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Baloch, Imran Abbas, Iscan, Imdat and Dragomir, Sever S (2016) Fejer Type Inequalities for Harmonically (s,m)-Convex Functions. International Journal of Analysis and Applications, 12 (2). pp. 188-197. ISSN 2291-8639

International Journal of Analysis and Applications
ISSN 2291-8639
Volume 12, Number 2 (2016), 188-197
http://www.etamaths.com

FEJÉR TYPE INEQUALITIES FOR HARMONICALLY (s,m)-CONVEX FUNCTIONS

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Abstract. In this paper, a new weighted identity involving harmonically symmetric functions and differentiable functions is established. By using the notion of harmonic symmetricity, harmonic (s,m)-convexity, analysis and some auxiliary results, some new Fejér type integral inequalities are presented for the class of harmonically (s,m)-convex functions.

1. Introduction

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called convex function if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in I$ and $\lambda \in [0,1]$. There are many results associated with convex functions in the area of inequalities, but one of them is the classical Hermite-Hadamard (see [21]) inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

for all $a, b \in I$, with $a < b$. The inequalities in (1.1) hold in reversed direction if f is a concave function. A vast literature have been produced by a number of mathematicians for convex functions but (1.1) is considered to be the most famous inequality for convex mappings due to its usefulness and many applications in various branches of pure and applied mathematics. The definition of classical or usual convex functions has been generalized in a variety of ways and as a consequence many researchers have established a number of Hermite-Hadamard type inequalities by using different generalizations of the classical convexity, see for instance [2]-[23] and the references mentioned in these papers.
One of the generalizations of classical convexity is the harmonic (s,m)-convexity in second sense, which unifies the notion of Harmonically convex [12] and Harmonically s-convex functions in second sense [13] introduced by Imdat Iscan, as stated in the definition below.

Definition 1. [1] The function $f : I \subset (0,\infty) \to \mathbb{R}$ is said to be harmonically (s,m)-convex in second sense, where $s \in (0,1]$ and $m \in (0,1]$ if

$$f\left(\frac{mxy}{mty+(1-t)x}\right) = f\left(\left(\frac{t}{x}+\frac{1-t}{my}\right)^{-1}\right) \leq t^s f(x) + m(1-t)^s f(y)$$

$\forall x, y \in I$ and $t \in [0,1]$.

Remark 1. *Note that for $s = 1$,harmonic (s,m)-convexity reduces to harmonic m-convexity and for $m = 1$, harmonic (s,m)-convexity reduces to harmonic s-convexity in second sense (see [13]) and for $s, m = 1$, harmonic (s,m)-convexity reduces to ordinary harmonic convexity (see [12]).*

Proposition 1. *Let $f : (0,\infty) \to \mathbb{R}$ be a function*
a) if f is (s,m)-convex function in second sense and non-decreasing, thenf is harmonically (s,m)-convex function in second sense.
b) if f is harmonically (s,m)-convex function in second sense and non-increasing, then f is (s,m)-convex function in second sense.

2010 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A51, 26E60, 41A55.
Key words and phrases. Hermite-Hadamard's inequality; Fejér's inequality; convex function; harmonically (s,m)-convex function; Hölder's inequality; power mean inequality.

Remark 2. *According to proposition 1, every non-decreasing (s,m)-convex function in second sense is also harmonically (s,m)-convex function in second sense.*

Example 1. (see[3]) Let $0 < s < 1$ and $a, b, c \in \mathbb{R}$, then function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} a, & x = 0 \\ bx^s + c, & x > 0 \end{cases}$$

is non-decreasing s-convex function in second sense for $b \geq 0$ and $0 \leq c \leq a$. Hence, by proposition 1, f is harmonically $(s, 1)$-convex function.

Proposition 2. *Let $s \in [0,1]$, $m \in (0,1]$, $f : [a, mb] \subset (0, \infty) \to \mathbb{R}$, be an increasing function and $g : [a, mb] \to [a, mb]$, $g(x) = \frac{mab}{a+mb-x}$, $a < mb$. Then f is harmonically (s,m)-convex in second sense on $[a, mb]$ if and only if fog is (s,m)-convex in second sense on $[a, mb]$.*

The following result of the Hermite-Hadamard type holds.

Theorem 1. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a harmonically (s,m)-convex function in second sense with $s \in [0,1]$ and $m \in (0,1]$. If $0 < a < b < \infty$ and $f \in L[a,b]$, then one has following inequality*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \min\Big[\frac{f(a) + mf(\frac{b}{m})}{s+1}, \frac{f(b) + mf(\frac{a}{m})}{s+1}\Big]$$

Corollary 1. *If we take $m = 1$ in Theorem 1, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a) + f(b)}{s+1}$$

Corollary 2. *If we take $s = 1$ in Theorem 1, then we get*

$$\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \min\Big[\frac{f(a) + mf(\frac{b}{m})}{2}, \frac{f(b) + mf(\frac{a}{m})}{2}\Big]$$

Chen and Wu [4], established the following weighted Fejér type inequality for the harmonically convex function as follow

Theorem 2. [4] *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L([a,b])$, then one has*

$$f\Big(\frac{2ab}{a+b}\Big)\int_a^b \frac{g(x)}{x^2}dx \leq \int_a^b \frac{g(x)f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx, \tag{1.2}$$

where $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and satisfies

$$g\Big(\frac{ab}{x}\Big) = g\Big(\frac{ab}{a+b-x}\Big)$$

The main purpose of the present paper is to introduce a new notion of harmonically symmetric functions and to establish an identity involving a harmonically symmetric function and a differentiable function. We will prove some Fejér type inequalities by using this identity related with the second part of the inequality given above by (1.2).We believe that our findings are novel, new and better than those already exist and will open new ways for further research in this field.

2. Main Results

Throughout this section, we take $L(t) = \frac{2ab}{(1-t)a+(1+t)b}$ and $U(t) = \frac{2ab}{(1+t)a+(1-t)b}$. The Beta function, the Gamma function and the integral form of the hypergeometric function are defined as follows to be used in the sequel of paper

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt,\ \alpha, \beta > 0$$

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt,\ \alpha > 0$$

and

$$ {}_2F_1(\alpha,\beta;\gamma,z)=\frac{1}{B(\beta,\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}dt,\ \gamma>\beta>0,\ |z|<1 $$

The notion of harmonically symmetric functions is defined as follows:

Definition 2. We say that a function $g:[a,b]\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ is harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$ g(x)=g(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}) $$

holds for all $x\in[a,b]$.

Now, we give the weighted integral equality by using which we establish our results in this article.

Lemma 1. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\to\mathbb{R}$ be a differentiable function on I° and $a,b\in I^\circ$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$. If $f'\in L([a,b])$, then the following identity holds*

$$ \begin{aligned} &\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx \\ &\qquad=\frac{b-a}{4ab}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)\Big[(U(t))^2f'(U(t))-(L(t))^2f'(L(t))\Big]dt \end{aligned} $$

Proof. Since, $g:[a,b]\to[0,\infty)$ is harmonically symmetric to $\frac{2ab}{a+b}$, then $g(U(t))=g(L(t))$. Consider

$$ \begin{aligned} I &= \frac{b-a}{4ab}\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)\Big[(U(t))^2f'(U(t))-(L(t))^2f'(L(t))\Big]dt \\ &= \frac{1}{2}\left[\int_0^1\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)d[f(U(t))+f(L(t))]\right] \\ &= \frac{1}{2}\left[\left(\int_{L(t)}^{U(t)}\frac{g(x)}{x^2}dx\right)(f(U(t))+f(L(t)))\Big|_0^1\right. \\ &- \left.\frac{b-a}{2ab}\int_0^1(g(U(t))+g(L(t)))(f(U(t))+f(L(t)))dt\right] \\ &= \frac{1}{2}\left[(f(a)+f(b))\left(\int_a^b\frac{g(x)}{x^2}dx\right)-\frac{b-a}{ab}\int_0^1 g(U(t)f(U(t)dt\right. \\ &- \left.\frac{b-a}{ab}\int_0^1 g(L(t)f(L(t)dt\right] \\ &= \frac{1}{2}\left[(f(a)+f(b))\int_a^b\frac{g(x)}{x^2}dx-2\int_a^{\frac{2ab}{a+b}}\frac{g(x)f(x)}{x^2}dx+2\int_{\frac{2ab}{a+b}}^a\frac{g(x)f(x)}{x^2}dx\right] \\ &= \frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx \end{aligned} $$

□

Now, we present new Fejér type inequalities for harmonically (s,m)-convex functions, which give the weighted generalization of some of the results established in resent literature.

Theorem 3. *Let $f:I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,\frac{b}{m}\in I^\circ$, $m\in(0,1]$ with $a<b$ and let $g:[a,b]\to[0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f'\in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a,\frac{b}{m}]$ for $q\geq1$, then the following inequality holds*

$$ \left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right| $$

$$\leq \frac{b-a}{8ab}a^{\frac{2}{q}}\|g\|_\infty\Big\{\lambda_1^{1-\frac{1}{q}}(a,b)\Big(\big\{2^2B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b})$$
$$- \quad 2^{1-s}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b})+\frac{1}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q$$
$$+ \quad \frac{m2^{2-s}b^2}{(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Big)^{\frac{1}{q}} + \quad \lambda_2^{1-\frac{1}{q}}(a,b)$$
$$\times\Big(\big\{2^2B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b})-\frac{2^{2-s}b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a})$$
$$- \quad \frac{2^{2-s}b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q$$
$$+\frac{m}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Big)^{\frac{1}{q}}\Big\} \tag{2.1}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{g(x)}{x^2}dx-\int_a^b\frac{g(x)f(x)}{x^2}dx\right| \leq \frac{b-a}{8ab}\|g\|_\infty$$
$$\times\Big\{\Big(\int_0^1(1-t)(U(t))^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)(U(t))^2|f'(U(t))|^qdt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^1(1-t)(L(t))^2dt\Big)^{1-\frac{1}{q}}\Big(\int_0^1(1-t)(L(t))^2|f'(L(t))|^qdt\Big)^{\frac{1}{q}}\Big\} \tag{2.2}$$

By the harmonic (s,m)-convexity of $|f'|^q$ on $[a,b]$ for $q\geq 1$, we have

$$\int_0^1(1-t)(U(t))^2|f'(U(t))|^qdt = \int_0^1(1-t)\big(\frac{2ab}{(1+t)a+(1-t)b}\big)^2$$
$$\times \quad \left|f'\big(\frac{2ab}{(1+t)a+(1-t)b}\big)\right|^qdt \leq \frac{1}{2^s}|f'(b)|^q\int_0^1(1-t)(1+t)^s\big(\frac{2ab}{(1+t)a+(1-t)b}\big)^2dt$$
$$+ \quad m\frac{1}{2^s}|f'(\frac{a}{m})|^q\int_0^1(1-t)^{s+1}\big(\frac{2ab}{(1+t)a+(1-t)b}\big)^2dt$$

$$= \big\{2^2a^2B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b})-\frac{a^2}{2^{s-1}}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b}) \tag{2.3}$$
$$+\frac{a^2}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q+\frac{ma^2b^2}{2^{s-2}(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q$$

and

$$\int_0^1(1-t)(L(t))^2|f'(L(t))|^qdt = \int_0^1(1-t)\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2$$
$$\times \quad \left|f'\big(\frac{2ab}{(1-t)a+(1+t)b}\big)\right|^qdt \leq \frac{1}{2^s}|f'(a)|^q\int_0^1(1-t)(1+t)^s\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2dt$$
$$+ \quad m\frac{1}{2^s}|f'(\frac{b}{m})|^q\int_0^1(1-t)^{s+1}\big(\frac{2ab}{(1-t)a+(1+t)b}\big)^2dt$$

$$= \big\{2^2a^2B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b})-\frac{2^{2-s}a^2b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a}) \tag{2.4}$$
$$-\frac{2^{2-s}a^2b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q+\frac{ma^2}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q$$

Moreover,

$$\int_0^1 (1-t)(U(t))^2 dt = \int_0^1 (1-t)\Big(\frac{2ab}{(1+t)a+(1-t)b}\Big)^2 dt = \Big(\frac{2ab}{b-a}\Big)^2 \ln\Big(\frac{a+b}{2a}\Big) - \frac{(2ab)^2}{b^2-a^2} := \lambda_1(a,b) \tag{2.5}$$

and

$$\int_0^1 (1-t)(L(t))^2 dt = \int_0^1 (1-t)\Big(\frac{2ab}{(1-t)a+(1+t)b}\Big)^2 dt = \frac{(2ab)^2}{b^2-a^2} + \Big(\frac{2ab}{b-a}\Big)^2 \ln\Big(\frac{a+b}{2b}\Big) := \lambda_2(a,b) \tag{2.6}$$

A combination of (2.2), (2.3), (2.4), (2.5) and (2.6) gives required result. This completes the proof. □

Corollary 3. *Suppose the assumptions of the Theorem 3 are satisfied. If $g(x) = \frac{ab}{b-a}$ for all $x \in [a,b]$, then one has the following inequality*

$$\Big|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\Big|$$

$$\begin{aligned}
\leq\ & \frac{a^{\frac{2}{q}}}{8}\Big\{\lambda_1^{1-\frac{1}{q}}(a,b)\Big(\big\{2^2 B(s+1,2).{}_2F_1(2,s+1,s+3;\frac{b-a}{b})\\
-\ & 2^{1-s}B(s+1,1).{}_2F_1(2,s+1,s+2;\frac{b-a}{2b}) + \frac{1}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})\big\}|f'(b)|^q\\
+\ & \frac{m2^{2-s}b^2}{(b+a)^2}B(1,s+2).{}_2F_1(2,1,s+3;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Big)^{\frac{1}{q}} + \lambda_2^{1-\frac{1}{q}}(a,b)\\
& \times\Big(\big\{2^2 B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b}) - \frac{2^{2-s}b^2}{(b+a)^2}B(1,s+1).{}_2F_1(2,1,s+2;\frac{b-a}{b+a})\\
-\ & \frac{2^{2-s}b^2}{(b+a)^2}B(2,s+1).{}_2F_1(2,2,s+3;\frac{b-a}{b+a})\big\}|f'(a)|^q\\
& + \frac{m}{2^s}B(s+2,1).{}_2F_1(2,s+2,s+3;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Big)^{\frac{1}{q}}\Big\}
\end{aligned} \tag{2.7}$$

Theorem 4. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, \frac{b}{m} \in I^\circ$, $m \in (0,1]$ with $a < b$ and let $g: [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a, \frac{b}{m}]$ for $q > 1$, then the following inequality holds*

$$\Big|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\Big| \leq \frac{a(b-a)}{8b}.\|g\|_\infty$$

$$\begin{aligned}
\times\Big\{&\Big(\big\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b}) - 2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\big\}|f'(b)|^q\\
& + m2^{2q-s}\Big(\frac{b}{b+a}\Big)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\Big)^{\frac{1}{q}}\\
+&\Big(\big\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b}) - 2^{2q-s}\Big(\frac{b}{b+a}\Big)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\big\}|f'(a)|^q\\
& + \frac{m}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\Big)^{\frac{1}{q}}
\end{aligned} \tag{2.8}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{g(x)f(x)}{x^2}dx\right| \le \frac{b-a}{8ab}\|g\|_\infty \left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}}+\left(\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}\right\} \tag{2.9}$$

By the harmonic (s,m)-convexity of $|f'|^q$ on $[a,b]$ for $q>1$, we have

$$\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt=\int_0^1 \left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q}$$

$$\times\left|f'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\right|^q dt \le \frac{1}{2^s}|f'(b)|^q\int_0^1 (1+t)^s\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q}dt$$

$$+\ m\frac{1}{2^s}|f'(\frac{a}{m})|^q\int_0^1 (1-t)^s\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^{2q}dt$$

$$=a^{2q}\left\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})-2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\right\}|f'(b)|^q$$
$$+m2^{2q-s}\left(\frac{ab}{b+a}\right)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q \tag{2.10}$$

and

$$\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt=\int_0^1 \left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q}$$

$$\times\left|f'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\right|^q dt \le \frac{1}{2^s}|f'(a)|^q\int_0^1 (1+t)^s\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q}dt$$

$$+\ m\frac{1}{2^s}|f'(\frac{b}{m})|^q\int_0^1 (1-t)^s\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^{2q}dt$$

$$=a^{2q}\left\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})-2^{2q-s}\left(\frac{b}{b+a}\right)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\right\}|f'(a)|^q$$
$$+\frac{ma^{2q}}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q \tag{2.11}$$

By putting (2.10) and (2.11) in (2.9), we get desired result. □

Corollary 4. *Suppose the assumptions of the Theorem 3 are satisfied. If $g(x)=\frac{ab}{b-a}$ for all $x\in[a,b]$, then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{g(x)f(x)}{x^2}dx\right| \le \frac{a^2}{8}$$

$$\times\left\{\left(\left\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})-2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\right\}|f'(b)|^q\right.\right.$$

$$\left.+m2^{2q-s}\left(\frac{b}{b+a}\right)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q\right)^{\frac{1}{q}}$$

$$+\left(\left\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})-2^{2q-s}\left(\frac{b}{b+a}\right)^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\right\}|f'(a)|^q\right.$$

$$\left.+\frac{m}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q\right)^{\frac{1}{q}} \tag{2.12}$$

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, \frac{b}{m} \in I^\circ$, $m \in (0,1]$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L([a,b])$. If $|f'|^q$ is harmonically (s,m)-convex on $[a, \frac{b}{m}]$ for $q > 1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq 2^{1-\frac{1}{q}}\frac{a(b-a)}{8b}\|g\|_\infty$$

$$\Bigg(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q + 2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q - |f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q - |f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Bigg)^{\frac{1}{q}} \tag{2.13}$$

Proof. From Lemma 1 and hölder's inequality, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq \frac{b-a}{8ab}\|g\|_\infty\left(\int_0^1 (1-t)^{\frac{q}{q-1}}dt\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}\right\} \tag{2.14}$$

By the power-mean inequality ($a^r + b^r \leq 2^{1-r}(a+b)^r$ for $a > 0$, $b > 0$ and $r < 1$), we have

$$\left(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt\right)^{\frac{1}{q}} + \left(\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}}$$

$$\leq 2^{1-\frac{1}{q}}\left(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt + \int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\right)^{\frac{1}{q}} \tag{2.15}$$

Since, $|f'|^q$ is harmonically (s,m)-convex on $[a,b]$ for $q > 1$, we obtain

$$\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt + \int_0^1 (L(t))^{2q}|f'(L(t))|^q dt$$

$$\leq \frac{1}{2^s}|f'(b)|^q \int_0^1 (1+t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$+m\frac{1}{2^s}|f'(\frac{a}{m})|^q \int_0^1 (1-t)^s(\frac{2ab}{(1+t)a+(1-t)b})^{2q}dt$$

$$+\frac{1}{2^s}|f'(a)|^q \int_0^1 (1+t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$+m\frac{1}{2^s}|f'(\frac{b}{m})|^q \int_0^1 (1-t)^s(\frac{2ab}{(1-t)a+(1+t)b})^{2q}dt$$

$$= a^{2q}\{2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b}) - 2^{1-s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})\}|f'(b)|^q$$

$$+m2^{2q-s}(\frac{ab}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})|f'(\frac{a}{m})|^q$$

$$+a^{2q}\{2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b}) - 2^{2q-s}(\frac{b}{b+a})^{2q}B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\}|f'(a)|^q$$

$$+\frac{ma^{2q}}{2^s}B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})|f'(\frac{b}{m})|^q$$

using (2.15) in (2.14), we get

$$\Big(\int_0^1 (U(t))^{2q}|f'(U(t))|^q dt\Big)^{\frac{1}{q}} + \Big(\int_0^1 (L(t))^{2q}|f'(L(t))|^q dt\Big)^{\frac{1}{q}}$$

$$\leq 2^{1-\frac{1}{q}}a^2\Big(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q + 2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q - |f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q - |f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Big)^{\frac{1}{q}} \tag{2.16}$$

Applying (2.17) in (2.14), we obtain the required inequality. □

Corollary 5. *Suppose the assumptions of the Theorem 3 are satisfied. If $g(x) = \frac{ab}{b-a}$ for all $x \in [a,b]$, then one has the following inequality*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq 2^{1-\frac{1}{q}}\frac{a^2}{8}$$

$$\Big(2B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{b})|f'(b)|^q + 2B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b})|f'(a)|^q$$

$$+\frac{m|f'(\frac{b}{m})|^q - |f'(b)|^q}{2^s}.B(s+1,1).{}_2F_1(2q,s+1,s+2;\frac{b-a}{2b})$$

$$+2^{2q-s}(\frac{b}{b+a})^{2q}(m|f'(\frac{a}{m})|^q - |f'(a)|^q)B(1,s+1).{}_2F_1(2q,1,s+2;\frac{b-a}{b+a})\Big)^{\frac{1}{q}} \tag{2.17}$$

Theorem 6. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $a, \frac{b}{m} \in I^\circ$, $m \in (0,1]$ with $a < b$ and let $g : [a,b] \to [0,\infty)$ be continuous positive mapping and harmonically symmetric to $\frac{2ab}{a+b}$ such that $f' \in L([a,b])$. If $|f'|$ is harmonically (s,m)-convex on $[a,\frac{b}{m}]$, then the following inequality holds for $q > 1$*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq \frac{(b-a)}{8ab}\|g\|_\infty\big(\frac{1}{sq+1}\big)^{\frac{1}{q}}$$

$$\times\Big\{2^{2-s}(\frac{ab}{b+a})^2\big((2^{sq+1}-1)|f'(b)| + m|f'(\frac{a}{m})|\big)\Big(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\Big)^{\frac{q-1}{q}}$$

$$+\frac{a^2}{2^s}\big((2^{sq+1}-1)|f'(a)| + m|f'(\frac{b}{m})|\big)\Big(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\Big)^{\frac{q-1}{q}}\Big\} \tag{2.18}$$

Proof. From Lemma 1 and by using the harmonic (s,m)-convexity of $|f'|$ on $[a,b]$, we get

$$\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\right| \leq \frac{b-a}{8ab}\|g\|_\infty$$

$$\times\Big[\int_0^1 (1-t)(U(t))^2|f'(U(t))|dt + \int_0^1 (1-t)(L(t))^2|f'(L(t))|dt\Big]$$

$$\leq \frac{b-a}{8ab}\|g\|_\infty\Big\{\int_0^1 (1-t)(U(t))^2\big[(\frac{1+t}{2})^s|f'(b)| + m(\frac{1-t}{2})^s|f'(\frac{a}{m})|\big]$$

$$+\int_0^1 (1-t)(L(t))^2\big[(\frac{1+t}{2})^s|f'(a)| + m(\frac{1-t}{2})^s|f'(\frac{b}{m})|\big]\Big\} \tag{2.19}$$

Now, by using hölder's inequality, we get

$$\int_0^1 (1-t)(U(t))^2\big[(\frac{1+t}{2})^s|f'(b)| + m(\frac{1-t}{2})^s|f'(\frac{a}{m})|\big]dt$$

$$\leq \Big(\int_0^1 (1-t)^{\frac{q}{q-1}} (U(t))^{\frac{2q}{q-1}} dt\Big)^{\frac{q-1}{q}}$$

$$\times\Big\{\Big(\int_0^1 (\frac{1+t}{2})^{sq} dt\Big)^{\frac{1}{q}} |f'(b)| + m\Big(\int_0^1 (\frac{1-t}{2})^{sq} dt\Big)^{\frac{1}{q}} |f'(\frac{a}{m})|\Big\}$$

(2.20)

$$= 2^{2-s}(\frac{ab}{b+a})^2(\frac{1}{sq+1})^{\frac{1}{q}}\big((2^{sq+1}-1)|f'(b)|+m|f'(\frac{a}{m})|\big)\Big(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\Big)^{\frac{q-1}{q}}.$$

Similarly, one has

$$\int_0^1 (1-t)(L(t))^2\big[(\frac{1+t}{2})^s|f'(a)| + m(\frac{1-t}{2})^s|f'(\frac{b}{m})|\big]$$

(2.21)

$$= \frac{a^2}{2^s}(\frac{1}{sq+1})^{\frac{1}{q}}\big((2^{sq+1}-1)|f'(a)| + m|f'(\frac{b}{m})|\big)\Big(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\Big)^{\frac{q-1}{q}}.$$

□

Corollary 6. *Suppose the assumptions of the Theorem 3 are satisfied. If* $g(x) = \frac{ab}{b-a}$ *for all* $x \in [a,b]$, *then one has the following inequality*

$$\Big|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{g(x)f(x)}{x^2}dx\Big| \leq \frac{1}{8}(\frac{1}{sq+1})^{\frac{1}{q}}$$

$$\times\Big\{2^{2-s}(\frac{ab}{b+a})^2\big((2^{sq+1}-1)|f'(b)| + m|f'(\frac{a}{m})|\big)\Big(B(1,\frac{2q-1}{q-1}).{}_2F_1(\frac{2q}{q-1},1,\frac{3q-2}{q-1};\frac{b-a}{b+a})\Big)^{\frac{q-1}{q}}$$

$$+\frac{a^2}{2^s}\big((2^{sq+1}-1)|f'(a)| + m|f'(\frac{b}{m})|\big)\Big(B(\frac{2q-1}{q-1},1).{}_2F_1(\frac{2q}{q-1},\frac{2q-1}{q-1},\frac{3q-2}{q-1};\frac{b-a}{b})\Big)^{\frac{q-1}{q}}\Big\} \quad (2.22)$$

3. Competing interests

The authors have no Competing interests regarding this article.

4. Funding

This research article is partially supported by Higher Education Commission of Pakistan.

5. Authors' contributions

Each author has equal contribution in this research article.

6. Acknowledgement

The authors wish to express their heartfelt thanks to the referees for their constructive comments and helpful suggestions to improve the final version of this paper.

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