Robust stability of min-max MPC controllers for nonlinear systems with bounded uncertainties

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Abstract

The closed loop formulation of the robust MPC has been shown to be a control technique capable of robustly stabilize uncertain nonlinear systems subject to constraints. Robust asymptotic stability of these controllers has been proved when the uncertainties are decaying. In this paper we extend the existing results to the case of uncertainties that decay with the state but do not tend to zero. This allows us to consider both plant uncertainties and external disturbances in a less conservative way.

First, we provide some results on robust stability under the considered kind of uncertainties. Based on these, we prove robust stability of the min-max MPC. In the paper we show how the robust design of the local controller is translated to the min-max controller and how the persistent term of the uncertainties determines the convergence rate of the closed-loop system.

1 Introduction

MPC is a control technique based on an associated optimization problem, which deals with constraints on the states and the inputs. This fact has provided a meaningful success in process industries. Furthermore, the theoretical framework to analyze topics as stability, robustness, optimality, etc. has been developed recently. See (Chen & Allgöwer 1998, Mayne, Rawlings, Rao & Scokaert 2000) for a survey, or (Camacho & Bordons 1999) for process industry application issues.

Model predictive control is a receding horizon strategy that requires the solution of a finite-horizon optimization problem at each sample time. This one can be posed as a mathematical programming problem. It is well known that considering a terminal cost and a terminal constraint in the optimization problem, the MPC stabilizes asymptotically the constrained system (Mayne et al. 2000). However, the system that is controlled by an MPC must be slow enough to compute the control action in one sampling time. Thus, recent papers are focused on reducing the computational burden of this problem (Scokaert, Mayne & Rawlings. 1999, Bemporad, Morari, Dua & Pistikopoulos 2002, Limon, Alamo & Camacho 2003).

If the system is uncertain, then the stabilizing properties of the MPC provides certain degree of robustness (Scokaert, Rawlings & Meadows 1997, Limon , Alamo & Camacho 2002*b*). One of the approaches to the design of MPC controllers incorporating the uncertainty is the so-called open loop formulations (Michalska & Mayne 1993, Limon , Alamo & Camacho 2002*a*). These controllers guarantee robust stability and constraint satisfaction but, they ends up being very conservative since they are likely to have a very small feasible region.

This conservativeness was overcome thanks to the closed-loop formulations (Scokaert & Mayne 1998, Mayne 2001, Kerrigan & Maciejowski 2004). In this case a sequence of control laws is computed instead of a sequence of control actions. By doing this, the reaction of the controller to the uncertainty is incorporated in the prediction and the conservativeness is mitigated. The closed-loop formulation of min-max MPC has been analyzed in (Mayne 2001). In that paper, sufficient conditions to design an stabilizing min-max MPC in case of uncertainties that decay with the state are given.

In our paper, we extend that result to the case of uncertainties that decay with the state, but tending to a constant. This allows us to model any bounded uncertainty: model mismatches as well as persistent disturbances. When the uncertainties are modelled as bounded, some controllers, as the min-max MPC, compute the control action considering the worst expected uncertainty. This fact makes the control law depend on the modelled bound of the uncertainties. Consider, for instance, a system with an external disturbance modelled as persistent and bounded by $\mu > 0$. If the disturbance never appears (and consequently there are not mismatches between the prediction model and the real plant), then the closed loop system does not evolve to the origin.

This property makes that the existing results on stability such as input to state stability can not be applied. Consequently, new sufficient conditions on the notion of robust Lyapunov function are established to achieve robust stability for this kind of systems. The main difference of the proposed conditions is that it is not necessary that the Lyapunov function is zero at the origin. Based on this result, the main contribution of this paper is given: sufficient conditions for robust stability of min-max MPC with bounded uncertainties. It is done assuming that the terminal cost is a robust Lyapunov function and proving that the optimal cost is also a robust Lyapunov function.

The paper is organized as follows: in section 2, some preliminary results are presented. In section 3 sufficient conditions for robust stability are given. The closed-loop min-max MPC is presented in section 4. The paper finishes with the proposed sufficient conditions of robust stability of min-max MPC under bounded uncertainties, which is the main contribution of the paper.

2 Preliminary results

2.1 Some definitions and properties

In this section, we present some well-established definitions and properties which will be used later.

Definition 1

- *A continuous function* $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is a* $\mathcal{K}$*-function if* $\alpha(0) = 0$, $\alpha(s) > 0$ *for all* $s > 0$ *and it is strictly increasing.*
- *A continuous function* $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is a* $\mathcal{K}_\infty$*-function if is a* $\mathcal{K}$*-function and* $\alpha(s) \to \infty$ *when* $s \to \infty$.
- *A continuous function* $\beta : \mathbb{R}_+ \times Z_+ \mapsto \mathbb{R}_+$ *is a* $\mathcal{KL}$*-function if* $\beta(s,t)$ *is a* $\mathcal{K}$*-function in* s *for any* $t \geq 0$ *and it is strictly decreasing in* t *for any* $s > 0$.

In what follows $\theta_1 \circ \theta_2(s)$ denotes the function $\theta_1(\theta_2(s))$ and $\theta^k(s) = \theta \circ \theta^{k-1}(s)$, with $\theta^0(s) = s$. This class of functions satisfy the following properties:

Property 1 *Let $\theta_1 : [0,a_1] \mapsto \mathbb{R}_+$ and $\theta_2 : [0,a_2] \mapsto \mathbb{R}_+$ be $\mathcal{K}$-functions, let $\theta_3(\cdot)$ and $\theta_4(\cdot)$ be $\mathcal{K}_\infty$-functions and let $\beta(\cdot,\cdot)$ be a $\mathcal{KL}$-function , then*

1. $\theta_1^{-1}(\cdot)$ *is a $\mathcal{K}$-function defined in* $[0,\theta_1(a_1)]$.
2. $\theta_1 \circ \theta_2(\cdot)$ *is a $\mathcal{K}$-function defined in* $[0,b]$, *with* $b = min(a_2, \theta_2^{-1}(a_1))$.
3. $\theta_3^{-1}(\cdot)$ *is a $\mathcal{K}_\infty$-function .*
4. $\theta_3 \circ \theta_4(\cdot)$ *is a $\mathcal{K}_\infty$-function .*
5. $\theta_1 \circ \beta(\cdot)$ *is a $\mathcal{KL}$-function .*
6. $max(\theta_1(s), \theta_2(s))$ *is a $\mathcal{K}$-function defined in* $[0,b]$ *with* $b = min(a_1,a_2)$.
7. $max(\theta_3(s), \theta_4(s))$ *is a $\mathcal{K}_\infty$-function .*
8. $min(\theta_1(s), \theta_2(s))$ *is a $\mathcal{K}$-function defined in* $[0,b]$ *with* $b = min(a_1,a_2)$.
9. $min(\theta_3(s), \theta_4(s))$ *is a $\mathcal{K}_\infty$-function .*
10. $\theta_1(s_1+s_2) \leq \theta_1(2\cdot s_1) + \theta_1(2\cdot s_2)$ *for all* $s_1, s_2 \in [0,a_1/2]$
11. $\theta_1(s_1) + \theta_2(s_2) \leq \theta_5(s_1+s_2)$, *where* $\theta_5(s) = \theta_1(s) + \theta_2(s)$, *for all* $s_1+s_2 \leq min(a_1,a_2)$.
12. $\theta_1(s_1) + \theta_2(s_2) \geq \theta_6(s_1+s_2)$, *where* $\theta_6(s) = min(\theta_1(s/2), \theta_2(s/2))$, *for all* $s_1 \in [0,a_1]$ *and* $s_2 \in [0,a_2]$ *such that* $s_1+s_2 \leq 2\cdot min(a_1,a_2)$.
13. *There exists a $\mathcal{K}_\infty$-function* $\theta_7(s)$ *such that* $\theta_7(s) \leq \theta_3(s)$ *for all* $s \geq 0$ *and* $\theta_8(s) = s - \theta_7(s)$ *is a $\mathcal{K}$-function .*

Note that $\mathcal{K}_\infty$-functions are a class of $\mathcal{K}$-functions; hence all the properties of the $\mathcal{K}$-functions (as properties 10, 11 and 12) can be extended to $\mathcal{K}_\infty$-functions.

2.2 System description

Consider a system described by an uncertain nonlinear time-invariant discrete time model

$$x^+ = f(x,u,w) \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the current control vector, the disturbance input $w \in \mathbb{R}^q$ models the uncertainty and x^+ is the successor state. It is assumed that the uncertainty is contained in a compact set,

$$w \in W \tag{2}$$

which may depend on the state and contains the origin. We consider that the uncertainty w is modelled by

$$\|w\| \leq \gamma(\|x\|) + \mu \tag{3}$$

where $\gamma(\cdot)$ is a $\mathcal{K}$-function . This description is suitable for modelling both plant uncertainties and persistent external disturbances.

The system is subject to constraints on both the state and the control action. These constraints are given by

$$u \in U \tag{4}$$

$$x \in X \tag{5}$$

where X and U are compact sets containing the origin.

In the sequel, x_k and u_k will denote the state and the control action applied to the system at sampling time k. Effective control in the presence of uncertainties requires a feedback structure. So, a sequence of control laws $\pi(x)$ to be applied to the system at current state x must be considered. This control policy for a prediction horizon N is given by

$$\pi(x) = \{\mu_0(x), \mu_1(\cdot), \cdots, \mu_{N-1}(\cdot)\}.$$

Note that, for a given state x, the first term is a control action, so it may be denoted as $u(0)$.

The evolution of the system controlled by $\pi(x)$ depends on the future values of the uncertainties. This sequence of future disturbances is denoted as future realization of uncertainties $\mathbf{w}$. The realization $\mathbf{w}$ is a possible realization of N uncertainties if $\mathbf{w} \in W^N$, where $W^N = W \times W \times \cdots \times W$, N times.

The solution to (1) at time j when the initial state is x at time 0, the uncertainty realization is $\mathbf{w}$ and the control policy π is applied will be denoted as $x(j) = \phi(j; x, \pi, \mathbf{w})$.

2.3 Some concepts on invariant sets

In this section, some well established definitions and results on invariant sets used in the paper are shown. See (Blanchini 1999, Kerrigan & Maciejowski 2000) for a compilation of definitions and results in set invariance theory.

Definition 1 *Consider the uncertain system $x^+ = \mathcal{F}(x,w)$, where $w \in \mathbb{R}^q$ models the uncertainty and $w \in W$. Then the set $\Omega \subset \mathbb{R}^n$ is a robust positively invariant set if $\mathcal{F}(x,w) \in \Omega$, for all $x \in \Omega$ and for all $w \in W$.*

Definition 2 *A set $\Omega \subset \mathbb{R}^n$ is a robust control invariant set for the system (1) subject to constraint (4) if for all $x \in \Omega$, there exists an admissible input $u = u(x) \in U$ such that $f(x,u,w) \in \Omega$ for all $w \in W$.*

Definition 3 *Let $\Omega \subset \mathbb{R}^n$ be a robust positively (or control) invariant set for system (1) subject to constraints (4) and (5), then the i-step robust stabilizable set $X_i(\Omega)$ is the set of admissible states which can be steered to the target set Ω in i steps or less by a sequence of admissible control laws $\pi(x)$ for all possible realization of the uncertainty $\mathbf{w} \in W^i$. This set is given by*

$$X_i(\Omega) = \{x \in \mathbb{R}^n : \exists \pi(x) \mid \mu_k(x(k)) \in U, x(k) \in X, \forall k = 0, \cdots, i-1, \text{ and } x(i) \in \Omega, \forall w(k) \in W\}$$

where $x(k) = \phi(k; x, \pi, \mathbf{w})$.

This set satisfies that $X_i(\Omega) \supseteq X_{i-1}(\Omega)$ and moreover $X_i(\Omega)$ is a robust control invariant set, for $i \geq 0$.

3 Sufficient conditions for robust stability

Consider a system given by

$$x_{k+1} = \mathcal{F}(x_k, w_k) \tag{6}$$

where x_k is the state of the system, and w_k is the uncertainty vector described by (3). Note that system (1) controlled by a given control law can be expressed by (6).

Definition 2 *The system (6) is robustly stable if there are a $\mathcal{KL}$-function $\beta(\cdot,\cdot)$ and a $\mathcal{K}$-function $\delta(\cdot)$ such that*

$$\|x_k\| \leq \beta(\|x_0\|, k) + \delta(\mu)$$

for all $\|w_k\| \leq \gamma(\|x_k\|) + \mu$.

This definition of stability is closely related with the notion of Input to State Stability (ISS) (Jiang & Wang 2001), but there exists some differences. In the given definition, the effect of the uncertainty depends on the modelled bound of uncertainties μ but not on the current disturbance w_k. This fact makes that if the uncertainty is modelled as tending to zero when the system tends to the origin (that is, $\mu = 0$) then robust stability implies asymptotic stability (as the input to state stability). But if we consider that $\mu > 0$, then the system might not evolve to the origin despite the evolution of the uncertainty w_k.

In the following definition, a robust Lyapunov function which provides a sufficient condition for robust stability is presented.

Definition 3 *Consider system (6) and suppose that the uncertainties vector w is bounded as in (3). A function $V(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}_+$ is called a robust Lyapunov function if there are some $\mathcal{K}_\infty$-functions $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\alpha_3(\cdot)$ and $\sigma(\cdot)$ and a $\mathcal{K}$-function $\rho(\cdot)$ such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) + \sigma(\mu)$$
$$V(\mathcal{F}(x,w)) - V(x) \leq -\alpha_3(\|x\|) + \rho(\mu)$$

for all $\|w\| \leq \gamma(\|x\|) + \mu$.

Note that the robust Lyapunov function may not be bounded above by a $\mathcal{K}_\infty$-function of the state. This happens, for instance, when the value of $V(0)$ is not zero because it depends on the modelled bound of the uncertainties.

Based on the previous definition, sufficient conditions for robust stability can be shown. Before stating these conditions, some technical lemmas are necessary.

Lemma 1 *Consider a $\mathcal{K}$-function $\psi(s) = s - \theta(s)$ where $\theta(\cdot)$ is a $\mathcal{K}_\infty$-function . Consider a $\mathcal{K}$-function given by $\phi(s) = s - 1/2 \cdot \theta(s)$, then $\psi(s_1 + s_2) \leq \phi(s_1) + \phi(s_2)$.*

Proof: First, we have that

$$\theta(s_1 + s_2) = 1/2 \cdot \theta(s_1 + s_2) + 1/2 \cdot \theta(s_1 + s_2) \geq 1/2 \cdot \theta(s_1) + 1/2 \cdot \theta(s_2)$$

Based on this result, we derive that

$$\psi(s_1 + s_2) = s_1 + s_2 - \theta(s_1 + s_2) \leq s_1 + s_2 - 1/2 \cdot \theta(s_1) - 1/2 \cdot \theta(s_2) = \phi(s_1) + \phi(s_2)$$

■

Lemma 2 *Let $\phi(\cdot)$ be a $\mathcal{K}$-function such that $\phi(s) < s$ for all $s > 0$, then the function $\delta(s,k) = \phi^k(s)$ is a $\mathcal{KL}$-function .*

Proof: It is immediate that $\phi^k(s)$ is a $\mathcal{K}$-function in s. The fact that $\phi^k(s)$ is decreasing in k for all $s > 0$ is proved by induction: by assumption, $\phi^1(s) = \phi(s) < s = \phi^0(s)$. Assume that $\phi^i(s) < \phi^{i-1}(s)$, then

$$\phi^{i+1}(s) = \phi \circ \phi^i(s) < \phi \circ \phi^{i-1}(s) = \phi^i(s)$$

which completes the proof. ■

Using these lemmas, we state sufficient conditions for robust stability in the following theorem.

Theorem 1 *If system (6) admits a robust Lyapunov function, then it is robustly stable.*

Proof:

Consider a $\mathcal{K}_\infty$-function $\bar{\alpha}_2(s) = \alpha_2(s) + \sigma(s)$. With this choice of $\bar{\alpha}_2(\cdot)$, it results that $\alpha_2(\|x\|) + \sigma(\mu) \leq \bar{\alpha}_2(\|x\| + \mu)$. Therefore, $V(x) \leq \bar{\alpha}_2(\|x\| + \mu)$ and hence, $\|x\| + \mu \geq \bar{\alpha}_2^{-1}(V(x))$.

Let $\varepsilon(\cdot)$ be a given $\mathcal{K}_\infty$-function, and consider the $\mathcal{K}_\infty$-function given by $\underline{\alpha}_3(s) = min(\alpha_3(s/2), \varepsilon(s/2))$, then

$$\alpha_3(\|x\|) + \varepsilon(\mu) \geq \underline{\alpha}_3(\|x\| + \mu) \geq \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}(V(x)) = \alpha_4(V(x))$$

where $\alpha_4(s) = \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}(s)$ is a $\mathcal{K}_\infty$-function . Then, we have that

$$V(\mathcal{F}(x,w)) \leq V(x) - \alpha_3(\|x\|) + \rho(\mu) \leq V(x) - \alpha_4(V(x)) + \varepsilon(\mu) + \rho(\mu)$$

In virtue of property 1, there exists a $\mathcal{K}_\infty$-function $\alpha_5(s)$ such that $\alpha_5(s) \leq \alpha_4(s)$ for all $s \geq 0$ and $\psi(s) = s - \alpha_5(s)$ is a $\mathcal{K}$-function . Then, denoting $\gamma(\mu) = \varepsilon(\mu) + \rho(\mu)$, we have that

$$V(\mathcal{F}(x,w)) \leq \psi(V(x)) + \gamma(\mu) \tag{7}$$

Consider the $\mathcal{K}$-function given by $\phi(s) = s - 1/2 \cdot \alpha_5(s)$, and consider that the initial state of the system is x_0, then we are going to prove that

$$V(x_{k+1}) \leq \phi^{k+1}(V(x_0)) + \delta_{k+1}(\mu) \tag{8}$$

where $\delta_k(\mu) = \phi(\delta_{k-1}(\mu)) + \gamma(\mu)$ with $\delta_1(\mu) = \gamma(\mu)$. This is proved by induction: in virtue of (7) and lemma 1 we have that

$$V(x_1) \leq \psi(V(x_0)) + \gamma(\mu) \leq \phi(V(x_0)) + \delta_1(\mu)$$

Assume that $V(x_k) \leq \phi^k(V(x_0)) + \delta_k(\mu)$, then in virtue of lemma 1 we have that

$$\begin{aligned} V(x_{k+1}) &\leq \psi(V(x_k)) + \gamma(\mu) \\ &\leq \psi(\phi^k(V(x_0)) + \delta_k(\mu)) + \gamma(\mu) \\ &\leq \phi(\phi^k(V(x_0))) + \phi(\delta_k(\mu)) + \gamma(\mu) \\ &= \phi^{k+1}(V(x_0))) + \delta_{k+1}(\mu)) \end{aligned}$$

We are going to show that the sequence $\delta_k(\mu)$ is strictly increasing for all $\mu > 0$ and it is bounded above, which implies that it is convergent. First, we have that $\delta_2(\mu) = \phi \circ \delta_1(\mu) + \gamma(\mu) > \gamma(\mu) = \delta_1(\mu)$. Assume that $\delta_k(\mu) > \delta_{k-1}(\mu)$, then

$$\delta_{k+1}(\mu) = \phi \circ \delta_k(\mu) + \gamma(\mu) > \phi \circ \delta_{k-1}(\mu) + \gamma(\mu) = \delta_k(\mu)$$

Moreover the sequence is bounded by a $\mathcal{K}_\infty$-function $\theta(s) = \alpha_5^{-1}(2\cdot\gamma(s))$. It is proved by induction: since $\alpha_5(s) < s$, then $\alpha_5^{-1}(s) > s$ and hence $\theta(\mu) \geq 2\cdot\gamma(\mu) \geq \delta_1(\mu)$; assume that $\delta_k(\mu) \leq \theta(\mu)$, then

$$\begin{aligned}
\delta_{k+1}(\mu) &= \phi\circ\delta_k(\mu) + \gamma(\mu) \\
&\leq \phi\circ\theta(\mu) + \gamma(\mu) \\
&= \theta(\mu) - 1/2\cdot\alpha_5\circ\theta(\mu) + \gamma(\mu) \\
&= \theta(\mu)
\end{aligned}$$

From this result and equation (8) we have that

$$\alpha_1(\|x_k\|) \leq V(x_k) \leq \phi^k(V(x_0)) + \theta(\mu)$$

From properties of the $\mathcal{K}$-functions and lemma 1 we derive that

$$\begin{aligned}
\alpha_1(\|x_k\|) &\leq \phi^k(\alpha_2(\|x_0\|) + \sigma(\mu)) + \theta(\mu) \\
&\leq \phi^k(2\cdot\alpha_2(\|x_0\|)) + \phi^k(2\cdot\sigma(\mu)) + \theta(\mu) \\
&\leq \phi^k(2\cdot\alpha_2(\|x_0\|)) + 2\cdot\sigma(\mu) + \theta(\mu)
\end{aligned}$$

From this inequality we have that

$$\begin{aligned}
\|x_k\| &\leq \alpha_1^{-1}\big(\phi^k(2\cdot\alpha_2(\|x_0\|)) + 2\cdot\sigma(\mu) + \theta(\mu)\big) \\
&\leq \alpha_1^{-1}\big(2\cdot\phi^k(2\cdot\alpha_2(\|x_0\|))\big) + \alpha_1^{-1}(4\cdot\sigma(\mu) + 2\cdot\theta(\mu)) \\
&= \beta(\|x_0\|, k) + \varphi(\mu)
\end{aligned}$$

From properties of $\mathcal{K}$-functions, it is easy to see that $\beta(\|x_0\|, k)$ is a $\mathcal{KL}$-function and $\varphi(\mu)$ is a $\mathcal{K}$-function , which completes the proof. ■

From this result, it is easy to see that if a system is ISS then it is robustly stable. Moreover, a given ISS-Lyapunov function is a robust Lyapunov function. However, a robustly stable system may not be ISS because the Lyapunov function may be not zero at the origin and because the convergence condition depends on the modelled bound of the uncertainty.

4 Robust model predictive control

As it is well known, the MPC control technique is able to asymptotically stabilize constrained systems under mild assumptions (Mayne et al. 2000). If the system to be controlled is uncertain, then stability and constraint satisfaction are not guaranteed. For uncertainties that are small enough, the MPC stabilizes the system in a neighborhood of the origin (Limon et al. 2002*b*). If this is not the case, it is necessary to use a robust formulation of the controller, that is, the effect of the uncertainties must be taken into account in the design of the controller. An approach to this problem is the so-called open loop robust MPC. In this case, a sequence of control actions is computed so that it guarantees both robust constraint satisfaction and convergence (Michalska & Mayne 1993, Limon et al. 2002*a*). This approach is based on the solution of a nonlinear mathematical programming problem, as in the nominal case. However, the open loop nature of the predictions makes the controller very conservative.

To overcome this drawback, the closed loop formulation is proposed (Scokaert & Mayne 1998, Mayne 2001, Kerrigan & Mayne 2002). In this case, a sequence of control laws is computed instead of a sequence of control actions. By doing this, the reaction of the controller to the uncertainty is incorporated in the prediction and the conservativeness is mitigated.

The cost associated to the future evolution of the system depends on the control policy and the future realization of the uncertainties

$$J_N(x,\pi,\mathbf{w}) = \sum_{i=0}^{N-1} \ell(x(i),\mu_i(x(i))) + F(x(N))$$

where $x(i) = \phi(i;x,\pi,\mathbf{w})$, $\pi = \{\mu_i(\cdot)\}$, and the stage cost $\ell(\cdot,\cdot)$ is a definite positive function. The control policy is derived from the solution of an optimization problem $P_N(x,\Omega)$ given by

$$\begin{aligned} J_N^*(x) \quad &= \quad \min_{\pi} \max_{\mathbf{w}} J_N(x,\pi,\mathbf{w}) \\ s.t. \quad & \quad \mu_i(x(i)) \in U, x(i) \in X,\ i = 0,\dots,N-1, \forall \mathbf{w} \in W^N \\ & \quad x(N) \in \Omega, \forall \mathbf{w} \in W^N \end{aligned}$$

Due to the receding horizon policy, the min-max MPC controller is given by $K_N(x) = \mu_0(x)$.

This problem is feasible in the region $X_N(\Omega)$. If the terminal set Ω is a robust invariant set, then robust feasibility all the time and hence robust constraint satisfaction are guaranteed. However, additional assumptions must be considered to ensure robust stability. In (Mayne 2001) sufficient conditions for asymptotic stability for the closed loop min-max controller in case of decaying uncertainties are given. It is proved that it suffices to choose an admissible robust invariant set as terminal set Ω and a terminal cost function such that

$$F(f(x,h(x),w)) - F(x) \leq -\ell(x,h(x)) \quad \forall x \in \Omega, \forall w \in W \tag{9}$$

where $u = h(x)$ is an admissible robustly stabilizing local control law in Ω.

5 Robust stability analysis

In this paper we extend the results presented in (Mayne 2001) to the case of not decaying uncertainties described by (3). This constitutes the main contribution of the paper.

Assumption 1 *Consider system (1) and suppose that the uncertainties vector w is modelled by (3). Let Ω be an admissible robust invariant set for the system controlled by the control law $u = h(x)$ such that the origin is in its interior. Let $F(x)$ be an associated robust Lyapunov function such that for all $x \in \Omega$ and for all $\|w\| \leq \gamma(\|x\|) + \mu$ we have that*

$$\begin{gathered} \alpha_1(\|x\|) \leq F(x) \leq \alpha_2(\|x\|) + \sigma(\mu) \\ F(f(x,h(x),w)) - F(x) \leq -\ell(x,h(x)) + \rho(\mu) \end{gathered}$$

where $\alpha_1(\cdot)$, $\alpha_2(\cdot)$, $\sigma(\cdot)$ and $\rho(\cdot)$ are $\mathcal{K}_\infty$-functions and the stage cost satisfies $\ell(x,u) \geq \alpha_3(\|(x,u)\|)$, being $\alpha_3(\cdot)$ a $\mathcal{K}_\infty$-function

First, an upper bound of the optimal cost is obtained in the following lemmas.

Lemma 3 *Consider system (1) and suppose that the uncertainties vector w is modelled by (3). Let Ω and $F(x)$ satisfy assumption 1, then for all $x \in \Omega$ we have that*

$$J_N^*(x) \leq F(x) + N \cdot \rho(\mu)$$

Proof: Let $\pi_h(x)$ be a control policy obtained from the local control law, that is, $\mu_i(x) = h(x)$. From assumption 1 we have that

$$F(x(k+1)) - F(x(k)) \leq -\ell(x(k), h(x(k))) + \rho(\mu)$$

where $x(k) = \phi(k; x, \pi_h, \mathbf{w})$. Summing this inequality from $k = 0$ to $N-1$ we have

$$F(x(N)) - F(x) \leq -\sum_{i=0}^{N-1} \ell(x(k), h(x(k))) + N \cdot \rho(\mu)$$

and hence

$$F(x) \geq \sum_{i=0}^{N-1} \ell(x(k), h(x(k))) + F(x(N)) - N \cdot \rho(\mu)$$

In virtue of assumption 1, the control policy π_h is feasible. By optimality, it is derived that

$$F(x) \geq J_N^*(x) - N \cdot \rho(\mu)$$

■

Based on this lemma, an upper bound of the optimal cost is obtained for all $x \in X_N(\Omega)$.

Lemma 4 *Consider system (1) and suppose that the uncertainty vector w is modelled by (3). Let Ω and $F(x)$ satisfy assumption 1, then there exists a couple of $\mathcal{K}_\infty$-functions $\alpha_s^J(\cdot)$ and $\sigma^J(\cdot)$ such that*

$$J_N^*(x) \leq \alpha_2^J(\|x\|) + \sigma^J(\mu)$$

for all $x \in X_N(\Omega)$ and for all $\|w\| \leq \gamma(\|x\|) + \mu$.

Proof: The compactness of X and U implies that $x(k)$, the predicted evolution of the system, and $u(k)$, the feasible control action, are bounded. This fact and assumption 1 guarantee that the optimal cost is upper bounded, that is, there exists a finite real number $\bar{J_N}$ such that $J_N^*(x) \leq \bar{J_N}$ for all $x \in X_N(\Omega)$.

Let $B_r \subset \mathbb{R}^n$ be a ball $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ such that $B_r \subseteq \Omega$. Note that this ball exists since the origin is in the interior of Ω.

Let ε be a positive constant $\varepsilon = max(1, \bar{J_N}/\alpha_2(r))$. Consider the $\mathcal{K}_\infty$-functions given by $\alpha_2^J(s) = \varepsilon \cdot \alpha_2(s)$ and $\sigma^J(s) = \sigma(s) + N \cdot \rho(s)$. Two cases must be taken into account:

- If $x \in \Omega$, then based on the previous lemma we have that

$$J_N^*(x) \leq F(x) + N \cdot \rho(\mu) \leq \alpha_2(\|x\|) + \sigma(\mu) + N \cdot \rho(\mu) \leq \alpha_2^J(\|x\|) + \sigma^J(\mu)$$

- If $x \notin \Omega$, then $x \notin B_r$ and hence $\alpha_2(\|x\|) > \alpha_2(r)$. Hence

$$J_N^*(x) \leq \bar{J_N} \leq \bar{J_N} \cdot \frac{\alpha_2(\|x\|)}{\alpha_2(r)} \leq \alpha_2^J(\|x\|) + \sigma^J(\mu)$$

■

In the following theorem, we present the main result of the paper: the optimal cost of the min-max MPC controller is a robust Lyapunov function. Hence, robust stability of the min-max MPC is proved.

Theorem 2 *Consider system (1) and suppose that the uncertainty vector w is modelled by (3). Let Ω and $F(x)$ satisfy assumption 1. Then the uncertain system controlled by the min-max MPC controller $u = K_N(x)$ is robustly stable for any initial state $x_0 \in X_N(\Omega)$ and for any uncertainty $\|w_k\| \leq \gamma(\|x_k\|)+\mu$. Furthermore, the optimal cost is a robust Lyapunov function.*

Proof: We are going to check that the optimal cost is a robust Lyapunov function. In virtue of the previous lemmas we have that

$$\alpha_3(\|x\|) \leq \ell(x, K_N(x)) \leq J_N^*(x) \leq \alpha_2^J(\|x\|) + \sigma^J(\mu)$$

which is the first property of the robust Lyapunov function. The decreasing property of the optimal cost is proved in what follows by means of the dynamic programming approach to the min-max problem (in an analogous way to the proof presented in (Mayne 2001)).

Thanks to the invariance of the terminal set, the feasible region of the controller $X_N(\Omega)$ is a robust invariant set for the closed loop system and the controller is well defined all the time (Mayne 2001, Kerrigan & Maciejowski 2001).

Define the optimal cost in i-steps:

$$J_i^*(x) = \min_{u\in U}\left\{\max_{w\in W}\{\ell(x,u)+J_{i-1}^*(f(x,u,w))\} \text{ such that } f(x,u,w)\in X_{i-1}(\Omega),\ \forall w\in W\right\}$$

where $J_0^*(x) = F(x)$ defined in $X_0(\Omega) = \Omega$. Define $u = K_i(x)$ as the argument of the optimal solution to this optimization problem.

For all $x \in \Omega$, it easy to check that

$$\begin{aligned}
J_1^*(x) - J_0^*(x) &= \min_{u\in U}\left\{\max_{w\in W}\{\ell(x,u)+F(f(x,u,w))\} \text{ such that } f(x,u,w)\in\Omega\,\forall w\in W\right\} - F(x) \\
&\leq \max_{w\in W}\{\ell(x,h(x))+F(f(x,h(x),w))\} - F(x) \\
&\leq \rho(\mu)
\end{aligned}$$

Assume that $J_i^*(x) - J_{i-1}^*(x) \leq \rho(\mu)$ for all $x \in X_{i-1}(\Omega)$. Consider any $x \in X_i(\Omega)$, then

$$J_{i+1}^*(x) - J_i^*(x) = \min_{u\in U}\left\{\max_{w\in W}\{\ell(x,u)+J_i^*(f(x,u,w))\} \text{such that } f(x,u,w)\in X_i(\Omega),\ \forall w\in W\right\} - J_i^*(x)$$

Since $x \in X_i(\Omega)$, the control action $u = K_i(x)$ is well defined and it is feasible for the optimization problem in $i+1$ steps since $f(x, K_i(x), w) \in X_{i-1}(\Omega) \subseteq X_i(\Omega)$. Then it follows that

$$J_{i+1}^*(x) \leq \max_{w\in W}\{\ell(x,K_i(x)) + J_i^*(f(x,K_i(x),w))\}$$

and we have that

$$\begin{aligned}
J_{i+1}^*(x) - J_i^*(x) &\leq \max_{w\in W}\{\ell(x,K_i(x))+J_i^*(f(x,K_i(x),w))\} - J_i^*(x) \\
&= \max_{w\in W}\{\ell(x,K_i(x))+J_i^*(f(x,K_i(x),w))\} - \max_{w\in W}\{\ell(x,K_i(x))+J_{i-1}^*(f(x,K_i(x),w))\} \\
&\leq \max_{w\in W}\left\{\ \{\ell(x,K_i(x))+J_i^*(f(x,K_i(x),w))\} - \{\ell(x,K_i(x))+J_{i-1}^*(f(x,K_i(x),w))\}\right\} \\
&= \max_{w\in W}\left\{J_i^*(f(x,K_i(x),w)) - J_{i-1}^*(f(x,K_i(x),w))\right\} \\
&\leq \rho(\mu)
\end{aligned}$$

Hence, by induction it is inferred that $J^*_{i+1}(x) - J^*_i(x) \leq \rho(\mu)$ for all $i \geq 0$ and $x \in X_i(\Omega)$.

Consider that the state of the system is x_k and that the min-max MPC control law $u_k = K_N(x_k)$ is applied, then the system evolves to $x_{k+1} = f(x_k, K_N(x_k), w_k)$. Since $x_k \in X_N(\Omega)$, it is clear that $x_{k+1} \in X_N(\Omega)$.

Based on the monotonicity result, it follows that

$$\begin{aligned}
J^*_N(x_{k+1}) - J^*_N(x_k) &\leq J^*_N(x_{k+1}) - \min_{u \in U}\left\{\max_{w \in W}\{\ell(x_k,u) + J^*_{N-1}(f(x_k,u,w))\} \text{ such that } f(x_k,u,w) \in X_{N-1}(\Omega), \forall w \in W\right\} \\
&= J^*_N(x_{k+1}) - \max_{w \in W}\{\ell(x_k, K_N(x_k)) + J^*_{N-1}(f(x_k, K_N(x_k), w))\} \\
&\leq J^*_N(x_{k+1}) - \ell(x_k, K_N(x_k)) - J^*_{N-1}(f(x_k, K_N(x_k), w_k)) \\
&= J^*_N(x_{k+1}) - \ell(x_k, K_N(x_k)) - J^*_{N-1}(x_{k+1}) \\
&\leq -\ell(x_k, K_N(x_k)) + \rho(\mu)
\end{aligned}$$

Consequently, the optimal cost is a robust Lyapunov function, and hence the closed loop system is robustly stable.

■

It is interesting to see that the robustness of the design of the terminal cost is translated to robustness of the optimal cost of the MPC: the effect of the decaying part of the uncertainties does not appear explicitly in the bounds of the robust Lyapunov function, and hence in the convergence rates derived from it; only the term of persistent disturbances (which is induced by the constant bound μ) has an effect on them. Moreover, the effect of the persistent uncertainty on the optimal cost is the same that the one on the terminal cost.

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