Core Percolation on Complex Networks

Yang-Yu Liu,[1,2,*] Endre Csóka,[3] Haijun Zhou,[4] and Márton Pósfai[1,5,6]

[1]Center for Complex Network Research and Department of Physics, Northeastern University, Boston, Massachusetts 02115, USA
[2]Center for Cancer Systems Biology, Dana-Farber Cancer Institute, Boston, Massachusetts 02115, USA
[3]Eötvös Loránd University, H-1053 Budapest, Hungary
[4]State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
[5]Department of Theoretical Physics, Budapest University of Technology and Economics, H-1521 Budapest, Hungary
[6]Department of Physics of Complex Systems, Eötvös Loránd University, H-1053 Budapest, Hungary

(Received 1 August 2012; published 14 November 2012)

We analytically solve the core percolation problem for complex networks with arbitrary degree distributions. We find that purely scale-free networks have no core for any degree exponents. We show that for undirected networks if core percolation occurs then it is continuous while for directed networks it is discontinuous (and hybrid) if the in- and out-degree distributions differ. We also find that core percolations on undirected and directed networks have completely different critical exponents associated with their critical singularities.

DOI: 10.1103/PhysRevLett.109.205703 PACS numbers: 64.60.aq, 64.60.ah

In the last decade structural transitions in complex networks were extensively studied [1–12] and found to affect many network properties, e.g., robustness and resilience to breakdowns [4,13], cascading failure in interdependent networks [14,15], and epidemic or information spreading on sociotechnical systems [16,17]. Recent work on network controllability reveals another interesting interplay between the structural and dynamical properties of complex networks [18,19]. It was found that the robustness of network controllability is closely related to the presence of the *core* in the network [18]. Historically, core percolation has been related to a wide range of important problems in random graphs, including combinatorial optimizations, e.g., maximum matching (a maximum set of edges without common vertices) [20–22], and minimum vertex cover (a minimum set of vertices containing at least one ending vertex of each edge of the graph) [23–28], and conductor-insulator transitions, where the adjacency matrix of the graph is used as a Hamiltonian describing the hopping of electrons from one node to another via edges [29].

The core of an undirected network is the remainder of the greedy leaf removal (GLR) procedure: leaves (nodes of degree one) and their neighbors are removed iteratively from the network. GLR was originally introduced to study the size of maximum matchings in the classical Erdős-Rényi (ER) random graphs [20]. Previous theoretical studies of core percolation also focused on ER random graphs and found that the core emerges at the critical mean degree $c^* = e = 2.7182818\ldots$ [20,30]. More interestingly, it was suggested that for the minimum vertex cover problem, which is one of the basic NP-complete problems [31], core percolation coincides with the changes of the solution-space structure and an "easy-hard transition" of the typical computational complexity [23–26]. In the conductor-insulator transition problem, it was found that core percolation coincides with the spectral singularity of the hopping Hamiltonian $\mathcal{H}$ of ER random graphs; i.e., the average height of the delta peak at zero energy in the spectrum of $\mathcal{H}$ is nonanalytical at $c = e$ [29].

With a generalized GLR procedure, core percolation in directed networks was numerically studied and found to coincide with the sudden decrease of the fraction of redundant edges, which can be safely removed without affecting our ability to control the linear time-invariant dynamics $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$ on a directed network $\mathcal{G}$ [18]. Here, $\mathbf{x}(t) \in \mathbb{R}^N$ captures the state of each node at time t. The state matrix $\mathbf{A} \in \mathbb{R}^{N\times N}$ describes the weighted wiring diagram of $\mathcal{G}$. The input matrix $\mathbf{B} \in \mathbb{R}^{N\times M}$ identifies the nodes that are controlled by the input vector $\mathbf{u}(t) \in \mathbb{R}^M$ ($M \le N$). Finding the minimum set of driver nodes, whose time-dependent control can guide the whole network $\mathcal{G}$ to any desired final state in finite time, is equivalent to finding the maximum matchings of $\mathcal{G}$, which can be efficiently calculated in $\mathcal{G}$'s bipartite representation $\mathcal{B}$ [18]. After the core develops, the number of maximum matchings starts increasing exponentially [18,22], implying the fraction of redundant edges decreases (because they do not belong to any maximum matchings).

Despite the intriguing relation between core percolation and other important problems, we lack a general theory of core percolation. Previous theoretical studies of core percolation have focused on ER random graphs with Poisson degree distribution. Yet, many real-world networks have scale-free or fat-tailed degree distributions. It is still unknown how degree distributions will affect core percolation.

To systematically study core percolation on random graphs with arbitrary degree distributions, we first categorize the nodes into four distinct categories according to how they can be removed during the GLR procedure: (i) α removable: nodes that can become isolated without

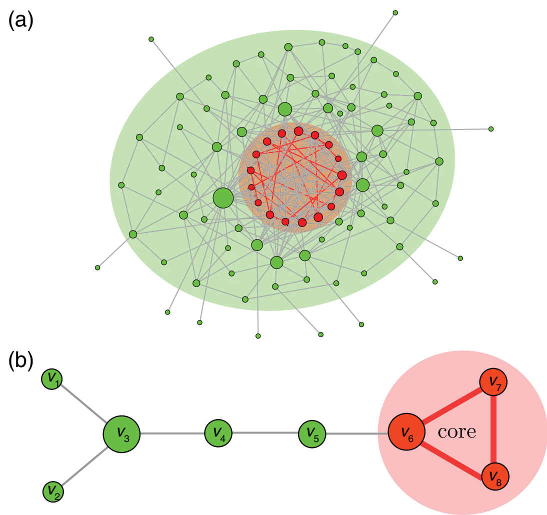

FIG. 1 (color online). The core of small networks. (a) The core (highlighted in dark red) obtained after the GLR procedure is fundamentally different from the two-core (highlighted in light green) obtained by iteratively removing nodes of degree less than 2. (b) Removal categories of nodes. Red nodes are nonremovable; i.e., they belong to the core (with shaded background). Green nodes are removable: nodes v_1 and v_2 are α removable; nodes v_3 and v_5 are β removable; node v_4 is γ removable.

directly removing themselves [e.g., v_1 and v_2 in Fig. 1(b)]; (ii) β removable: nodes that can become a neighbor of a leaf [e.g., v_3 and v_5 in Fig. 1(b)]; (iii) γ removable: nodes that can become leaves but are neither α nor β removable [e.g., v_4 in Fig. 1(b)]; (iv) nonremovable: nodes that cannot be removed and hence belong to the core [e.g., v_6, v_7 and v_8 in Fig. 1(b)]. Note that the core is independent of the order the leaves are removed [30]. Now we consider a large uncorrelated random network $\mathcal{G}$ with arbitrary degree distribution $P(k)$ [5,32]. Assuming that in each removable category the removal of a random node can be made locally (see Supplemental Material [33]), we can determine the category of a node v in $\mathcal{G}$ by the categories of its neighbors in $\mathcal{G} \setminus v$, i.e., the subgraph of $\mathcal{G}$ with node v and all its edges removed, using the following rules: (i) α removable: all neighbors are β removable; (ii) β removable: at least one neighbor is α removable; (iii) nonremovable: no neighbor is α removable, and at least two neighbors are not β removable. (We omit the rule for γ removable nodes because it is not useful to determine the core's size.) Let α (or β) denote the probability that a random neighbor of a random node v in a network $\mathcal{G}$ is α removable (or β removable) in $\mathcal{G} \setminus v$. Rules (i) and (ii) enable us to derive two self-consistent equations

$$\alpha = \sum_{k=1}^{\infty} Q(k)\beta^{k-1} = A(1-\beta), \qquad (1)$$

$$1-\beta = \sum_{k=1}^{\infty} Q(k)(1-\alpha)^{k-1} = A(\alpha), \qquad (2)$$

where $Q(k) \equiv kP(k)/c$ is the degree distribution for the node at a random end of a randomly chosen edge, $c \equiv \sum_{k=0}^{\infty} kP(k)$ is the mean degree, and $A(x) \equiv \sum_{k=0}^{\infty} Q(k+1)(1-x)^k$. Note that $Q(k+1)$ is also called the excess degree distribution [5,34] and $A(1-x)$ is its generating function. The derivations of Eqs. (1) and (2) follow the generating function formalism developed in the theory of random graphs with arbitrary degree distributions [5]. Equations (1) and (2) indicate that α satisfies $x = A(A(x))$. It can be shown that α is the smallest fix point of $A(A(x))$, i.e., the smallest root of the function $f(x) \equiv A(A(x)) - x$ [33]. By invoking rule (iii) we can calculate the normalized core size ($n_{\text{core}} \equiv N_{\text{core}}/N$)

$$n_{\text{core}} = \sum_{k=0}^{\infty} P(k) \sum_{s=2}^{k} \binom{k}{s} \beta^{k-s}(1-\beta-\alpha)^s, \qquad (3)$$

which can be simplied to $n_{\text{core}} = G(1-\alpha) - G(\beta) - c(1-\beta-\alpha)\alpha$, where $G(x) \equiv \sum_{k=0}^{\infty} P(k)x^k$ is the generating function of $P(k)$. The normalized number of edges in the core ($l_{\text{core}} \equiv L_{\text{core}}/N$) can also be calculated: $l_{\text{core}} = \frac{c}{2}(1-\alpha-\beta)^2$ [33]. Clearly, both $n_{\text{core}} > 0$ and $l_{\text{core}} > 0$ if and only if $1-\beta-\alpha > 0$.

The above approach can be readily generalized for directed networks with given in- and out-degree distributions [denoted by $P^-(k)$ and $P^+(k)$, respectively]. We first generalize the GLR procedure for directed networks. Motivated by the relation between maximum matchings and controllability of directed networks [18], we transform a directed network $\mathcal{G}$ to its bipartite graph representation $\mathcal{B}$ by splitting each node v into two nodes v^+ (upper) and v^- (lower), and we connect v_1^+ to v_2^- in $\mathcal{B}$ if there is an edge $(v_1 \to v_2)$ in $\mathcal{G}$. The core of $\mathcal{G}$ can then be defined as the core of $\mathcal{B}$ obtained by applying GLR to $\mathcal{B}$ as if $\mathcal{B}$ is a unipartite undirected graph and node v_i belongs to the core of $\mathcal{G}$ provided that either v_i^+ or v_i^- belongs to the core of $\mathcal{B}$. Let c denote the mean degree of each partition in $\mathcal{B}$, i.e., the mean in-degree (or out-degree) of $\mathcal{G}$. Define $Q^{\pm}(k) \equiv kP^{\pm}(k)/c$, which is the degree distribution of the upper or lower partition of a random edge in $\mathcal{B}$. Define $A^{\pm}(x) \equiv \sum_{k=0}^{\infty} Q^{\pm}(k+1)(1-x)^k$. Then the same argument as we used in the undirected case yields

$$\alpha^{\pm} = A^{\pm}(1-\beta^{\mp}), \qquad (4)$$

$$1-\beta^{\pm} = A^{\pm}(\alpha^{\mp}), \qquad (5)$$

and $\alpha^{\pm}$ is the smallest fix point of $A^{\pm}(A^{\mp}(x))$. The normalized core size for each partition in $\mathcal{B}$ is

$$n_{\text{core}}^{\pm} = \sum_{k=0}^{\infty} P^{\pm}(k) \sum_{s=2}^{k} \binom{k}{s} (\beta^{\mp})^{k-s}(1-\beta^{\mp}-\alpha^{\mp})^s \qquad (6)$$

and the normalized core size of $\mathcal{G}$ is $n_{\text{core}} = (n_{\text{core}}^+ + n_{\text{core}}^-)/2$. The normalized number of edges in the core is given by $l_{\text{core}} = c(1-\alpha^+-\beta^+)(1-\alpha^--\beta^-)$. We confirmed the above analytical results with extensive numerical calculations [33].

Except for some special cases (e.g., no cycles or no leaves), no rules were proposed to predict the core's existence for general complex networks. Our framework, established in this Letter, allows us to derive the condition for core percolation. Since the core in an undirected network with degree distribution $P(k)$ is similar to a directed network with the same out- and in-degree distributions, i.e., $P^+(k) = P^-(k) = P(k)$, we can deal with directed networks without loss of generality. As $n_{\rm core}$ is a continuous function of $\alpha^\pm$, we focus on $\alpha^\pm$, which is the smallest root of the function $f^\pm(x) \equiv A^\pm(A^\mp(x)) - x$. There are several interesting facts about $f^\pm(x)$. First, since $A^\pm(x)$ is a monotonically decreasing function for $x \in [0, 1]$ and $A^\pm(0) = 1$ is the maximum (see Figs. 2 and 3), we have $f^\pm(0) > 0$ and $f^\pm(1) < 0$ [see Figs. 3(c) and 3(d)]. Consequently, the number of roots of $f^\pm(x)$ in [0, 1] is odd (including multiplicity). Numerical calculations suggest that the number of roots is 1 or 3 (see Figs. 2 and 3). Second, if $f^\pm(x_0) = 0$ then $f^\mp(A^\mp(x_0)) = 0$, which means $A^\mp(x)$ transforms the roots of $f^\pm(x)$ to the roots of $f^\mp(x)$. This also suggests that $f^\pm(x)$ *always* has a trivial root $\alpha^\pm = A^\pm(\alpha^\mp) = 1 - \beta^\pm$. $A^\mp(x)$ is a monotonically decreasing function and $\alpha^\pm$ is the smallest root of $f^\pm(x)$, $A^\mp(\alpha^\pm) = 1 - \beta^\mp$ is therefore the largest root of $f^\mp(x)$. Hence $1 - \beta^\pm - \alpha^\pm$ is the difference between the largest and the smallest roots of $f^\pm(x)$ (see Fig. 2). Consequently, if $f^\pm(x)$ has only one root (which must be the trivial root $\alpha^\pm = A^\pm(\alpha^\mp) = 1 - \beta^\pm$), then $1 - \beta^\pm - \alpha^\pm = 0$ and there is no core according to Eq. (6). If multiple different roots coexist then $1 - \beta^\pm - \alpha^\pm > 0$ and the core emerges.

We apply the above condition to a series of random undirected networks with specific degree distributions [33]. Surprisingly, we find that for purely scale-free (SF) networks with $P(k) = k^{-\gamma}/\zeta(\gamma)$ and $\zeta(\gamma)$ the Riemann ζ function, the core does not exist for any $\gamma > 2$. While for asymptotically SF networks generated by the static model with $P(k) \sim k^{-\gamma}$ for large k [35,36], the core develops when the mean degree c is larger than a threshold value c^*.

Hereafter, we systematically study the net effect of increasing mean degree c on core percolation. ER networks and the asymptotically SF networks generated by the static model naturally serve this purpose, since their mean degree c is an independent and explicitly tunable parameter. We observe that if the mean degree c is small, then $f^\pm(x)$ has one root, but if c is large, $f^\pm(x)$ has three roots [see Figs. 3(c) and 3(d)]. At the critical point $c = c^*$, the number of roots jumps from 1 to 3; one new root with multiplicity 2 appears. According to the transformation from the roots of $f^\pm(x)$ to the roots of $f^\mp(x)$ through $A^\mp(x)$, for either $f^+(x)$ or $f^-(x)$ its new root at $c = c^*$ is smaller than its original root, and for either $f^-(x)$ or $f^+(x)$ the new root at $c = c^*$ is larger than the original root, or there is a degenerate case when this new root is the same as the original root for both $f^+(x)$ and $f^-(x)$. For example, for directed asymptotically SF networks generated by the static model with $\gamma_{\rm in} = 2.7$, $\gamma_{\rm out} = 3.0$, the new root (marked as green dot) of $f^+(x)$ at $c = c^*$ is smaller than the original root (green square) of $f^+(x)$ [see Fig. 3(c)], and the new root (green solid square) of $f^-(x)$ at $c = c^*$ is

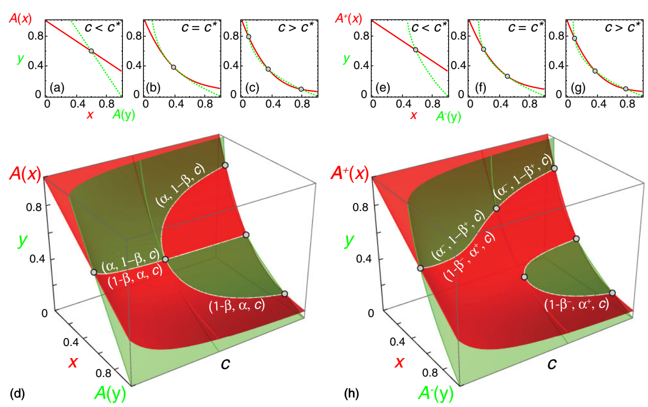

FIG. 2 (color online). Graphical solution of the self-consistent equations. By plotting $A^+(x)$ vs x (red solid line) and y vs $A^-(y)$ (green dashed line) in the same coordinate system, the x or y coordinate of the two curves' intersection points give the solutions of $f^-(x) = 0$ or $f^+(x) = 0$, respectively. Plotting the two curves as a function of c yields two surfaces and the intersection curve gives the solutions of $f^\pm(x) = 0$. For $c < c^*$, the intersection curve has one branch given by $(\alpha^-, 1 - \beta^+, c) = (1 - \beta^-, \alpha^+, c)$. For $c > c^*$, it has three branches with the top and bottom branches given by $(\alpha^-, 1 - \beta^+, c)$ and $(1 - \beta^-, \alpha^+, c)$, respectively. The above description for directed networks (e–g) can be similarly reproduced for undirected networks (a–d).

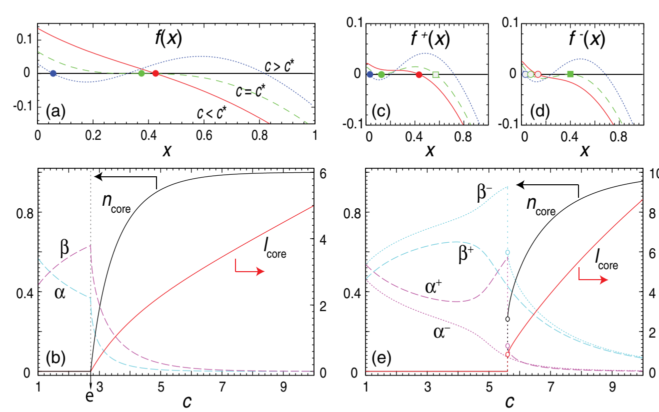

FIG. 3 (color online). Analytical solution of core percolation. (a–b) Undirected Erdős-Rényi (ER) random networks. (a) α is the smallest root of the function $f(x)$. (b) α, β, $n_{\rm core}$ and $l_{\rm core}$ as functions of c. (c–e) Directed asymptotically scale-free (SF) random networks generated by the static model. Both the in-degree and out-degree distributions are asymptotically scale-free with degree exponents $\gamma_{\rm in} = 2.7$ and $\gamma_{\rm out} = 3.0$. (c–d) $\alpha^\pm$ is the smallest root of the function $f^\pm(x)$. Note that $f^\pm(x)$ cannot immediately intersect the x axis at two new points, but it touches first. (e) $\alpha^\pm$, $\beta^\pm$, $n_{\rm core}$, and $l_{\rm core}$ as functions of c. The jumps in α^+ and β^- result in the jumps in $n_{\rm core}$ and $l_{\rm core}$.

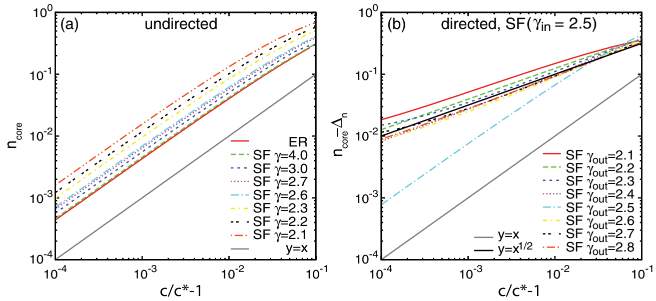

FIG. 4 (color online). Critical behavior of core percolation. The critical exponent η (or η') can be read from the slope of the curve $n_{\rm core} - \Delta_{\rm n} \sim (c - c^*)^\eta$ (or $n_{\rm core} \sim (c - c^*)^{\eta'}$) in the critical regime. (a) $\eta' = 1$ for undirected Erdős-Rényi (ER) random networks and asymptotically scale-free (SF) random networks generated by the static model. (b) For directed asymptotically SF random networks with $\gamma_{\rm in} = 2.5$, $\eta = 1/2$ if $\gamma_{\rm out} \neq \gamma_{\rm in}$; $\eta' = 1$ if $\gamma_{\rm out} = \gamma_{\rm in}$.

larger than the original root (green circle) of $f^-(x)$ [see Fig. 3(d)]. In other words, at the critical point, for either $f^+(x)$ or $f^-(x)$, its smallest two roots are the same, and for either $f^-(x)$ or $f^+(x)$ its largest two roots are the same. While for the degenerate case $[P^+(k) = P^-(k) = P(k)]$ we have $f^+(x) = f^-(x) = f(x)$ and the new root of $f(x)$ at $c = c^*$ has to be the same as the the original root of $f(x)$, indicating all three roots must be the same [see Fig. 3(a)]. Therefore, at the critical point, except in the degenerate case, α^+ together with β^- (or α^- together with β^+) decrease discontinuously. To sum up, in the degenerate case $[P^+(k) = P^-(k) = P(k)]$ core percolation is continuous, but for the nondegenerate case $[P^+(k) \neq P^-(k)]$ we have a discontinuous transition in both $n_{\rm core}$ and $l_{\rm core}$ [see Figs. 3(b) and 3(e)].

We can further show that in the nondegenerate case, core percolation is actually a *hybrid* phase transition [7,37,38]. More specifically, $n_{\rm core}$ (or $l_{\rm core}$) has a jump at the critical point as a first-order phase transition but also has a critical singularity as a continuous transition [33]. We find that in the critical regime $\epsilon = c - c^* \to 0^+$, $n_{\rm core} - \Delta_{\rm n} \sim (c - c^*)^\eta$ and $l_{\rm core} - \Delta_{\rm l} \sim (c - c^*)^\theta$ with the critical exponents $\eta = \theta = \frac{1}{2}$ and $\Delta_{\rm n}$ (or $\Delta_{\rm l}$) presents the discontinuity in $n_{\rm core}$ (or $l_{\rm core}$) at c^*. Interestingly, in the degenerate case, one has a continuous phase transition ($\Delta_{\rm n} = \Delta_{\rm l} = 0$) but with a completely different set of critical exponents [30]: $\eta' = \theta' = 1$ (see Fig. 4).

The results presented here vividly illustrate that core percolation is a fundamental structural transition in complex networks. Its implication on other problems (e.g., combinatorial optimizations, conductor-insulator transition, and network controllability) deserves further exploration. The analytical framework developed here also raises a number of questions, answers to which would further improve our understanding of core percolation on complex networks. For example, we find directed real-world networks usually have larger core sizes than our theoretical predictions based on uncorrelated random networks [33]. We leave the systematic studies of the effects of higher order correlations as future work.

We thank B. A. W. Brinkman, N. Azimi, C. Song, G. Tsekenis, and Y. Li for discussions. This work was supported by NS-CTA sponsored by US Army Research Laboratory under Agreement No. W911NF-09-2-0053; DARPA under Agreement No. 11645021; DTRA award WMD BRBAA07-J-2-0035; the generous support of Lockheed Martin; MTA Rényi "Lendület" Groups and Graphs Research Group; ERC Advanced Research Grant No. 227701 and KTIA-OTKA Grant No. 77780; and NSFC Grant No. 11121403. All authors have contributed equally to this work.

*Corresponding author.
ya.liu@neu.edu

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[3] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. **80**, 1275 (2008).
[4] D. S. Callaway, M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. Lett. **85**, 5468 (2000).
[5] M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
[6] B. Pittel, J. Spencer, and N. Wormald, J. Comb. Theory Ser. A **67**, 111 (1996).
[7] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **96**, 040601 (2006).
[8] G. Palla, I. Dernyi, I. Farkas, and T. Vicsek, Nature (London) **435**, 814 (2005).
[9] I. Derényi, G. Palla, and T. Vicsek, Phys. Rev. Lett. **94**, 160202 (2005).
[10] D. Achlioptas, R. M. D'Souza, and J. Spencer, Science **323**, 1453 (2009).
[11] R. A. da Costa, S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Phys. Rev. Lett. **105**, 255701 (2010).
[12] O. Riordan and L. Warnke, Science **333**, 322 (2011).
[13] R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
[14] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature (London) **464**, 1025 (2010).
[15] R. Parshani, S. V. Buldyrev, and S. Havlin, Phys. Rev. Lett. **105**, 048701 (2010).
[16] R. Pastor-Satorras and A. Vespignani, Phys. Rev. Lett. **86**, 3200 (2001).
[17] M. Kitsak, L. K. Gallos, S. Havlin, F. Liljeros, L. Muchnik, H. E. Stanley, and H. A. Makse, Nat. Phys. **6**, 888 (2010).
[18] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabási, Nature (London) **473**, 167 (2011).
[19] T. Nepusz and T. Vicsek, Nat. Phys. **8**, 568 (2012).
[20] R. M. Karp and M. Sipser, *Proceedings of the 22nd Annual IEEE Symposium on Foundations of Computer Science* p. 364 (Wiley-IEEE Computer Society Press, Hoboken, NJ, 1981).

[21] H. Zhou and Z. Ou-Yang, arXiv:cond-mat/0309348v1 [Europhys. Lett. (to be published)].
[22] L. Zdeborová and M. Mézard, J. Stat. Mech. 2006, P05003.
[23] M. Weigt and A. K. Hartmann, Phys. Rev. Lett. **84**, 6118 (2000).
[24] A. K. Hartmann and M. Weigt, J. Phys. A **36**, 11 069 (2003).
[25] W. Barthel and A. K. Hartmann, Phys. Rev. E **70**, 066120 (2004).
[26] A. K. Hartmann, A. Mann, and W. Radenbach, J. Phys. Conf. Ser. **95**, 012011 (2008).
[27] H. Zhou, Eur. Phys. J. B **32**, 265 (2003).
[28] A. K. Hartmann and M. Weigt, *Phase Transitions in Combinatorial Optimization Problems* (Wiley-VCH Verlag GmbH & Co. KGaA, Weinheim, 2005).
[29] M. Bauer and O. Golinelli, Phys. Rev. Lett. **86**, 2621 (2001).
[30] M. Bauer and O. Golinelli, Eur. Phys. J. B **24**, 339 (2001).
[31] M. Garey and D. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness* (W. H. Freeman, New York, 1979).
[32] M. Molloy and B. Reed, Random Struct. Algorithms **6**, 161 (1995).
[33] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevLett.109.205703 for proof or detailed discussion.
[34] M. E. J. Newman, Phys. Rev. E **76**, 045101 (2007).
[35] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[36] M. Catanzaro and R. Pastor-Satorras, Eur. Phys. J. B **44**, 241 (2005).
[37] G. Parisi and T. Rizzo, Phys. Rev. E **78**, 022101 (2008).
[38] A. V. Goltsev, S. N. Dorogovtsev, and J. F. F. Mendes, Phys. Rev. E **73**, 056101 (2006).