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A correspondence between voting procedures and stochastic orderings

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ABSTRACT

In voting theory, two different settings are commonplace: either voters express a preference ordering on the set of candidates or they express an individual evaluation of each candidate. In either case, the aim may be to obtain a global ranking of the candidates and, in particular, to determine the winner of the election. We introduce a probabilistic framework that allows us to explore a correspondence between some usual voting procedures based on either preference orderings (e.g. the Borda count and the Condorcet procedure) or individual evaluations (e.g. the Borda majority count and the majority judgment) and some classical stochastic orderings (e.g. comparison of expected values, comparison of medians and statistical preference). We also consider a recently-introduced multivariate stochastic ordering, called probabilistic preference, and show its connection with the plurality and veto procedures.

1. Introduction

Voting procedures and stochastic orderings are two commonly used tools in different branches of operations research, such as decision support for the former and stochastic modelling for the latter. Just as illustrative recent examples, voting procedures have been used in the aggregation of rankings (Aledo, Gámez, & Rosete, 2018; Ding, Han, & Yang, 2018), social welfare (Darmann & Schauer, 2015), and decision making (García-Lapresta & del Pozo, 2019; Kolgour & Vetschera, 2018), whereas stochastic orderings have been applied in decision making (Jiang, Lian, Liang, & Yang, 2018; Montes, Miranda, & Montes, 2014) and in reliability theory (Navarro, Arriaza, & Suárez-Llorens, 2019), among many others.

Formally, voting theory is the subfield of social sciences devoted to the study and development of mathematical tools (voting procedures) used to deduce the winner of an election. There are two main settings in voting theory: the framework in which voters express a preference ordering on the set of candidates (hereinafter referred to as Arrow's framework (Arrow, 1950; 1963)[1]) and the framework in which voters express individual evaluations of the candidates on a given (linearly-ordered) linguistic scale (hereinafter referred to as Balinski and Laraki's framework (Balinski & Laraki, 2007; 2011a; 2011b)[2]). In both frameworks, the final objective may be to obtain a global ranking of the candidates and, in this way, determine the winner of the election according to the opinions of the voters.

Probably the most prominent and often used voting procedures in Arrow's framework are the Borda count (de Borda, 1784), which determines a winner based on numerical values assigned to each candidate according to its position in the preference orderings given by the voters, the Condorcet procedure (Condorcet, 1785), which determines a winner based on pairwise comparisons between candidates, as well as the plurality and veto procedures, which consider the winner to be the candidate that appears with the highest frequency at the first and last positions, respectively. In Balinski and Laraki's framework, the most common voting procedures are majority judgment (Balinski & Laraki, 2007; 2011a; 2011b) and the Borda majority count (Zahid & de Swart, 2015) (or range voting (Smith, 2000)), in which the individual evaluations of each candidate are aggregated and, subequently, ranked according to these aggregated evaluations. However, using the Borda majority count in this setting has some significant problems, as argued in Zahid and de Swart (2015). In this direction, an alternative to the Borda majority count and majority judgment was recently presented in Ngoie, Savadogo, and Ulungu (2015).

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[1] Arrow's framework could be traced back in time much further. Some authors refer to the eighteenth century and the discussions between Jean Charles de Borda and Nicolas de Condorcet, whereas some others refer to Ramon Llull in the thirteenth century and Nicolas Cusanus in the fifteenth century.

[2] Balinski and Laraki's framework could actually be traced back to Laplace in the nineteenth century and to Galton in the early twentieth century. Approval voting (Brams & Fishburn, 1983) could also be argued to fit within this framework.

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In this work, we define a probability space that allows to link the most prominent voting procedures and the comparison of some appropriately defined random variables. Recall that the comparison of random variables is usually performed in terms of stochastic orderings (Müller & Stoyan, 2002; Shaked & Shanthikumar, 2006). Some of the most common stochastic orderings are the *comparison of expected values* and the *comparison of medians*, both based on the comparison of location parameters. *Statistical preference* (De Schuymer, De Meyer, & De Baets, 2003a; De Schuymer, De Meyer, De Baets, & Jenei, 2003b) is another common stochastic ordering that is based on a reciprocal relation computed from the bivariate marginal distributions of pairs of random variables. A recently-proposed approach based on multivariate distributions is *probabilistic preference* (Montes, Montes, & De Baets, 2019), allowing for the simultaneous comparison of all the random variables.

In Arrow's framework we associate a random variable with each candidate such that for any voter the random variable expresses the position of the candidate in the preference ordering of this voter. We will prove that the Borda count and the Condorcet procedure are equivalent to the comparison of the random variables associated with the candidates in terms of expected values and statistical preference, respectively. Since there is a close connection between voting procedures and stochastic orderings, we will also compare the random variables associated with the candidates in terms of probabilistic preference, and we will investigate the winner of the election in this setting, which will turn to be closely related to the plurality and veto winners.

In Balinski and Laraki's framework, we also associate a random variable with each candidate expressing for any voter the individual evaluation of the candidate given by this voter. In this setting, the evaluations are usually given on a (linearly-ordered) linguistic scale, and therefore we are dealing with qualitative random variables. We will show that if we apply the comparison of expected values or medians to the random variables associated with the candidates, we obtain the same result as the Borda majority count and the majority judgment, respectively. We will also consider probabilistic preference in this framework.

The idea of connecting voting procedures and stochastic orderings already appeared in the literature in Stein, Mizzi, and Pfaffenberger (1994), where the authors proposed a voting procedure in the spirit of stochastic dominance as an alternative to the usual Borda count or Condorcet procedures. However, as we will discuss later on in Remark 1, this approach could lead to incomparability.

It must be noted that we do stick to classical deterministic ranking voting problems and that the here-defined probabilistic space does not aim at defining a probabilistic voting procedure (see, e.g. Fishburn, 1972; Fishburn, 1984 for some introductory papers on probabilistic social choice). In addition, the objective of this paper is not to compare the mentioned voting procedures to determine the most adequate one. Instead, the main goal of this contribution is to show that similar ideas arise in two apparently separated fields when a voting problem is seen as a result of comparing random variables defined on a probabilistic space with a uniform distribution over the voters. Interestingly, the uniform distribution over the voters aligns with the standard assumption in social choice theory in which all voters are assumed to be equally important (neutrality). We end this introduction by noting that this paper shows a correspondence between the most prominent stochastic orderings and the classics of social choice theory, and does not cover the closely related field of (group/multiattribute) decision making (Huynh & Nakamori, 2005; Yan & Ma, 2015; Yan, Ma, & Huynh, 2017).

The remainder of this paper is organized as follows. Section 2 introduces basic notions related to voting theory. In Section 3, we develop an approach that allows to express voting procedures in terms of a probability space and stochastic orderings in case the voters express their opinions in terms of preference orderings. Section 4 introduces the probabilistic approach to ranking candidates in case the voters express their opinions in terms of individual evaluations. As in the preceding section, we will show that the usual procedures for dealing with this kind of voting problems can be expressed in terms of stochastic orderings. Finally, Section 5 ends the paper with some concluding remarks.

2. Basics of voting theory

In the framework of voting theory, a number of candidates, denoted $C_1, \ldots, C_m$, participate in an election. The voters, denoted $v_1, \ldots, v_n$, usually give their *preference orderings* over the candidates, denoted $e_1, \ldots, e_n$, each of these preference orderings establishing a total order on the set of candidates. A voter v_i thus expresses the preference ordering e_i as $C_{\sigma_i(1)} \succ \ldots \succ C_{\sigma_i(m)}$, where σ_i is a permutation of $\{1, \ldots, m\}$. We say that $C_{\sigma_i(1)}$ is the most preferred candidate for voter v_i and $C_{\sigma_i(m)}$ is the least preferred candidate for voter v_i. The aim of voting procedures is to establish a global ranking on the set of candidates to be able to decide which is(are) the preferred candidate(s) according to the given preference orderings.

Once we apply a voting procedure, we obtain a global ranking $C_{\sigma(1)} \succ \cdots \succ C_{\sigma(m)}$, where σ again denotes a permutation of $\{1, \ldots, m\}$. $C_{\sigma(1)}$ will be called the *winner* of the election, while $C_{\sigma(m)}$ will be called the *loser*. Also, a candidate $C_{\sigma(i)}$ is preferred to another candidate $C_{\sigma(j)}$ in the global ranking if $C_{\sigma(i)} \succ C_{\sigma(j)}$. One should note that the global ranking might not be unique and/or some candidates might be considered to be tied for some collections of preference orderings given by the voters. In the latter case, we will denote the fact that the candidates at the jth and $(j+1)$th positions are tied by $C_{\sigma(j)} \sim C_{\sigma(j+1)}$.

Probably the procedure most often used for aggregating a collection of preference orderings is the *Borda count procedure* (de Borda, 1784). If voter v_i gives her preference ordering e_i as $C_{\sigma_i(1)} \succ \cdots \succ C_{\sigma_i(m)}$, then the Borda count procedure assigns $m-1$ points to $C_{\sigma_i(1)}$, $m-2$ points to $C_{\sigma_i(2)}$,..., and 0 points to $C_{\sigma_i(m)}$. To obtain the global ranking, the Borda count procedure computes the sum of the points any candidate receives in each of the preference orderings. For a candidate C, this sum is usually referred to as the Borda count of C and is denoted by $\mathrm{B}(C)$. Naturally, it holds that the greater the Borda count of a certain candidate is, the more preferred the candidate is according to the Borda count procedure, yielding a global ranking on the set of candidates. However, since two (or more) candidates could have the same Borda count, ties are allowed.

The Borda count procedure belongs to the oldest and most prominent family of voting rules: scoring rules (Saari, 2000; Young, 1975). Other classical examples in this family are the plurality rule and the veto rule (also known as the antiplurality rule or the inverse plurality rule), which respectively rank the candidates according to the number of times that they appear at the first or last position. Different elimination procedures have been combined with scoring rules, to reduce the effect of unimportant candidates within the election (Richelson, 1980).

Another common procedure for ranking the candidates given the preference orderings is that of Condorcet (1785): for any two candidates C_i and C_j we denote by $n_{C_i \succ C_j}$ the number of voters that consider C_i preferred to C_j. In this way, C_i is preferred to C_j whenever $n_{C_i \succ C_j} > n_{C_j \succ C_i}$.

Note that this procedure might not lead to a global ranking since cycles might arise. The presence of such cycles is usually referred to as the voting paradox or, more precisely, the Condorcet paradox. For avoiding such cycles, different procedures respecting as far as possible the rationale of Condorcet have been proposed (e.g. the Kemeny procedure (Kemeny, 1959)). These procedures are commonly referred to as Condorcet procedures. Note that the Borda count, the plurality rule and the veto rule are not Condorcet procedures since they might yield a different winner than the Condorcet procedure, even in the absence of cycles.

A slightly different approach to voting theory is based on individual *evaluations* of candidates instead of preference orderings (Balinski & Laraki, 2007; 2011a; 2011b). In this framework, each voter assigns a label to each candidate, indicating her evaluation. These evaluations are usually expressed on a (linearly-ordered) linguistic scale $\mathcal{L} = \{l_1, \ldots, l_s\}$, where the linguistic terms are ordered from the worst to the best, i.e., $l_1 < \cdots < l_s$. This approach is in some sense more expressive than the previous one, since a preference ordering can be obtained from the evaluations (yet allowing for ties), however, the converse does not hold.

The evaluations provided by the voters can be summarized in a matrix:

$$\mathcal{G} = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{pmatrix}, \tag{1}$$

where $g_{ij} \in \mathcal{L}$. For each candidate C_i, $(g_{i1}, \ldots, g_{in})$ denotes the evaluations given by the n voters. The main problem here is to aggregate the evaluations of the voters in order to obtain the final evaluation of every candidate.

Two of the most commonly used procedures for solving this problem are the *majority judgment* (Balinski & Laraki, 2007; Galton, 1907) and the *Borda majority count* (Zahid & de Swart, 2015) procedures. The first is defined in the following way:

1. If n is odd and the evaluations of a candidate C can be ordered as $g_1 \geq \cdots \geq g_n$, then her majority grade is defined as $\mathrm{MG}(C) = g_{(n+1)/2}$. Note that in this case the majority grade is a unique value and amounts to the median or (lower/upper) middlemost.
2. If n is even and the evaluations of a candidate C can be ordered as $g_1 \geq \cdots \geq g_n$, then her majority grade is defined as $\mathrm{MG}(C) = [g_{n/2}, g_{(n+2)/2}]$, where $g_{n/2}$ is the lower middlemost grade and $g_{(n+2)/2}$ is the upper middlemost grade.

The candidates are ordered by comparing majority grades: the greater the majority grade is, the more preferred the candidate is. Balinski and Laraki (2011b) argue that when n is even, $\mathrm{MG}(C)$ must be defined as the lower middlemost grade $g_{n/2}$ and not as the interval $[g_{n/2}, g_{(n+2)/2}]$. In this way, the problem of comparing intervals of grades is avoided. In case two or more candidates have the same majority grade, one single appearance of this value is removed from the evaluations of both candidates and the newly-obtained majority grade is used. This procedure is followed until both majority grades differ or until there are no more evaluations, in which case the candidates are tied.

The Borda majority count procedure (Zahid & de Swart, 2015) works as follows: first of all, the linguistic labels $\mathcal{L} = \{l_1, \ldots, l_s\}$ are transformed into a numerical scale $\mathcal{L}^* = \{0, 1, \ldots, s-1\}$. This assumes that all linguistic terms are equally spaced. The matrix $\mathcal{G}$ is transformed into a matrix $\mathcal{G}^*$ such that any element $g_{ij} = l_k$ is transformed into $g^*_{ij} = k - 1$. Then the Borda majority count of candidate C_i is defined as:

$$\mathrm{B}_{\mathrm{MC}}(C_i) = \sum_{j=1}^{n} g^*_{ij}.$$

As before, the greater the Borda majority count is, the more preferred the candidate is.

3. A probabilistic approach to the ranking of candidates based on preference orderings

In this section we introduce a probabilistic approach that allows to express the voting procedures for ranking candidates based on preference orderings in terms of stochastic orderings.

3.1. Probability space and random variables

By $V = \{v_1, \ldots, v_n\}$ we denote the set of all voters. We define a probability space $(V, \mathcal{P}(V), P)$, where P is the uniform distribution over the voters. Note that the choice of the uniform distribution follows the same line of thought as the standard requirement of the neutrality property in social choice theory assuring that all voters are considered to be equally important. Then any candidate C_i has an associated random variable X_i defined on $(V, \mathcal{P}(V), P)$ that assigns for any $v_j \in V$ the position of the candidate C_i in the preference ordering e_j.

Similarly, for any candidate C_i we can also define another random variable $\tilde{X}_i$ on $(V, \mathcal{P}(V), P)$ that assigns for any $v_j \in V$ the number of candidates ranked at a worse position than C_i in the preference ordering e_j. Of course, there is a linear relation between the random variables X_i and $\tilde{X}_i$, because if a voter ranks C_i at the jth position, there are $m - j$ candidates ranked at a worse position than C_i, that is: $\tilde{X}_i = m - X_i$, for any $i = 1, \ldots, m$. Let us stress that neither the random variables $X_1, \ldots, X_m$, nor the random variables $\tilde{X}_1, \ldots, \tilde{X}_m$ are independent, because the position of one candidate obviously influences the position of the other candidates. Furthermore, both $\{X_1, \ldots, X_m\}$ and $\{\tilde{X}_1, \ldots, \tilde{X}_m\}$ are sets of distinct random variables.

Example 1. Consider an election with three candidates C_1, C_2 and C_3 and five voters $v_1, \ldots, v_5$, whose preference orderings are given by:

$e_1 : C_1 \succ C_3 \succ C_2$
$e_2 : C_3 \succ C_1 \succ C_2$
$e_3 : C_2 \succ C_1 \succ C_3$
$e_4 : C_3 \succ C_1 \succ C_2$
$e_5 : C_2 \succ C_3 \succ C_1$

The random variables X_1, X_2, X_3 and $\tilde{X}_1, \tilde{X}_2, \tilde{X}_3$ associated with the candidates C_1, C_2, C_3, are given by:

	v_1	v_2	v_3	v_4	v_5
X_1	1	2	2	2	3
X_2	3	3	1	3	1
X_3	2	1	3	1	2

	v_1	v_2	v_3	v_4	v_5
$\tilde{X}_1$	2	1	1	1	0
$\tilde{X}_2$	0	0	2	0	2
$\tilde{X}_3$	1	2	0	2	1

Clearly, all the information given by the voters is captured by the random variables.

In this probabilistic approach, candidates are represented by random variables, and therefore the global ranking of the candidates should be obtained by means of the comparison of their associated random variables. Stochastic ordering is the subfield of probability theory that allows for the comparison of random variables (see, for instance, Müller & Stoyan, 2002; Shaked & Shanthikurmar, 2006). In the remainder of this section, we are going to see that some procedures for ranking candidates can be expressed in terms of stochastic orderings applied to the random variables of the candidates. For this aim, we consider three different cases: stochastic orderings that only consider the univariate distribution of the random variables, stochastic orderings based on the bivariate distributions and stochastic orderings that consider the joint distribution of all the random variables.

3.2. Univariate stochastic orderings

Probably the most basic stochastic ordering we can find in the literature is the *comparison of expected values*. The main characteristic of this stochastic ordering is that it is based on the univariate marginal distributions of the random variables. This fact has advantages and disadvantages: on the one hand, it is much simpler to handle univariate distributions than bivariate or multivariate ones; on the other hand, bivariate or multivariate distributions are much more informative than the univariate marginal distributions, which do not take into account the possible dependence between the random variables.

Recall that given two random variables X and Y, X is preferred to Y with respect to *expected value* (Müller & Stoyan, 2002), denoted by $X \succeq_{\mathrm{EV}} Y$, if $E(X) \geq E(Y)$, where $E(X)$ and $E(Y)$ denote the expected values of X and Y. Next, we investigate the meaning of the comparison of expected values when applied to the comparison of random variables associated with candidates.

We have already mentioned the Borda count, which probably is the most commonly used procedure for ranking candidates when voters express preference orderings. For any candidate, this procedure only considers its position in each preference ordering, regardless of the position of the other candidates. It will thus be not surprising that the Borda count and the univariate stochastic ordering are closely related.

Proposition 1. *A candidate C_i is preferred to a candidate C_j in the global ranking induced by the Borda count if and only if $X_j \succeq_{\mathrm{EV}} X_i$, or, equivalently, if and only if $\tilde{X}_i \succeq_{\mathrm{EV}} \tilde{X}_j$.*

Proof. Denote by $n(C_i, k)$ the number of voters $v \in V$ that rank C_i at the kth position. The Borda count of C_i takes the following value:

$$\mathrm{B}(C_i) = \sum_{k=1}^{m} (m-k) \cdot n(C_i, k) = \sum_{k=0}^{m-1} k \cdot n(C_i, m-k)\,.$$

Considering the relationship $\tilde{X}_i = m - X_i$, it holds that:

$$E(\tilde{X}_i) = \sum_{k=0}^{m-1} k \cdot P(\tilde{X}_i = k) = \sum_{k=0}^{m-1} k \cdot P(X_i = m-k)$$
$$= \sum_{k=0}^{m-1} k \cdot \frac{n(C_i, m-k)}{n} = \frac{\mathrm{B}(C_i)}{n}\,.$$

Furthermore, $E(\tilde{X}_i) = m - E(X_i)$, and therefore:

$$\mathrm{B}(C_i) \geq \mathrm{B}(C_j) \Leftrightarrow E(X_j) \geq E(X_i) \Leftrightarrow E(\tilde{X}_i) \geq E(\tilde{X}_j)\,.$$

□

Note that the expected value is quite sensitive to extreme values. The strong connection between the expected value and the Borda count implies that the same holds for the Borda count. This translates to possibilities for manipulation by voters: suppose two candidates would have a very close Borda count, then by (insincerely) assigning these two candidates positions 1 and m, a voter could attempt to produce a big difference between these two candidates. There is quite some literature on this topic in the field of voting theory (Chamberlin, 1985; Kelly, 1993; Saari, 1990).

Remark 1. Another univariate stochastic ordering is stochastic dominance, which is probably the most common stochastic ordering that can be found in the literature (Lehmann, 1955). It compares the cumulative distribution functions of the random variables X and Y, which are given by:

$$F_X(t) = P(X \leq t), \quad F_Y(t) = P(Y \leq t) \quad \forall t \in \mathbb{R}\,.$$

We say X is stochastically preferred to Y, denoted by $X \succeq_{\mathrm{FSD}} Y$, if $F_X(t) \leq F_Y(t)$ for any $t \in \mathbb{R}$.

For the random variables $X_1, \ldots, X_m$ associated with the candidates, $F_{X_i}(k)$ denotes the number of voters that rank candidate C_i at the first k positions of their preference orderings. With this very meaning, stochastic dominance has been used in the field of voting theory under the name of Borda-dominance (Fishburn, 1974; Stein et al., 1994). Its main drawback is that it might lead to incomparability. In our framework, this means that stochastic dominance cannot be counted on to produce a ranking on the set of candidates.

3.3. Bivariate stochastic orderings

We have mentioned in the previous subsection that one of the main drawbacks of the expected value is that it only takes into account univariate marginal distributions, so it ignores the possible dependence between the random variables. One possible way of dealing with the dependence between the random variables is to use stochastic orderings based on the bivariate marginal distributions (or even the entire multivariate distribution, as will be discussed in the next subsection). *Statistical preference* is one approach to do so, and it is based on a reciprocal relation (Bezdek, Spilman, & Spilman, 1978; De Baets, De Loof, & De Meyer, 2015).

Definition 1. Given a set of alternatives $\mathcal{A}$, a reciprocal relation is a function $Q : \mathcal{A} \times \mathcal{A} \to [0, 1]$ such that $Q(a, b) + Q(b, a) = 1$, for any $a, b \in \mathcal{A}$.

Reciprocal relations are a very important tool because $Q(a, b)$ can be interpreted as a measure of the strength of the preference for a over b on the scale [0,1], where $\frac{1}{2}$ is understood as indifference, 1 is understood as total preference for a over b and 0 is understood as total preference for b over a.

When the set of alternatives is formed by random variables defined on the same probability space (Ω, Σ, P), it is possible to define the following reciprocal relation:

$$Q(X, Y) = P(X > Y) + \frac{1}{2} P(X = Y)\,. \tag{2}$$

The value $Q(X, Y)$, called *winning probability* of X over Y, measures the strength of the preference for X over Y, and statistical preference is defined by considering the strong $\frac{1}{2}$-cut of the relation Q (De Schuymer et al., 2003a; De Schuymer et al., 2003b).

Definition 2. X is *statistically preferred* to Y when $Q(X, Y) > \frac{1}{2}$, and this is denoted by $X \succ_{\mathrm{SP}} Y$.

It is obvious that $X \succ_{\mathrm{SP}} Y$ if and only if $P(X \geq Y) > P(X \leq Y)$. Furthermore, when $P(X = Y) = 0$, like for instance for continuous random vectors or discrete random variables with disjoint supports, the reciprocal relation given in Eq. (2) becomes $Q(X, Y) = P(X > Y)$.

Statistical preference can be understood as a stochastic ordering based on the bivariate marginal distributions. It takes into account the possible dependence between the random variables and yields winning probabilities, in the sense that the greater the winning probability is, the more preferred is one random variable over the other. Moreover, since it only takes into account the order between the values $X(\omega)$ and $Y(\omega)$ for any $\omega \in \Omega$ instead of their values, statistical preference can also be applied to qualitative random variables.

Let us now consider the random variables X_i associated with the candidates C_i, for $i = 1, \ldots, m$. First of all, note that these random variables cannot take the same value simultaneously, and hence $Q(X_i, X_k) = P(X_i > X_k)$ for any $i \neq k$.

Proposition 2. *A candidate C_i is a Condorcet winner if and only if $X_k \succ_{\mathrm{SP}} X_i$ for any $k \neq i$.*

Proof. Let us develop the expression $Q(X_k, X_i)$:

$$Q(X_k, X_i) = P(X_k > X_i) = P(\{v \in V \mid X_k(v) > X_i(v)\}) = \frac{|\{v \in V \mid X_k(v) > X_i(v)\}|}{n} = \frac{|\{v_j \in V \mid \sigma_j(i) < \sigma_j(k)\}|}{n}.$$

Hence, $X_k \succ_{SP} X_i$ means that C_i is preferred to C_k in pairwise comparisons. Then, if such preference holds for any $k \neq i$, this is equivalent to being a Condorcet winner. □

Of course, the previous result can also be expressed in terms of the random variables $\tilde{X}_i$, because $X_k \succ_{SP} X_i$ is equivalent to $\tilde{X}_i \succ_{SP} \tilde{X}_k$.

We have already mentioned that the Condorcet procedure might lead to cycles. This is related to the lack of transitivity of statistical preference (De Baets & De Meyer, 2005; 2008; De Baets, De Meyer, & De Loof, 2010; De Baets, De Meyer, De Schuyer, & Jenei, 2006; Martinetti, Montes, Díaz, & Montes, 2011): it is possible to find random variables X, Y, Z such that $X \succ_{SP} Y$, $Y \succ_{SP} Z$, but $Z \succ_{SP} X$.

3.4. Multivariate stochastic orderings

3.4.1. Probabilistic preference on a set of candidates

As mentioned above, one of the most important drawbacks of statistical preference is its lack of transitivity. This is due to the fact that statistical preference is based on bivariate marginal distributions only, so it does not consider all the information given by the multivariate distribution. Hence, in order to construct a voting procedure giving a ranking, one could use the multivariate distribution of all the random variables to be compared, rather than only the bivariate marginal distributions. One possible way of doing so is by using *probabilistic preference* (Montes et al., 2019). This procedure was introduced as an extension of statistical preference allowing for the comparison of more than two random variables simultaneously.

Definition 3. Let $\mathcal{A}$ be a finite set of distinct random variables defined on the same probability space (Ω, Σ, P). For every $X \in \mathcal{A}$, the *multivariate winning probability* of X in $\mathcal{A}$ is defined as[3]:

$$\Pi_{\mathcal{A}}(X) = \sum_{\mathcal{Y} \subseteq \mathcal{A} \setminus \{X\}} \frac{1}{1+|\mathcal{Y}|} P\Big(\big(\forall Z \in \mathcal{Y}\big)\big(\forall W \in \mathcal{A} \setminus (\{X\} \cup \mathcal{Y})\big)\big(X = Z > W\big)\Big). \tag{3}$$

As we can see, the multivariate winning probabilities preserve the idea of the (pairwise) winning probabilities in Eq. (2). In fact, considering a set of random variables $\mathcal{A} = \{X, Y\}$, we obtain the reciprocal relation in Eq. (2):

$$\Pi_{\mathcal{A}}(X) = Q(X, Y) \text{ and } \Pi_{\mathcal{A}}(Y) = Q(Y, X).$$

In this way, the function $\Pi_{\mathcal{A}}$ also measures the intensity of preference for one random variable in the set $\mathcal{A}$, preserving the property that

$$\sum_{X \in \mathcal{A}} \Pi_{\mathcal{A}}(X) = 1.$$

Using these multivariate winning probabilities, we can define a weak ordering (allowing for ties) on the random variables in $\mathcal{A}$.

Definition 4. Let $\mathcal{A}$ be a finite set of distinct random variables defined on the same probability space. Given $X, Y \in \mathcal{A}$, X is *probabilistically preferred* to Y in $\mathcal{A}$ if $\Pi_{\mathcal{A}}(X) \geq \Pi_{\mathcal{A}}(Y)$. Similarly, given $X \in \mathcal{A}$, X is the *probabilistically preferred* random variable in $\mathcal{A}$ if:

$$\Pi_{\mathcal{A}}(X) \geq \max_{Y \in \mathcal{A} \setminus \{X\}} \Pi_{\mathcal{A}}(Y).$$

As we can see, probabilistic preference reduces to statistical preference when $\mathcal{A} = \{X, Y\}$. For an in-depth study on this stochastic ordering, we refer to Montes et al. (2019).

When we deal with a set of discrete and distinct random variables $\mathcal{A}$ whose supports are pairwisely disjoint, it holds that $P(X = Y) = 0$ for every $X, Y \in \mathcal{A}$, and the expression of the multivariate winning probabilities becomes simpler, more precisely (see Montes et al., 2019, Prop. 13):

$$\Pi_{\mathcal{A}}(X) = Q\Big(X, \max_{Y \in \mathcal{A} \setminus \{X\}} Y\Big) = P\Big(X > \max_{Y \in \mathcal{A} \setminus \{X\}} Y\Big). \tag{4}$$

We have seen that by computing the expected value of the random variables associated with the preference orderings given by the voters (either the random variables X_i or $\tilde{X}_i$) we obtain the Borda count, and when applying statistical preference, we obtain the Condorcet procedure. In the remainder of this section, we apply the notion of probabilistic preference to the comparison of the random variables X_i and $\tilde{X}_i$ associated with the candidates to obtain a ranking of them.

Although for univariate or bivariate stochastic orderings, we have seen that the use of either the random variables X_i or the random variables $\tilde{X}_i$ is equivalent, this is no longer the case for probabilistic preference, and two possible ways of ranking the candidates arise.

On the one hand, the random variable $\tilde{X}_i$ indicates the number of candidates ranked at a worse position than C_i for any voter. This means that the greater the value of $\tilde{X}_i$ is, the more preferred the candidate C_i is. Applying the notion of probabilistic preference to the set of random variables $\mathcal{A} = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$, we can rank the candidates from the winner to the loser. This approach will be called the *top-down probabilistic preference procedure*.

On the other hand, the random variable X_i indicates the position of C_i in the preference orderings given by the voters. This means that the smaller the value of X_i is, the more preferred the candidate C_i is. Hence, the greater the multivariate winning probability is, the less preferred the candidate is. So if we apply the notion of probabilistic preference to the set of random variables $\mathcal{A} = \{X_1, \ldots, X_m\}$, we will obtain a ranking from the loser to the winner. This approach will be called the *bottom-up probabilistic preference procedure*.

Next, we study both approaches in detail.

3.4.2. Top-down probabilistic preference procedure

Let us consider the set of random variables $\mathcal{A} = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$ associated with the candidates. Recall that all the random variables in $\mathcal{A}$ are distinct, so we can apply the notion of probabilistic preference to them. If we apply the top-down probabilistic preference procedure to this set of distinct random variables $\mathcal{A} = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$, we obtain that not all the candidates have a strictly positive multivariate winning probability.

Proposition 3. *Consider the set of distinct random variables $\mathcal{A} = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$ associated with the candidates. A candidate has a strictly positive top-down multivariate winning probability if and only if it is the most preferred candidate in at least one preference ordering.*

Proof. If there is a voter $v_j \in V$ for which C_i is considered the most preferred candidate, then $\tilde{X}_i(v_j) = m - 1$. This implies that:

$$\Pi_{\mathcal{A}}(\tilde{X}_i) \geq P(\{v_j\}) = \frac{1}{m} > 0.$$

Conversely, if $\Pi_{\mathcal{A}}(\tilde{X}_i) > 0$, this means that there is at least one v_j with $\tilde{X}_i(v_j) = m - 1$, which means that C_i is the most preferred candidate for the voter v_j. □

[3] Note that the notation $\big\{\big(\forall Z \in \mathcal{Y}\big)\big(\forall W \in \mathcal{A} \setminus (\{X\} \cup \mathcal{Y})\big)\big(X = Z > W\big)\big\}$ in Eq. (3) is a shorthand for $\big\{\omega \in \Omega \mid \big(\forall Z \in \mathcal{Y}\big)\big(\forall W \in \mathcal{A} \setminus (\{X\} \cup \mathcal{Y})\big)\big(X(\omega) = Z(\omega) > W(\omega)\big)\big\}$.

Using this property, we can easily identify which candidate will be the winner of the election.

Proposition 4. *Consider the set of distinct random variables $\mathcal{A} = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$ associated with the candidates. A candidate is the winner of the election with respect to the top-down probabilistic preference procedure if it is the most preferred candidate in at least as many preference orderings as any other candidate.*

Proof. Let us compute $\Pi_{\mathcal{A}}(\tilde{X}_i)$:

$$\Pi_{\mathcal{A}}(\tilde{X}_i) = P\Big(\tilde{X}_i > \max_{Y \in \mathcal{A}\setminus\{\tilde{X}_i\}} Y\Big) = P(\tilde{X}_i = m-1)$$
$$= \frac{|\{v \in V \mid \tilde{X}_i(v) = m-1\}|}{n} = \frac{|\{v_j \in V \mid \sigma_j(i) = 1\}|}{n}\,.$$

Then a winner of the election is the most preferred candidate in at least as many preference orderings as any other candidate. □

This result allows to relate top-down probabilistic preference to the plurality rule. Moreover, the previous results show that if there is a subset of candidates $\mathcal{C}$ that contains only the candidates that appear at the first position of at least one preference ordering, the ranking given by the top-down probabilistic preference procedure ranks them at the first $|\mathcal{C}|$ positions, and all of them have positive multivariate winning probabilities. The top-down probabilistic preference procedure thus creates a ranking of the candidates in the set $\mathcal{C}$, but it does not rank the other candidates. If we want to obtain a ranking of all the candidates, we can apply the top-down probabilistic preference procedure recursively:

Step 0. Let $\mathcal{A}_1 = \{\tilde{X}_1, \ldots, \tilde{X}_m\}$ be the set of distinct random variables associated with the candidates.

Step i. For any $\tilde{X} \in \mathcal{A}_i$, compute the multivariate winning probability $\Pi_{\mathcal{A}_i}(\tilde{X})$.

- i.1: Rank the candidates such that their associated random variables $\tilde{X} \in \mathcal{A}_i$ carry a positive multivariate winning probability $\Pi_{\mathcal{A}_i}(\tilde{X}) > 0$ according to decreasing $\Pi_{\mathcal{A}_i}(\tilde{X})$.
- i.2: Let $\mathcal{A}_{i+1} = \mathcal{A}_i \setminus \{\tilde{X} \in \mathcal{A}_i \mid \Pi_{\mathcal{A}_i}(\tilde{X}) > 0\}$.
- i.3: If $\mathcal{A}_{i+1} = \emptyset$, then all the candidates are ranked. Otherwise, go to step $i+1$.

In the first step, this procedure considers the candidates that are ranked at the first position of the preference ordering for at least one voter. Then, it ranks those candidates according to their frequency at the first position. In the next step, we remove the candidates that have already been ranked and we iterate the procedure.

Example 2. Consider an election with five candidates C_1, C_2, C_3, C_4 and C_5, and 100 voters. The following table summarizes the preference orderings:

Number of votes	Preference orderings
40	$C_1 \succ C_2 \succ C_3 \succ C_4 \succ C_5$
35	$C_2 \succ C_3 \succ C_4 \succ C_5 \succ C_1$
25	$C_4 \succ C_5 \succ C_2 \succ C_3 \succ C_1$

Consider the random variables $\tilde{X}_i$ associated with the candidates, which are defined by:

	$v_1, \ldots, v_{40}$	$v_{41}, \ldots, v_{75}$	$v_{76}, \ldots, v_{100}$
$\tilde{X}_1$	4	0	0
$\tilde{X}_2$	3	4	2
$\tilde{X}_3$	2	3	1
$\tilde{X}_4$	1	2	4
$\tilde{X}_5$	0	1	3

Applying the top-down probabilistic preference procedure to the set of distinct random variables $\mathcal{A}_1 = \{\tilde{X}_1, \tilde{X}_2, \tilde{X}_3, \tilde{X}_4, \tilde{X}_5\}$ associated with the candidates, we obtain the following multivariate winning probabilities:

$$\Pi_{\mathcal{A}_1}(\tilde{X}_1) = 0.4,\quad \Pi_{\mathcal{A}_1}(\tilde{X}_2) = 0.35,$$
$$\Pi_{\mathcal{A}_1}(\tilde{X}_4) = 0.25,\quad \Pi_{\mathcal{A}_1}(\tilde{X}_3) = \Pi_{\mathcal{A}_1}(\tilde{X}_5) = 0\,.$$

This means that C_1, C_2 and C_4 are the candidates that are ranked at the first position in at least one preference ordering, and the multivariate winning probabilities express the frequency with which they appear at the first position. Then, C_1 is the winner with multivariate winning probability 0.4, C_2 is the candidate ranked second and C_4 is ranked third. Next, in order to rank $\tilde{X}_3$ and $\tilde{X}_5$, we consider the set of random variables $\mathcal{A}_2 = \{\tilde{X}_3, \tilde{X}_5\}$ and we compute the multivariate winning probabilities in $\mathcal{A}_2$:

$$\Pi_{\mathcal{A}_2}(\tilde{X}_3) = 0.75 \text{ and } \Pi_{\mathcal{A}_2}(\tilde{X}_5) = 0.25\,.$$

Thus, the ranking given by the top-down probabilistic preference procedure is $C_1 \succ C_2 \succ C_4 \succ C_3 \succ C_5$.

Remark 2. The top-down probabilistic preference procedure can be seen as a slightly modified version of the plurality rule. The difference lies in the tie-breaker used to rank the candidates that do not appear at the first position in at least one preference ordering.

Note that the Borda count and the top-down probabilistic preference procedures could give different winners. In fact, if we consider the previous example, it holds that:

$$\mathrm{B}(C_1) = 27\,,\quad \mathrm{B}(C_2) = 21\,,\quad \mathrm{B}(C_3) = 22\,,\quad \mathrm{B}(C_4) = 8\,.$$

C_1 is the Borda winner, while the winner with respect to the top-down probabilistic preference procedure is C_2. This is quite reasonable, because, as we have shown in Montes et al. (2019), the ranking with respect to the comparison of expected values and the probabilistic preference may give different results.

We have seen that the top-down probabilistic preference procedure uses all the available information about the random variables $\tilde{X}_i$ (or, in this framework, about the preference orderings given by the voters). Since the Condorcet procedure only considers bivariate marginal distributions, it is reasonable that the top-down probabilistic preference procedure does not necessarily give the same winner as the Condorcet procedure, as the following example shows.

Example 3. Consider an election with three candidates C_1, C_2 and C_3 and 100 voters. The preference orderings given by the voters are summarized in the following table:

Number of votes	Preference ordering
49	$C_1 \succ C_2 \succ C_3$
49	$C_3 \succ C_2 \succ C_1$
2	$C_2 \succ C_1 \succ C_3$

C_2 is the winner with respect to the Condorcet procedure because:

$$n_{C_2 \succ C_1} = \frac{51}{100} > \frac{49}{100} = n_{C_1 \succ C_2}\,,\quad n_{C_3 \succ C_1} = \frac{51}{100} > \frac{49}{100} = n_{C_1 \succ C_3}\,,$$

but according to the top-down probabilistic preference procedure C_1 and C_3 are considered the winners with multivariate winning probability 0.49, while the multivariate winning probability of C_2 is only 0.02.

The fact that the winner according to the top-down probabilistic preference procedure might differ from the winners according to the Borda count and Condorcet procedures was to be expected since it is widely known in the field of voting theory that the plurality rule might yield a different winner than the Borda count and the Condorcet procedure.

$$\begin{tikzcd}
\begin{array}{c} C_{\sigma_i(1)} \succ \ldots \succ C_{\sigma_i(m)} \\ i \in \{1,\ldots,n\} \end{array} \arrow[r, "\text{Bottom-up}", "\text{probabilistic preference}"'] \arrow[d, "\begin{array}{l}\text{Reversed}\\ \text{preference}\\ \text{orderings}\end{array}"] & C_{\sigma(1)} \succ \ldots \succ C_{\sigma(m)} \\
\begin{array}{c} C_{\sigma_i(m)} \succ \ldots \succ C_{\sigma_i(1)} \\ i \in \{1,\ldots,n\} \end{array} \arrow[r, "\text{Top-down}", "\text{probabilistic preference}"'] & C_{\sigma(m)} \succ \ldots \succ C_{\sigma(1)} \arrow[u, "\begin{array}{l}\text{Reversed}\\ \text{preference}\\ \text{ordering}\end{array}"']
\end{tikzcd}$$

Fig. 1. Graphical explanation of the relationship between the top-down and bottom-up probabilistic preference procedures.

3.4.3. Bottom-up probabilistic preference procedure

Next, let us consider the bottom-up probabilistic preference procedure, which ranks the candidates from the loser to the winner. For this aim, we consider the set of distinct random variables $\mathcal{A} = \{X_1, \ldots, X_m\}$ associated with the candidates and perform the following steps:

Step 0. Let $\mathcal{A}_1 = \{X_1, \ldots, X_m\}$ be the set of distinct random variables associated with the candidates.

Step i. For any $X \in \mathcal{A}_i$, compute the multivariate winning probability $\Pi_{\mathcal{A}_i}(X)$.

- *i*.1: Rank the candidates such that their associated random variables $X \in \mathcal{A}_i$ carry a positive multivariate winning probability $\Pi_{\mathcal{A}_i}(X) > 0$ according to increasing $\Pi_{\mathcal{A}_i}(X)$.
- *i*.2: Let $\mathcal{A}_{i+1} = \mathcal{A}_i \setminus \{X \in \mathcal{A}_i \mid \Pi_{\mathcal{A}_i}(X) > 0\}$.
- *i*.3: If $\mathcal{A}_{i+1} = \emptyset$, then all the candidates are ranked. Otherwise, go to step $i + 1$.

In the first step, this procedure considers the candidates that are ranked at the last position of the preference ordering for at least one voter. Then, it ranks those candidates according to their frequency at the last position. In the next step, we remove the candidates that have already been ranked and we iterate the procedure.

Remark 3. The bottom-up probabilistic preference procedure can also be understood as a slightly modified version of the veto rule. The difference lies in the tie-breaker used to rank the candidates that do not appear at the last position in at least one preference ordering.

A sharp reader would note that there exists a connection between the top-down and bottom-up probabilistic preference procedures. The reason is that, due to the relationship between the random variables X_i and $\tilde{X}_i$, the reversed ranking of the one obtained with the bottom-up probabilistic preference procedure coincides with the ranking given by the top-down probabilistic preference applied to the reversed preference orderings. This fact is graphically explained in Fig. 1.

Example 4. Consider again the election of Example 2. Let us now apply the bottom-up probabilistic preference procedure to the set of distinct random variables $\mathcal{A}_1 = \{X_1, X_2, X_3, X_4, X_5\}$ associated with the candidates, which are given by:

	$v_1, \ldots, v_{40}$	$v_{41}, \ldots, v_{75}$	$v_{76}, \ldots, v_{100}$
X_1	1	5	5
X_2	2	1	3
X_3	3	2	4
X_4	4	3	1
X_5	5	4	2

We obtain the following multivariate winning probabilities:

$$\Pi_{\mathcal{A}_1}(X_1) = 0.6, \quad \Pi_{\mathcal{A}_1}(X_5) = 0.4, \quad \Pi_{\mathcal{A}_1}(X_2) = \Pi_{\mathcal{A}_1}(X_3) = \Pi_{\mathcal{A}_1}(X_4) = 0 .$$

According to *Step 1.1*, this means that C_1 is the loser, followed by C_5. Next, we consider the set of unranked random variables $\mathcal{A}_2 = \{X_2, X_3, X_4\}$, and compute again the multivariate winning probabilities:

$$\Pi_{\mathcal{A}_2}(X_2) = 0, \quad \Pi_{\mathcal{A}_2}(X_3) = 0.25, \quad \Pi_{\mathcal{A}_2}(X_4) = 0.75 .$$

Then, of these random variables, X_4 is the least preferred, followed by X_3 and X_2. The final ranking is given by: $C_2 \succ C_3 \succ C_4 \succ C_5 \succ C_1$.

If we compare this result with the one obtained in Example 2, the ranking is quite different, because C_1 moves from the first to the last position.

Remark 4. When there are only two candidates, both top-down and bottom-up probabilistic preference procedures are equivalent and coincide with statistical preference. This is related to the well-known fact in voting theory that the only meaningful procedure for ranking candidates in a two-candidate election is simple majority, to which all among the plurality rule, the veto rule, the Borda count and the Condorcet procedure reduce in two-candidate elections.

4. A probabilistic approach to the ranking of candidates based on individual evaluations

We now consider Balinski and Laraki's framework, where the voters express their evaluations on a (linearly-ordered) linguistic scale. In this case, we also define a probability space and a random variable for each candidate, and we express the voting procedures in terms of the comparison of those random variables. Also, we apply the probabilistic preference procedure and analyze the winner of the election following this procedure.

4.1. Probability space and random variables

Let us now consider that voters express an individual evaluation of the candidates instead of a preference ordering. In this framework, as we have seen in Eq. (1), these evaluations can be collected in a matrix $\mathcal{G} = (g_{ij})_{m,n}$ such that g_{ij} denotes the evaluation of candidate C_i given by voter v_j. In this situation, we can also consider a probability space $(V, \mathcal{P}(V), P)$, where $V = \{v_1, \ldots, v_n\}$ is the set of voters and P is the uniform distribution on V. Then, we can define a random variable Y_i associated with a given candidate C_i by $Y_i(v_j) = g_{ij}$.

We can consider the scale $\mathcal{L}' = \{l'_1, \ldots, l'_s\}$ such that $l'_1 > \ldots > l'_s$, and a function $N : \mathcal{L} \to \mathcal{L}'$ that assigns $N(l_i) = l'_{s-i+1}$ for any $i = 1, \ldots, s$. This means that for any candidate C_i, we can also consider the random variable $\tilde{Y}_i$ given by

$$\tilde{Y}_i(v_j) = N(Y_i(v_j)) = N(g_{ij}).$$

In this section we will try to express the procedures for ranking the candidates based on individual evaluations in terms of stochastic orderings, and in particular we will apply the probabilistic preference procedure to the sets of random variables $\mathcal{A} = \{Y_1, \ldots, Y_m\}$ and $\mathcal{A} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$ and investigate the rankings they yield.

4.2. Univariate stochastic orderings

The use of univariate stochastic orderings is closely related to the use of an aggregation function for aggregating the evaluations assigned to each of the candidates and, subsequently, to compare these results of the aggregation. In the following, we consider the median and the mean and, tie-breakers aside, relate them to the majority judgment and the Borda majority count.

Remark 5. Since stochastic dominance only requires the random variables to be defined on a linearly-ordered scale, we can also apply it in this framework. However, as we have already mentioned in Remark 1 in Arrow's framework, it can give rise to incomparabilities, so it will rarely give a ranking on the set of candidates. Note that if the evaluations of each candidate are aggregated by means of an increasing and symmetric aggregation function, any ranking obtained by comparing these aggregated results will be a refinement of the ordering given by stochastic dominance.

4.2.1. Comparison of medians

First of all, let us consider a stochastic ordering based on the comparison of the medians. In this way, X is preferred to Y with respect to the *median criterion*, denoted by $X \succ_{\mathrm{Me}} Y$, when $\inf \mathrm{Me}(X) > \inf \mathrm{Me}(Y)$, where the median of a random variable Z is defined as:

$$\mathrm{Me}(Z) = \left\{ t \mid P(Z \geq t) \geq \frac{1}{2}, P(Z \leq t) \geq \frac{1}{2} \right\}.$$

When applying the median criterion to the random variables associated with evaluations of candidates, we obtain the most primitive version of majority judgment due to Galton (1907) in which only the median evaluations are used for comparing the candidates. The introduction of a suitable tie-breaker is due to Balinski and Laraki (2007, 2011b).

Proposition 5. *Let $Y_1, \ldots, Y_m$ be the random variables associated with the candidates $C_1, \ldots, C_m$. Then $Y_i \succ_{\mathrm{Me}} Y_j$ for any $j \neq i$ if and only if C_i is the winner with respect to the majority grade.*

Proof. The main point is that $\mathrm{Me}(Y_i) = \mathrm{MG}(C_i)$, and then the result trivially follows. □

4.2.2. Comparison of means

In the case of ranking candidates based on preference orderings, we have seen that the Borda count procedure can be expressed in terms of the comparison of the expected values of the random variables. When dealing with evaluations of candidates, we also obtain a connection between the comparison of expected values and the Borda majority count.

Proposition 6. *Let $Y_1, \ldots, Y_m$ be the random variables associated with the candidates $C_1, \ldots, C_m$. Consider the mapping $f : \mathcal{L} = \{l_1, \ldots, l_s\} \to \{0, \ldots, s-1\}$ given by $f(l_k) = k - 1$. Then $E(f(Y_i)) > E(f(Y_j))$ for any $j \neq i$ if and only if C_i is the winner with respect to the Borda majority count.*

Proof. Let us compute $E(f(Y_i))$:

$$E(f(Y_i)) = \sum_{j=1}^{n} \frac{1}{n} f(g_{ij}) = \frac{1}{n} \sum_{k=0}^{s-1} k \cdot |\{j \mid g(i, j) = l_{k+1}\}| = \frac{1}{n} B_{\mathrm{MC}}(C_i).$$

We conclude that $E(f(Y_i)) = \frac{1}{n} B_{\mathrm{MC}}(C_i)$, and therefore Y_i is the random variable with the greatest expected value if and only if C_i is the candidate with the greatest Borda majority count. □

We have to point out that the Borda majority count can be criticized from several points of view (see, for instance, Balinski & Laraki, 2011b; Zahid & de Swart, 2015). In our probabilistic framework, the most important critique is that the expected value is not an adequate location parameter for describing qualitative random variables. Instead, it seems more natural to use the median or, as we will see in Section 4.4, a multivariate stochastic ordering that could be used for this kind of random variables.

4.3. Bivariate stochastic orderings

Statistical preference is a stochastic ordering that can be applied whenever the random variables are defined on a linearly-ordered scale. Hence, as we did in Section 3.3, we can also apply it in this framework. Thus, a candidate C_i is the winner of the election if $Q(Y_i, Y_j) \geq 0.5$ for every $j \neq i$. Of course, this is exactly the Condorcet procedure where we take into account possible ties. As was the case in Arrow's framework, the main drawback of statistical preference is that it might lead to cycles.

4.4. Multivariate stochastic orderings

Just as statistical preference, probabilistic preference is a stochastic ordering that allows for the comparison of qualitative random variables. In this framework, we are going to apply the notion of probabilistic preference to the sets of random variables $\mathcal{A} = \{Y_1, \ldots, Y_m\}$ and $\mathcal{A} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$.

4.4.1. Distinction between both approaches

As in Section 3.4, we have to distinguish two cases. On the one hand, for the random variables Y_i, the greater the value is, the more preferred the candidate is. This means that when applying the notion of probabilistic preference to $\mathcal{A} = \{Y_1, \ldots, Y_m\}$, we rank the random variables from the winner to the loser. On the other hand, for the random variables $\tilde{Y}_i$, the greater the value is, the less preferred the candidate is. This means that when applying the notion of probabilistic preference to $\mathcal{A} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$, we rank the candidates from the loser to the winner. Following the terminology from Section 3.4, these procedures will be called the *top-down* probabilistic preference procedure and the *bottom-up* probabilistic preference procedure, respectively.

Note that one of the main differences with respect to ranking candidates based on preference orderings is that the random variables considered could take the same value, i.e., ties are allowed, in contrast with the random variables considered in Section 3. This means that for computing the multivariate winning probabilities, we have to use Eq. (3), but we cannot use the simplified formula in Eq. (4). Additionally, unlike the random variables defined in Section 3, $\mathcal{A} = \{Y_1, \ldots, Y_m\}$ and $\mathcal{A} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$ do not necessarily need to be sets of distinct random variables. Thus, for computing the multivariate winning probabilities, we assume in the remainder of this section that the random variables $Y_1, \ldots, Y_m$, and, hence, $\tilde{Y}_1, \ldots, \tilde{Y}_m$, are all distinct. Note that this is not a heavy restriction because, if two random variables Y_i and Y_j are equal, this means that the candidates Y_i and Y_j have the same evaluations for all the voters, so we can simple remove one of them and, at the end, assign the same position in the global ranking to both candidates.

4.4.2. Top-down probabilistic preference procedure

For the set of distinct random variables $\mathcal{A} = \{Y_1, \ldots, Y_m\}$ expressing the evaluations of the candidates by the voters, the greater the evaluation (with respect to the qualitative scale) is, the more preferred the candidate is. Hence, we will compute the multivariate winning probabilities using Eq. (3), and the greater the multivariate winning probability is, the more preferred the candidate is. In order to rank all the candidates, we follow a procedure that is quite similar to that of Section 3.4.2:

Step 0. Let $\mathcal{A}_1 = \{Y_1, \ldots, Y_m\}$ be the set of distinct random variables associated with the candidates.

Step i. For any $Y \in \mathcal{A}_i$, compute the multivariate winning probability $\Pi_{\mathcal{A}_i}(Y)$.

- $i.1$: Rank the candidates such that their associated random variables $Y \in \mathcal{A}_i$ carry a positive multivariate winning probability $\Pi_{\mathcal{A}_i}(Y) > 0$ according to decreasing $\Pi_{\mathcal{A}_i}(Y)$.
- $i.2$: Let $\mathcal{A}_{i+1} = \mathcal{A}_i \setminus \{Y \in \mathcal{A}_i \mid \Pi_{\mathcal{A}_i}(Y) > 0\}$.
- $i.3$: If $\mathcal{A}_{i+1} = \emptyset$, then all the candidates are ranked. Otherwise, go to step $i + 1$.

As we can see, this procedure is quite similar to that of Section 3.4.2, since we rank the candidates starting from the winner to the loser according to their frequency at the first position.

Therefore, we also name it *top-down probabilistic preference procedure*.

Example 5. Consider an election with five candidates C_1, C_2, C_3, C_4 and C_5 and ten voters that evaluate each candidate on the following linguistic scale:

$$\mathcal{L} = \{\text{poor (po), acceptable (ac), good (go), very good (vg), excellent (ex)}\}.$$

Their evaluations are given in matrix form as in Eq. (1).

$$\mathcal{G} = \begin{pmatrix} go & ac & ac & go & go & ex & vg & go & ac & go \\ vg & go & go & ac & po & ac & go & vg & ac & go \\ ac & vg & go & ac & ac & go & go & go & go & ac \\ po & ac & po & po & ac & vg & go & ac & po & po \\ ac & po & po & po & ac & go & ac & ac & po & po \end{pmatrix}$$

Let us apply the top-down probabilistic preference procedure to the set of distinct random variables $\mathcal{A}_1 = \{Y_1, Y_2, Y_3, Y_4, Y_5\}$ associated with the candidates. First, let us compute the multivariate winning probabilities:

$$\Pi_{\mathcal{A}_1}(Y_1) = P(\{v_4\}) + P(\{v_5\}) + P(\{v_6\}) + P(\{v_7\}) + \frac{1}{2}P(\{v_{10}\}) = 0.45;$$
$$\Pi_{\mathcal{A}_1}(Y_2) = P(\{v_1\}) + \frac{1}{2}P(\{v_3\}) + P(\{v_8\}) + \frac{1}{2}P(\{v_{10}\}) = 0.3;$$
$$\Pi_{\mathcal{A}_1}(Y_3) = P(\{v_2\}) + \frac{1}{2}P(\{v_3\}) + P(\{v_9\}) = 0.25;$$
$$\Pi_{\mathcal{A}_1}(Y_4) = \Pi_{\mathcal{A}_1}(Y_5) = 0.$$

We conclude that Y_1 is the winner, with multivariate winning probability 0.45, followed by Y_2 and Y_3, with multivariate winning probabilities 0.3 and 0.25, respectively.

Next, consider the set of random variables $\mathcal{A}_2 = \{Y_4, Y_5\}$, and let us compute their multivariate winning probabilities:

$$\Pi_{\mathcal{A}_2}(Y_4) = \frac{1}{2}\big(P(\{v_2\}) + P(\{v_3\}) + P(\{v_4\}) + P(\{v_5\}) + P(\{v_8\}) + P(\{v_9\}) + P(\{v_{10}\})\big) + P(\{v_1\}) + P(\{v_6\}) + P(\{v_7\}) = 0.65;$$
$$\Pi_{\mathcal{A}_2}(Y_5) = 1 - \Pi_{\mathcal{A}_2}(Y_4) = 0.35.$$

The global ranking obtained from the top-down probabilistic preference procedure is:

$$C_1 \succ C_2 \succ C_3 \succ C_4 \succ C_5.$$

The following result is quite similar to Proposition 3. Its proof is obvious and therefore omitted.

Proposition 7. *Let $\mathcal{A} = \{Y_1, \ldots, Y_m\}$ be the set of distinct random variables associated with the candidates $C_1, \ldots, C_m$. A candidate has a strictly positive top-down probability if and only if it is a most preferred candidate for at least one voter.*

4.4.3. Bottom-up probabilistic preference procedure

As we did in Section 3.4.3, we can also apply the notion of probabilistic preference to the set of random variables $\mathcal{A} = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$. In that case, taking into account that the greater the value of $\tilde{Y}_i$ is, the worse the evaluation of the candidate is, the multivariate winning probabilities establish a ranking from the loser to the winner. This means that the greater the multivariate winning probability is, the less preferred the evaluation of the candidate is. Taking this comment into account, we obtain the following procedure:

Step 0. Let $\mathcal{A}_1 = \{\tilde{Y}_1, \ldots, \tilde{Y}_m\}$ be the set of distinct random variables associated with the candidates.

Step i. For any $\tilde{Y} \in \mathcal{A}_i$, compute the multivariate winning probability $\Pi_{\mathcal{A}_i}(\tilde{Y})$.

i.1: Rank the candidates such that their associated random variables $\tilde{Y} \in \mathcal{A}_i$ carry a positive multivariate winning probability $\Pi_{\mathcal{A}_i}(\tilde{Y}) > 0$ according to increasing $\Pi_{\mathcal{A}_i}(\tilde{Y})$.

i.2: Let $\mathcal{A}_{i+1} = \mathcal{A}_i \setminus \{\tilde{Y} \in \mathcal{A}_i \mid \Pi_{\mathcal{A}_i}(\tilde{Y}) > 0\}$.

i.3: If $\mathcal{A}_{i+1} = \emptyset$, then all the candidates are ranked. Otherwise, go to step $i + 1$.

In this framework, there is also a relationship between the top-down and bottom-up probabilistic preference procedures. The ranking obtained from the bottom-up probabilistic preference procedure is the reversed ranking of the one we obtain if we apply the top-down probabilistic preference procedure to the random variables $\tilde{Y}_1, \ldots, \tilde{Y}_m$.

Example 6. Let us apply this approach to the evaluations given in Example 5. First of all, consider the set of distinct random variables $\mathcal{A}_1 = \{\tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \tilde{Y}_4, \tilde{Y}_5\}$, and let us compute the multivariate winning probabilities:

$$\Pi_{\mathcal{A}_1}(\tilde{Y}_1) = \Pi_{\mathcal{A}_1}(\tilde{Y}_3) = 0;$$
$$\Pi_{\mathcal{A}_1}(\tilde{Y}_2) = P(\{v_5, v_6\}) = 0.2;$$
$$\Pi_{\mathcal{A}_1}(\tilde{Y}_4) = P(\{v_1\}) + \tfrac{1}{2}P(\{v_3, v_4, v_8, v_9, v_{10}\}) = 0.35;$$
$$\Pi_{\mathcal{A}_1}(\tilde{Y}_5) = P(\{v_2, v_7\}) + \tfrac{1}{2}P(\{v_3, v_4, v_8, v_9, v_{10}\}) = 0.45.$$

$\tilde{Y}_2, \tilde{Y}_4$ and $\tilde{Y}_5$ are the random variables with positive multivariate winning probability. According to *Step 1.1*, C_5 is the loser, followed by C_4 and C_2.

Next, we compute the multivariate winning probabilities on the set of random variables $\mathcal{A}_2 = \{\tilde{Y}_1, \tilde{Y}_3\}$ of the still unranked candidates C_1 and C_3:

$$\Pi_{\mathcal{A}_2}(\tilde{Y}_1) = 0.35 \text{ and } \Pi_{\mathcal{A}_2}(\tilde{Y}_3) = 0.65.$$

The global ranking obtained from the bottom-up probabilistic preference procedure is:

$$C_1 \succ C_3 \succ C_2 \succ C_4 \succ C_5.$$

As we can see, the ranking given by the bottom-up probabilistic preference procedure is slightly different from the one obtained in Example 5, but in both cases C_1 is the winner of the election.

5. Conclusions

In the framework of voting theory, there are two common scenarios: voters either give their preference orderings or their individual evaluations of the candidates. In this work, we have considered a probabilistic approach to model both situations.

In the first scenario, any candidate defines a random variable that expresses the position in the preference ordering of this candidate for any voter. We have seen that the most common procedure used for solving this kind of problems, the Borda count, is related to the comparison of expected values of those random variables. Furthermore, the Condorcet procedure is also connected to the comparison of the random variables through the notion of statistical preference. The recently-proposed notion of probabilistic preference is proved to be linked to the plurality and veto rules. More precisely, top-down probabilistic preference relates to the former and bottom-up probabilistic preference relates to the latter. The choice between the top-down and the bottom-up approaches is similar to that of decision making with *maximax* and *maximin* criteria, and depends on the interpretation. If we are looking for the candidate that is ranked as the most preferred candidate for most voters, we should use the top-down probabilistic preference procedure; on the other hand, if we are looking for the candidate that is not ranked as the least preferred candidate for most voters, we will apply the bottom-up probabilistic preference procedure. A summary of the connections between stochastic orderings and voting procedures in Arrow's framework is shown in the next table:

Arrow's framework	
Voting procedure	Stochastic ordering
Borda	Expected value
Borda-dominance	Stochastic dominance
Condorcet	Statistical preference
Plurality	Probabilistic preference (top-down)
Veto	Probabilistic preference (bottom-up)

In the second scenario, any candidate has an associated random variable that expresses the evaluation of this candidate for any voter. In this case, we have seen that the two most common procedures – the majority judgment and the Borda majority count – are equivalent to the comparison of the medians and expected values of the random variables, respectively, whereas the Condorcet procedure, allowing ties, is again equivalent to applying statistical preference. In this second scenario, we have also analyzed probabilistic preference, both the top-down and the bottom-up procedure. To the best of our knowledge, there is no connection between these two stochastic orderings and any existing voting procedure in the framework of Balinski and Laraki. Again, the choice of procedure depends on the kind of winner we are looking for. The next table summarizes the connection between the voting procedures in this framework and stochastic orderings.

Balinski and Laraki's framework	
Voting procedure	Stochastic ordering
Majority judgement	Comparison of medians
Borda majority count	Expected value
Condorcet	Statistical preference
	Probabilistic preference (top-down)
	Probabilistic preference (bottom-up)

We end by recalling that the aim of this paper was not to compare the different voting procedures but, instead, showing a correspondence between voting procedures and stochastic orderings. Determining which is the most suitable procedure for solving voting problems seems to be closely related to the question of which is the most suitable stochastic ordering for comparing random variables: it depends on the available information and the interpretation we are adopting. A conclusion is clear nevertheless: both voting theory and the comparison of random variables build upon some common concepts.

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