An ideal toy model for confining, walking and conformal gauge theories: the $O(3)$ sigma model with ϑ-term

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ABSTRACT: A toy model is proposed for four dimensional non-abelian gauge theories coupled to a large number of fermionic degrees of freedom. As the number of flavors is varied the gauge theory may be confining, walking or conformal. The toy model mimicking this feature is the two dimensional $O(3)$ sigma model with a ϑ-term. For all ϑ the model is asymptotically free. For small ϑ the model is confining in the infra red, for $\vartheta = \pi$ the model has a non-trivial infra red fixed point and consequently for ϑ slightly below π the coupling walks. The first step in investigating the notoriously difficult systematic effects of the gauge theory in the toy model is to establish non-perturbatively that the ϑ parameter is actually a relevant coupling. This is done by showing that there exist quantities that are entirely given by the total topological charge and are well defined in the continuum limit and are non-zero, despite the fact that the topological susceptibility is divergent. More precisely it is established that the differences of connected correlation functions of the topological charge (the cumulants) are finite and non-zero and consequently there is only a single divergent parameter in $Z(\vartheta)$ but otherwise it is finite. This divergent constant can be removed by an appropriate counter term rendering the theory completely finite even at $\vartheta > 0$.

KEYWORDS: Lattice Gauge Field Theories, Sigma Models

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Contents

1 Introduction

Lattice simulations of technicolor inspired models are plagued by known systematic uncertainties [1–5]. Although the models under consideration are QCD-like in that they are four dimensional non-abelian gauge theories coupled to dynamical fermions the systematic effects of the interesting models (those that are either conformal or walking) are much more difficult to control than in actual QCD. As a result currently there are disagreements between various approaches, discretizations, etc, and universality is not immediately evident [6–10]. Clearly the general expectation is that once all systematic effects are controlled and taken into account the results from different approaches and regularizations will agree as they should.

In this paper a toy model is proposed which mimics many of the features of non-abelian gauge theories in the hope that systematic effects can be fully explored. Hopefully these will help controlling the corresponding effects in the much more complicated gauge theories. The proposed model is the two dimensional $O(3)$ non-linear sigma model with a ϑ term. At $\vartheta = 0$ the model served as a toy model of QCD for a long time since it is asymptotically free, features instantons, confinement and dimensional transmutation [11]. It is exactly solvable [12] even at finite volume [13–15]. Since the topological term is invisible in perturbation theory the model is asymptotically free for arbitrary ϑ. The dynamics in the infra red is however expected to be very sensitive to ϑ.

At $\vartheta = \pi$ the model is conjectured [16, 17] to have a non-trivial infra red fixed point governed by the SU(2) WZNW model at level $k = 1$ and, if the conjecture holds, is also exactly solvable. Some numerical evidence in support of the conjecture has been presented in [18] and a recent very detailed study confirming it in [19]. The infra red fixed point implies a zero of the β-function. This situation is analogous to gauge theories in the conformal window.

For $0 < \vartheta < \pi$ exact solvability is lost but based on continuity one expects that for ϑ not much below π the β-function develops a near zero and the renormalized coupling will walk. This arrangement is analogous to gauge theories just below the conformal window. Hence dialing ϑ corresponds to dialing the number of flavors N_f in the gauge theory.

In all three scenarios (confining, walking, conformal) one may also introduce an external magnetic field to mimic the effect of a finite quark mass.

Before exploring the analogies further and investigating the origins of the severe systematic effects the first task is to establish non-perturbatively that the ϑ-term is actually a relevant operator and also what the singularity structure of the theory is for $\vartheta > 0$. This is not immediately obvious largely because of the unusual scaling properties of the topological susceptibility and a class of similar observables.

It is well known that small size instantons render the topological susceptibility $\chi = \langle Q^2 \rangle / V$ ill defined in the semi-classical approximation [20]. Going beyond the semi-classical approximation fully non-perturbative lattice studies have shown that regardless how one improves the details of the lattice implementation a logarithmically divergent susceptibility is obtained at finite physical volume in the continuum limit. Moreover, all even moments of the total topological charge distribution $\langle Q^{2m} \rangle / V$ have the same property.

However, the model at $\vartheta = 0$ is exactly solvable and both the exact solution and the continuum limit of lattice simulations agree that correlators of the topological charge density, e.g. $\langle q(x)q(0) \rangle$ are finite. The above two observations, namely that certain statistical properties of the total charge distribution $P(Q)$ are ill defined while at the same time correlators of $q(x)$ are finite, might make one wonder whether the total charge operator Q is an irrelevant operator while $q(x)$ is not. If so, the only consistent continuum value of $\langle Q^{2m} \rangle$ would be zero and the apparent divergences in the lattice calculations would be regarded as artifacts. This scenario would imply that the theory defined on the lattice at non-zero ϑ leads to an identical continuum theory as the one defined at $\vartheta = 0$. Equivalently, the total charge operator inserted into any correlation function would be zero in the continuum theory $\langle Q \ldots \rangle = 0$, while correlation functions of the type $\langle q(x) \ldots \rangle$ are finite. This scenario would of course invalidate Haldane's conjecture about the equivalence of the $\vartheta = \pi$ theory with a non-trivial interacting conformal field theory.

In this work it is shown that there exist quantities built out of the total topological charge operator Q which have well defined continuum limits and are non-zero. These observables are differences of connected correlation functions of the topological charge, in other words the cumulants. Each term is logarithmically divergent but the divergence cancels in the difference and moreover they scale correctly in the continuum limit to non-zero values. Showing correct scaling towards the continuum limit in itself would not be sufficient to prove that the ϑ-term is a relevant operator because the continuum limit value could be zero. Since all cumulant differences are finite there is only a single UV-divergent parameter in the partition function $Z(\vartheta)$ but otherwise it is finite.

While preparing this manuscript the preprint [19] appeared also with the conclusion that ϑ is a relevant coupling. The method was different though, in [19] it was shown to high precision that a well defined observable is different in the continuum limit for three different values of ϑ implying that ϑ can not be irrelevant. In the current work all simulations are carried out at $\vartheta = 0$ and the same conclusion is reached by showing that certain combinations of the topological charge operator are non-zero in the continuum.

2 $O(3)$ sigma model with a ϑ-term

The model in Euclidean continuum notation is defined by the action

$$S = \frac{1}{2g_0^2} \int d^2x \partial_\mu s_a \partial_\mu s_a \tag{2.1}$$

for the unit 3-vectors s, $s_1^2 + s_2^2 + s_3^2 = 1$, where g_0 is the bare coupling. Only a torus geometry will be considered corresponding to a box of finite linear size L which will be regularized by a symmetric lattice.

The corresponding partition function, free energy per unit volume and topological charge distribution of the model at non-zero ϑ and volume V is given by

$$Z(\vartheta) = \langle e^{i\vartheta Q} \rangle = e^{-Vf(\vartheta)} = \sum_Q P(Q)e^{i\vartheta Q} \ , \tag{2.2}$$

with the normalization $Z(0) = \sum_Q P(Q) = 1$. Since physics is periodic with period 2π in ϑ and $\vartheta \to -\vartheta$ is a symmetry the free energy per unit volume can be Fourier expanded

$$f(\vartheta) = \sum_{n=1}^{\infty} (1 - \cos(n\vartheta)) f_n \ . \tag{2.3}$$

It has been pointed out in [21] that in the semi-classical or dilute gas approximation all f_n coefficients vanish except for f_1 which is UV divergent due to instantons of size $a \ll \rho \ll \xi$ where a is the lattice cut-off and ξ is the physical correlation length. The remaining coefficients come from interactions between instantons. Semi-classical arguments also suggest that for instantons causing the UV divergence in f_1 the ratio between their size and their average separation goes to zero in the continuum limit. This would imply that the interactions responsible for the $f_{n>1}$ coefficients are small in the continuum limit hence will not cause them to diverge.

To summarize, the semi-classical approximation accounts for a UV divergent f_1 and finite $f_{n>1}$ coefficients. A suitable way of addressing whether this statement is true beyond the semi-classical approximation is to consider observables that can be expressed by the $f_{n>1}$ coefficients only and calculating them fully non-perturbatively. The simplest choice is to take the connected correlation functions of the topological charge,

$$\chi_{2m} = (-1)^{m+1} \left. \frac{d^{2m}f}{d\vartheta^{2m}} \right|_{\vartheta=0} \tag{2.4}$$

and consider their differences,

$$\Delta\chi_{2m} = \chi_{2m} - \chi_{2m+2} = \sum_{n=2}^{\infty} f_n n^{2m}(1 - n^2) \tag{2.5}$$

from which f_1 drops out. The first few such correlation functions are

$$\begin{aligned} \chi_2 &= \frac{\langle Q^2 \rangle}{V} \\ \chi_4 &= \frac{\langle Q^4 \rangle - 3\langle Q^2 \rangle^2}{V} \\ \chi_6 &= \frac{\langle Q^6 \rangle - 15\langle Q^4 \rangle\langle Q^2 \rangle + 30\langle Q^2 \rangle^3}{V} \ . \end{aligned} \tag{2.6}$$

All of these are expected to diverge in the continuum limit but their differences are expected to be finite. Some numerical evidence has been presented in [21] in favor of correct scaling behavior for $\Delta\chi_2$ but whether the continuum value is zero or non-zero has not been discussed.

In the following it will be shown to high precision that the expectations from the semi-classical analysis indeed hold non-perturbatively and all moments $\langle Q^{2m}\rangle$ and all cumulants χ_{2m} are logarithmically divergent but the differences $\Delta\chi_{2m}$ are finite. This implies that there is a single ill-defined constant in $f(\vartheta)$ namely f_1 but otherwise it is finite. The constant f_1 can be removed by an appropriate renormalization condition leading to a finite and universal free energy and partition function for arbitrary ϑ.

3 Numerical simulation

It is convenient to take the continuum limit on a symmetric periodic lattice L^2 of fixed physical volume. Physical length and mass is defined by the second moment correlation length ξ_2 [22],

$$\frac{1}{\xi_2(L)^2} = \left(\frac{\sin\frac{\pi a}{L}}{\frac{\pi a}{L}}\right)^2 \left(2\frac{M_0}{M_2} - \frac{4\pi^2}{L^2}\right) \tag{3.1}$$

where

$$M_{2n} = \left(\frac{L}{2\pi}\right)^{2n} \sum_t \left(2\sin\frac{\pi t}{L}\right)^{2n} C(t) \tag{3.2}$$

is given in terms of the zero spatial momentum projection of the 2-point correlation function $C(t) = \sum_x \langle s_a(t,x) s_a(0,0)\rangle$ of the field s. Let us introduce $m(L) = 1/\xi_2(L)$. Note that in this notation $m(L)$ is not the mass gap in finite volume but rather is simply defined as the inverse of ξ_2 (which for $L\to\infty$ agrees with the mass gap but not for finite L). The physical volume is fixed to $m(L)L = 4$. A novel topological lattice action [23–25] is used for the simulations,

$$S = \sum_{\langle i,j\rangle} S(s_i, s_j) \tag{3.3}$$

where the sum is over all neighboring sites and

$$S(s_i, s_j) = \begin{cases} 0 & \text{if } s_i \cdot s_j > \cos\delta \\ \infty & \text{otherwise} \end{cases} \tag{3.4}$$

In other words the action is zero for two neighboring vectors if their relative angle is smaller than δ and infinite otherwise. The continuum limit is taken by tuning the bare coupling δ towards zero. This action is topological because small perturbations of the field s do not change the action nevertheless it has been shown that it is in the right universality class [25].

If $\delta < \pi/2$ powerful improvements exist for the measurement of the topological charge distribution [18] based on a generalization of the usual cluster algorithms [26, 27]. The

L/a	δ/π	$m(L)L$	$L^2\chi_2$	$L^2\chi_4$	$L^2\chi_6$	$L^2\Delta\chi_2$	$L^2\Delta\chi_4$
60	0.48490	4.0017(14)	1.2957(2)	0.8812(8)	-0.019(5)	0.4145(8)	1.069(5)
80	0.47260	4.0032(19)	1.4651(2)	1.0292(8)	-0.011(6)	0.4359(7)	1.143(6)
100	0.46370	4.0007(19)	1.6018(3)	1.1512(9)	-0.035(8)	0.4507(9)	1.186(7)
120	0.45680	3.9939(20)	1.7155(3)	1.257(1)	0.033(9)	0.459(1)	1.224(8)
160	0.44680	4.0011(14)	1.9214(4)	1.444(1)	0.16(1)	0.477(1)	1.28(1)
200	0.43950	4.0015(17)	2.0836(3)	1.596(1)	0.24(1)	0.488(1)	1.35(1)
240	0.43385	3.9998(14)	2.2208(3)	1.729(1)	0.40(1)	0.492(1)	1.33(1)
320	0.42545	4.0010(17)	2.4476(4)	1.946(1)	0.57(1)	0.502(1)	1.37(1)
400	0.41930	3.9983(14)	2.6259(4)	2.118(2)	0.71(2)	0.508(2)	1.41(2)
480	0.41455	4.0014(19)	2.7845(4)	2.274(2)	0.88(2)	0.511(2)	1.39(2)
640	0.40740	4.0021(18)	3.0347(4)	2.521(2)	1.07(3)	0.514(2)	1.45(3)
800	0.40210	3.9952(19)	3.2221(3)	2.704(2)	1.20(3)	0.518(2)	1.50(3)

Table 1. Results for the first few cumulants and their differences for fixed physical volume $m(L)L = 4$. The bare parameters δ are taken from [25].

topological charge operator from [28] is used assigning an integer charge to each configuration even at finite lattice spacing.

The continuum extrapolation of the cumulant differences will be done through 12 lattice spacings using the parameter values from [25] listed in table 1. The measured correlation lengths and topological susceptibilities are in agreement with those in [25]. In the present work $O(10^8)$ configurations were generated at each volume and every 10th was measured for the topological charge distribution and correlation length. The large number of configurations was necessary because there are huge cancellations between the various terms in the difference of cumulants, especially for $\Delta\chi_4$. The third difference, $\Delta\chi_6$, was already impossible to obtain with the current statistics.

The results for the cumulant differences $\Delta\chi_2$ and $\Delta\chi_4$ are shown on figure 1. Obtaining continuum estimates is not entirely trivial since the precise form of the leading and sub leading cut-off effects is not known a priori. Using the results of [29, 30] one may expect the leading corrections to be $O((a/L)^2)$ with possibly large logarithmic corrections. Fits of the form

$$C + (a/L)^2 \left(\sum_{j=n}^{m} A_j \log^j(L/a) \right) \tag{3.5}$$

with $(n, m) = (0, 3), (1, 3), (2, 3), (0, 2)$ all work quite well with χ^2/dof values close to unity for $\Delta\chi_2$ and slightly higher, around 1.8 for $\Delta\chi_4$. The continuum extrapolated values agree in both cases among the four fit function choices and the four curves lie almost entirely on
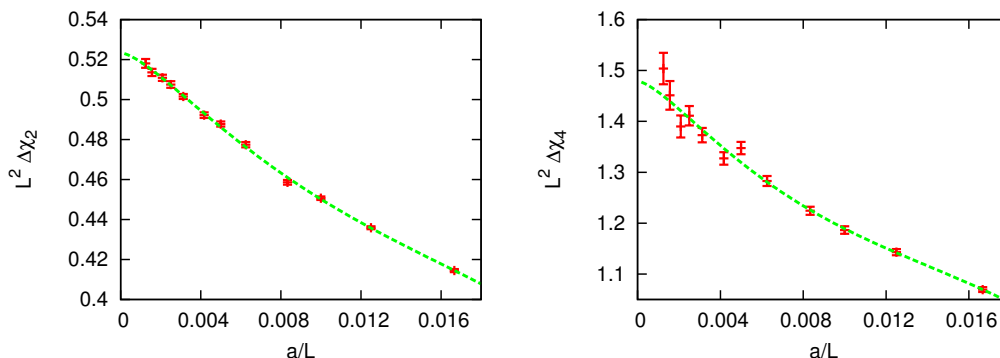
Figure 1. Continuum extrapolation for the first two cumulant differences multiplied by the volume, $L^2 \Delta \chi_2$ and $L^2 \Delta \chi_4$.

top of each other. In both cases the $(n, m) = (0, 2)$ choice is shown on the plots leading to continuum estimates $C = 0.523(2)$ and $1.48(2)$ for $L^2 \Delta \chi_2$ and $L^2 \Delta \chi_4$, respectively. Clearly, both values are non-zero.

4 Summary and conclusion

It has been known for a long time that the topological susceptibility in the two dimensional $O(3)$ model is ill-defined in the continuum. Consequently the topological charge distribution $P(Q)$ does not have a finite continuum limit. The semi-classical analysis predicts precisely what part of $P(Q)$ is actually divergent and what part of it is finite. In this work non-perturbative evidence has been presented supporting the semi-classical result. The only divergent quantity is the first Fourier coefficient of the free energy density,

$$f_1 = - \int_0^\pi f(\vartheta) \cos(\vartheta) \frac{d\vartheta}{2\pi} \, , \tag{4.1}$$

while the remaining part $\sum_{n>1}(1 - \cos(n\vartheta))f_n$ is finite and non-zero. Hence the quantity

$$f_R(\vartheta) = f(\vartheta) - (1 - \cos(\vartheta))f_1 \tag{4.2}$$

is finite and universal and one may consider the subtraction an additive renormalization. Similarly the renormalized partition function $Z_R(\vartheta) = \exp(-V f_R(\vartheta))$ is finite and universal and related to the bare partition function by a multiplicative renormalization. Instead of subtracting f_1 it is sufficient to subtract only its divergent piece. The logarithmic singularity is expected to be volume independent.[1] Let us then denote this singular quantity by f_{1s}. Since $\chi_2 = f_1 + \sum_{n>1} n^2 f_n$ a suitable definition of f_{1s} is the logarithmic singularity in the topological susceptibility which can directly be measured in lattice calculations. A natural renormalization procedure is then the following: one defines the theory for non-zero ϑ by the action

$$S(\vartheta) = S(\vartheta = 0) - i\vartheta Q - (1 - \cos(\vartheta))V f_{1s} \tag{4.3}$$

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[1]I thank Ferenc Niedermayer for pointing this out.

and all resulting correlation functions related to topology (i.e. derivatives with respect to ϑ) become finite. The last term in the full action above is a non-perturbatively generated counter term. It is important to note that the above renormalization does not mean that ϑ itself gets renormalized, the bare ϑ is still a physical quantity which does not require renormalization. It would of course be very interesting to check the volume independence of f_{1s} in lattice simulations.

The finite quantities $f_{n>1}$ and $\Delta\chi_{2m}$ are not volume independent and are non-trivial functions of $z = m(L)L$. Since the model is exactly solvable at $\vartheta = 0$ it would be interesting to derive the first few cumulant differences $\Delta\chi_{2m}(z)$ from the exact solution or at least their value in the infinite volume limit.

In any case the finite and non-zero cumulant differences naturally lead to the conclusion that ϑ is a relevant coupling of the theory and the total topological charge operator Q is a relevant operator despite the ill-defined nature of the moments $\langle Q^{2m}\rangle$.

The original motivation was the study of a toy model mimicking confining, walking and conformal behavior in four dimensional gauge theories in order to study the severe systematic effects of the latter. It was proposed that increasing ϑ is analogous to increasing the number of flavors N_f because as ϑ goes from zero to π the model goes from confining to walking and to conformal. In the toy model a suitable renormalized coupling is $g_R^2(L) = m(L)L$ which would then run with the finite volume L. A necessary condition for this analogy to hold was establishing precisely the divergence structure of the partition function at non-zero ϑ.

A particular difficulty of the gauge theory calculation can also be studied in the toy model. It is very difficult to distinguish numerically the following two cases: the theory with zero quark mass just below the conformal window and the theory with a small but non-zero quark mass just inside the conformal window. Both theories walk, the former for the usual reason of being just below the conformal window while the latter because even though it would be conformal for zero quark mass, the non-zero mass drives the coupling away from the would-be fixed point as soon as the running scale goes below the massive fermionic states. This phenomenon can be mimicked in the toy model by considering it at zero external magnetic field and $\vartheta = \pi - \varepsilon$ and also at a small but non-zero external magnetic field and $\vartheta = \pi$. Both theories are expected to walk and it would be interesting to explore in the toy model what intrinsic features are different despite the similar behavior of the walking coupling constant.

There are a couple of differences between the toy model and gauge theory though. Less important is the fact that while ϑ does not enter the perturbative β-function, N_f does. More significant is the fact that due to the $\vartheta \to -\vartheta$ symmetry and periodicity by 2π the two values $\vartheta - \varepsilon$ and $\vartheta + \varepsilon$ lead to the same continuum theory and it does not have an infra red fixed point (for non-zero ε the coupling walks). This means that the zero of the β-function at $\vartheta = \pi$ is eliminated by arbitrary perturbations of ϑ meaning that this zero is a second order zero, unlike in the gauge theory where generically the zero is expected to be first order and is preserved by small perturbations. Hence the $\vartheta = \pi$ model is really analogous to a gauge theory which is exactly at the lower edge of the conformal window. It would be interesting to find a simple toy model which possesses all essential features and

in addition the infra red fixed point is a first order zero of the β-function and disappears by joining with a non-trivial UV fixed point as expected in gauge theory [31–33].

Acknowledgments

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