphs are nonrepetitively 12-colourable [5, 36], and more generally, every graph with treewidth k is nonrepetitively 4^k-colourable [36]. Graphs with maximum degree Δ are nonrepetitively $O(\Delta^2)$-colourable [3, 17, 27, 32], and graphs excluding a fixed immersion have bounded nonrepetitive chromatic number [53].

It is widely recognised that the most important open problem in the field of nonrepetitive graph colouring is whether planar graphs have bounded nonrepetitive chromatic number. It was first asked by Alon et al. [3]. The best known lower bound is 11, due to Ochem (see [16]). The best known upper bound is $O(\log n)$ where n is the number of vertices, due to Dujmović, Frati, Joret, and Wood [16]. Note that several works have studied colourings of planar graphs in which only facial paths are required to be nonrepetitively coloured [4, 8, 33, 34, 44, 45, 48].

This paper proves that planar graphs have bounded nonrepetitive chromatic number.

Theorem 1. *Every planar graph G satisfies $\pi(G) \leqslant 768$.*

We generalise this result for graphs of bounded Euler genus, for graphs excluding any fixed minor, and for graphs excluding any fixed topological minor.[1] The result for graphs excluding a fixed minor confirms a conjecture of Grytczuk [27, 28].

Theorem 2. *Every graph G with Euler genus g satisfies $\pi(G) \leqslant 256 \max\{2g, 3\}$.*

Theorem 3. *For every graph X, there is an integer k such that every X-minor-free graph G satisfies $\pi(G) \leqslant k$.*

Theorem 4. *For every graph X, there is an integer k such that every X-topological-minor-free graph G satisfies $\pi(G) \leqslant k$.*

The proofs of Theorems 1 and 2 are given in Section 3, and the proofs of Theorems 3 and 4 are given in Section 4. Before that in Section 2 we introduce the tools used in our proofs, namely so-called strongly nonrepetitive colourings, tree-decompositions and treewidth, and strong products. With these tools in hand, the above theorems quickly follow from recent results of Dujmović, Joret, Micek, Morin, Ueckerdt, and Wood [18] that show that planar graphs and other graph classes are subgraphs of certain strong products.

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[1]The *Euler genus* of the orientable surface with h handles is $2h$. The *Euler genus* of the non-orientable surface with c cross-caps is c. The *Euler genus* of a graph G is the minimum integer k such that G embeds in a surface of Euler genus k. Of course, a graph is planar if and only if it has Euler genus 0; see [39] for more about graph embeddings in surfaces. A graph X is a *minor* of a graph G if a graph isomorphic to X can be obtained from a subgraph of G by contracting edges. A graph X is a *topological minor* of a graph G if a subdivision of X is a subgraph of G. If G contains X as a topological minor, then G contains X as a minor. If G contains no X minor, then G is *X-minor-free*. If G contains no X topological minor, then G is *X-topological-minor-free*.

2 Tools

Undefined terms and notation can be found in [15].

2.1 Strongly Nonrepetitive Colourings

A key to all our proofs is to consider a strengthening of nonrepetitive colouring defined below.

For a graph G, a *lazy walk* in G is a sequence of vertices $v_1, \ldots, v_k$ such that for each $i \in \{1, \ldots, k\}$, either $v_i v_{i+1}$ is an edge of G, or $v_i = v_{i+1}$. A lazy walk can be thought of as a walk in the graph obtained from G by adding a loop at each vertex. For a colouring ϕ of G, a lazy walk $v_1, \ldots, v_{2k}$ is ϕ-*repetitive* if $\phi(v_i) = \phi(v_{i+k})$ for each $i \in \{1, \ldots, k\}$.

A colouring ϕ is *strongly nonrepetitive* if for every ϕ-repetitive lazy walk $v_1, \ldots, v_{2k}$, there exists $i \in \{1, \ldots, k\}$ such that $v_i = v_{i+k}$. Let $\pi^*(G)$ be the minimum number of colours in a strongly nonrepetitive colouring of G. Since a path has no repeated vertices, every strongly nonrepetitive colouring is nonrepetitive, and thus $\pi(G) \leqslant \pi^*(G)$ for every graph G.

2.2 Layerings

A *layering* of a graph G is a partition $(V_0, V_1, \ldots)$ of $V(G)$ such that for every edge $vw \in E(G)$, if $v \in V_i$ and $w \in V_j$, then $|i - j| \leqslant 1$. If r is a vertex in a connected graph G and V_i is the set of vertices at distance exactly i from r in G for all $i \geqslant 0$, then the layering $(V_0, V_1, \ldots)$ is called a *BFS layering* of G.

Consider a layering $(V_0, V_1, \ldots)$ of a graph G. Let H be a connected component of $G[V_i \cup V_{i+1} \cup \cdots]$, for some $i \geqslant 1$. The *shadow* of H is the set of vertices in V_{i-1} adjacent to some vertex in H. The layering is *shadow-complete* if every shadow is a clique. This concept was introduced by Kündgen and Pelsmajer [36] and implicitly by Dujmović, Morin, and Wood [19].

We will need the following result.

Lemma 5 ([19, 36]). *Every BFS-layering of a connected chordal graph is shadow-complete.*

2.3 Treewidth

A *tree-decomposition* of a graph G consists of a collection $\{B_x \subseteq V(G) : x \in V(T)\}$ of subsets of $V(G)$, called *bags*, indexed by the vertices of a tree T, and with the following properties:
- for every vertex v of G, the set $\{x \in V(T) : v \in B_x\}$ induces a non-empty (connected) subtree of T, and
- for every edge vw of G, there is a vertex $x \in V(T)$ for which $v, w \in B_x$.

The *width* of such a tree-decomposition is $\max\{|B_x| : x \in V(T)\} - 1$. The *treewidth* of a graph G is the minimum width of a tree-decomposition of G. Tree-decompositions were introduced by Robertson and Seymour [46]. Treewidth measures how similar a given graph is to a tree, and is particularly important in structural and algorithmic graph theory.

Barát and Varjú [5] and Kündgen and Pelsmajer [36] independently proved that graphs of bounded treewidth have bounded nonrepetitive chromatic number. Specifically, Kündgen and Pelsmajer [36] proved that every graph with treewidth k is nonrepetitively 4^k-colourable, which is the best known

bound. Theorem 7 below strengthens this result. The proof is almost identical to that of Kündgen and Pelsmajer [36] and depends on the following lemma. A lazy walk $v_1, \ldots, v_{2k}$ is *boring* if $v_i = v_{i+k}$ for each $i \in \{1, \ldots, k\}$.

Lemma 6 ([36]). *Every path P has a 4-colouring ϕ such that every ϕ-repetitive lazy walk is boring.*

Theorem 7. *For every graph G of treewidth at most $k \geqslant 0$, we have $\pi^*(G) \leqslant 4^k$.*

Proof. The proof proceeds by induction on k. If $k = 0$, then G has no edges, so assigning the same colour to all the vertices gives a strongly nonrepetitive colouring. For the rest of the proof, assume that $k \geqslant 1$. Consider a tree-decomposition of G of width at most k. By adding edges if necessary, we may assume that every bag of the tree-decomposition is a clique. Thus, G is connected and chordal, with clique-number at most $k + 1$.

Let $(V_0, V_1, \ldots)$ be a BFS-layering of G. We refer to V_i as the set of vertices at *depth i*. Note that the subgraph $G[V_i]$ of G induced by each layer V_i has treewidth at most $k - 1$.[2] Thus the spanning subgraph H of G induced by all edges whose endpoints have the same depth also has treewidth at most $k - 1$. By the induction hypothesis, H has a strongly nonrepetitive colouring ϕ_1 with 4^{k-1} colours. The graph P obtained from G by contracting each set V_i (which might not induce a connected graph) into a single vertex x_i is a path, and thus, by Lemma 6, P has a 4-colouring ϕ_2 such that every ϕ_2-repetitive walk is boring. For each $i \geqslant 0$ and each vertex $u \in V_i$, set $\phi(u) := (\phi_1(u), \phi_2(x_i))$. The colouring ϕ of G uses at most $4 \cdot 4^{k-1} = 4^k$ colours.

We now prove that ϕ is strongly nonrepetitive. Let W be a ϕ-repetitive lazy walk $v_1, \ldots, v_{2k}$. Our goal is to prove that $v_j = v_{j+k}$ for some $j \in \{1, \ldots, k\}$. Let d be the minimum depth of a vertex in W.

Let W' be the sequence of vertices obtained from W by removing all vertices at depth greater than d. We claim that W' is a lazy walk. To see this, consider vertices $v_i, v_{i+1}, \ldots, v_{i+t}$ of W such that v_i and v_{i+t} have depth d but $v_{i+1}, \ldots, v_{i+t-1}$ all have depth greater than d; thus, $v_{i+1}, \ldots, v_{i+t-1}$ were removed when constructing W'. Then, the vertices $v_{i+1}, \ldots, v_{i+t-1}$ lie in a connected component of the graph induced by the vertices of depth greater than d, thus it follows that v_i and v_{i+t} are adjacent or equal by Lemma 5. This shows that W' is a lazy walk in $G[V_d]$.

The projection of W on P is a ϕ_2-repetitive lazy walk in P, and is thus boring by Lemma 6. It follows that the vertices v_j and v_{j+k} of W have the same depth for every $j \in \{1, \ldots, k\}$. In particular, v_j was removed from W' if and only if v_{j+k} was. Hence, there are indices $1 \leqslant i_1 < i_2 < \cdots < i_\ell \leqslant k$ such that $W' = v_{i_1}, v_{i_2}, \ldots, v_{i_\ell}, v_{i_1+k}, v_{i_2+k}, \ldots, v_{i_\ell+k}$. Since W is (ϕ_1, ϕ_2)-repetitive, it follows that W' is also (ϕ_1, ϕ_2)-repetitive and in particular W' is ϕ_1-repetitive. By the definition of ϕ_1, there is an index i_r such that $v_{i_r} = v_{i_r+k}$, which completes the proof. $\square$

2.4 Strong Products

The *strong product* of graphs A and B, denoted by $A \boxtimes B$, is the graph with vertex set $V(A) \times V(B)$, where distinct vertices $(v, x), (w, y) \in V(A) \times V(B)$ are adjacent if (1) $v = w$ and $xy \in E(B)$, or (2) $x = y$ and $vw \in E(A)$, or (3) $vw \in E(A)$ and $xy \in E(B)$. Nonrepetitive colourings of graph products have been

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[2]This is clear for $i = 0$ (since $k \geqslant 1$), and for $i \geqslant 1$ this follows from the fact that the graph $G[V_i]$ plus a universal vertex is a minor of G (contract $V_0 \cup \cdots \cup V_{i-1}$ into a single vertex and remove $V_{i+1}, V_{i+2}, \ldots$), and thus has treewidth at most k. Since removing a universal vertex decreases the treewidth by exactly one, it follows that $G[V_i]$ has treewidth at most $k - 1$.

studied in [7, 35, 36, 42]. Indeed, Kündgen and Pelsmajer [36] note that their method shows that the strong product of k paths is nonrepetitively 4^k-colourable, which is a precursor to the following results.

Lemma 8. *Let H be a graph with an ℓ-colouring ϕ_2 such that every ϕ_2-repetitive lazy walk is boring. For every graph G, we have $\pi^*(G \boxtimes H) \leqslant \ell \pi^*(G)$.*

Proof. Consider a strongly nonrepetitive colouring ϕ_1 of G with $\pi^*(G)$ colours. For any two vertices $u \in V(G)$ and $v \in V(H)$, we define the colour $\phi(u,v)$ of the vertex $(u,v) \in V(G \boxtimes H)$ by $\phi(u,v) := (\phi_1(u), \phi_2(v))$. We claim that this is a strongly nonrepetitive colouring of $G \boxtimes H$. To see this, consider a ϕ-repetitive lazy walk $W = (u_1, v_1), \ldots, (u_{2k}, v_{2k})$ in $G \boxtimes H$. By the definition of the strong product and the definition of ϕ, the projection $W_G = u_1, u_2, \ldots, u_{2k}$ of W on G is a ϕ_1-repetitive lazy walk in G and the projection $W_H = v_1, v_2, \ldots, v_{2k}$ of W on H is a ϕ_2-repetitive lazy walk in H. By the definition of ϕ_1, there is an index i such that $u_i = u_{i+k}$. By the definition of ϕ_2, we have $v_j = v_{j+k}$ for every $j \in \{1, \ldots, k\}$. In particular, $v_i = v_{i+k}$ and $(u_i, v_i) = (u_{i+k}, v_{i+k})$, which completes the proof. $\square$

Applying Lemma 6, we obtain the following immediate corollary.

Corollary 9. *For every graph G and every path P, we have $\pi^*(G \boxtimes P) \leqslant 4\pi^*(G)$.*

By taking $H = K_\ell$ and a proper ℓ-colouring ϕ_2 of K_ℓ, we also obtain the following direct corollary to Lemma 8.

Corollary 10. *For every graph G and every integer $\ell \geqslant 1$, we have $\pi^*(G \boxtimes K_\ell) \leqslant \ell \pi^*(G)$.*

3 Planar Graphs and Graphs of Bounded Genus

The following recent result by Dujmović et al. [18] is a key theorem.

Theorem 11 ([18]). *Every planar graph is a subgraph of $H \boxtimes P \boxtimes K_3$ for some graph H with treewidth at most 3 and some path P.*

Corollary 9 and Theorems 7 and 11 imply that for every planar graph G,

$$\pi(G) \leqslant \pi^*(G) \leqslant \pi^*(H \boxtimes P \boxtimes K_3) \leqslant 3\pi^*(H \boxtimes P) \leqslant 3 \cdot 4\pi^*(H) \leqslant 3 \cdot 4 \cdot 4^3 = 768,$$

which proves Theorem 1.

For graphs of bounded Euler genus, Dujmović et al. [18] proved the following strengthening of Theorem 11.

Theorem 12 ([18]). *Every graph G of Euler genus g is a subgraph of $H \boxtimes P \boxtimes K_{\max\{2g,3\}}$ for some graph H with treewidth at most 3 and some path P.*

Corollary 9 and Theorems 7 and 12 imply that for every graph G with Euler genus g,

$$\pi(G) \leqslant \pi^*(G) \leqslant \pi^*(H \boxtimes P \boxtimes K_{\max\{2g,3\}}) \leqslant \max\{2g,3\} \cdot \pi^*(H \boxtimes P) \leqslant \max\{2g,3\} \cdot 4 \cdot \pi^*(H)$$
$$\leqslant \max\{2g,3\} \cdot 4^4$$
$$= 256 \max\{2g,3\},$$

which proves Theorem 2.

4 Excluded Minors

Our results for graphs excluding a minor depend on the following version of the graph minor structure theorem of Robertson and Seymour [47]. A tree-decomposition $(B_x : x \in V(T))$ of a graph G is *r-rich* if $B_x \cap B_y$ is a clique in G on at most r vertices, for each edge $xy \in E(T)$.

Theorem 13 ([20]). *For every graph X, there are integers $r \geqslant 1$ and $k \geqslant 1$ such that every X-minor-free graph G_0 is a spanning subgraph of a graph G that has an r-rich tree-decomposition such that each bag induces a k-almost-embeddable subgraph of G.*

We omit the definition of k-almost embeddable from this paper, since we do not need it. All we need to know is the following theorem of Dujmović et al. [18].

Theorem 14 ([18]). *Every k-almost embeddable graph is a subgraph of $H \boxtimes P \boxtimes K_{\max\{6k,1\}}$ for some graph H with treewidth at most $11k + 10$.*

Theorems 7 and 14 and Corollary 9 imply that for every k-almost embeddable graph G,

$$\pi(G) \leqslant \pi^*(G) \leqslant \pi^*(H \boxtimes P \boxtimes K_{\max\{6k,1\}}) \leqslant 6k\,\pi^*(H \boxtimes P) \leqslant 6k \cdot 4\pi^*(H) \leqslant 6k \cdot 4^{11(k+1)}. \qquad (4.1)$$

Dujmović et al. [20] proved the following lemma, which generalises a result of Kündgen and Pelsmajer [36].

Lemma 15 ([20]). *Let G be a graph that has an r-rich tree-decomposition such that the subgraph induced by each bag is nonrepetitively c-colourable. Then $\pi(G) \leqslant c\,4^r$.*

Theorem 13 and Lemma 15 and (4.1) with $c = 6k \cdot 4^{11(k+1)}$ imply that for every graph X and every X-minor-free graph G,

$$\pi(G) \leqslant \pi^*(G) \leqslant 6k \cdot 4^{11(k+1)} \cdot 4^r,$$

which implies Theorem 3 since k and r depend only on X.

To obtain our result for graphs excluding a fixed topological minor, we use the following version of the structure theorem of Grohe and Marx [24].

Theorem 16 ([20]). *For every graph X, there are integers $r \geqslant 1$ and $k \geqslant 1$ such that every X-topological-minor-free graph G_0 is a spanning subgraph of a graph G that has an r-rich tree-decomposition such that the subgraph induced by each bag is k-almost-embeddable or has at most k vertices with degree greater than k.*

Alon et al. [3] proved that graphs with maximum degree Δ are nonrepetitively $O(\Delta^2)$-colourable. The best known bound is due to Dujmović et al. [17].

Theorem 17 ([17]). *Every graph with maximum degree $\Delta \geqslant 2$ is nonrepetitively $(\Delta^2 + O(\Delta^{5/3}))$-colourable.*

Theorem 17 implies that if a graph has at most k vertices with degree greater than k, then it is nonrepetitively c'-colourable for some constant $c' = k^2 + O(k^{5/3}) + k$. Theorem 16 and Lemma 15 and (4.1) with $c = \max\{6k \cdot 4^{11(k+1)}, c'\}$ imply that for every graph X, every X-topological-minor-free graph G satisfies $\pi(G) \leqslant \pi^*(G) \leqslant c \cdot 4^r$, which implies Theorem 4, since c and r depend only on X.

Acknowledgements

References

[1] NOGA ALON AND JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs. *Electronic Notes in Discrete Math.*, 22:353–355, 2005. https://dx.doi.org/10.1016/j.endm.2005.06.077. 2

[2] NOGA ALON AND JAROSŁAW GRYTCZUK. Breaking the rhythm on graphs. *Discrete Math.*, 308:1375–1380, 2008. https://dx.doi.org/10.1016/j.disc.2007.07.063. MR: 2392054. 2

[3] NOGA ALON, JAROSŁAW GRYTCZUK, MARIUSZ HAŁUSZCZAK, AND OLIVER RIORDAN. Nonrepetitive colorings of graphs. *Random Structures Algorithms*, 21(3–4):336–346, 2002. https://dx.doi.org/10.1002/rsa.10057. MR: 1945373. 2, 6

[4] JÁNOS BARÁT AND JÚLIUS CZAP. Facial nonrepetitive vertex coloring of plane graphs. *J. Graph Theory*, 74(1):115–121, 2013. https://dx.doi.org/10.1002/jgt.21695. 2

[5] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free vertex colorings of graphs. *Studia Sci. Math. Hungar.*, 44(3):411–422, 2007. https://dx.doi.org/10.1556/SScMath.2007.1029. MR: 2361685. 2, 3

[6] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free edge colorings of graphs. *Ars Combin.*, 87:377–383, 2008. MR: 2414029. 2

[7] JÁNOS BARÁT AND DAVID R. WOOD. Notes on nonrepetitive graph colouring. *Electron. J. Combin.*, 15:R99, 2008. https://doi.org/10.37236/823. MR: 2426162. 2, 5

[8] PROSENJIT BOSE, VIDA DUJMOVIĆ, PAT MORIN, AND LUCAS RIOUX-MALDAGUE. New bounds for facial nonrepetitive colouring. *Graphs Combin.*, 33(4):817–832, 2017. https://dx.doi.org/10.1007/s00373-017-1816-1. 2

[9] BOŠTJAN BREŠAR, JAROSŁAW GRYTCZUK, SANDI KLAVŽAR, STANISŁAW NIWCZYK, AND IZTOK PETERIN. Nonrepetitive colorings of trees. *Discrete Math.*, 307(2):163–172, 2007. https://dx.doi.org/10.1016/j.disc.2006.06.017. MR: 2285186. 2

[10] BOŠTJAN BREŠAR AND SANDI KLAVŽAR. Square-free colorings of graphs. *Ars Combin.*, 70:3–13, 2004. MR: 2023057. 2

[11] PANAGIOTIS CHEILARIS, ERNST SPECKER, AND STATHIS ZACHOS. Neochromatica. *Comment. Math. Univ. Carolin.*, 51(3):469–480, 2010. http://www.dml.cz/dmlcz/140723. MR: 2741880. 2

[12] JAMES D. CURRIE. There are ternary circular square-free words of length n for $n \geqslant 18$. *Electron. J. Combin.*, 9(1):#N10, 2002. https://doi.org/10.37236/1671. MR: 1936865. 2

[13] JAMES D. CURRIE. Pattern avoidance: themes and variations. *Theoret. Comput. Sci.*, 339(1):7–18, 2005. https://dx.doi.org/10.1016/j.tcs.2005.01.004. MR: 2142070. 2

[14] SEBASTIAN CZERWIŃSKI AND JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs. *Electronic Notes in Discrete Math.*, 28:453–459, 2007. https://dx.doi.org/10.1016/j.endm.2007.01.063. MR: 2324051. 2

[15] REINHARD DIESTEL. *Graph theory*, vol. 173 of *Graduate Texts in Mathematics*. Springer, 4th edn., 2010. `http://diestel-graph-theory.com/`. MR: 2744811. 3

[16] VIDA DUJMOVIĆ, FABRIZIO FRATI, GWENAËL JORET, AND DAVID R. WOOD. Nonrepetitive colourings of planar graphs with $O(\log n)$ colours. *Electron. J. Combin.*, 20(1):#P51, 2013. `https://doi.org/10.37236/3153`. 2

[17] VIDA DUJMOVIĆ, GWENAËL JORET, JAKUB KOZIK, AND DAVID R. WOOD. Non-repetitive colouring via entropy compression. *Combinatorica*, 36(6):661–686, 2016. https://dx.doi.org/10.1007/s00493-015-3070-6. 2, 6

[18] VIDA DUJMOVIĆ, GWENAËL JORET, PIOTR MICEK, PAT MORIN, TORSTEN UECKERDT, AND DAVID R. WOOD. Planar graphs have bounded queue-number. In *Proc. 60th Annual Symp. Foundations Comput. Sci.* (FOCS '19), pp. 862–875. 2019. https://dx.doi.org/10.1109/FOCS.2019.00056. To appear in *J. ACM*, arXiv: 1904.04791. 2, 5, 6

[19] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layout of graphs with bounded tree-width. *SIAM J. Comput.*, 34(3):553–579, 2005. https://dx.doi.org/10.1137/S0097539702416141. MR: 2137079. 3

[20] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layered separators in minor-closed graph classes with applications. *J. Combin. Theory Ser. B*, 127:111–147, 2017. https://dx.doi.org/10.1016/j.jctb.2017.05.006. MR: 3704658. 2, 6

[21] FRANCESCA FIORENZI, PASCAL OCHEM, PATRICE OSSONA DE MENDEZ, AND XUDING ZHU. Thue choosability of trees. *Discrete Applied Math.*, 159(17):2045–2049, 2011. https://dx.doi.org/10.1016/j.dam.2011.07.017. MR: 2832329. 2

[22] ADAM GĄGOL, GWENAËL JORET, JAKUB KOZIK, AND PIOTR MICEK. Pathwidth and nonrepetitive list coloring. *Electron. J. Comb.*, 23(4):P4.40, 2016. `https://doi.org/10.37236/5855`. 2

[23] DANIEL GONÇALVES, MICKAËL MONTASSIER, AND ALEXANDRE PINLOU. Entropy compression method applied to graph colorings, 2014. arXiv: 1406.4380. 2

[24] MARTIN GROHE AND DÁNIEL MARX. Structure theorem and isomorphism test for graphs with excluded topological subgraphs. *SIAM J. Comput.*, 44(1):114–159, 2015. https://dx.doi.org/10.1137/120892234. MR: 3313569. 6

[25] JAROSŁAW GRYTCZUK. Thue-like sequences and rainbow arithmetic progressions. *Electron. J. Combin.*, 9(1):R44, 2002. `https://doi.org/10.37236/1660`. MR: 1946146. 2

[26] JAROSŁAW GRYTCZUK. Thue type colorings of graphs. In *Graph Theory 2004: A conference in memory of Claude Berge*. 2004. 2

[27] JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs—a survey. *Int. J. Math. Math. Sci.*, 74639, 2007. https://dx.doi.org/10.1155/2007/74639. MR: 2272338. 2

[28] JAROSŁAW GRYTCZUK. Thue type problems for graphs, points, and numbers. *Discrete Math.*, 308(19):4419–4429, 2008. https://dx.doi.org/10.1016/j.disc.2007.08.039. MR: 2433769. 2

[29] JAROSŁAW GRYTCZUK, KAROL KOSIŃSKI, AND MICHAŁ ZMARZ. Nonrepetitive colorings of line arrangements. *Eur. J. Combin.*, 51:275–279, 2016. https://dx.doi.org/10.1016/j.ejc.2015.05.013. 2

[30] JAROSŁAW GRYTCZUK, JAKUB KOZIK, AND PIOTR MICEK. A new approach to nonrepetitive sequences. *Random Structures Algorithms*, 42(2):214–225, 2013. https://dx.doi.org/10.1002/rsa.20411.

2

[31] JAROSŁAW GRYTCZUK, JAKUB PRZYBYŁO, AND XUDING ZHU. Nonrepetitive list colourings of paths. *Random Structures Algorithms*, 38(1-2):162–173, 2011. https://dx.doi.org/10.1002/rsa.20347. MR: 2768888. 2

[32] JOCHEN HARANTA AND STANISLAV JENDROĽ. Nonrepetitive vertex colorings of graphs. *Discrete Math.*, 312(2):374–380, 2012. https://dx.doi.org/10.1016/j.disc.2011.09.027. MR: 2852595. 2

[33] FRÉDÉRIC HAVET, STANISLAV JENDROĽ, ROMAN SOTÁK, AND ERIKA ŠKRABUĽÁKOVÁ. Facial non-repetitive edge-coloring of plane graphs. *J. Graph Theory*, 66(1):38–48, 2011. https://dx.doi.org/10.1002/jgt.20488. 2

[34] STANISLAV JENDROĽ AND ERIKA ŠKRABUĽÁKOVÁ. Facial non-repetitive edge colouring of semiregular polyhedra. *Acta Univ. M. Belii Ser. Math.*, 15:37–52, 2009. http://actamath.savbb.sk/acta1503.shtml. MR: 2589669. 2

[35] BALÁZS KESZEGH, BALÁZS PATKÓS, AND XUDING ZHU. Nonrepetitive colorings of lexicographic product of graphs. *Discrete Math. Theor. Comput. Sci.*, 16(2):97–110, 2014. http://dmtcs.episciences.org/2077. 2, 5

[36] ANDRÉ KÜNDGEN AND MICHAEL J. PELSMAJER. Nonrepetitive colorings of graphs of bounded tree-width. *Discrete Math.*, 308(19):4473–4478, 2008. https://dx.doi.org/10.1016/j.disc.2007.08.043. MR: 2433774. 2, 3, 4, 5, 6

[37] ANDRÉ KÜNDGEN AND TONYA TALBOT. Nonrepetitive edge-colorings of trees. *Discrete Math. Theor. Comput. Sci.*, 19(1), 2017. http://dmtcs.episciences.org/3724. 2

[38] DÁNIEL MARX AND MARCUS SCHAEFER. The complexity of nonrepetitive coloring. *Discrete Appl. Math.*, 157(1):13–18, 2009. https://dx.doi.org/10.1016/j.dam.2008.04.015. MR: 2479374. 2

[39] BOJAN MOHAR AND CARSTEN THOMASSEN. *Graphs on surfaces*. Johns Hopkins University Press, 2001. MR: 1844449. 2

[40] JAROSLAV NEŠETŘIL, PATRICE OSSONA DE MENDEZ, AND DAVID R. WOOD. Characterisations and examples of graph classes with bounded expansion. *Eur. J. Combin.*, 33(3):350–373, 2011. https://dx.doi.org/10.1016/j.ejc.2011.09.008. MR: 2864421. 2

[41] WESLEY PEGDEN. Highly nonrepetitive sequences: winning strategies from the local lemma. *Random Structures Algorithms*, 38(1-2):140–161, 2011. https://dx.doi.org/10.1002/rsa.20354. MR: 2768887. 2

[42] IZTOK PETERIN, JENS SCHREYER, ERIKA FECKOVÁ ŠKRABUĽÁKOVÁ, AND ANDREJ TARANENKO. A note on the Thue chromatic number of lexicographic products of graphs. *Discuss. Math. Graph Theory*, 38(3):635–643, 2018. https://dx.doi.org/10.7151/dmgt.2032. MR: 3811958. 2, 5

[43] ANDRZEJ PEZARSKI AND MICHAŁ ZMARZ. Non-repetitive 3-coloring of subdivided graphs. *Electron. J. Combin.*, 16(1):N15, 2009. https://doi.org/10.37236/253. MR: 2515755. 2

[44] JAKUB PRZYBYŁO. On the facial Thue choice index via entropy compression. *J. Graph Theory*, 77(3):180–189, 2014. https://dx.doi.org/10.1002/jgt.21781. 2

[45] JAKUB PRZYBYŁO, JENS SCHREYER, AND ERIKA ŠKRABUĽÁKOVÁ. On the facial Thue choice number of plane graphs via entropy compression method. *Graphs Combin.*, 32(3):1137–1153, 2016. https://dx.doi.org/10.1007/s00373-015-1642-2. 2

[46] NEIL ROBERTSON AND PAUL SEYMOUR. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986. https://dx.doi.org/10.1016/0196-6774(86)90023-4. MR: 0855559.

3

[47] NEIL ROBERTSON AND PAUL SEYMOUR. Graph minors. XVI. Excluding a non-planar graph. *J. Combin. Theory Ser. B*, 89(1):43–76, 2003. https://dx.doi.org/10.1016/S0095-8956(03)00042-X. MR: 1999736. 6

[48] JENS SCHREYER AND ERIKA ŠKRABUĽÁKOVÁ. On the facial Thue choice index of plane graphs. *Discrete Math.*, 312(10):1713–1721, 2012. https://dx.doi.org/10.1016/j.disc.2012.01.023. 2

[49] JENS SCHREYER AND ERIKA ŠKRABUĽÁKOVÁ. Total Thue colourings of graphs. *Eur. J. Math.*, 1(1):186–197, 2015. https://dx.doi.org/10.1007/s40879-014-0016-2. MR: 3386233. 2

[50] ERIKA ŠKRABUĽÁKOVÁ. The Thue choice number versus the Thue chromatic number of graphs, 2015. arXiv: 1508.02559. 2

[51] AXEL THUE. Über unendliche Zeichenreihen. *Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiania*, 7:1–22, 1906. 2

[52] PRZEMYSŁAW WENUS AND KRZYSZTOF WĘSEK. Nonrepetitive and pattern-free colorings of the plane. *Eur. J. Combin.*, 54:21–34, 2016. https://dx.doi.org/10.1016/j.ejc.2015.12.002. 2

[53] PAUL WOLLAN AND DAVID R. WOOD. Nonrepetitive colourings of graphs excluding a fixed immersion or topological minor. *J. Graph Theory*, 91(3):259–266, 2019. https://dx.doi.org/10.1002/jgt.22430. 2

[54] HUANHUA ZHAO AND XUDING ZHU. (2+ε)-nonrepetitive list colouring of paths. *Graphs Combin.*, 32(4):1635–1640, 2016. https://dx.doi.org/10.1007/s00373-015-1652-0. 2

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