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Relation between Zitterbewegung and the charge conductivity, Berry curvature, and the Chern number of multiband systems

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We show that the charge conductivity for impurity-free multiband electronic systems can be expressed in terms of the nondiagonal elements of the Zitterbewegung amplitudes while the Berry curvature and the Chern number is related only to the diagonal elements. Thus, the phenomenon of the Zitterbewegung can no longer be viewed just as an interesting consequence of quantum physics but it has also an experimental relevance. Moreover, through several examples we demonstrate how efficient our approach is in the analytical calculation of the charge conductivity.

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Introduction. The Zitterbewegung predicted originally by Schrödinger for Dirac electron is a "trembling" or in other words a rapid oscillatory motion of the center of the free wave packet for relativistic electron.[1] Most recently, Schliemann *et al.*[2] predicted the Zitterbewegung in spintronic systems, where the experimental observation of the effect is more realistic due to the much smaller frequency of the oscillatory motion. Thus, the Zitterbewegung can, in principle, be observed not only in the relativistic regime but for spintronic systems[2] and graphene[3] as well. This seminal paper has initiated many other works with an aim to demonstrate the appearance of the Zitterbewegung not only for Dirac electrons but for quasiparticles in condensed-matter physics (see references in our recent work on the general theory of Zitterbewegung,[4] where alternative expressions for the Zitterbewegung amplitudes and explicit forms of the position operator for several systems are presented).

In connection with graphene Katsnelson has already pointed out that the Zitterbewegung resulting in an oscillating term in the current operator is responsible for the nontrivial behavior of the conductivity at zero temperature and zero chemical potential.[5] In this Rapid Communication we show generally that the charge conductivity for a impurity-free multiband system is related to Zitterbewegung. Such a relation is expected since in the Kubo formula for the charge conductivity one needs to calculate the velocity operator in Heisenberg picture which includes an oscillatory term due to the Zitterbewegung in the case of multiband systems. Moreover, we clarify the relationship between Zitterbewegung and the Hall conductivity of insulators with noninteracting Bloch electrons.[6,7]

To show the subtle relation between the Zitterbewegung and the charge conductivity we start with a multiband system described by the most general matrix Hamiltonian in a Bloch wave-function basis: $H_{ab}(\mathbf{k})$, where $a,b=1,2,\ldots,N$ are the band indices (here N is the number of bands of the system). Here each matrix element $H_{ab}(\mathbf{k})$ is a differentiable function of the wave number $\mathbf{k}$ corresponding to the Bloch states. In Ref. 4 we calculated the time dependence of the position operator $\mathbf{x}(t)=e^{i/\hbar Ht}\mathbf{x}(0)e^{-i/\hbar Ht}$ of the quasiparticle by decomposing the Hamiltonian into a sum of projection operators: $H=\Sigma_a E_a(\mathbf{k})Q_a(\mathbf{k})$, where $E_a(\mathbf{k})$ are the distinct eigenvalues of the Hamiltonian at a given wave number $\mathbf{k}$, and $Q_a(\mathbf{k})$ are projection operators ($N\times N$ matrix operator) satisfying the usual relations: $Q_aQ_b=\delta_{ab}Q_a$ and $\Sigma_aQ_a=I_N$, where I_N is the $N\times N$ unit matrix. Note that $a=1,2,\ldots,s$, where $s\leq N$ (for degenerate case $s<N$). We found that the time dependence of the position operator $\mathbf{x}(t)$ in Heisenberg picture becomes

$$\mathbf{x}(t)=\mathbf{x}(0)+\mathbf{W}t+\sum_{a,b}\mathbf{Z}^{ab}(e^{i\omega_{ab}t}-1), \tag{1a}$$

$$\mathbf{W}=\frac{1}{\hbar}\sum_a\frac{\partial E_a(\mathbf{k})}{\partial\mathbf{k}}Q_a,\quad \mathbf{Z}^{ab}=iQ_a\frac{\partial Q_b}{\partial\mathbf{k}}, \tag{1b}$$

and $\omega_{ab}=\frac{E_a-E_b}{\hbar}$. Here $\mathbf{W}$ is the *drift velocity* and $\mathbf{Z}^{ab}$ are the *Zitterbewegung amplitudes*. This is a general result for the phenomenon of the Zitterbewegung (a simple derivation of the above result and explicit examples are presented in our recent work[4]). In what follows, these Zitterbewegung amplitudes $\mathbf{Z}^{ab}$ will play a crucial role in the charge conductivity and the Chern number.

Using the Kubo formula we show that the frequency-dependent charge conductivity (often called optical conductivity) of a multiband system can be expressed in terms of the Zitterbewegung amplitudes $\mathbf{Z}^{ab}$

$$\sigma_{ij}(\omega)=-\frac{e^2}{\hbar^2}\frac{1}{\omega}\lim_{\delta\to0}\mathrm{Im}\{\Pi_{ij}(\hbar\omega+i\delta)\}, \tag{2a}$$

$$\Pi_{ij}(i\nu_m)=-\frac{1}{V}\sum_{\mathbf{k}}\sum_{\substack{a,b\\a\neq b}}K_{ab}(i\nu_m)(E_a-E_b)^2\mathrm{Tr}(Z_i^{ab}Z_j^{ab\dagger}), \tag{2b}$$

$$K_{ab}(i\nu_m)=\frac{n_F(E_a-\mu)-n_F(E_b-\mu)}{i\nu_m+E_a-E_b}. \tag{2c}$$

Here the Zitterbewegung amplitudes Z_i^{ab} (i denotes the components x,y,z) are given by Eq. (1b), $\nu_m=2\pi m/\beta$ [m is an integer, $\beta=1/(k_BT)$] are the bosonic Matsubara's frequencies, $n_F(E)=1/(e^{\beta E}+1)$ is the usual Fermi distribution, μ is the Fermi energy, $\mathrm{Im}\{\cdot\}$ is the imaginary part of the argu-

ment, † stands for the conjugate transpose, and finally V is the volume of the sample. The trace is taken over the band indices. Note that $K_{ab}(z)=K_{ba}(-z)$ and $K_{aa}(z)=0$ for $z\neq 0$. Equation (2) is one of our central results in this Rapid Communication. Here we omit the calculation of the Drude peak, and focus on the interband contribution. One can see clearly from Eq. (2) that the Zitterbewegung amplitudes describing the interband interference manifest in the interband contribution of the charge conductivity. The Zitterbewegung amplitudes are indeed probed every time when one measures the conductivity of a scattering-free multiband system.

Derivation of Eq. (2). First consider the response function for operators A and B given by[7–10]

$$\tilde{\Pi}_{AB}(i\nu_m) = -\frac{1}{V}\int_0^\beta \langle \hat{T}A(\tau)B(0)\rangle e^{i\nu_m\tau}d\tau$$

$$= \frac{1}{V\beta}\sum_{\mathbf{k},n} \mathrm{Tr}[AG(\mathbf{k},i\omega_n+i\nu_m)BG(\mathbf{k},i\omega_n)], \quad (3)$$

where $G(\mathbf{k},z)=[z+\mu-H]^{-1}$ is the one-particle Green's function, $\hat{T}$ is the time-ordering operator, and $\omega_n=(2n+1)\pi/\beta$ are the fermionic Matsubara's frequencies (n is an integer). Using the projector decomposition of the Hamiltonian $H=\Sigma_a E_a Q_a$ the Green's function takes the following form: $G(\mathbf{k},z)=\Sigma_a \frac{Q_a}{z+\mu-E_a(\mathbf{k})}$. Then substituting this Green's function into Eq. (3) one can find

$$\tilde{\Pi}_{AB}(i\nu_m) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{a,b} K_{ba}(i\nu_m)\mathrm{Tr}(AQ_aBQ_b), \quad (4a)$$

$$K_{ab}(i\nu_m) = \frac{1}{\beta}\sum_n \frac{1}{i\omega_n+i\nu_m+\mu-E_b}\frac{1}{i\omega_n+\mu-E_a}$$

$$= \frac{n_F(E_a-\mu)-n_F(E_b-\mu)}{i\nu_m+E_a-E_b}, \quad (4b)$$

where in the last step of the calculation of the function $K_{ab}(i\nu_m)$ we have used the usual summation technics over the Matsubara's frequencies.[8]

Now using Eq. (4) the current-current correlation function with current operator $\mathbf{J}=\frac{\partial H}{\partial \mathbf{k}}$ (in units of $e/\hbar$, which is taken into account in the expression of the conductivity) reads

$$\Pi_{ij}(i\nu_m) \equiv \tilde{\Pi}_{J_iJ_j}(i\nu_m) = \frac{1}{V}\sum_{\mathbf{k}}\sum_{a,b} K_{ba}\,\mathrm{Tr}\left(\frac{\partial H}{\partial k_i}Q_a\frac{\partial H}{\partial k_j}Q_b\right). \quad (5)$$

Applying the well-known relations $Q_a\frac{\partial H}{\partial \mathbf{k}}Q_b=\delta_{ab}\frac{\partial E_a}{\partial \mathbf{k}}Q_a+(E_b-E_a)Q_a\frac{\partial Q_b}{\partial \mathbf{k}}$ and $Q_aQ_b=\delta_{ab}Q_a$, and the fact that $K_{aa}(z)=0$ for $z\neq 0$, it is easy to obtain the current-current response function Π_{ij} given by Eq. (2b). Finally, the conductivity in Eq. (2a) can be calculated by analytic continuation $i\nu_m \to \hbar\omega+i\delta$, where δ is a positive infinitesimal.

Berry curvature and the first Chern number. Now, we show that the first Chern number characterizing the Hall conductivity of insulators with noninteracting Bloch electrons[6,7] can also be expressed in terms of the Zitterbewegung amplitudes in Eq. (1b).

From Eqs. (2a) and (5) we find that the intrinsic Hall conductivity (here we focus on two-dimensional systems) for $\omega\to 0$ (dc conductivity, for $j\neq l$) is

$$\sigma_{jl}(\omega=0) = \frac{e^2}{h}\frac{1}{A}\sum_{\mathbf{k}}\sum_a n_a\Omega_{jl}^{(a)}(\mathbf{k}), \quad (6a)$$

$$\Omega_{jl}^{(a)}(\mathbf{k}) = 2\pi i\sum_{b\neq a}\frac{\mathrm{Tr}\left(\frac{\partial H}{\partial k_j}Q_a\frac{\partial H}{\partial k_l}Q_b\right)-\text{c.c.}}{(E_a-E_b)^2} \quad (6b)$$

$$= -2\pi i\,\mathrm{Tr}\left(Q_a\left[\frac{\partial Q_a}{\partial k_j},\frac{\partial Q_a}{\partial k_l}\right]\right), \quad (6c)$$

where $n_a=n_F[E_a(\mathbf{k})-\mu]$, $\Omega_{jl}^{(a)}(\mathbf{k})$ is the Berry curvature [Eq. (6b) can be casted to that derived by Thouless *et al.* in Ref. 11], A is the area of the sample and c.c. and $[\cdot,\cdot]$ stand for the complex conjugation and the commutator, respectively. To get Eq. (6c) we used again $Q_a\frac{\partial H}{\partial \mathbf{k}}Q_b=\delta_{ab}\frac{\partial E_a}{\partial \mathbf{k}}Q_a+(E_b-E_a)Q_a\frac{\partial Q_b}{\partial \mathbf{k}}$ and $Q_a+\Sigma_{b\neq a}Q_b=I_N$. Now replacing Q_a by $Q_a=Q_a^2$ in Eq. (6c) we obtain a very simple expression for the Berry curvature $\Omega_{jl}^{(a)}$ in terms of the Zitterbewegung amplitudes

$$\Omega_{jl}^{(a)}(\mathbf{k}) = 2\pi i\,\mathrm{Tr}([Z_j^{aa},Z_l^{aa\dagger}]). \quad (7)$$

It is interesting to note that in contrast to the Zitterbewegung, where only the nondiagonal elements $\mathbf{Z}^{ab}(a\neq b)$ appear in Eq. (1b), in the Hall effect only the diagonal ones $\mathbf{Z}^{aa}$ play the role. However, in the charge conductivity for finite frequencies given by Eq. (2) the nondiagonal elements $\mathbf{Z}^{ab}$ are present.

The Hall conductivity for band insulator in which the Fermi energy μ is located inside the energy gap between conduction and valence subbands and at zero temperature reads as $\sigma_H=\frac{e^2}{h}C_1$, where $C_1=1/(2A)\Sigma_{\mathbf{k}}\Sigma_{E_a<\mu}\varepsilon_{jl}\Omega_{jl}^{(a)}(\mathbf{k})$ is the first Chern number (here ε_{jl} is the fully antisymmetric tensor and the summation is assumed on indices j and l) and it can also be expressed with the Zitterbewegung amplitudes as

$$C_1 = -\frac{1}{2\pi i}\int d^2k \sum_{\substack{a\\ E_a<\mu}} \varepsilon_{jl}\,\mathrm{Tr}(Z_j^{aa}Z_l^{aa\dagger}). \quad (8)$$

Note that for the first Chern number one can obtained the same result starting from Eq. (3) given by Avron *et al.* in Ref. 12.

Important mathematical theorem. Now, we recall a less known mathematical theorem which enables us to calculate the projector operators $Q_a=|a\rangle\langle a|$ without calculating the eigenvectors $|a\rangle$ of the Hamiltonian H. Let H be an $N\times N$

Hermitian matrix with $s \leq N$ distinct eigenvalues, $E_a, \ldots, E_s$. Then the matrix H can be decomposed in terms of projector matrices as $H=\Sigma_a E_a Q_a$, where the projector matrix Q_a for $a=1,\ldots,s$ (in the mathematical literature called Frobenius covariant[13]) is given by

$$Q_a = \prod_{\substack{b=1 \\ b\neq a}}^{s} \frac{1}{E_a - E_b}(H - E_b I_N). \tag{9}$$

The proof of this theorem is based on the Cayley-Hamilton theorem.[13,14]

Applications. In the following we show a few examples how the phenomenon of the Zitterbewegung is related to the charge conductivity or the Chern number for specific multiband systems.

(i) Consider the most general noninteracting two-band model[7,9] with Hamiltonian

$$H = \varepsilon(\mathbf{k}) I_2 + \mathbf{h}(\mathbf{k})\boldsymbol{\sigma}, \tag{10}$$

where I_2 is the two by two unit matrix in spin or pseudospin space, and the system is characterized by the one-particle energy dispersion $\varepsilon(\mathbf{k})$ and an effective magnetic field $\mathbf{h}$ depending on the wave number $\mathbf{k}$ while $\boldsymbol{\sigma}=(\sigma_x,\sigma_y,\sigma_z)$ is a vector formed from the Pauli matrices. The two eigenvalues are $E_\pm(\mathbf{k})=\varepsilon(\mathbf{k})\pm h$, where $h=\sqrt{\mathbf{h}^2}$, and the two corresponding projectors obtained from Eq. (9) are $Q_\pm=\frac{1}{2}(I_2 \pm \hat{\mathbf{h}}\boldsymbol{\sigma})$, where $\hat{\mathbf{h}}=\mathbf{h}/h$ is a unit vector. The conductivity for impurity-free samples can be obtained from Eq. (2) and after a little algebra the response function becomes

$$\Pi_{ij}(i\nu_m) = -\frac{2}{V}\sum_{\mathbf{k}} \frac{n_+ - n_-}{(i\nu_m)^2 - (2h)^2} h^2 \times \left(\nu_m \epsilon_{\alpha\beta\gamma} \frac{\partial \hat{h}_\alpha}{\partial k_i}\frac{\partial \hat{h}_\beta}{\partial k_j}\hat{h}_\gamma + 2h \frac{\partial \hat{h}_\alpha}{\partial k_i}\frac{\partial \hat{h}_\alpha}{\partial k_j} \right), \tag{11}$$

where $n_\pm = n_F(E_\pm - \mu)$ and we sum on any repeated index. For two-dimensional samples this result agrees with that obtained by Bernevig.[9]

(ii) For a two-band model with Hamiltonian (10) using Eq. (8) we easily find the first Chern number: $C_1 = -\frac{1}{4\pi}\int d^2\mathbf{k}\ \hat{\mathbf{h}}\cdot\left(\frac{\partial \hat{\mathbf{h}}}{\partial k_x}\times\frac{\partial \hat{\mathbf{h}}}{\partial k_y}\right)$, which is a well-known result.[6,7]

(iii) We now consider the Luttinger-type systems[6,15] for which the Hamiltonian is given by

$$H = \frac{\hbar^2}{2m}\left[\left(\gamma_1 + \frac{5}{2}\gamma_2\right)\mathbf{k}^2 - 2\gamma_2(\mathbf{kS})^2\right], \tag{12}$$

where $\mathbf{k}=(k_x,k_y,k_z)$ is the wave number and $\mathbf{S}=(S_x,S_y,S_z)$ represents the spin operator with spin 3/2 while $\gamma_{1,2}$ and m are parameters of the model. The Hamiltonian can be expressed in terms of the projection operators Q_+ and Q_- as[6]

$$H = E_-(\mathbf{k})Q_-(\mathbf{k}) + E_+(\mathbf{k})Q_+(\mathbf{k}), \tag{13a}$$

$$Q_+(\mathbf{k}) = \frac{9}{8}I_4 - \frac{1}{2\mathbf{k}^2}(\mathbf{kS})^2, \tag{13b}$$

$$Q_-(\mathbf{k}) = I_4 - Q_+(\mathbf{k}), \tag{13c}$$

where I_4 is the 4×4 unit matrix, and the double degenerate eigenvalues are $E_\pm(\mathbf{k})=\frac{\gamma_1 \pm 2\gamma_2}{2m}(\hbar\mathbf{k})^2$ corresponding to the light-hole (+) and the heavy-hole (−) bands. The projection operators $Q_\pm$ can be obtained from Eq. (9). In earlier calculations of the response function Π_{ij} the SO(5) Clifford algebra has been invoked.[6,9] In our approach the response function can be obtained without using the Clifford algebra. Indeed, it is easy to calculate the current-current response function using Eqs. (1b) and (2b) or Eq. (5), and the commutation relations $[S_j,S_k]=i\varepsilon_{jkl}\hbar S_l$ for the spin operator $\mathbf{S}$. The Zitterbewegung amplitudes have already been calculated in Ref. 4. After some algebra we have

$$\Pi_{ij}(i\nu_m) = \frac{24\hbar^6\gamma_2^3}{m^3(2\pi)^3}\int d^3k \frac{(n_+ - n_-)(k^2\delta_{ij} - k_i k_j)}{\left(\frac{2\gamma_2\hbar^2k^2}{m}\right)^2 - (i\nu_m)^2}, \tag{14}$$

where $i,j=x,y,z$ and we used $\Sigma_{\mathbf{k}} \to V\int \frac{d^3k}{(2\pi)^3}$. The integration over the polar angles of $\mathbf{k}$ can be done analytically and at zero temperature we have the same result as that, e.g., in Ref. 9.

(iv) Consider the spin-orbit interaction in two-dimensional electron gas in a fully symmetric quantum well investigated recently by Bernardes *et al.*[16] The Hamiltonian of this system is given by

$$H = \begin{pmatrix} \frac{\hbar^2\mathbf{k}^2}{2m} + \varepsilon_e & -i\eta k_- & 0 & 0 \\ i\eta k_+ & \frac{\hbar^2\mathbf{k}^2}{2m} + \varepsilon_o & 0 & 0 \\ 0 & 0 & \frac{\hbar^2\mathbf{k}^2}{2m} + \varepsilon_o & -i\eta k_- \\ 0 & 0 & i\eta k_+ & \frac{\hbar^2\mathbf{k}^2}{2m} + \varepsilon_e \end{pmatrix}, \tag{15}$$

where $k_\pm = k_x \pm ik_y$ while $\varepsilon_{e,o}$, η, and m are parameters of the model. The two double degenerate eigenvalues are $E_\pm = \varepsilon_{\mathbf{k}} \pm f_k$, where $\varepsilon_{\mathbf{k}}=\hbar^2\mathbf{k}^2/(2m)+\varepsilon_+$, $f_k=\sqrt{\varepsilon_-^2+\mathbf{k}^2\eta^2}$, $\varepsilon_\pm = (\varepsilon_e \pm \varepsilon_o)/2$. Again one can show that $H=E_+Q_+ + E_-Q_-$, where the two projection operators obtained from Eq. (9) are

$$Q_\pm = \frac{1}{2f_k}\begin{pmatrix} h_k^\mp & \mp i\eta k_- & 0 & 0 \\ \pm i\eta k_+ & h_k^\pm & 0 & 0 \\ 0 & 0 & h_k^\pm & \mp i\eta k_- \\ 0 & 0 & \pm i\eta k_+ & h_k^\mp \end{pmatrix} \tag{16}$$

and $h_k^\pm = \pm\varepsilon_- + f_k$. To calculate the current-current response function we again use Eqs. (1b) and (2). After some algebra we have

$$\Pi_{ij}(i\nu_m) = 2\eta^2 \int \frac{d^2k}{(2\pi)^2} \frac{(n_+ - n_-)(f_k^2 \delta_{ij} - k_i k_j \eta^2)}{f_k \left[f_k^2 - \left(\frac{i\nu_m}{2} \right)^2 \right]}, \quad (17)$$

where $i,j=x,y$. Performing the integration over the polar angle of $\mathbf{k}$ one can see that $\Pi_{ij}(i\nu_m)$ is a diagonal matrix. To our best knowledge this result is distinct in the literature. The charge conductivity can be obtained from Eq. (2a) and it will be published elsewhere.

(v) For single-layer graphene the most general Hamiltonian in tight-binding approximation can be given by a 2×2 matrix in which the diagonal elements H_{AA} and H_{BB} include second, fourth, etc., neighbor hopping terms while the off-diagonal elements H_{AB} and H_{BA} contain the first, third, etc., hopping terms.[17] Thus, the Hamiltonian can be mapped to Eq. (10). Note that the same is true even for a strained graphene. The response function can be obtained from Eq. (11). In particular, taking into account only first-nearest neighbors, we have $H_{AA}=H_{BB}=\varepsilon_0$ and $H_{AB}=H_{BA}^*=f(\mathbf{k})$, where $f(\mathbf{k})=\gamma_0(1+e^{-i\mathbf{k}\mathbf{a}_1}+e^{-i\mathbf{k}\mathbf{a}_2})$, and γ_0 and ε_0 are parameters of the model, and $\mathbf{a}_1$ and $\mathbf{a}_2$ are the unit vectors of the unit cell in the honeycomb lattice.[18] The two eigenvalues of the Hamiltonian are $E_\pm=\varepsilon_0 \pm |f(\mathbf{k})|$ and $\hat{\mathbf{h}}=(\mathrm{Re}\{f(\mathbf{k})\}, -\mathrm{Im}\{f(\mathbf{k})\},0)/|f(\mathbf{k})|$. Using Eq. (11) the current-current response function is

$$\Pi_{ij}(i\nu_m) = -\frac{1}{A}\sum_{\mathbf{k}} \frac{n_+ - n_-}{|f(\mathbf{k})|^2 - \left(\frac{i\nu_m}{2} \right)^2} F_{ij}(\mathbf{k}), \quad (18a)$$

$$F_{ij}(\mathbf{k}) = \frac{\mathrm{Im}\left\{ f^* \frac{\partial f}{\partial k_i} \right\} \mathrm{Im}\left\{ f^* \frac{\partial f}{\partial k_j} \right\}}{|f(\mathbf{k})|}, \quad (18b)$$

where the summation in $\mathbf{k}$ is over the entire Brillouin zone of the honeycomb lattice, A is the area of the sample, $\mathrm{Re}\{\cdot\}$ is the real part of the argument, and * denotes the complex conjugation. Note that Eq. (18) is valid not only in the usual Dirac cone approximation. Our result agrees with that obtained by Zhang *et al.*[19] and by Yuan *et al.*[20] However, according to our numerical calculations it slightly differs from that obtained by Stauber *et al.*[21] for the high-frequency region.

(vi) Finally, we give at least one example in which the projector decomposition of the Hamiltonian involves not only two projector operators (as in the previous cases) but four projector operators. Such a system is, e.g., the bilayer graphene.[22,23] Our general framework presented in this work for calculating the charge conductivity can also be applied to bilayer graphene. The Zitterbewegung amplitudes have already been given in Ref. 4 for bilayer graphene excluding trigonal warping. Using this result we obtained the same analytical expression for the frequency-dependent optical conductivity as that by Nicol and Carbotte in Ref. 24 using the spectral-function representation of the Green's function. We would like to stress that our approach for calculating the optical conductivity is a convenient and very efficient method even for more complex systems. The study of strained bilayer graphene with/without trigonal warping is in progress.

Conclusions. In this Rapid Communication we derived an explicit expression for the charge conductivity, the Berry curvature, and the Chern number in terms of the Zitterbewegung amplitudes. Our results show that the Zitterbewegung is not just an interesting phenomenon in quantum physics but it is closely related to measurable effect.

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