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THE BURR XII-BURR XII DISTRIBUTION: MATHEMATICAL PROPERTIES AND CHARACTERIZATIONS

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ABSTRACT

We introduce a new continuous distribution called the Burr XII-Burr XII distribution. Some of its properties are derived. The method of maximum likelihood is used to estimate the unknown parameters. An application is provided with details to illustrate the importance of the new. The new model provides adequate fits as compared to other related models with smallest values for A-IC, B-IC, CA-IC and HQ-IC. Characterization results are presented based on two truncated moments, hazard function as well as based on the conditional expectation.

KEYWORDS

Burr XII Distribution; Burr XII Family; Characterizations; Modeling.

1. INTRODUCTION

Analogous to the Pearson system of distributions (see Elderton, (1953) and Elderton and Johnson, (1969)), Burr (1942) introduced another system of distributions that includes twelve types of cumulative distribution functions (CDFs) which yield a variety of density shapes. This system is obtained by considering CDFs satisfying a certain differential equation whose solution is given by

$$G(x) = \frac{1}{1 + e^{-\int \Phi(x)dx}},$$

where $\Phi(x)$ is chosen such that $G(x)$ is a CDF on the real line. Twelve choices for $\Phi(x)$, made by Burr, resulted in twelve distributions which might be useful for fitting data. For more details see Burr (1942), Burr (1968), Burr (1973), Burr and Cislak (1968), Hatke (1949) and Rodriguez (1977). Special attention has been paid to one of these forms called Type XII. The CDF and probability density function (PDF) of the three-parameter BXII distribution are given, respectively, by

$$G_{a,b,c}(x) = 1 - \frac{1}{\left(\frac{x^a}{c^a}+1\right)^b}, \mathrm{x} \geq 0, \tag{1}$$

and

$$g_{a,b,c}(x) = abc^{-a}\frac{x^{a-1}}{\left(\frac{x^a}{c^a}+1\right)^{b+1}}, \tag{2}$$

where both $a > 0$ and $b > 0$ are shape parameters, $c > 0$ is the scale parameter. For $a = 1$, the BXII model reduces to the Lomax (Lx) or Pareto type II (PaII) model. For $b = 1$, the BXII model reduces to the log-logistic (LL) model. For $a = c = 1$ the BXII model reduces to the one-parameter Lx or one-parameter PaII model. For $b = c = 1$ the BXII model reduces to the one-parameter LL model. For $c = 1$ the BXII model reduces to two-parameter BuXII model. When $b \to \infty$ the BXII model reduces the two-parameter Weibull (W) model, for $c = 1$ and $b \to \infty$ the BXII model reduces the one-parameter W model. Tadikamalla (1980) studied the BXII model and its associated models, namely: PaII (Lx), LL, compound Weibull, gamma (Ga) and Weibull exponential (WE) distributions. Recently, Cordeiro et al. (2018) defined the Burr XII-G (BXII-G) family of distributions with CDF

$$F_\Theta(x) = 1 - \left\{\left[G_\psi(x) - 1\right]^{-\alpha} + 1\right\}^{-\beta}|_{x\in R}. \tag{3}$$

The PDF corresponding to (3) is

$$f_\Theta(x) = \alpha\beta\frac{g_\psi(x)G_\psi(x)^{\alpha-1}}{\left[1-G_\psi(x)\right]^{\alpha+1}}\left\{\left[G_\psi(x) - 1\right]^{-\alpha} + 1\right\}^{-\beta-1}|_{x\in R}, \tag{4}$$

where $g_\psi(x)$ is the baseline PDF and $\Theta = (\alpha, \beta, a, b, c)$. The new model (BXII-BXII) has CDF

$$F_\Theta(x) = 1 - \left\{\left[\left(1+\frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}|_{x\geq 0}, \tag{5}$$

and PDF

$$f_\Theta(x) = \alpha\beta abc^{-a}x^{a-1}\left(\frac{x^a}{c^a}+1\right)^{b-1}\frac{\left[-\left(\frac{x^a}{c^a}+1\right)^{-b}+1\right]^{\alpha-1}}{\left\{\left[\left(\frac{x^a}{c^a}+1\right)^b-1\right]^\alpha+1\right\}^{\beta+1}}|_{x>0}. \tag{6}$$

Recently, many authors considered the extension of the BXII model such as Shao (2004), Altun et al. (2017), Alizadeh et al. (2017a and b). Afify et al. (2018), Yousof et al. (2018), Cordeiro et al. (2018), Altun et al. (2018a and b) and Yousof et al. (2019). From Figure 1(a) we see that the new PDF can be left skewed and right skewed (see also Table 2 for more details about the skewness and kurtosis of the new model). It is clear from Figure 1(b) that the HRF can be upside down and decreasing. Equation (5) contains as sub-models several generated models. Generally, for $\beta = 1$ (b=1), we can change the first (second) name of the model by LL. The log-logistic-log-logistic (LLLL) model follows when β = b = 1. For α = 1 (a=1), we can change the first (second) name by PaII. For α = 1 and b=1, we obtain the PaII-LL model. If β → ∞ (or b→∞), the first (second) name can be changed by Weibull (W). For c=1 we have same model with a smaller number of parameters. If we combine these conditions, we can generate at least 40 special cases of (5).

Table 1
Some Special Submodels of BXII-BXII Model

N	α	β	a	b	c	model	CDF
1		$\to\infty$				W-BXII	$1-\left\{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta\to\infty}$
2		$\to\infty$			1	W-BXII	$1-\left\{\left[\left(1+x^a\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta\to\infty}$
3		$\to\infty$	1			W-Lx	$1-\left\{\left[\left(1+c^{-1}x\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta\to\infty}$
4		$\to\infty$	1		1	W-Lx	$1-\left\{\left[\left(1+x\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta\to\infty}$
5		$\to\infty$		1		W-LL	$1-\left[\left(\frac{x^a}{c^a}\right)^\alpha+1\right]^{-\beta}\Big\vert_{\beta\to\infty}$
6		$\to\infty$		1	1	W-LL	$1-\left(x^{a\alpha}+1\right)^{-\beta}\vert_{\beta\to\infty}$
7		$\to\infty$		$\to\infty$		W-W	$1-\left\{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta,b\to\infty}$
8		$\to\infty$		$\to\infty$	1	W-W	$1-\left\{\left[\left(1+x^a\right)^b-1\right]^\alpha+1\right\}^{-\beta}\Big\vert_{\beta,b\to\infty}$
9	1					Lx-BXII	$1-\left[\left(1+\frac{x^a}{c^a}\right)^b\right]^{-\beta}$
10	1			$\to\infty$		Lx-W	$1-\left[\left(1+\frac{x^a}{c^a}\right)^b\right]^{-\beta}\Big\vert_{\beta\to\infty}$
11	1			$\to\infty$	1	Lx-W	$1-\left[\left(1+x^a\right)^b\right]^{-\beta}\Big\vert_{b\to\infty}$
12	1		1			Lx-Lx	$1-\left[\left(1+c^{-1}x\right)^b\right]^{-\beta}$
13	1		1		1	Lx-Lx	$1-\left[\left(1+x\right)^b\right]^{-\beta}$
14	1			1		Lx-LL	$1-\left[\left(1+\frac{x^a}{c^a}\right)^b\right]^{-1}$
15	1			1	1	Lx-LL	$1-\left[\left(1+x^a\right)^b\right]^{-1}$
16	1				1	Lx-BXII	$1-\left[\left(1+x^a\right)^b\right]^{-\beta}$
17		1				LL-BXII	$1-\left\{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1\right\}^{-1}$
18		1		1		LL-LL	$1-\left\{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1\right\}^{-1}$
19		1		1	1	LL-LL	$1-\left\{\left[\left(1+x^a\right)^b-1\right]^\alpha+1\right\}^{-1}$
20		1		$\to\infty$		LL-W	$1-\left\{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1\right\}^{-1}\Big\vert_{b\to\infty}$

N	α	β	a	b	c	model	CDF
21		1		$\to\infty$	1	LL-W	$1-\left\{\left[(1+x^{a})^{b}-1\right]^{\alpha}+1\right\}^{-1}\Big\vert_{b\to\infty}$
22		1	1			LL-Lx	$1-\left\{\left[(1+\frac{x^{a}}{c^{a}})^{b}-1\right]^{\alpha}+1\right\}^{-1}$
23		1	1		1	LL-Lx	$1-\left\{\left[(1+x^{a})^{b}-1\right]^{\alpha}+1\right\}^{-1}$
24		1			1	LL-BXII	$1-\left\{\left[(1+x^{a})^{b}-1\right]^{\alpha}+1\right\}^{-1}$
25	1	1				BXII	$1-(1+\frac{x^{a}}{c^{a}})^{-b}$
26	1	1		$\to\infty$		W	$1-(1+\frac{x^{a}}{c^{a}})^{-b}\vert_{b\to\infty}$
27	1	1			1	BXII	$1-(1+x^{a})^{-b}$
28	1	1		$\to\infty$	1	W	$1-(1+x^{a})^{-b}\vert_{b\to\infty}$
29	1	1	1			Lx	$1-(1+c^{-1}x)^{-b}$
30	1	1		1		LL	$1-(1+\frac{x^{a}}{c^{a}})^{-1}$
31	1	1	1		1	Lx	$1-(1+x)^{-b}$
32	1	1		1	1	LL	$1-(1+x^{a})^{-1}$
33					1	BXII-BXII	$1-\left\{\left[(1+x^{a})^{b}-1\right]^{\alpha}+1\right\}^{-\beta}$
34				$\to\infty$		BXII-W	$1-\left\{\left[(1+\frac{x^{a}}{c^{a}})^{b}-1\right]^{\alpha}+1\right\}^{-\beta}\Big\vert_{b\to\infty}$
35				$\to\infty$	1	BXII-W	$1-\left\{\left[(1+x^{a})^{b}-1\right]^{\alpha}+1\right\}^{-\beta}\Big\vert_{b\to\infty}$
36			1			BXII-Lx	$1-\left\{\left[(1+c^{-1}x)^{b}-1\right]^{\alpha}+1\right\}^{-\beta}$
37			1		1	BXII-Lx	$1-\left\{\left[(1+x)^{b}-1\right]^{\alpha}+1\right\}^{-\beta}$
38				1		BXII-LL	$1-\left[(\frac{x^{a}}{c^{a}})^{\alpha}+1\right]^{-\beta}$
39				1	1	BXII-LL	$1-(x^{a\alpha}+1)^{-\beta}$
40	1	$\to\infty$	1	$\to\infty$		W-W	$1-(1+c^{-1}x)^{-b\beta}\vert_{b,\beta\to\infty}$

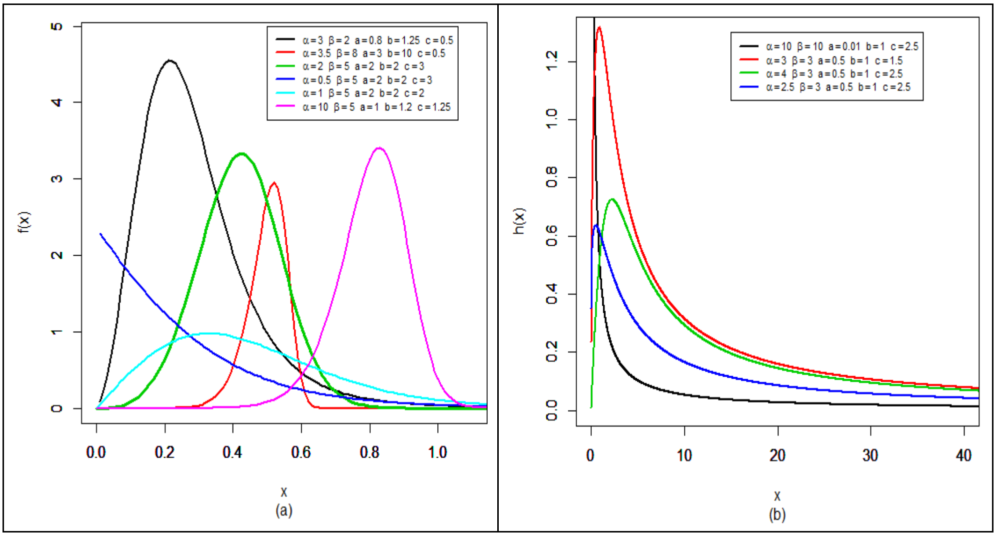

Figure 1: Plots of the PDF and HRF for Selected Values of the Parameters.

2. PROPERTIES

2.1 Linear Representation

We denote by $X \sim$ BXII-BXII (α, β, a, b, c) a random variable (rv) with PDF (6). In this subsection, we provide a useful linear representation for the density of X, which can be used to derive some mathematical properties of the BXII-BXII model. The CDF (5) can be expressed as

$$F_{\Theta}(x) = 1 - \underbrace{\left\{1 + \left[\frac{1 - \left(\frac{x^a}{c^a} + 1\right)^{-b}}{\left(\frac{x^a}{c^a} + 1\right)^{-b}}\right]^{\alpha}\right\}^{-\beta}}_{A}. \tag{7}$$

First, we shall consider the three series

$$(1+z)^{-a} = \sum_{k=0}^{\infty} (-1+z)^k \left(\frac{1}{2}\right)^{a+k} \binom{-a}{k}, \tag{8}$$

$$(1-z)^{-a} = \sum_{j=0}^{\infty} \frac{\Gamma(a+j)}{j!\,\Gamma(a)} z^j \big|_{(|z|<1,\ a>0)} \tag{9}$$

and the generalized binomial series given by

$$(1-z)^{\alpha-1} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(\alpha)}{r!\,\Gamma(\alpha-r)} z^r \big|_{(|z|<1 \text{and } \alpha>0 \text{ realnon-integer})}. \tag{10}$$

Applying (8) for A in (7), we obtain

$$F_{\Theta}(x) = 1 - \sum_{k=0}^{\infty} \left\{ \left[\frac{1 - \left(\frac{x^a}{c^a} + 1\right)^{-b}}{\left(\frac{x^a}{c^a} + 1\right)^{-b}} \right]^{\alpha} - 1 \right\}^{k} \left(\frac{1}{2}\right)^{a+k} \binom{-\beta}{k}.$$

Second, using the binomial expansion, the last equation can be expressed as

$$F_{\Theta}(x) = 1 - \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{(-1)^i \left(\frac{1}{2}\right)^{a+k} \binom{k}{i} \binom{-\beta}{k}}{\left[1 - \left(\frac{x^a}{c^a} + 1\right)^{-b}\right]^{-\alpha(k-i)}} \underbrace{\left\{1 - \left[1 - \left(\frac{x^a}{c^a} + 1\right)^{-b}\right]\right\}^{-\alpha(k-i)}}_{B}.$$

Third, applying (9) for B in the last equation, we can have

$$F_{\Theta}(x) = 1 - \sum_{j,k=0}^{\infty} \sum_{i=0}^{k} \tau_{i,j,k} \Pi_{\alpha(k-i)+j}(x), \tag{11}$$

where

$$\Pi_{\alpha(k-i)+j}(x) = \left[1 - \left(\frac{x^a}{c^a} + 1\right)^{-b}\right]^{\alpha(k-i)+j}$$

is the CDF of the exponentiated BXII $[\alpha(k\text{-}i) + j] > 0$and

$$\tau_{i,j,k} = \frac{(-1)^i \left(\frac{1}{2}\right)^{a+k} \Gamma(\alpha(k-i)+j)}{j!\, \Gamma\big(\alpha(k-i)\big)} \binom{k}{i} \binom{-\beta}{k}.$$

Upon differentiating (11) and applying (10), we obtain

$$f_{\Theta}(x) = \sum_{r=0}^{\infty} \zeta_r g_{a,b(1+r),c}(x), \tag{12}$$

where $g_{a,b(1+r),c}(x)$ is the BXII density with parameters $a, b(1+r)$, c and

$$\zeta_r = \frac{(-1)^{r+1}}{r!\,(1+r)\Gamma(\alpha(k-i)+j-r)} \sum_{j,k=0}^{\infty} \sum_{i=0}^{k} \tau_{i,j,k} (\alpha(k-i)+j)\Gamma(\alpha(k-i)+j)|_{(j+k\geq 1)}.$$

Equation (12) reveals that the BXII-BXII density is a linear combination of BXII densities. So, some of its mathematical properties can be determined from those of the BXII distribution.

2.2 Ordinary Moment

The nth ordinary moment of X is given by

$$\mu_n' = E(X^n) = \sum_{r=0}^{\infty} \zeta_r \int_0^{\infty} x^n \; g_{a,b(1+r),c}(x)dx.$$

or

$$\mu_n' = E(X^n) = \sum_{r=0}^{\infty} \zeta_r b(1+r)c^n B\left(b(1+r) - \frac{n}{a}, \frac{n}{a} + 1\right)|_{(n<ab(1+r))}. \qquad (13)$$

Setting $n = 1$ in (13), we have the mean of X. The sth central moment (M_s) and cumulants (κ_s) of X, are given, respectively, by

$$M_s = E(X - \mu_1')^s = \sum_{i=0}^{s} (-1)^i \binom{s}{i} (\mu_1')^s \mu_{s-i}'$$

and

$$\kappa_s = \mu_s' - \sum_{i=1}^{s-1} \binom{s-1}{i-1} \kappa_r \mu_{s-r}',$$

where $\kappa_1 = \mu_1'$.

2.3 Numerical Analysis for Expected Value, Variance, Skewness and Kurtosis

The skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships. The effects of the parameters α, β, a, b and c on the mean, variance, skewness and kurtosis, for given values, are displayed in Table 2. For Table 2 we have the following results:

1- The BXII-BXII density can be right skewed or left skewed.
2- The kurtosis can be more than 3 or less than 3.
3- The parameters α and β have minimal effect on the expected value.
4- The parameters α and β have less effect on the kurtosis.

Table 2
Numerical Results for Expected Value, Variance, Skewness, Kurtosis

α	β	a	b	c	μ_1'	Variance	Skewness	Kurtosis
10	1.5	0.5	1.5	1	0.3212691	0.0084360	0.8594555	5.804449
15					0.3261923	0.0038352	0.4311338	4.372474
50					0.3380326	0.00037647	−0.1321425	3.904742
100					0.3413825	$9.671\times e^{-05}$	−0.2547767	4.01435
150					0.3425669	$4.340\times e^{-05}$	−0.2962296	4.063467
15	0.25	1.5	1.5	1.5	0.5724742	0.3891802	0.3360489	1.623289
	0.5				0.5493357	0.3148919	0.08904296	1.113732
	1				0.5359509	0.2792599	−0.00669279	1.043676
	10				0.5092085	0.2199492	−0.153066	1.04575
	50				0.4936705	0.1907251	−0.2340034	1.077233
	100				0.4870896	0.179339	−0.2687802	1.095213
	200				0.4805218	0.168537	−0.3038457	1.115903
	500				0.4718379	0.155084	−0.3507943	1.147568
	1000				0.4652581	0.145494	−0.3868153	1.174919
5	10	0.25	1.25	1.25	0.0621626	0.002981	1.731309	8.047693
		1			0.4696441	0.053484	−0.9693768	3.00728
		2			0.5777699	0.148855	−0.7355838	1.723454
		3			0.5335119	0.221480	−0.2212407	1.10564
		4			0.4487744	0.252950	0.2427352	1.086138
5	3	2	0.25	1.25	2.252816	3.899049	0.03982338	1.639265
			0.5		1.118607	0.824826	−0.2744446	1.357711
			1		0.7022615	0.278060	−0.4786052	1.426896
			2		0.4804141	0.113090	−0.6390002	1.58383
			3		0.3901852	0.069949	−0.7104736	1.676719
5	3	2	0.25	1	3.001298	0.609069	0.606196	4.310098
				1.5	1.761198	5.289711	0.7573032	2.061007
				3	0.653981	5.765541	3.684258	15.82487
				10	0.109151	3.426382	18.05618	348.9902
				20	0.038636	2.432213	43.07911	1980.257

2.4 Moment Generating Function

The moment generating function (MGF) of X, say $M_X(t) = E[exp(tX)]$, can be obtained from (12) as

$$M_X(t) = \sum_{r=0}^{\infty} \zeta_r M_{a,b(1+r),c}(t),$$

where $M_{a,b(1+r),c}(t)$ is the MGF of the BXII distribution with parameters $a, b(1+r)$ and c. Next, we require the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x\middle|\begin{matrix}a_1,\dots,a_p\\ b_1,\dots,b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j+t)\prod_{j=1}^{n}\Gamma(1-a_j-t)}{\prod_{j=n+1}^{p}\Gamma(a_j+t)\prod_{j=m+1}^{p}\Gamma(1-b_j-t)} x^{-t}dt,$$

where $i=\sqrt{-1}$ is the complex unit and L denotes an integration path (Gradshteyn and Ryzhik, 2000). The Meijer **G**-function contains, as particular cases, many integrals with elementary and special functions (Prudnikov et al. (1986)). We now assume that $a = \frac{m}{b}$ where m and b are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. We have the following result, which holds for m and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov et al., 1992).

$$I\left(p,\mu,\frac{m}{\beta},v\right)|_0^\infty = \int_0^\infty exp(-px)\,x^\mu\left(1+x^{\frac{m}{b}}\right)^v dx$$
$$= C_{(p,\mu,m,v)} G_{b+m,b}^{b,b+m}\left(\frac{m^m}{p^m}\,|\begin{matrix}\Delta(m,-\mu),\Delta(b,v+1)\\ \Delta(b,0)\end{matrix}\right),$$

where

$$C_{(p,\mu,m,v)} = \frac{b^{-v}m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{m-1}{2}}\Gamma(-v)p^{\mu+1}},$$

and

$$\Delta(a,b) = \frac{a}{b},\frac{a+1}{b},\frac{a+2}{b},\ldots,\frac{a+b}{b}.$$

We can write (for $t < 0$)

$$M(t) = m \times I\left(-ct,\frac{m}{b}-1,\frac{m}{b},-b-1\right).$$

Hence, the MGF of X can be expressed as

$$M_X(t) = m \times \sum_{r=0}^{\infty}\zeta_r I\left(-ct,\frac{m}{b(1+r)}-1,\frac{m}{b(1+r)},-[1+b(1+r)]\right).$$

2.5 Incomplete Moments

The sth incomplete moment, say $\phi_s(t)$, of the BXII-BXII distribution is given by $\phi_s(t) = \int_0^t x^s f(x)dx$. From equation (12), we have

$$I_s(t) = \sum_{r=0}^{\infty}\zeta_r\int_0^t x^s g_{a,b(1+r),c}(x)dx,$$

and using the lower incomplete gamma function, we obtain

$$I_s(t) = \sum_{r=0}^{\infty}\zeta_r b(1+r)c^n B\left(t^\alpha; b\,(1+r)\ -\frac{s}{a},\frac{s}{a}+1\right).$$

The first incomplete moment of X, denoted by $\phi_1(t)$, is simply determined from the above equation by setting $s = 1$. The first incomplete moment has important applications related to the Bonferroni and Lorenz curves and the mean residual life and the mean waiting time. Furthermore, the amount of scatter in a population is evidently measured, to some extent, by the totality of deviations from the mean and median. The mean deviations, about the mean and about the median of X, depend on $\phi_1(t)$.

2.6 Residual and Reversed Residual Life Functions

The nth moment of the residual life (RL), denoted by

$$m_n(t) = E[(X-t)^n]|_{(X>t,\ n=1,2,\dots)}.$$

The nth moment of the residual life of X is given by

$$m_n(t) = \frac{1}{1-F_\Theta(t)}\int_t^\infty (x-t)^n dF_\Theta(x).$$

Then, we can write

$$m_n(t) = \frac{1}{1-F(t)}\sum_{i=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^{n-i} n!\, t^{n-i}}{i!\,\Gamma(n-i+1)}\zeta_r b(1+r)c^n B\left(t^\alpha; b\,(1+r)-\frac{n}{a},\frac{n}{a}+1\right).$$

The nth moment of the reversed residual life, say

$$M_n(t) = E[(t-X)^n]|_{(X\le t, t>0\ and\ n=1,2,\dots,)}.$$

Then, $M_n(t)$ is defined by

$$M_n(t) = \frac{1}{F_\Theta(t)}\int_0^t (t-x)^n dF_\Theta(t).$$

The nth moment of the reversed residual life of X

$$M_n(t) = \frac{1}{F(t)}\sum_{i=0}^{n}\sum_{r=0}^{\infty}\frac{(-1)^{i} n!}{i!\,(n-i)!}\zeta_r b(1+r)c^n B\left(t^\alpha; b\,(1+r)-\frac{n}{a},\frac{n}{a}+1\right).$$

3. CHARACTERIZATION RESULTS

This section deals with the characterizations of the BXII-BXII distribution based on: (i) a simple relation between two truncated moments; (ii) hazard function and (iii) the conditional expectation of a function of the random variable. We like to mention that the characterization (i) can be employed when the CDF does not have a closed form. We present our characterizations $(i)-(iii)$ in three subsections.

3.1 Characterizations based on Two Truncated Moments

In this subsection we present characterizations of BXII-BXII distribution in terms of a simple relationship between two truncated moments. The first characterization result employs a theorem due to Glänzel (1987), see Theorem 3.1.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the CDF F does not have a closed form. As shown in Glänzel (1990), this characterization is stable in the sense of weak convergence.

Theorem 3.1.1.

Let (Ω, F, P) be a given probability space and let $H=[d,e]$ be an interval for some $d<e(d=-\infty, e=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let g and h be two real functions defined on H such that

$$E[g(X)\mid X\ge x] = E[h(X)\mid X\ge x]\xi(x),\quad x\in H,$$

is defined with some real function ξ. Assume that g, $h \in C^1(H)$, $\xi \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $\xi h = g$ has no real solution in the interior of H. Then F is uniquely determined by the functions g, h and ξ, particularly

$$F(x) = \int_a^x C \left|\frac{\xi'(u)}{\xi(u)h(u) - g(u)}\right| exp\big(-s(u)\big)\ du,$$

where the function s is a solution of the differential equation $s' = \frac{\xi' h}{\xi\, h - g}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 3.1.1.

Let $X: \Omega \to (0, \infty)$ be a continuous random variable and let, $h(x) \equiv 1$ and $g(x) = \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}$ for $x > 0$. The random variable X has PDF (6) if and only if the function ξ defined in Theorem 3.1.1 has the form

$$\xi(x) = \frac{1}{2}\left\{1 + \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\}, x > 0.$$

Proof:

Let X be a random variable with PDF (6), then

$$\big(1 - F(x)\big)E[h(X)|X \geq x] = 1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}, x > 0,$$

and

$$\big(1 - F(x)\big)E[g(X) \mid X \geq x] = \frac{1}{2}\left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-2\beta}\right\}, x > 0,$$

and finally

$$\xi(x)h(x) - g(x) = \frac{1}{2}\left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\} > 0 for x > 0.$$

Conversely, if ξ is given as above, then

$$\begin{aligned} s'(x) &= \frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} \\ &= \frac{\alpha\beta abc^{a-1}x^{a-1}\left(1 + \frac{x^a}{c^a}\right)^{b-1}\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^{\alpha-1}}{\left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{\beta+1}\left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\}}, \end{aligned}$$

x > 0, and hence

$$s(x) = -\log\left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\}, x > 0.$$

Now, in view of Theorem 3.1.1, X has density (6).

Corollary 3.1.1.

Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 3.1.1. The PDF of X is (6) if and only if there exist functions g and ξ defined in Theorem 3.1.1 satisfying the differential equation

$$\frac{\xi'(x)h(x)}{\xi(x)h(x) - g(x)} = \frac{\alpha\beta abc^{a-1}x^{a-1}\left(1 + \frac{x^a}{c^a}\right)^{b-1}\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^{\alpha-1}}{\left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{\beta+1}\left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\}}, \quad x > 0.$$

The general solution of the differential equation in Corollary 3.1.1 is

$$\xi(x) = \left\{1 - \left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-\beta}\right\}^{-1} \times \left\{\int \frac{\alpha\beta abc^{a-1}x^{a-1}\left(1 + \frac{x^a}{c^a}\right)^{b-1}\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^{\alpha-1}}{\left\{\left[\left(1 + \frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{\beta+1}} \left(h(x)\right)^{-1} g(x)dx + D\right\},$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 3.1.1 with $D = \frac{1}{2}$. However, it should be also noted that there are other triplets (h, g, ξ) satisfying the conditions of Theorem 3.1.1.

3.2 Characterization based on Hazard Function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)} = \frac{h_F'(x)}{h_F(x)} - h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following proposition establish a non-trivial characterization of BXII-BXII distribution in terms of the hazard function, which is not of the above trivial form.

Proposition 3.2.1.

Let $X: \Omega \rightarrow (0, \infty)$ be a continuous random variable. The PDF of X is (6) if and only if its hazard function $h_F(x)$ satisfies the differential equation

$$h'_F(x) - \frac{a-1}{x} h_F(x)$$
$$= \alpha\beta abc^{-a}x^{a-1}\frac{d}{dx}\left\{\frac{\left(1+\frac{x^a}{c^a}\right)^{b-1}\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^{\alpha-1}}{\left[\left(1+\frac{x^a}{c^a}\right)^b-1\right]^\alpha+1}\right\}, x > 0.$$

Proof:

Is straightforward and hence omitted.

Remark 3.2.1.

For $\alpha = 1$, the above differential equation has the following simple form

$$h'_F(x) - \frac{a-1}{x} h_F(x) = -\beta a^2 bc^{-2a}x^{2(a-1)}\left(1+\frac{x^a}{c^a}\right)^{-2}, x > 0.$$

3.3 Characterizations based on Conditional Expectation

The following proposition has already appeared in Hamedani, (2013), so we will just state it here which can be used to characterize the BXII-BXII distribution.

Proposition 3.3.1.

Let $X: \Omega \to (a,b)$ be a continuous random variable with CDF F. Let $\psi(x)$ be a differentiable function on (a,b) with $lim_{x\to a^+}\psi(x) = 1$. Then for $\delta \neq 1$

$$E[\psi(X) \mid X \geq x] = \delta\psi(x), \quad x \in (a,b),$$

if and only if

$$\psi(x) = \left(1-F(x)\right)^{\frac{1}{\delta}-1}, \quad x \in (a,b).$$

Remarks 3.3.1.

a) For $(a,b) = (0,\infty), \psi(x) = \left\{\left[\left(1+\frac{x^a}{c^a}\right)^b - 1\right]^\alpha + 1\right\}^{-1}$ and $\delta = \frac{\beta}{\beta+1}$, Proposition 3.3.1 provides a characterization of BXII-BXII distribution.

b) For $(a,b) = (0,\infty)$, $\alpha = 1$, $\psi(x) = \left(1+\frac{x^a}{c^a}\right)$ and $= \frac{b\beta}{b\beta+1}$, Proposition 3.3.1 provides a characterization of BXII-BXII distribution.

c) Of course, there are other suitable functions than the ones we mentioned above, which are chosen for simplicity.

4. ESTIMATION

In this Section we shall estimate the unknown parameters with the method the maximum likelihood (ML). The log likelihood function, $\log L$, will be

$$\log L = \ell_n(\Theta) = n \log \alpha + n \log \beta \ + n \log a + n \log b - na \log c$$
$$+(\alpha-1)\sum_{i=1}^{n} \log(x_i) + (b\alpha-1)\sum_{i=1}^{n} \log \zeta_i$$
$$+(\alpha-1)\sum_{i=1}^{n} \log\left(-\zeta_i^{-b}+1\right) - (\beta+1)\sum_{i=1}^{n} \log\left[\left(\zeta_i^{b}-1\right)^{\alpha}+1\right],$$

where

$$x_i^{a} + 1 = \zeta_i.$$

The above log-likelihood function can be maximized numerically by R (optim), SAS (PROC NLMIXED) or Ox program (sub-routine MaxBFGS), among others. The elements of the 5 × 5 observed information matrix can be evaluated numerically, where

$$\frac{\partial \log L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(x_i) + b\sum_{i=1}^{n} \log\zeta_i$$
$$+\sum_{i=1}^{n} \log\left(-\zeta_i^{-b}+1\right) - (\beta+1)\sum_{i=1}^{n} \frac{\left(\zeta_i^{b}-1\right)^{\alpha} \log\left(\zeta_i^{b}-1\right)}{(\zeta_i^{b}-1)^{\alpha}+1},$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} \log\left[\left(\zeta_i^{b}-1\right)^{\alpha}+1\right],$$

$$\frac{\partial \log L}{\partial a} = \frac{n}{a} - n \log c + (b\alpha-1)\sum_{i=1}^{n} \frac{\left(\frac{x_i}{c}\right)^{a} \log\left(\frac{x_i}{c}\right)}{\zeta_i}$$
$$+(\alpha-1)\sum_{i=1}^{n} \frac{b\zeta_i^{-b-1}\left(\frac{x_i}{c}\right)^{a} \log\left(\frac{x_i}{c}\right)}{-\zeta_i^{-b}+1}$$
$$-(\beta+1)\sum_{i=1}^{n} \frac{\alpha b\zeta_i^{b-1}\left(\zeta_i^{b}-1\right)^{\alpha-1}\left(\frac{x_i}{c}\right)^{a} \log\left(\frac{x_i}{c}\right)}{(\zeta_i^{b}-1)^{\alpha}+1},$$

$$\frac{\partial \log L}{\partial b} = \frac{n}{b} + \alpha\sum_{i=1}^{n} \log\zeta_i + (\alpha-1)\sum_{i=1}^{n} \frac{\zeta_i^{-b} \log(\zeta_i)}{-\zeta_i^{-b}+1}$$
$$-(\beta+1)\sum_{i=1}^{n} \frac{\alpha\zeta_i^{b} \log(\zeta_i)\left(\zeta_i^{b}-1\right)^{\alpha-1}}{(\zeta_i^{b}-1)^{\alpha}+1},$$

and

$$\frac{\partial \log L}{\partial c} = -\frac{na}{c} + (b\alpha-1)\sum_{i=1}^{n} \frac{-ac^{-a-1}x_i^{a}}{\zeta_i}$$
$$+(\alpha-1)\sum_{i=1}^{n} \frac{-abc^{-a-1}x_i^{a}\zeta_i^{-b-1}}{-\zeta_i^{-b}+1}$$
$$-(\beta+1)\sum_{i=1}^{n} \frac{-a\alpha bc^{-a-1}x_i^{a}\zeta_i^{b-1}\left(\zeta_i^{b}-1\right)^{\alpha-1}}{(\zeta_i^{b}-1)^{\alpha}+1}.$$

5. APPLICATIONS

We provide four applications to illustrate the importance, potentiality and flexibility of the DBXII model. For these data, we compare the DBXII distribution, with BXII, Marshall-Olkin BXII (MOBXII), Topp Leone BXII (TLBXII), Zografos-Balakrishnan BXII (ZBBXII), Five Parameters beta BXII (FBBXII), BBXII, B exponentiated BXII (BEBXII), Five Parameters Kumaraswamy BXII (FKwBXII) and KwBXII distributions given in Afify et al. 2018, Yousof et al. 2018a,b and Altun et al. 2018a,b. For illustrating the importance of the new model we shall use data set called taxes revenue data {5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9,16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8}. The actual taxes revenue data (in 1000 million Egyptian pounds). We consider the following goodness-of-fit statistics: the Akaike information criterion (A-IC), Bayesian information criterion (B-IC), Hannan-Quinn information criterion (HQ-IC), consistent Akaike information criterion (CA-IC), where

$$A-IC=-2\,\ell\left(\widehat{\Theta}\right)+2k,$$

$$B-IC=-2\,\ell\left(\widehat{\Theta}\right)+k\log(n),$$

$$HQ-IC=-2\,\ell\left(\widehat{\Theta}\right)+2k\log[log(n)],$$

and

$$CA-IC=-2\,\ell\left(\widehat{\Theta}\right)+2kn(n-k-1),$$

where k is the number of parameters, n is the sample size and $-2\ell(\widehat{\Theta})$ is the maximized log-likelihood. Generally, the smaller these statistics are, the better the fit. Table 3 gives the maximum likelihood estimations (MLEs), standard errors (SEs) (in parentheses), confidence interval (CIs) (in parentheses) for taxes revenue data. Table 4 gives A-IC, B-IC, CA-IC and HQ-IC values for taxes revenue data. Based on the values in Tables 4 and Figure 2 the BXII-BXII model provides the best fits as compared to other BXII models with small values for B-IC, A-IC, CA-IC and HQ-IC.

Table 3
MLEs, SEs and CIs.

Model	$\widehat{\alpha}, \widehat{\beta}, \widehat{a}, \widehat{b}, \widehat{c}$
BXII	—,—, 5.615, 0.072,—
	—,—, (15.048), (0.194),—
	—,—, (0, 35.11), (0, 0.45),—
MOBXII	—,—, 8.017, 0.419, 70.359
	—,—, (22.083), (0.312), (63.831)
	—,—, (0, 51.29), (0, 1.03), (0, 195.47)
TLBXII	—,—, 91.320, 0.012, 141.073
	—,—, (15.071), (0.002), (70.028)
	—,—, (61.78,120.86) (0.008, 0.02) (3.82,278.33)
KwBXII	18.130, 6.857, 10.694, 0.081,—
	(3.689), (1.035), (1.166), (0.012),—
	(10.89,25.36), (4.83,8.89), (8.41,12.98), (0.06,0.10),—
BBXII	26.725, 9.756, 27.364, 0.020,—
	(9.465), (2.781), (12.351), (0.007),—
	(8.17,45.27), (4.31,15.21), (3.16,51.57), (0.006,0.03),—
BEBXII	2.924, 2.911, 3.270, 12.486, 0.371
	(0.564), (0.549), (1.251), (6.938), (0.788)
	(1.82,4.03), (1.83,3.99), (0.82,5.72), (0, 26.08), (0, 1.92)
FBBXII	30.441, 0.584, 1.089, 5.166, 7.862
	(91.745), (1.064), (1.021), (8.268), (15.036)
	(0, 210.26), (0, 2.67), (0, 3.09), (0, 21.37), (0, 37.33)
FKwBXII	12.878, 1.225, 1.665, 1.411, 3.732
	(3.442), (0.131), (0.034), (0.088), (1.172)
	(6.13,19.62), (0.97,1.48), (1.56,1.73), (1.24,1.58), (1.43,6.03),—
BXII-BXII	35.61, 0.374, 0.129, 0.72, 0.18
	(5.95), (0.08), (0.021), (0.014), (0.0001)
	(23.6, 47.6), (0.21, 0.53), (0.17, 0.09), (0.67,0.73), (0.1798, 0.1802)

Table 4
A-IC, B-IC, CA-IC and HQ-IC values for taxes revenue data.

Model	A-IC, B-IC, CA-IC, HQ-IC
BXII	518.46, 522.62, 518.67, 520.08
MOBXII	387.22, 389.38, 387.66, 389.68
TLBXII	385.94, 392.18, 386.38, 388.40
KwBXII	385.58, 393.90, 386.32, 388.86
BBXII	385.56, 394.10, 386.30, 389.10
BEBXII	387.04, 397.42, 388.17, 391.09
FBBXII	386.74, 397.14, 387.87, 390.84
FKwBXII	386.96, 397.36, 388.09, 391.06
BXII-BXII	**367.6, 368.75, 378, 371.7**

Figure 2 gives the estimated PDF, P-P plot, estimated HRF, estimated CDF and the Kaplan-Meier survival plot. Figure 2 show that the new model provides an adequate fit to the used data set.

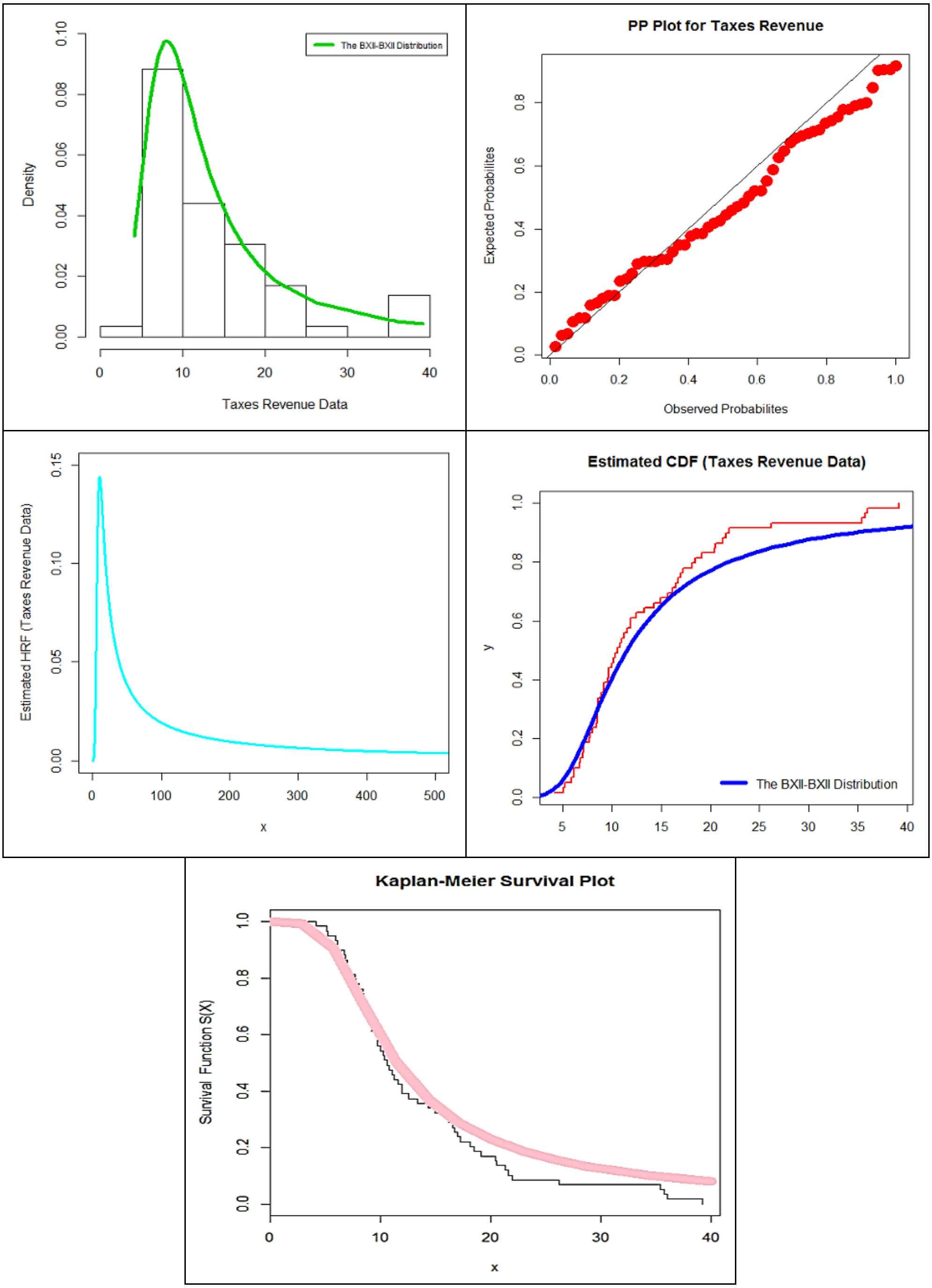

Figure 2: Estimated PDF, P-P Plot, estimated HRF, estimated CDF and Kaplan-Meier Survival Plot for Taxes Revenue Data.

6. CONCLUSIONS

In this work, we introduce a new continuous distribution called the Burr XII-Burr XII distribution. Some of its properties are derived. The method of maximum likelihood is used to estimate the unknown parameters. The new model provides adequate fits as compared to other related models with smallest values for A-IC, B-IC, CA-IC and HQ-IC. Certain characterization results are presented as well.

REFERENCES

1. Afify, A.Z., Cordeiro, G.M., Ortega, E.M.M. Yousof, H.M. and Butt, N.S. (2018). The four-parameter Burr XII distribution: properties, regression model and applications. *Communications in Statistics-Theory and Methods*, 47(11), 2605-2624.
2. Alizadeh, M., Altun, E., Ozel, G. (2017a). Odd Burr Power Lindley Distribution with Properties and Applications, Gazi University. *Journal of Science*, 30(3), 139-159.
3. Alizadeh, M., Cordeiro, G.M., Nascimento, A.D.C., Lima, M.C.S. and Ortega, E.G.G. (2017b). Odd-Burr generalized family of distributions with some applications. *Journal of Statistical Computation and Simulation*, 87(2), 367-389.
4. Altun, G., Alizadeh, M., Altun, E. and Ozel, G. (2017). Odd Burr Lindley Distribution with Properties and Applications. *Hacettepe Journal of Mathematics and Statistics*, 46(2), 255- 276.
5. Altun, E., Yousof, H.M. and Hamedani G.G. (2018a). A new log-location regression model with influence diagnostics and residual analysis. *Facta Universitatis, Series: Mathematics and Informatics*, 417-449. https://doi.org/10.22190/FUMI1803417A.
6. Altun, E., Yousof, H.M., Chakraborty, S. and Handique, L. (2018b). Zografos-Balakrishnan Burr XII distribution: regression modeling and applications. *International Journal of Mathematics and Statistics*, 19(3), 46-70.
7. Burr, I.W. (1942). Cumulative frequency functions. *Annals of Mathematical Statistics*, 13, 215-232.
8. Burr, I.W. (1968). On a general system of distributions, III. The simple range. *Journal of the American Statistical Association*, 63, 636-643.
9. Burr, I.W. (1973). Parameters for a general system of distributions to match a grid of 3 and 4. *Communications in Statistics*, 2, 1-21.
10. Burr, I.W. and Cislak, P.J. (1968). On a general system of distributions: I. Its curve-shaped characteristics; II. The sample median. *Journal of the American Statistical Association*, 63, 627-635.
11. Cordeiro, G.M., Yousof, H.M., Ramires, T.G. and Ortega, E.M.M. (2018). The Burr XII system of densities: properties, regression model and applications. *Journal of Statistical Computation and Simulation*, 88(3), 432-456.
12. Elderton, W.P. (1953). *Frequency Curves and Correlation*. 4th ed. Washington: Harren.
13. Elderton, W.P. and Johnson, N.L. (1969). *Frequency Curves and Correlation*. Cambridge, London: Cambridge University Press.
14. Glänzel, W. (1987). A characterization theorem based on truncated moments and its application to some distribution families. *Mathematical Statistics and Probability Theory* (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75-84.

15. Glänzel, W. (1990). Some consequences of a characterization theorem based on truncated moments. *Statistics: A Journal of Theoretical and Applied Statistics*, 21(4), 613-618.
16. Hamedani, G.G. (2013). On certain generalized gamma convolution distributions. *Technical Report*, No. 484, MSCS, Marquette University.
17. Gradshteyn, I.S. and Ryzhik, I.M. (2000). *Table of Integrals, Series and Products* (sixth edition). San Diego: Academic Press.
18. Paranaíba, P.F.P., Ortega, E.M.M., Cordeiro, G.M. and Pescim, R.R. (2011). The beta Burr XII distribution with application to lifetime data. *Computation Statistics and Data Analysis*, 55, 1118-1136.
19. Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I. (1986). *Integrals and Series*, 1. Gordon and Breach Science Publishers, Amsterdam.
20. Prudnikov, A.P., Brychkov, Y.A. and Marichev, O.I. (1992). *Integrals and Series*, 4. Gordon and Breach Science Publishers, Amsterdam.
21. Rodriguez, R.N. (1977). A guide to the Burr type XII distributions. *Biometrika*, 64, 129-134.
22. Tadikamalla, P.R. (1980). A look at the Burr and related distributions. *International Statistical Review*, 48, 337-344.
23. Yousof, H.M., Altun, E., Ramires, T.G., Alizadeh, M. and Rasekhi, M. (2018a). A new family of distributions with properties, regression models and applications. *Journal of Statistics and Management Systems*, 21, 163-188.
24. Yousof, H.M., Rasekhi, M., Altun, E., Alizadeh, M. Hamedani G.G. and Ali M.M. (2019). A new lifetime model with regression models, characterizations and applications. *Communications in Statistics-Simulation and Computation*, 48(1), 264-286.