Observability of nonlinear time–delay systems and its application to their state realization

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***Abstract*—In this paper, it is shown that the two notions of weak observability and strong observability may not be sufficient to describe the link between the input/output equation associated to the behaviour of a system and its state space realization. A new notion, called regular observability, is introduced, which is shown to capture essential features of nonlinear time delay systems and the existence of some realization.**

***Index Terms*—Nonlinear time–delay systems, observability, realization problem.**

I. Introduction

Weak observability and strong observability are well established notions for both linear and nonlinear time delay systems. They have been widely used to address the observer design problem which was investigated in [7], [8], [11], [12] and [15] between the others.

In this paper a third notion, denoted regular observability, is introduced, which is motivated by the search of a state realization derived from a higher order input-output delay differential equation. This new notion of regular observability appears to be stronger than weak observability and weaker than strong observability, and it is shown to play a key role in the search of a state realization for nonlinear time delay systems. The latter is far from being granted and does not rely only on properness conditions as it is instead the case for standard delay free linear systems.

The conditions under which a state realization exists are investigated in this paper. Early (and partial) results on the realization of delay-free nonlinear systems can be found in [5], [13]. On one hand, the most advanced results on the existence of a state realization for non linear time-delay systems are reported in [10] where necessary and sufficient conditions are given for the existence of a retarded type state space realization for a retarded type input-output equation. On the other hand, it is known [1] that the state elimination in a retarded type state space may yield a neutral type input-output equation of the same order, or a higher order retarded type input-output equation as shown in [9].

Thus, open problems include the following.

- Under which conditions does there exist a state space realization for a given neutral type input-output equation?
- Which is the minimal realization for a nonlinear time-delay system?

The new notion of regular observability is introduced in Section II where its connection with the realization problem is discussed. Some preliminary results are derived in Section III on the realization problem and state elimination for linear time-delay systems as well as on state elimination in the nonlinear case. Section IV concerns the main results. Some conclusions are given in Section V.

II. Three notions of observability

Consider a nonlinear system affected by constant commensurate delays, which can be described by the equations

$$\begin{aligned} \dot{x}(t) &= f(x(t),\cdots,x(t-s\tau),u(t),\cdots,u(t-s\tau)) \\ y(t) &= h(x(t),\cdots,x(t-s\tau)) \end{aligned} \tag{1}$$

with $\tau = const.$, $x(t) \in R^n$ and $y(t) \in R$, $f(\cdot)$ and $h(\cdot)$ meromorphic functions in their arguments. For notational simplicity we will assume in the following that $\tau = 1$.

In the sequel we will consider the notations and algebraic approach introduced in [14]. To take into account the link between the delayed variables, we need to consider the backward shift operator δ. Let $\mathcal{K}$ denote the field of causal meromorphic functions $f(x(t),\cdots,x(t-s\tau),u(t),\cdots,u(t-s\tau))$, with $s \in I\!N$. Given a function $\gamma(x(t),\cdots,x(t-j\tau)) \in \mathcal{K}$, $\gamma(-1)$ denotes the function shifted by τ, that is $\gamma(-1) := \gamma(x(t-\tau),\cdots,x(t-j\tau-\tau))$. Let $dx(t)$ denote the differential of x. Then, thanks to the back shift operator δ, $dx(t-s\tau) = \delta^s dx$. Accordingly, given the function

$$y(t) = h(x(t),\cdots,x(t-s\tau))$$

its differential form $dy(t) = \sum_{j=0}^{s} \frac{\partial h}{\partial x(t-j\tau)} dx(t-j\tau)$ can be written in concise form as

$$dy = \left[\frac{\partial h}{\partial x(t)} + \cdots + \frac{\partial h}{\partial x(t-s\tau)}\delta^s\right] dx.$$

Given $a(\cdot), f(\cdot)) \in \mathcal{K}$:

$$\delta[\,a(\cdot)\,df(\cdot)] \quad = \quad a(-1)\,df(-1).$$

Finally $\mathcal{K}(\delta]$ is the (left) ring of non commutative polynomials in δ with coefficients in $\mathcal{K}$. A general module spanned by the differentials of function in $\mathcal{K}$ is then defined over the ring $\mathcal{K}(\delta]$ [14]. In particular, the following filtration of accessibility submodules is defined: setting $\mathcal{H}_1 = \text{span}_{\mathcal{K}(\delta]}\{dx\}$, iteratively $\mathcal{H}_{i+1} = \{\omega \in \mathcal{H}_i | \dot{\omega} \in \mathcal{H}_i\}$, for $i \geq 1$, where ω denotes a one-form that is $\omega = a(x,\delta)dx$. In the special case of a delay free linear time invariant system $\dot{x} = Ax + Bu$, the module $\mathcal{H}_2$ reduces to the left annihilator of B, the module $\mathcal{H}_3$ is the left annihilator of $[B\ AB]$ and the limit $\mathcal{H}_\infty$ reduces to the left annihilator of the controllability matrix. This accessibility filtration is derived in Section IV for time–delay systems in a slightly more general case.

A. Weak Observability and Strong Observability

Weak and strong observability were introduced in the delay context to describe the possibility of reconstructing the state starting from the measure of the output and its derivatives. It was shown already in the linear case, that two notions were necessary due to a pathology which may arise for the presence of the delay. More precisely, the following definitions are given.

Definition 1 (Weak observability): System (1) with $x(t) \in R^n$ is said to be weakly observable if, setting

$$\begin{pmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(n-1)} \end{pmatrix} = \mathcal{O}(\cdot,\delta)dx + \mathcal{G}(\cdot,\delta) \begin{pmatrix} du \\ d\dot{u} \\ \vdots \\ du^{(n-2)} \end{pmatrix},$$

the observability matrix $\mathcal{O}(x, u, \cdots, u^{(n-2)}, \delta)$ has rank n over $K(\delta]$.

Proposition 1: System (1) with $x(t) \in R^n$ is weakly observable if and only if

$$rank\,(\mathrm{span}_{\mathcal{K}(\delta]}\{dx\} \cap \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-2)}\}) = n$$

Definition 2 (Strong observability): System (1) with $x(t) \in R^n$ is said to be strongly observable if the observability matrix $\mathcal{O}$ is unimodular, that is it has an inverse polynomial matrix in δ.

Proposition 2: System (1) with $x(t) \in R^n$ is strongly observable if and only if

$$dx \subset \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-2)}\}$$

Obviously, strong observability yields weak observability.

Example 1: Consider the system taken from [6] (recall that $\tau = 1$)

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & x(t-1)u(t) \\ y(t) & = & x(t) + x(t-1) \end{array} \right. \tag{2}$$

Compute the observability matrix, which is

$$(\,dy\,) = (1+\delta)\mathrm{d}x = \mathcal{O}(x,\delta)dx$$

It is easily seen that the given system is weakly observable, but not strongly observable, since $1+\delta$ does not have a polynomial inverse.

Remark. Note that the state elimination of a linear retarded type system always yields a retarded type input-output equation of the same order as shown in [9]. This is no more the case for nonlinear systems as shown by Example 1. In fact in this case one gets that, due to (2)

$$y(-1) = x(-1) + x(-2) \tag{3}$$
$$y(-2) = x(-2) + x(-3) \tag{4}$$

and accordingly,

$$\dot{y} = x(-1)u + x(-2)u(-1) \tag{5}$$
$$\dot{y}(-1) = x(-2)u(-1) + x(-3)u(-2). \tag{6}$$

Using respectively equations (4) and (6) in order to eliminate $x(-3)$, and equations (3) and (5) to eliminate $x(-2)$, one first gets

$$\dot{y}(-1) = x(-2)(u(-1) - u(-2)) + y(-2)u(-2) \tag{7}$$
$$\dot{y} = x(-1)(u - u(-1)) + y(-1)u(-1) \tag{8}$$

so that using (3)

$$\dot{y}(-1) = (y(-1) - x(-1))(u(-1) - u(-2)) + y(-2)u(-2),$$

and finally for $u \neq u(-1)$

$$(u - u(-1))\dot{y}(-1) - (u - u(-1))y(-2)u(-2) + (\dot{y} - y(-1)u)(u(-1) - u(-2)) = 0, \tag{9}$$

which is a first order neutral input–output differential equation.

B. A new notion of Observability

The weak observability and strong observability properties are important since they allow to distinguish between the case in which the state of the system at time t can be expressed or not as a causal function of the output, its derivatives up to order $n-1$, the input and its derivatives up to order $n-2$ and their delays.

With reference to Example 1, it is easily seen that, since the system is not strongly observable, then one cannot express $x(t)$ as a function of $y(t)$ and its delays.

Example 1 shows however that the definitions of strong and weak observability are not exhaustive if one aims at understanding if the state can be recovered as a combination of the output variables, input variables and their derivatives up to some order, eventually delayed.

In fact an easy elimination of $x(-1)$ between equations (8) and the expression of the output $y = x + x(-1)$ yields for $u \neq u(-1)$,

$$x = y + \frac{\dot{y} - u(-1)y(-1)}{u(-1) - u}. \tag{10}$$

If follows that even if the system is not strongly observable, the state can still be reconstructed starting from the output and its derivatives, provided one considers in this case also $\dot{y}$ and with an input which is not a repeating sequence of period τ. As a consequence a new definition of *Regular Observable system* must be considered, which characterizes the case in which the state of the given system at time t can be recovered as a causal function of the output, the input, their derivatives of some order and eventually the delayed variables.

Definition 3: (Regular observability): System (1) is said to be regularly observable if there exists an integer $N \geq n$ such that setting

$$\begin{pmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(N-1)} \end{pmatrix} = \mathcal{O}_e(\cdot,\delta)dx + \mathcal{G}_e(\cdot,\delta) \begin{pmatrix} du \\ d\dot{u} \\ \vdots \\ du^{(N-2)} \end{pmatrix}, \quad (11)$$

the extended observability matrix $\mathcal{O}_e(x, u, \cdots, u^{(N-2)}, \delta)$ has rank n and admits a polynomial left-inverse.

Note that strong observability implies regular observability and the latter yields weak observability.

Remark. It should be underlined that regular observability not only implies that the state can be recovered from the output, the input and their derivatives, but also that the obtained function is causal. Consider for instance the following system

$$\begin{aligned} \dot{x}(t) &= u(t) \\ y(t) &= x(t-1). \end{aligned}$$

This system is weakly observable, but not regularly observable, because $x(t) = y(t+1)$, which is not causal.

Proposition 3: System (1) is regularly observable if and only if there exists an integer $N \geq n$ such that

$$dx \subset \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(N-1)}, du, \cdots du^{(N-2)}\}.$$

Proof. **Sufficiency**- Assume that there exists and integer $N \geq n$ such that Proposition 3 is satisfied. Then there exist matrices $T_0(\cdot,\delta)$ and $T_1(\cdot,\delta)$ such that

$$dx = T_0(\cdot,\delta) \begin{pmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(N-1)} \end{pmatrix} + T_1(\cdot,\delta) \begin{pmatrix} du \\ d\dot{u} \\ \vdots \\ du^{(N-2)} \end{pmatrix}.$$

Since

$$\begin{pmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(N-1)} \end{pmatrix} = \mathcal{O}_e(\cdot,\delta)dx + \mathcal{G}_e(\cdot,\delta) \begin{pmatrix} du \\ d\dot{u} \\ \vdots \\ du^{(N-2)} \end{pmatrix},$$

one immediately gets that

$$T_0(\cdot,\delta) \times \mathcal{O}_e(\cdot,\delta) = Id_{n\times n},$$

that is $\mathcal{O}_e(\cdot,\delta)$ has a polynomial left-inverse of rank n.

Necessity- Since the system is regularly observable, then there exists an integer $N \geq n$ such that the matrix $\mathcal{O}_e$ has rank n and has a polynomial left-inverse. Let $T_0(\cdot,\delta)$ be such a matrix. Then multiplying on the left equation (11) by $T_0(\cdot,\delta)$ one has that

$$T_0(\cdot,\delta) \begin{pmatrix} dy \\ \vdots \\ dy^{(N)} \end{pmatrix} = dx + T_0(\cdot,\delta)\mathcal{G}_e(\cdot,\delta) \begin{pmatrix} du \\ d\dot{u} \\ \vdots \\ du^{(N-2)} \end{pmatrix} \quad (12)$$

which immediately proves the result. ◁

Example 2: Consider again system (2) in Example 1. In this case $N = 2$ and equations (2) and (5) yield

$$\mathcal{O}_e(\cdot,\delta) = \begin{pmatrix} 1+\delta \\ u\delta + u(-1)\delta^2 \end{pmatrix}$$

and accordingly for $u \neq u(-1)$,

$$T_0(\cdot,\delta) = \begin{pmatrix} 1 + \frac{u(-1)}{u-u(-1)}\delta & \frac{1}{u(-1)-u} \end{pmatrix},$$

which proves that (2) is regularly observable as Proposition 3 holds true. In fact (10) holds for $u \neq u(-1)$. Of course if $u = u(-1)$ then one can easily verify the system is only weakly observable.

Corollary 1: System (1) is regularly observable, and not strongly observable only if

$$dy^{(n)} \notin \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-1)}\}$$

Proof. If

$$dy^{(n)} \in \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-1)}\}$$

then for any index $i \geq 0$

$$dy^{(n+i)} \in \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n+i-1)}\}$$

As a consequence after standard computations one gets that

$$\begin{aligned} \begin{pmatrix} dy \\ \vdots \\ dy^{(N-1)} \end{pmatrix} &= \mathcal{O}_e(\cdot,\delta)dx + \mathcal{G}_e(\cdot,\delta) \begin{pmatrix} du \\ \vdots \\ du^{(N-2)} \end{pmatrix} \\ &= \mathcal{L}(\cdot,\delta)\mathcal{O}(\cdot,\delta)dx + \bar{\mathcal{G}}_e(\cdot,\delta) \begin{pmatrix} du \\ \vdots \\ du^{(N-2)} \end{pmatrix} \end{aligned}$$

Since by assumption the system is regularly observable, then there exists a matrix $T_0(\cdot,\delta)$ such that

$$T_0(\cdot,\delta) \begin{pmatrix} dy \\ \vdots \\ dy^{(N-1)} \end{pmatrix} = dx + T_0(\cdot,\delta)\mathcal{G}_e(\cdot,\delta) \begin{pmatrix} du \\ \vdots \\ du^{(N-2)} \end{pmatrix}$$

necessarily $T_0(\cdot,\delta)\mathcal{L}(\cdot,\delta)\mathcal{O}(\cdot,\delta) = Id_{n\times n}$, that is $T_0(\cdot,\delta)\mathcal{L}(\cdot,\delta)$ would be the inverse of $\mathcal{O}(\cdot,\delta)$, and accordingly the system would be strongly observable, which contradicts the assumption the system is not strongly observable. ◁

III. Preliminary Results on SISO systems

In the sequel, we will first analyze the properties of a linear system and then move to the nonlinear case, to clarify the main differences with respect to the two cases and better highlight the role of regular observability in the realization problem.

A. Realization and state elimination in linear time delay systems

First note that for a linear weakly observable time delay single input system, there always exists also a strongly observable realization of the same order. The main point is that it is not possible to move from one realization to the other via a bicausal change of coordinates, that is causal, with a causal inverse. In fact the following result holds true.

Theorem 1: Any SISO (weakly) observable linear time-delay system of order n

$$\begin{aligned} \dot{x}(t) &= \sum_{j=0}^{s} A_j x(t-j\tau) + \sum_{j=0}^{s} B_j u(t-j\tau) \\ y(t) &= \sum_{j=0}^{s} C_j x(t-j\tau) \end{aligned} \tag{13}$$

with $\tau = const.$, admits a strongly observable realization of order n.

Proof. Let us first consider the differential representation of the given dynamics. Then one has

$$\begin{aligned} d\dot{x} &= A(\delta)dx + B(\delta)du \\ dy &= C(\delta)dx. \end{aligned}$$

The statement is straightforward to prove when $n = 1$, and is left to the reader. When $n > 1$, then non commutative matrices are involved. In this case the observability matrix is in general such that

$$\begin{bmatrix} dy \\ d\dot{y} \\ \vdots \\ dy^{(n-1)} \end{bmatrix} = \begin{bmatrix} C(\delta) \\ C(\delta)A(\delta) \\ \vdots \\ C(\delta)A^{(n-1)}(\delta) \end{bmatrix} dx + \sum_{i=0}^{n-2} G_i(\delta) du^{(i)}. \tag{14}$$

It is easily seen that there exists $d\tilde{x} = U(\delta)dx$, where U is unimodular, so that the observability matrix becomes lower triangular in the coordinates $\tilde{x}$. Accordingly, in the new coordinates the matrices $\tilde{A}(\delta), \tilde{C}(\delta)$ have the form

$$\begin{aligned} \tilde{A}(\delta) &= \begin{pmatrix} \tilde{a}_{11}(\delta) & \tilde{a}_{12}(\delta) & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ \tilde{a}_{n-1,1}(\delta) & \tilde{a}_{n-1,2}(\delta) & \cdots & \tilde{a}_{n-1,n}(\delta) \\ \tilde{a}_{n1}(\delta) & \tilde{a}_{n2}(\delta) & \cdots & \tilde{a}_{nn}(\delta) \end{pmatrix} \\ \tilde{C}(\delta) &= \begin{pmatrix} \tilde{c}_{11}(\delta) & 0 & 0 & \cdots & 0 \end{pmatrix}. \end{aligned}$$

Now, define the following change of coordinates, which is causal, but not bicausal,

$$dz = \begin{pmatrix} \tilde{c}_{11}(\delta)d\tilde{x}_1 \\ \tilde{c}_{11}(\delta)\tilde{a}_{12}(\delta)d\tilde{x}_2 \\ \vdots \\ \tilde{c}_{11}(\delta)\tilde{a}_{12}(\delta)\cdots\tilde{a}_{n-1,n}(\delta)d\tilde{x}_n \end{pmatrix}$$

In the new coordinates the system reads

$$\begin{aligned} d\dot{z}_1 &= \hat{a}_{11}(\delta)dz_1 + dz_2 + \hat{b}_1(\delta)du \\ d\dot{z}_2 &= \hat{a}_{21}(\delta)dz_1 + \hat{a}_{22}(\delta)dz_2 + dz_3 + \hat{b}_2(\delta)du \\ &\vdots \\ d\dot{z}_{n-1} &= \sum_{l=1}^{n-1} \hat{a}_{n-1,l}(\delta)dz_l + dz_n + \hat{b}_{n-1}(\delta)du \\ d\dot{z}_n &= \sum_{l=1}^{n} \hat{a}_{n,l}(\delta)dz_l + \hat{b}_n(\delta)du \\ dy &= dz_1 \end{aligned}$$

which is strongly observable. $\triangleleft$

An immediate corollary is the following.

Corollary 2: A SISO linear time–delay system of the form (13), which is weakly observable, always admits a retarded type input–output equation of order n.

Surprisingly, this is no more true in the nonlinear case as it will be shown in Section IV.

B. State elimination in nonlinear time delay systems

State elimination for delay–free nonlinear systems is found in [4]. Time-delay nonlinear systems are considered next, in the flavor of the results above.

Theorem 2: A strongly observable SISO system with state space realization of order n admits a retarded type input–output equation of order n.

Proof. On one hand, since the system is strongly observable, the state $x(t)$ can be expressed as a function of the output, the input eventually delayed and their derivatives, that is

$$x(t) = \psi(y^{(l)}(t-j), u^{(l)}(t-j), j \in [0,k],\ l \in [0, n-1]).$$

On the other hand, since the system is strongly observable and of order n, we get

$$y^{(n)} = \varphi(x(t-j), u(t-j), \cdots, u^{(n-1)}(t-j), j \in [0,k])$$

and by substitution one gets that

$$y^{(n)} = \varphi(\psi(t-j), u(t-j), \cdots, u^{(n-1)}(t-j), j \in [0,k])$$

which proves the result. $\triangleleft$

Theorem 3: A weakly observable SISO retarded type system with state space realization of order n, which satisfies

$$dy^{(n)} \in \mathrm{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-1)}\}, \tag{15}$$

admits a retarded type input–output equation of order n.

Proof. By assumption the system is weakly observable and satisfies (15). As a consequence

$$y^{(n)} = \varphi(y^{(\ell)}(t-j), u^{(\ell)}(t-j), j \in [0,s], \ell \in [0, n-1])$$

which proves the result. $\triangleleft$

Proposition 4: A weakly observable SISO retarded type system with a state space realization of order n, such that

$$dy^{(n)} \notin \text{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-1)}\},$$

admits a neutral type input–output equation of order n, and a retarded type input–output equation of order $n+1$ (and not smaller).

Proof. The result follows immediately from [9] where it is stated that a retarded type system admits a retarded type input/output equation of at most degree $n+1$ and the previous results which shows that it can be of degree n if (15) is satisfied. Since the system is assumed to be weakly observable, if (15) is not fulfilled, then there exists a polynomial $a(\cdot, \delta)$ such that

$$a(\cdot, \delta)dy^{(n)} \in \text{span}_{\mathcal{K}(\delta]}\{dy, \cdots, dy^{(n-1)}, du, \cdots du^{(n-1)}\}.$$

which also shows that the system admits a neutral type input–output equation of order n. $\triangleleft$

Remark. It should be noted that, based on the previous discussion, the result of Proposition 4 is peculiar of the nonlinear case.

IV. Main results. Different realizations

In the present Section, starting from Example 1 it will be shown that different realizations may be associated to the given system. However, differently from the delay–free case, these realizations should be considered with caution, since some information may be missed.

A. Neutral type input–output equations

As we have already shown, to the given retarded time system

$$\begin{aligned} \dot{x} &= x(-1)u \\ y &= x + x(-1), \end{aligned}$$

which is weakly observable and strongly controllable for $x(-1) \neq 0$, we can associate, for $u \neq u(-1)$, the first order neutral type equation (9), that is

$$\begin{aligned} (u - u(-1))\dot{y}(-1) \; &- \; (u - u(-1))y(-2)u(-2) \\ &+ (\dot{y} - y(-1)u)(u(-1) - u(-2)) = 0. \quad (16) \end{aligned}$$

This neutral type input–output equation is thus linked to the first order retarded type differential representation.

B. Retarded type input–output equations

As already shown in [9], if we go one step further, we will get a second order retarded type equation associated to the given system.

As a matter of fact if we compute the second order derivative of the output for system (2), we get that

$$\begin{aligned} \ddot{y} &= x(-1)\dot{u} + x(-2)\dot{u}(-1) \\ &\quad + x(-2)u(-1)u + x(-3)u(-2)u(-1), \end{aligned} \quad (17)$$

and after eliminating $x(-3)$ between equations (17) and (4):

$$\begin{aligned} \ddot{y} &= y(-2)u(-2)u(-1) + x(-1)\dot{u} + \\ &+ x(-2)[\dot{u}(-1) + u(-1)u] - x(-2)u(-2)u(-1). \quad (18) \end{aligned}$$

Finally, after a few more computations one gets that the given system is represented by the *second order* retarded type input–output equation

$$\begin{aligned} \ddot{y} &= y(-2)u(-1)u + (\dot{u}(-1) - \dot{u})\frac{\dot{y} - y(-1)u}{u(-1) - u} \\ &+ y(-1)\dot{u} + u(-1)[u(-2) - u]\frac{\dot{y}(-1) - y(-2)u(-1)}{u(-2) - u(-1)} \end{aligned} \quad (19)$$

To compute a realization, we may use the procedure in [10]. If n is the order of the differential equation, we have to compute the sequence of submodules derived from the filtration of submodules introduced in Section 3 of [14] that is

$$\mathcal{H}_{i+1} = \{\omega \in \mathcal{H}_i | \dot{\omega} \in \mathcal{H}_i\}$$

with $\mathcal{H}_1 = \text{span}_{\mathcal{K}(\delta]}\{dy^{(j)}, du^{(j)}, j \in [0, n-1]\}$, and check if $\mathcal{H}_{n+1}$ is integrable. In our case $n = 2$, and we then have to verify if $\mathcal{H}_3$ is integrable.

We have for $u \neq u(-1)$

$$\begin{aligned} \mathcal{H}_1 &= \text{span}_{\mathcal{K}(\delta]}\{dy, d\dot{y}, du, d\dot{u}\} \\ \mathcal{H}_2 &= \text{span}_{\mathcal{K}(\delta]}\left\{dy, d\left(\frac{\dot{y} - y(-1)}{u(-1) - u}\right), du\right\} \\ \mathcal{H}_3 &= \text{span}_{\mathcal{K}(\delta]}\left\{dy, d\left(\frac{\dot{y} - y(-1)}{u(-1) - u}\right)\right\}. \end{aligned}$$

Since $\mathcal{H}_3$ is integrable we can set

$$x_1 = y, \qquad x_2 = \frac{\dot{y} - uy(-1)}{u(-1) - u}. \quad (20)$$

Accordingly, after standard computations, we get that

$$\left\{ \begin{aligned} \dot{x}_1 &= x_1(-1)u + x_2[u(-1) - u] \\ \dot{x}_2 &= x_2(-1)u(-2) \\ y &= x_1. \end{aligned} \right. \quad (21)$$

The observability matrix is $\mathcal{O} = \begin{pmatrix} 1 & 0 \\ u\delta & u(-1) - u \end{pmatrix}$, which is unimodular provided again $u(-1) \neq u$, that is the system is not solicited with a periodic input of period τ.

Similarly setting $\alpha = x_1(-1) - x_2$ and

$$r_{12} = u(-1)(\alpha(-1) - x_2(-1)\delta^2) + (x_2(-1)u(-2) - \alpha(-1)u)\delta,$$

the controllability matrix is

$$\mathcal{R} = \begin{pmatrix} \alpha + x_2\delta & r_{12}(\cdot, \delta) \\ x_2(-1)\delta^2 & x_2(-2)(u(-3) - u(-2)\delta)\delta^2 \end{pmatrix}.$$

To check if the controllability matrix has full rank, we can use Smith decomposition. Standard computations show that setting $p(\cdot, \delta) = x_2 - \alpha(-1)$, and $\beta(u, \delta) = u(-1) - u\delta$ one gets that

$$\mathcal{R} = T^{-1}(\delta)\begin{pmatrix} p(-1)\delta^2 & p(-2)\delta^2 \\ \alpha + p\delta & \alpha(-1) + p(-1)\delta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \beta(u, \delta) \end{pmatrix}$$

with $T(\delta) = \begin{pmatrix} -\delta^2 & 1+\delta \\ 1-\delta & 1 \end{pmatrix}$. Consequently, it can be easily verified that $\mathcal{R}$ has full rank if and only if $u \neq 0$, $p \neq 0$ and $\alpha \neq 0$. In this case, since $\mathcal{R}$ will not be unimodular the system will be weakly controllable.

This fact highlights an important issue: we have a retarded type realization of the first order which is weakly observable and strongly controllable, and a retarded type realization of the second order which is strongly observable and weakly controllable provided $u \neq 0$ and not periodic of period τ, $p \neq 0$ and $\alpha \neq 0$ while we would have expected a loss in controllability or observability. This pathology is due to the fact that we are neglecting the relation established by the neutral type input–output equation. In the x coordinates, equation (16) reads

$$x_2 + x_2(-1) = x_1(-2) \tag{22}$$

which corresponds to $p = 0$. If this condition is satisfied, while $u \neq 0$ and $\alpha \neq 0$, then $\mathcal{R}$ has rank 1, and, after standard computations, one gets that $d\varphi(x) = (1+\delta)dx_2 - \delta^2 dx_1 = dp$ is in the left annihilator of $\mathcal{R}$. Let us then consider the bicausal change of coordinates $z_1 = x_2 + x_2(-1) - x_1(-2), z_2 = x_1 - x_1(-1) + x_2$. In these coordinates the system then reads

$$\begin{aligned} \dot{z}_1 &= z_1(-1)u(-2) \\ \dot{z}_2 &= (z_2(-1) - z_1)u + z_1 u(-1) \\ y &= z_2 + z_2(-1) - z_1 \end{aligned}$$

and since by assumption $z_1 = 0$ we get

$$\begin{aligned} \dot{z}_1 &= 0 \\ \dot{z}_2 &= z_2(-1)u \\ y &= z_2 + z_2(-1), \end{aligned}$$

which highlights our weakly (and for $u \neq u(-1)$ also regularly) observable and strongly controllable subsystem of dimension 1.

Similarly if we assume $p \neq 0$, $u \neq 0$ but $\alpha = 0$, the matrix $\mathcal{R}$ has again rank 1 and $d\bar{\varphi}(x) = dx_2 - \delta dx_1 = d\alpha$ is in the left annihilator of $\mathcal{R}$. Let us then consider the bicausal change of coordinates $\chi_1 = x_2 - x_1(-1)$, $\chi_2 = x_1$. In these coordinates the system then reads

$$\begin{aligned} \dot{\chi}_1 &= \chi_1(-1)u(-1) \\ \dot{\chi}_2 &= -\chi_1 u + [\chi_1 + \chi_2(-1)]u(-1) \\ y &= \chi_2. \end{aligned}$$

Since by assumption $\chi_1 = 0$, we get

$$\begin{aligned} \dot{\chi}_1 &= 0 \\ \dot{\chi}_2 &= \chi_2(-1)u(-1) \\ y &= \chi_2, \end{aligned}$$

which highlights another subsystem of dimension 1, which is strongly observable but weakly controllable, satisfies the same second order differential equation (19), but is characterized by the different first order input–output equation

$$\dot{y} = y(-1)u(-1).$$

V. Conclusion and perspectives

The realizability of a general nonlinear delay differential input-output equation remains widely open. Some of the most important open questions are related to neutral type input-output equations. Based on some academic example, it was shown that in general an (observable) retarded type nonlinear SISO system with state space realization of order n yields an input-output equation of the same order n which is of neutral type. Any retarded type input-output equation will then be of order larger than or equal to n. When considering MIMO systems, with or without delays, the input-output equations can be written in different ways. The observability indices are useful to write input-output equations in some canonical way. The results in this paper suggest the possibility of considering different sets of observability indices associated to the same dynamics; their impact on the realizations requires further investigations.

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