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Admissible invariant canonical quantizations of classical mechanics

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Dopuszczalne niezmiennicze kanoniczne kwantowania mechaniki klasycznej

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If anybody says he can think about quantum physics without getting giddy, that only shows he has not understood the first thing about them.

Niels Bohr

God used beautiful mathematics in creating the world.

Paul Dirac

Abstract

In the thesis is developed an invariant quantization procedure of classical Hamiltonian mechanics. The procedure is based on a deformation quantization theory, which is used to introduce quantization in arbitrary canonical coordinates as well as in a coordinate independent way. In this approach to quantization a classical Poisson algebra of a classical system is deformed to an appropriate non-commutative algebra of smooth functions on a phase space. The non-commutative product from this quantum Poisson algebra is called a star-product. In addition to the star-product, on the quantum Poisson algebra is introduced a deformed Poisson bracket and an involution being a deformation of the complex-conjugation of functions. To each measurable quantity corresponds a function from the quantum Poisson algebra, self-conjugated with respect to the quantum involution, i.e. quantum observable. Thus, a quantization is fixed by a choice of a deformation of the classical Poisson algebra, and an assignment to measurable quantities quantum observables. It is discussed that for a given classical system its quantization is not specified uniquely and there may exist many different quantizations. A notion of equivalent quantizations is introduced, which allows for a systematic characterization of different quantizations.

The developed formalism of quantum mechanics uses a mathematical language similar to that of classical Hamiltonian mechanics. This allows to introduce in quantum theory analogs of many concepts from classical theory. For instance, in the thesis are introduced quantum canonical (Darboux) coordinates and transformations between them. Moreover, a notion of almost global coordinates is defined. These are the only coordinates in which it is meaningful to consider quantum systems.

For particular examples of phase spaces are introduced canonical star-products. In particular, on a cotangent bundle to a general Riemannian manifold is defined a two-parameter family of star-products, which reproduces most of the results received by different approaches to quantization found in the literature. The introduced star-products were written in a covariant form. Moreover, it was proved that for a given coordinate system, which is at the same time classical and quantum canonical, a general star-product on a general phase space is equivalent with the Moyal product.

The operator representation of quantum mechanics is constructed for a general quantization and arbitrary canonical coordinates. A very general family of orderings of operators of position and momentum (containing all orderings found in the literature) is introduced. It is shown that for different quantizations and canonical coordinates correspond different orderings. This fact allowed to construct an operator representation of quantum mechanics in a consistent way for any canonical coordinates as well as in a coordinate independent way. The construction is illustrated with examples of quantum mechanical operators corresponding to observables linear, quadratic and cubic in momenta. Moreover, as an another example, a quantization of the hydrogen atom is presented.

Finally, using the developed formalism, a quantum analog of classical trajectories in phase space is introduced. Quantum trajectories are defined as integral curves of quantum Hamiltonian vector fields. A quantum action of a quantum flow on observables, which is a deformation of the respective classical action, is presented in an explicit form. Then, it is shown that a set of quantum flows has a structure of a group with multiplication being a deformation of the ordinary composition of flows. The theory of quantum trajectories is illustrated with examples of quantum systems.

Streszczenie — Abstract in Polish

W pracy rozwijana jest niezmiennicza procedura kwantowania klasycznych układów hamiltonowskich. Procedura ta bazuje na teorii kwantyzacji deformacyjnej, która została użyta do wprowadzenia kwantyzacji w dowolnych współrzędnych kanonicznych, jak również w sposób niezależny od układu współrzędnych. W tym podejściu do kwantyzacji klasyczna algebra Poissona układu klasycznego jest deformowana do odpowiedniej niekomutatywnej algebry funkcji gładkich na przestrzeni fazowej. Niekomutatywny iloczyn z tej kwantowej algebry Poissona nazywany jest gwiazdka-iloczynem. Poza gwiazdka-iloczynem na kwantowej algebrze Poissona wprowadzany jest zdeformowany nawias Poissona i inwolucja będąca deformacją sprzężenia zespolonego funkcji. Każdej wielkości mierzalnej odpowiada funkcja z kwantowej algebry Poissona, samosprzężona ze względu na kwantową inwolucję, tzn. kwantowa obserwabla. Tak więc kwantyzacja jest zadana poprzez wybór deformacji klasycznej algebry Poissona oraz przyporządkowania wielkościom mierzalnym obserwabli kwantowych. Dyskutowane jest, że dla danego układu klasycznego jego kwantyzacja nie jest określona jednoznacznie i może istnieć wiele różnych kwantyzacji. Ponadto wprowadzone zostało pojęcie równoważnych kwantyzacji, pozwalające na systematyczną charakteryzację różnych kwantowań.

O strukturze algebraicznej kwantowej algebry Poissona można myśleć jak o wyznaczającej kwantową geometrię przestrzeni fazowej, podobnie jak klasyczna algebra Poissona wyznacza klasyczną przestrzeń fazową. Ponadto struktura kwantowej algebry Poissona użyta została do zdefiniowania stanów kwantowych oraz ewolucji czasowej układów kwantowych, poprzez analogię z przypadkiem klasycznym.

Rozwijany formalizm mechaniki kwantowej używa języka matematyki podobnego do tego opisującego klasyczną mechanikę hamiltonowską. Pozwala to wprowadzić w teorii kwantów analogi wielu pojęć z teorii klasycznej. Przykładowo w pracy wprowadzone zostały kwantowo kanoniczne współrzędne (kwantowe współrzędne Darboux) oraz transformacje pomiędzy nimi. Ponadto zdefiniowane zostało pojęcie prawie globalnego układu współrzędnych. Są to jedyne współrzędne, w których ma sens rozpatrywać układy kwantowe.

Dla szczególnych przykładów przestrzeni fazowych skonstruowane zostały kanoniczne gwiazdka-iloczyny. Jako pierwszy przykład rozpatrzona została przestrzeń $\mathbb{R}^{2N}$ z gwiazdka-iloczynem Moyala zdefiniowanym na niej. Następnie wiązka kostyczna do przestrzeni Euklidesowej, na której wprowadzona została rodzina gwiazdka-iloczynów. W dalszej kolejności rozważona została wiązka kostyczna do płaskiej rozmaitości Riemanna z kanonicznym gwiazdka-iloczynem zadanym poprzez koneksję liniową Levi-Civita. Ostatecznie, dla wiązki kostycznej do ogólnej rozmaitości Riemanna wprowadzona została dwu-parametrowa rodzina gwiazdka-iloczynów, która odtwarza większość rezultatów otrzymanych różnymi podejściami do kwantyzacji spotykanymi w literaturze. Skonstruowane gwiazdka-iloczyny zostały zapisane w postaci kowariantnej. Ponadto udowodniono, że dla układu współrzędnych, który jest jednocześnie klasycznie i kwantowo kanoniczny, ogólny gwiazdka-iloczyn na ogólnej przestrzeni fazowej jest równoważny z iloczynem Moyala.

W dalszej części pracy skonstruowana została operatorowa reprezentacja dla ogólnej kwantyzacji i dowolnych współrzędnych kanonicznych. Punktem wyjścia była konstrukcja operatorowej reprezentacji w przestrzeni Hilberta nad przestrzenią fazową. Pozwoliło to uzyskać w naturalny sposób bardzo ogólną rodzinę uporządkowań operatorów położenia i pędu (zawierającą wszystkie porządki spotykane w literaturze). W następnym kroku zaprezentowana została konstrukcja operatorowej reprezentacji w przestrzeni Hilberta nad przestrzenią konfiguracyjną. Odtworzony został w ten sposób standardowy opis mechaniki kwantowej w ujęciu przestrzeni Hilberta. Pokazane zostało, że różnym kwantowaniom i współrzędnym kanonicznym odpowiadają różne porządki operatorów położenia i pędu. Ten fakt pozwolił na konstrukcję operatorowej reprezentacji mechaniki kwantowej w spójny sposób, dla dowolnych współrzędnych kanonicznych. Mianowicie operatory odpowiadające danej obserwabli kwantowej zapisanej w dwóch różnych kanonicznych układach współrzędnych będą unitarnie równoważne. Ponadto uzyskane rezultaty wyrażone zostały w sposób niezależny od układu współrzędnych. Konstrukcja zilustrowana została przykładami kwantowo-mechanicznych operatorów odpowiadających obserwablom liniowym, kwadratowym i kubicznym w pędach. Co więcej, jako kolejny przykład, zaprezentowana została kwantyzacja atomu wodoru.

Na zakończenie, używając rozwijanego formalizmu, wprowadzony został kwantowy analog klasycznych trajektorii na przestrzeni fazowej. Kwantowe trajektorie zdefiniowane zostały jako krzywe całkowe kwantowych pól hamiltonowskich. Zaprezentowana została postać kwantowego działania kwantowych potoków fazowych na obserwable, które jest deformacją klasycznego działania. Następnie pokazane zostało, że zbiór kwantowych potoków fazowych posiada strukturę grupy z mnożeniem będącym deformacją zwykłego składania potoków. Teoria trajektorii kwantowych zilustrowana została różnymi przykładami układów kwantowych.

Contents

Chapter 1

Introduction

Quantum mechanics proved to be a proper theory to describe physical systems in a micro scale. However, after over 100 years of development there is still lack of a consistent quantization procedure of classical systems. The most common approach to quantum theory is the Hilbert space approach. In this approach we associate with every measurable quantity a self-adjoint operator defined on a Hilbert space. If we have some classical system and we would like to quantize it, then first we have to find a correspondence between classical observables and operators on a certain Hilbert space. In a Hamiltonian description of classical mechanics observables are defined as real-valued functions on a phase space, and the passage to quantum mechanics is done using Weyl quantization rule. The Weyl quantization rule states that to functions on a phase space one associates operators by formally replacing q^i and p_j coordinates in classical observable with operators $\hat{q}^i$, $\hat{p}_j$ of position and momentum, and symmetrically ordering them. By such procedure one can quantize every classical Hamiltonian system. Note however, that this procedure works only for systems whose phase space is $\mathbb{R}^{2N}$. Moreover, quantization has to be performed in Cartesian coordinates. Even in that well recognized case a natural question appears: whether the Weyl quantization is a unique choice? In other words, whether there are other quantization procedures which are consistent with physical experiments.

The proper quantization procedure should be possible to perform for a system defined on a general phase space and in any coordinate system. However, if we would take a classical system and naively perform a quantization according to the Weyl quantization rule, for two different canonical coordinates, then in general we would not get equivalent quantum systems. As an example let us consider a hydrogen atom which Hamiltonian in Cartesian coordinates is given by the formula

$$H(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{4\pi\epsilon_0} \frac{e^2}{\sqrt{x^2 + y^2 + z^2}}.$$

In accordance to the Weyl quantization rule to this function will correspond the following operator

$$H(\hat{q}_x, \hat{q}_y, \hat{q}_z, \hat{p}_x, \hat{p}_y, \hat{p}_z) = -\frac{\hbar^2}{2m}\Delta - \frac{1}{4\pi\epsilon_0} \frac{e^2}{\sqrt{x^2 + y^2 + z^2}},$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator in the Cartesian coordinates. If we will now consider this system in spherical polar coordinates then the Hamiltonian

H takes the form

$$H(r,\theta,\phi,p_r,p_\theta,p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r},$$

and operators of position and momentum corresponding to spherical polar coordinates are given by

$$\begin{aligned} &\hat{q}_r = r, && \hat{q}_\theta = \theta, && \hat{q}_\phi = \phi, \\ &\hat{p}_r = -i\hbar\left(\partial_r + \frac{1}{r}\right), && \hat{p}_\theta = -i\hbar\left(\partial_\theta + \frac{1}{2\tan\theta}\right), && \hat{p}_\phi = -i\hbar\partial_\phi. \end{aligned}$$

The function H of symmetrically ordered operators $\hat{q}_r, \hat{q}_\theta, \hat{q}_\phi, \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi$ of position and momentum will not be an operator unitarily equivalent with the operator $H(\hat{q}_x, \hat{q}_y, \hat{q}_z, \hat{p}_x, \hat{p}_y, \hat{p}_z)$ derived for Cartesian coordinates.

As we will show later on this apparent inconsistency of quantization can be solved by a proper choice of quantum observables in new coordinates, i.e. by performing an appropriate deformation of classical observables written in new coordinates, or alternatively by using different ordering rules of position and momentum operators for different coordinates. The situation gets even more complicated when we consider non-flat configuration spaces. In such case there are very few experiments which could distinguish quantization rules.

The problem of quantization in arbitrary coordinates on a configuration space was evident in early days of quantum mechanics. The majority of efforts was related to an invariant quantization of Hamiltonians quadratic in momenta. The construction of a quantum Hamiltonian in flat and non-flat cases was considered by many authors (see for example several relevant papers [1–9]). Much less results concern an invariant quantization of Hamiltonians cubic in momenta [10, 11]. However, to our knowledge, there does not exist general solution valid for any classical observable and canonical coordinates.

Possibility of considering quantum systems in different canonical coordinates is connected with the theory of canonical transformations in quantum mechanics. The development of the theory of canonical transformations of coordinates in quantum mechanics is mainly contributed to Jordan, London and Dirac back in 1925 [12–18] and it is still an area of intense research.

In the usual approach to canonical transformations in quantum mechanics one identifies canonical transformations with unitary operators defined on a Hilbert space. Such approach was used by Mario Moshinsky and his collaborators in a series of papers [19–23]. Also other researchers used such approach [24–26]. Worth noting are also papers of Anderson [27, 28] where an extension of canonical transformations to non-unitary operators is presented. Nevertheless, after so many years of efforts, there is still lack of a general theory of coordinate transformations in quantum mechanics, including a satisfactory complete theory of canonical transformations.

Although, the usual Hilbert space approach to quantum mechanics is very popular it is not the best approach for developing a theory of quantization in any canonical coordinate systems and to characterize different quantizations. It seems that the bast approach to quantum theory to achieve these tasks is the phase space quantum mechanics. This approach is also very natural for introducing quantization and is described by a mathematical language similar to that of a classical Hamiltonian mechanics. This allows to introduce many concepts from classical theory to its quantum counterpart, like coordinate systems, coordinate transformations and trajectories on phase space. The standard Hilbert space approach to quantum mechanics is then reproduced as an appropriate operator representation of phase space quantum mechanics.

The theory of trajectories on phase space plays an important role in a description of time evolution of classical systems. From the very beginning of quantum physics, efforts have been taken to formulate some kind of an analogue of phase space trajectories in quantum mechanics [18]. The most common approaches to quantum dynamics are the de Broglie-Bohm approach [29–33] and the average value approach [34, 35]. Worth noting is also the paper [36] written by Rieffel where he considers a classical limit of a quantum time evolution in the framework of a strict deformation quantization.

Furthermore, the phase space approach to quantum mechanics makes it possible to introduce, in a natural way, an analog of classical trajectories in quantum mechanics (see [37, 38] and references therein). In this approach one considers the Heisenberg evolution of fundamental observables of position and momentum, being $\hbar$-deformation of the classical Hamiltonian evolution. Moreover, the deformation to an arbitrary order can be calculated by an $\hbar$-hierarchy of recursive first order linear partial differential equations [37–39]. The time evolution of observables cannot be given as a simple composition of observables with a quantum flow. For this reason Dias and Prata [37], and Krivoruchenko and Faessler [38] considered observables as $\star$-functions and a quantum phase space as a plane of non-commuting variables. Then the action of a flow on observables was given as a $\star$-composition.

The thesis is organized as follows. In Chapter 2 we review classical Hamiltonian mechanics. The theory is described in a language of differential geometry. The definitions of basic objects of the theory are given including a phase space, Poisson algebra, observables, states, and canonical coordinates. We present characterization of states which will be used when defining quantum states. Also we introduce a concept of almost global coordinates which will be intensively used during a quantization process. Moreover, the thorough description of time evolution of classical systems is presented including a definition of trajectories on a phase space which quantum counterpart will be developed in Chapter 5.

In Chapter 3 we present the general theory of quantization base on deformation of classical Hamiltonian mechanics. Although the deformation approach to quantization is not new and has a long history, usually in the literature one does not find quantum mechanics introduced in a fully invariant form. The deformation quantization is considered either from purely mathematical perspective, or in some particular coordinates, usually Cartesian on $\mathbb{R}^{2N}$. In the thesis we develop a fully invariant deformation quantization procedure of classical mechanics.

In the first section of Chapter 3 we review the theory of deformations of symplectic manifolds. A symplectic manifold represents a phase space of the system. The geometric structure of a symplectic manifold M is fully specified by its Poisson algebra $C^\infty(M)$. By deforming the algebra $C^\infty(M)$ to some non-commutative algebra we can think of it as describing a non-commutative symplectic manifold (non-commutative phase space). By a deformation of the Poisson algebra $C^\infty(M)$

is understood a space $C^\infty(M)[\![\nu]\!]$ of formal power series in ν with coefficients in $C^\infty(M)$, together with a non-commutative product $\star$, called a star-product, which in the limit $\nu \to 0$ reduces to the ordinary point-wise product of functions. In addition, on the space $C^\infty(M)[\![\nu]\!]$ we introduce a deformed Poisson bracket given by

$$[\![f, g]\!]_\star = \frac{1}{\nu}(f \star g - g \star f) = \{f, g\} + o(\nu),$$

and an involution $*$ which in the limit $\nu \to 0$ reduces to the complex-conjugation of functions. As the deformation parameter ν is taken $i\hbar$. The deformation of a phase space is the main ingredient of the process of quantization.

A star-product on a given symplectic manifold is not defined uniquely. This is one of the sources of the existence of different quantizations of a given classical system. However, some star-products are equivalent in the sense that there exists a morphism S on $C^\infty(M)[\![\nu]\!]$ intertwining them.

Section 3.2 contains a detailed description of a quantization procedure. A quantization of a given classical Hamiltonian system is performed first by deforming a phase space of the system to a non-commutative phase space in accordance to the theory of deformations of symplectic manifolds described in the previous section. That is, the classical Poisson algebra $\mathcal{A}_C(M) = (C^\infty(M), \cdot, \{\,\cdot\,,\,\cdot\,\}, {}^-)$ is deformed to a quantum Poisson algebra $\mathcal{A}_Q(M) = (C^\infty(M)[\![\hbar]\!], \star, [\![\,\cdot\,,\,\cdot\,]\!], *)$. The second step of the quantization process is assignment to every measurable quantity an element of $C^\infty(M)[\![\hbar]\!]$ self-adjoint with respect to the involution $*$ from $\mathcal{A}_Q(M)$, i.e. an observable. Usually in the literature as observables are taken the same functions as in the classical case, even when the involution $*$ is not the complex-conjugation. However, we use a different approach and take as quantum observables $\hbar$-deformations of classical observables. This crucial innovation allowed to characterize quantizations in a concise way. In particular, equivalent star-products can give equivalent quantizations if we appropriately assign to measurable quantities elements of $C^\infty(M)[\![\hbar]\!]$. Also, as an interesting consequence, for some involutions $*$ observables may be complex-valued functions.

Quantum states and time evolution of a quantum system are defined in an analogical way as in the classical case. The point-wise product $\cdot$ of functions and the Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$ have to be replaced by the $\star$-product and the deformed Poisson bracket $[\![\,\cdot\,,\,\cdot\,]\!]$. This is a consequence of the fact that the algebraic structure of the algebra of observables (Poisson algebra) defines states and time evolution.

The mathematical language used to introduce quantum mechanics is similar to that of classical Hamiltonian mechanics. As a consequence we can introduce to quantum theory coordinate systems and coordinate transformations in a straightforward way. All this is described in Section 3.3. Moreover, in this section are introduced quantum canonical coordinates and transformations in a total analogy with the classical case.

In Section 3.4 are constructed canonical star-products on particular examples of symplectic manifolds. We start with a simplest symplectic manifold, $\mathbb{R}^{2N}$, and introduce on it a Moyal star-product. It is well known how in this simplest case create an operator representation of a quantum system. We also prove that a wide family of star-products on a general symplectic manifold is equivalent with the Moyal product, for a given classical and quantum canonical coordinate system. This observation and the fact that the operator representation for the Moyal product is known is a key point for introducing an operator representation of a general quantum system for arbitrary canonical coordinates.

Next we move to a symplectic manifold in the form of a cotangent bundle T^*E^N to an Euclidean space E^N and introduce on it a family of star-products. Each star-product is parametrized by a sequence of pair-wise commuting vector fields $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ from a decomposition of a Poisson tensor $\mathcal{P}$ on T^*E^N

$$\mathcal{P} = \sum_{i=1}^{N} X_i \wedge Y_i.$$

One of the star-products from this family is distinguished, namely the one for which the vector fields X_i, Y_j in Cartesian coordinates are coordinate vector fields. We then write this canonical star-product in a covariant form. The covariant form of the star-product is given in terms of a linear connection on E^N.

The equation for the star-product on T^*E^N written in the covariant form can be generalized in a straightforward way to a case of a symplectic manifold $T^*\mathcal{Q}$ over a flat Riemannian manifold $\mathcal{Q}$. That way we introduced a canonical star-product on $T^*\mathcal{Q}$. We also derived the form (to the second order in $\hbar$) of the equivalence morphism S intertwining this star-product with the Moyal product, for a given classical and quantum canonical coordinate system.

Finally, we consider a general symplectic manifold $T^*\mathcal{Q}$ over a non-flat Riemannian manifold $\mathcal{Q}$ and propose a two-parameter family of star-products defined on it. In this general case there is no single distinguished star-product, which shows that in the non-flat case there is a problem of choosing a physically admissible quantization. In Section 4.3 we show that for this general case to functions quadratic in momenta correspond operators with an extra term added to the potential and dependent on the curvature tensor. The form of this operator, for particular values of the quantization parameters, was received by many authors using different approaches to quantization. The approach to quantization developed in the thesis reproduces all results present in the literature.

In Chapter 4 we describe the construction of an operator representation of quantum mechanics for an arbitrary canonical coordinate system, as well as, in a coordinate independent way. In the first section of this chapter we consider a quantum system over a phase space $\mathbb{R}^{2N}$ with the Moyal product defined on it. We construct a representation of the algebra $\mathcal{A}_Q(\mathbb{R}^{2N}) = (C^\infty(\mathbb{R}^{2N})[\![\hbar]\!], \star_M)$ in the Hilbert space $L^2(\mathbb{R}^{2N})$ according to the formula

$$f \mapsto f \star_M .$$

We show that operators $f \star_M$ can be written as functions f of symmetrically ordered operators $\hat{q}^i_{\star_M} = q^i \star_M$, $\hat{p}_{\star_M j} = p_j \star_M$ of position and momentum in accordance to a Weyl correspondence rule:

$$f \star_M = f(\hat{q}_{\star_M}, \hat{p}_{\star_M}).$$

Next we propose a generalization of the ordering of operators of position and momentum. The introduced generalization covers all orderings found in the literature, including symmetric, normal, and anti-normal orderings, as well as a wide family of orderings considered by L. Cohen. But it also extends to types of orderings not considered before. Using this general concept of the ordering we show that for every $\star$-product on $\mathbb{R}^{2N}$ operators $f\,\star$ can be written as appropriately ordered functions f of operators of position and momentum. As a result every star-product on $\mathbb{R}^{2N}$ gives rise to an ordering of operators $\hat{q}^i_\star, \hat{p}_{\star j}$ and a quantization can be fixed either by choosing a star-product on a phase space $\mathbb{R}^{2N}$ or equivalently, on a level of the operator representation, by choosing an ordering.

Section 4.2 contains a description of the λ-Weyl correspondence rule for a case of a symplectic manifold $T^*\mathcal{Q}$ over a general Riemannian manifold $\mathcal{Q}$, and for a Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. The results received in this section are used in the next section when introducing an operator representation of quantum mechanics in the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$.

In Section 4.3 we present a detailed description of the operator representation of quantum mechanics over a configuration space. We start with a Moyal quantization of a system defined over a phase space T^*U where U is some open subset of $\mathbb{R}^N$. First we construct a tensor product $\otimes_W$ of the Hilbert space $L^2(T^*U)$ in terms of Hilbert spaces $(L^2(U, \mathrm{d}\mu))^*$ and $L^2(U, \mathrm{d}\mu)$. Then we show that for every element f of $C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ and state ρ the operators $f \star_M$ and $\rho \star_M$ take the form

$$
\begin{aligned}
f \star_M &= \hat{1} \otimes_W f(\hat{q}, \hat{p}), \\
\rho \star_M &= \hat{1} \otimes_W \hat{\rho},
\end{aligned}
$$

where $\hat{q}^i, \hat{p}_j$ are canonical operators of position and momentum, and $\hat{\rho}$ is a density operator. This way we received an operator representation in the Hilbert space $L^2(U, \mathrm{d}\mu)$:

$$
f \mapsto f(\hat{q}, \hat{p}), \quad \rho \mapsto \hat{\rho}.
$$

Next we move to a general quantum system. Using the fact that such system in some classical and quantum canonical coordinates is equivalent with the Moyal quantization of the corresponding classical system we received the operator representation of the given quantum system. Similarly as in the operator representation over a phase space also in this case the symmetric ordering had to be replaced by some other ordering of operators $\hat{q}^i, \hat{p}_j$. The received theory allowed to describe quantum mechanics in the Hilbert space formalism in a consistent way for any coordinate system on the configuration space, something which was not done before. Furthermore, an invariant form of the operator representation is presented.

We end up this chapter with examples of quantum mechanical operators corresponding to observables linear, quadratic and cubic in momenta. Moreover, the developed theory of quantization is illustrated with an example of the hydrogen atom.

Finally, Chapter 5 presents a theory of quantum trajectories based on the developed formalism. The quantum trajectories are defined, in an analogy with the classical case, as integral curves of quantum Hamiltonian vector fields. We present in explicit form a quantum action of a quantum flow on observables, which is a deformation of the respective classical action. The resulting time dependence of observables gives an appropriate solution of a quantum time evolution equation for observables (Heisenberg's representation on a phase space). Then, we show that a set of quantum symplectomorphisms (quantum flows) has a structure of a group with multiplication (quantum composition) being a deformation of the ordinary composition considered as a multiplication in a group of classical symplectomorphisms (classical flows). The explicit form of the quantum composition law is presented. Such approach to quantum trajectories have a benefit in that it is not needed to calculate the form of observables as $\star$-functions, but only a quantum action of a given trajectory needs to be found.

In Chapter 6 is given a summary of the thesis and an outlook on a further development of the received results.

Throughout the thesis we will use the Einstein summation convention over any twice repeated index if it appears once as a subscript and once as a superscript. By Latin letters $i, j, k, \ldots$ we will denote indices ranging from 0 to N and by Greek letters $\alpha, \beta, \gamma, \ldots$ indices ranging from 0 to $2N$. The complex-conjugation of f will be denoted by $\bar{f}$. Often partial derivatives ∂_{q^i} of tensors $t^{k\ldots l}_{m\ldots n}$ will be denoted by $t^{k\ldots l}_{m\ldots n,i}$ and covariant derivatives ∇_i by $t^{k\ldots l}_{m\ldots n;i}$.

The results presented in Chapters 3–5 are published in our papers [40–45].

Chapter 2

Classical mechanics

2.1 Phase space

The theory of classical Hamiltonian mechanics is described in an elegant language of differential geometry. The central role in this description is played by a symplectic manifold. The symplectic manifold represents a phase space of the system, which points are interpreted as states of the system. More details about classical Hamiltonian mechanics the reader can find in [46, 47].

Definition 2.1.1. A symplectic manifold is a smooth manifold M endowed with a 2-form ω which is closed ($\mathrm{d}\omega = 0$) and non-degenerate.

It can be proved that every symplectic manifold (M, ω) is necessarily even-dimensional.

Let us denote by $C^\infty(M)$ the space of all smooth complex-valued functions defined on a manifold M. On $C^\infty(M)$ we can introduce a point-wise product of functions

$$(f \cdot g)(x) \equiv (fg)(x) = f(x)g(x), \tag{2.1.1}$$

which will make from $C^\infty(M)$ a commutative algebra.

The symplectic structure distinguishes a class of vector fields on a symplectic manifold. Namely, for every $f \in C^\infty(M)$ we define a vector field ζ_f, called a Hamiltonian field, by the formula

$$\omega(\zeta_f) = \mathrm{d}f, \tag{2.1.2}$$

(here ω is treated as a map $\mathfrak{X}(M) \to \Omega^1(M)$, where $\mathfrak{X}(M)$ and $\Omega^1(M)$ denote the spaces of all smooth vector fields and 1-forms on M respectively, which is given by the formula $V \mapsto \omega(\,\cdot\,, V)$, i.e. $V^\mu \mapsto \omega_{\mu\nu}V^\nu$). On the space $C^\infty(M)$ can be defined a bilinear map $\{\,\cdot\,,\,\cdot\,\}$, called a Poisson bracket, by the formula

$$\{f, g\} = \omega(\zeta_g, \zeta_f) = \mathrm{d}f(\zeta_g) = \zeta_g f. \tag{2.1.3}$$

The Poisson bracket satisfies the following properties:

$$\{f, g\} = -\{g, f\} \qquad \text{(antisymmetry)}, \tag{2.1.4a}$$

$$\{f, gh\} = \{f, g\}h + g\{f, h\} \qquad \text{(Leibniz's rule)}, \tag{2.1.4b}$$

$$0 = \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} \qquad \text{(Jacobi's identity)}. \tag{2.1.4c}$$

Property (2.1.4a) is a consequence of the antisymmetry of the symplectic form ω. Property (2.1.4b) follows from the fact that ζ_f is a derivation of the algebra $C^\infty(M)$. Property (2.1.4c) is a consequence of the closedness of the symplectic form ω. Properties (2.1.4a) and (2.1.4c) state that the Poisson bracket is a Lie bracket on $C^\infty(M)$. The space $C^\infty(M)$ together with the point-wise product of functions, the Poisson bracket, and an involution being the complex-conjugation of functions $f \mapsto \bar{f}$, will be denoted by $\mathcal{A}_C(M)$ and called a Poisson algebra.

In the theory of classical Hamiltonian mechanics to every measurable quantity, like energy, momentum, position, etc., corresponds a smooth real-valued function in $C^\infty(M)$. Thus, elements of the Poisson algebra $\mathcal{A}_C(M)$, self-conjugated with respect to the involution in $\mathcal{A}_C(M)$, are called observables.

Note, that Hamiltonian fields satisfy the following properties

$$\zeta_{f+\mathrm{const}} = \zeta_f, \tag{2.1.5a}$$
$$\zeta_f + \lambda\zeta_g = \zeta_{f+\lambda g}, \tag{2.1.5b}$$
$$[\zeta_f, \zeta_g] = \zeta_{\{g,f\}}, \tag{2.1.5c}$$

for $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{C}$. Thus a space $\mathrm{Ham}(M)$ of all Hamiltonian fields is a Lie algebra and the map $\zeta\colon \mathcal{A}_C(M) \to \mathrm{Ham}(M)$, $f \mapsto \zeta_f$ is a homomorphism of Lie algebras whose kernel being constituted by the constant functions on M. Moreover, observe that Hamiltonian fields preserve the symplectic form ω:

$$\mathcal{L}_{\zeta_f}\omega = 0, \tag{2.1.6}$$

where $\mathcal{L}_{\zeta_f}$ denotes a Lie derivative in the direction ζ_f.

The symplectic form ω on a manifold M induces a two-times contravariant antisymmetric and non-degenerate tensor field $\mathcal{P}$ through the formula

$$\mathcal{P} \circ \omega = \hat{1} \quad \text{i.e. in local coordinates} \quad \mathcal{P}^{\alpha\gamma}\omega_{\gamma\beta} = \delta^\alpha_\beta, \tag{2.1.7}$$

(here $\mathcal{P}$ is treated as a map $\Omega^1(M) \to \mathfrak{X}(M)$ given by the formula $\alpha \mapsto \mathcal{P}(\,\cdot\,, \alpha)$, i.e. $\alpha_\mu \mapsto \mathcal{P}^{\mu\nu}\alpha_\nu$). Thus, $\mathcal{P}$ is the inverse of the symplectic form ω and often the components $\mathcal{P}^{\alpha\beta}$ of the tensor field $\mathcal{P}$ will be denoted by $\omega^{\alpha\beta}$. The tensor $\mathcal{P}$ satisfies the equality

$$\mathcal{L}_{\zeta_f}\mathcal{P} = 0 \tag{2.1.8}$$

and is called a Poisson tensor. In general, a two-times contravariant antisymmetric tensor field $\mathcal{P}$ satisfying (2.1.8) is called a Poisson tensor and a smooth manifold M endowed with a Poisson tensor is called a Poisson manifold. Note, that there is a one-to-one correspondence between symplectic forms and non-degenerate Poisson tensors on a given manifold M.

The definition of the Hamiltonian fields and the Poisson bracket can be restated in terms of the Poisson tensor:

$$\zeta_f = \mathcal{P}(\mathrm{d}f), \tag{2.1.9}$$
$$\{f, g\} = \mathcal{P}(\mathrm{d}f, \mathrm{d}g). \tag{2.1.10}$$

On a symplectic manifold (M, ω) there exists another useful structure, namely a distinguished volume form Ω_ω defined, up to a multiplicative constant, as an N-fold exterior product of the symplectic forms ω

$$\Omega_\omega \equiv \Omega = (-1)^{N(N+1)/2} \frac{1}{N!} \underbrace{\omega \wedge \cdots \wedge \omega}_{N}. \tag{2.1.11}$$

The volume form Ω_ω is called a Liouville form or phase volume form.

An example of a symplectic manifold, on which we will mainly focus in the rest of the thesis, is a cotangent bundle to a smooth manifold. Let $\mathcal{Q}$ be a smooth N-dimensional manifold, then we define a set

$$T^*\mathcal{Q} = \bigcup_{q \in \mathcal{Q}} T^*_q \mathcal{Q}. \tag{2.1.12}$$

Each point x in $T^*\mathcal{Q}$ can be parametrized by a pair (q, p) for some $q \in \mathcal{Q}$ and $p \in T^*_q\mathcal{Q}$. We can also define a canonical projection $\pi\colon T^*\mathcal{Q} \to \mathcal{Q}$, $x \mapsto q$ for $x = (q, p)$. The set $T^*\mathcal{Q}$ can be naturally endowed with a structure of a smooth $2N$-dimensional manifold. Indeed, an atlas on $\mathcal{Q}$ naturally induces an atlas on $T^*\mathcal{Q}$. If $(\mathcal{O}, \psi)$, $\psi\colon q \mapsto (q^1, \dots, q^N)$ is a chart on $\mathcal{Q}$, then for every $x = (q, p)$ in $\hat{\mathcal{O}} = \pi^{-1}(\mathcal{O})$ we can decompose $p \in T^*_q\mathcal{Q}$ with respect to the coordinate basis

$$p = p_i \, \mathrm{d}q^i\big|_q, \quad (p_1, \dots, p_N) \in \mathbb{R}^N \tag{2.1.13}$$

and a map $\hat{\psi}\colon x \mapsto (q^1, \dots, q^N, p_1, \dots, p_N)$ is a chart on $\hat{\mathcal{O}} \subset T^*\mathcal{Q}$ induced by the chart ψ on $\mathcal{O} \subset \mathcal{Q}$. The chart $(\hat{\mathcal{O}}, \hat{\psi})$ is called a canonical coordinate system on $T^*\mathcal{Q}$ and the manifold $T^*\mathcal{Q}$ is called a cotangent bundle to the manifold $\mathcal{Q}$.

On $T^*\mathcal{Q}$ we can define a canonical 1-form θ by the formula

$$\langle \theta_x, w \rangle = \langle p, \mathrm{d}\pi(x) w \rangle \tag{2.1.14}$$

for $w \in T_x T^*\mathcal{Q}$ and $x = (q, p)$. The form θ in canonical coordinates on $T^*\mathcal{Q}$ reads

$$\theta = p_i \, \mathrm{d}q^i. \tag{2.1.15}$$

Moreover, on $T^*\mathcal{Q}$ there exists a natural exact symplectic form ω given by $\omega = \mathrm{d}\theta$ or in canonical coordinates on $T^*\mathcal{Q}$

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i. \tag{2.1.16}$$

Thus, $T^*\mathcal{Q}$ is always a symplectic manifold. Usually in classical mechanics as the manifold $\mathcal{Q}$ is taken a Riemannian manifold. The manifold $\mathcal{Q}$ represents a configuration space of the system.

If $(q^1, \dots, q^N)$ and $(q'^1, \dots, q'^N)$ are two coordinate systems on $\mathcal{Q}$, (q^i, p_j) and (q'^i, p'_j) are two corresponding canonical coordinate systems on $T^*\mathcal{Q}$, and a map $\phi\colon (q'^1, \dots, q'^N) \mapsto (q^1, \dots, q^N)$ is a transformation between the two coordinate systems on $\mathcal{Q}$, then a corresponding transformation $T\colon (q'^i, p'_j) \mapsto (q^i, p_j)$ between the canonical coordinate systems on $T^*\mathcal{Q}$ is of the form

$$\begin{aligned} q^i &= \phi^i(q'), \\ p_i &= [(\phi'(q'))^{-1}]^j_i p'_j, \end{aligned} \tag{2.1.17}$$

where $[(\phi'(q'))^{-1}]^j_i$ denotes an inverse matrix to the Jacobian matrix $[\phi'(q')]^i_j = \dfrac{\partial \phi^i}{\partial q^j}(q')$ of ϕ. The transformation T is called a point transformation.

Example 2.1.1. Let us take as the manifold $\mathcal{Q}$ an Euclidean space E^N. An N-dimensional Euclidean space E^N is defined as a non-empty set E^N together with an N-dimensional real vector space V endowed with a scalar product $(\,\cdot\,,\,\cdot\,)$, and an operation (called addition or translation)

$$E^N \times V \ni (q, v) \mapsto q + v \in E^N, \tag{2.1.18}$$

satisfying the following conditions

(i) for $q \in E^N$ and $v, w \in V$ holds the equality

$$(q + v) + w = x + (v + w), \tag{2.1.19}$$

(ii) for $q_1, q_2 \in E^N$ there exists exactly one vector $v \in V$ such that $q_2 = q_1 + v$.

The space V is called a space of free vectors of E^N.

On an Euclidean space E^N we can introduce a Cartesian coordinate system. Let us choose a point $q_0 \in E^N$ and an orthonormal basis $e_1, \ldots, e_N$ on a space V of free vectors of E^N. Define a map $\psi\colon \mathbb{R}^N \to E^N$ by the formula

$$\psi(q^1, \ldots, q^N) = q_0 + q^i e_i. \tag{2.1.20}$$

The map ψ is called a Cartesian coordinate system on the Euclidean space E^N. The point q_0 is called an origin, and the vectors $e_1, \ldots, e_N$ axis vectors of the coordinate system.

An Euclidean space E^N is naturally endowed with a structure of an N-dimensional Riemannian manifold. Indeed, a set of all Cartesian coordinate systems (defined for different origins $q_0 \in E^N$ and axis vectors $e_1, \ldots, e_N \in V$) constitutes a smooth atlas on E^N. Moreover, the scalar product $(\,\cdot\,,\,\cdot\,)$ on the space V of free vectors induces a metric tensor g on E^N. Note, that tangent spaces $T_q E^N$ are naturally isomorphic to the space V of free vectors. Thus, the tangent and cotangent bundles TE^N and T^*E^N can be identified with Cartesian products $E^N \times V$ and $E^N \times V^*$ respectively.

Let ψ be a Cartesian coordinate system on E^N with an origin $q_0 \in E^N$ and axis vectors $e_1, \ldots, e_N \in V$. A canonical coordinate system on $T^*E^N = E^N \times V^*$ induced by ψ is a map $\hat{\psi}\colon T^*E^N \to \mathbb{R}^{2N}$, $x = (q, p) \mapsto (q^1, \ldots, q^N, p_1, \ldots, p_N)$, for $q = q_0 + q^i e_i$ and $p = p_i e^i$ where $e^1, \ldots, e^N$ is a dual basis to $e_1, \ldots, e_N$. The canonical coordinate system $\hat{\psi}$ will be called a Cartesian coordinate system on T^*E^N.

2.2 Coordinate systems

On a symplectic manifold (M, ω) there exists a distinguished class of coordinate systems, namely local coordinates $(q^1, \ldots, q^N, p_1, \ldots, p_N)$ in which the symplectic form takes the canonical form

$$\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i \quad \text{i.e.} \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}. \tag{2.2.1}$$

These coordinates are called canonical coordinates or Darboux coordinates and they always exist on a symplectic manifold, which is guaranteed by the Darboux theorem.

In canonical coordinates all objects introduced in the previous section take the form

$$\mathcal{P} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i} = \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_i} \otimes \frac{\partial}{\partial q^i}, \tag{2.2.2a}$$

$$\zeta_f = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}, \tag{2.2.2b}$$

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}, \tag{2.2.2c}$$

$$\Omega_\omega = \mathrm{d}q^1 \wedge \cdots \wedge \mathrm{d}q^N \wedge \mathrm{d}p_1 \wedge \cdots \wedge \mathrm{d}p_N. \tag{2.2.2d}$$

Note, that canonical coordinates on a cotangent bundle $T^*\mathcal{Q}$ to a manifold $\mathcal{Q}$ are example of canonical coordinates in the sense of the definition in this section.

Canonical coordinates can be equivalently defined in the following way. Coordinates $(x^1, \dots, x^{2N}) = (q^1, \dots, q^N, p_1, \dots, p_N)$ are canonical iff

$$\{x^\alpha, x^\beta\} = \mathcal{J}^{\alpha\beta}, \tag{2.2.3}$$

where

$$(\mathcal{J}^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \tag{2.2.4}$$

or equivalently

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{q^i, p_j\} = \delta^i_j. \tag{2.2.5}$$

The functions q^i and p_j are observables of position and momentum associated with the coordinate system $(q^1, \dots, q^N, p_1, \dots, p_N)$.

In classical statistical mechanics appear integrals over a phase space (cf. Section 2.3), which cannot be considered in arbitrary local coordinates, since doing this would change the values of integrals. For example, if $\psi\colon M \supset \mathcal{O} \to \mathbb{R}^{2N}$, $\psi\colon x \mapsto (x^1, \dots, x^{2N})$ is some coordinate chart, then in general

$$\int_M f \, \mathrm{d}\Omega \neq \int_{\psi(\mathcal{O})} f(\psi^{-1}(x)) \, \mathrm{d}x \tag{2.2.6}$$

where f is some function defined on M and $\mathrm{d}\Omega$ is a measure induced by the Liouville form Ω_ω. These integrals will be equal only when $M \setminus \mathcal{O}$ is of measure zero. For this reason we introduce the following definition. A coordinate system $\psi\colon M \supset \mathcal{O} \to \mathbb{R}^{2N}$ on a symplectic manifold (M, ω) is called almost global if $M \setminus \mathcal{O}$ is of measure zero with respect to the measure $\mathrm{d}\Omega$. Similarly, if $(\mathcal{Q}, g)$ is a Riemannian manifold representing a configuration space, then by an almost global coordinate system on $\mathcal{Q}$ we mean a coordinate system defined on an open subset $U \subset \mathcal{Q}$ such that $\mathcal{Q} \setminus U$ is of measure zero with respect to the measure induced by the metric volume form ω_g. It can be proved that an almost global coordinate system on $\mathcal{Q}$ induces a canonical coordinate system on $T^*\mathcal{Q}$ with the same property. In what follows we will mainly focus on almost global coordinate systems and consider only such manifolds which admit such coordinates.

Example 2.2.1. Let $\mathcal{Q} = E^3$ and consider on E^3 a Cartesian coordinates (x, y, z). Consider also on E^3 a spherical polar coordinates (r, θ, ϕ) related to the Cartesian coordinates by a transformation $\phi\colon (0,\infty) \times (0,\pi) \times (0,2\pi) \to \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^3 \setminus \{(x,y,z) \in \mathbb{R}^3 \mid x \geq 0, y = 0\}$, $\phi\colon (r,\theta,\phi) \mapsto (x,y,z)$,

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned} \tag{2.2.7}$$

In the Cartesian coordinates (x,y,z) the metric volume form ω_g on E^3 is equal $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, and the corresponding measure $\mathrm{d}\omega_g$ takes the form of the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. It can be seen that a set $\mathbb{R}^3 \setminus \mathcal{O}$ is of Lebesgue-measure zero, hence the spherical polar coordinates (r,θ,ϕ) are almost global on E^3.

Let (x,y,z,p_x,p_y,p_z) be canonical coordinates on T^*E^3 induced by the Cartesian coordinates (x,y,z) on E^3. In accordance to (2.1.17) canonical coordinates $(r,\theta,\phi,p_r,p_\theta,p_\phi)$ on T^*E^3 induced by the spherical polar coordinates (r,θ,ϕ) are related to the Cartesian coordinates (x,y,z,p_x,p_y,p_z) by a transformation $T\colon (0,\infty) \times (0,\pi) \times (0,2\pi) \times \mathbb{R}^3 \to \hat{\mathcal{O}} = \mathcal{O} \times \mathbb{R}^3$, $T\colon (r,\theta,\phi,p_r,p_\theta,p_\phi) \mapsto (x,y,z,p_x,p_y,p_z)$,

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta, \\ p_x &= \frac{r p_r \sin^2\theta \cos\phi + p_\theta \sin\theta \cos\theta \cos\phi - p_\phi \sin\phi}{r \sin\theta}, \\ p_y &= \frac{r p_r \sin^2\theta \sin\phi + p_\theta \sin\theta \cos\theta \sin\phi + p_\phi \cos\phi}{r \sin\theta}, \\ p_z &= \frac{r p_r \cos\theta - p_\theta \sin\theta}{r}. \end{aligned} \tag{2.2.8}$$

In the Cartesian coordinates (x,y,z,p_x,p_y,p_z) the Liouville form Ω_ω on T^*E^3 is equal $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}p_x \wedge \mathrm{d}p_y \wedge \mathrm{d}p_z$, and the corresponding measure $\mathrm{d}\Omega$ takes the form of the Lebesgue measure $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z$. It can be seen that a set $\mathbb{R}^6 \setminus \hat{\mathcal{O}}$ is of Lebesgue-measure zero, hence the canonical coordinates $(r,\theta,\phi,p_r,p_\theta,p_\phi)$ are almost global on T^*E^3.

2.3 Classical states

The points in a phase space (M,ω) represent states of the system. Each point in M can be interpreted as generalized positions and momenta of particles composing the classical system. Values of generalized positions and momenta of the particles can be extracted from a point in M by writing this point in canonical coordinates (q^i,p_j). Then, q^i are values of generalized positions and p_j are values of generalized momenta.

When the exact state of the system is not known, but only a probability that the state is in a given region of the phase space, then there is a need to extend the concept of a state to take into account such situation. The most natural way to do this is to define states as probabilistic measures μ defined on a σ-algebra $\mathfrak{B}(M)$ of

Borel subsets of M. In such setting points x of the phase space can be identified with Dirac measures δ_x

$$\delta_x(E) = \begin{cases} 1 & \text{for } x \in E \\ 0 & \text{for } x \notin E \end{cases}, \quad E \in \mathfrak{B}(M). \tag{2.3.1}$$

Dirac measures will be called pure states and other probabilistic measures mixed states.

Some probabilistic measures μ can be written in a form $\mathrm{d}\mu = \rho\,\mathrm{d}\Omega$, where ρ is some integrable function on M satisfying

$$\int_M \rho\,\mathrm{d}\Omega = 1 \qquad \text{(normalization)}, \tag{2.3.2a}$$

$$\rho \geq 0 \qquad \text{(positive-definiteness)}, \tag{2.3.2b}$$

and thus can be identified with functions ρ. In what follows every probabilistic measure μ we will formally write in the form $\mathrm{d}\mu = \rho\,\mathrm{d}\Omega$. In particular, for Dirac measures we will use a notation $\mathrm{d}\delta_x(y) = \delta(x, y)\,\mathrm{d}\Omega(y)$.

Observe, that states can be alternatively defined as those "functions" ρ which satisfy

(i) $\rho = \bar{\rho}$ (self-conjugation),

(ii) $\int_M \rho\,\mathrm{d}\Omega = 1$ (normalization),

(iii) $\int_M \bar{f} \cdot f \cdot \rho\,\mathrm{d}\Omega \geq 0$ for $f \in C_0^\infty(M)$ (positive-definiteness),

where $C_0^\infty(M)$ denotes a space of all smooth functions with compact support defined on M. Indeed, (iii) is equivalent with $\rho \geq 0$.

Classical states form a convex set. Pure states can be defined as extreme points of the set of states, i.e. as those states which cannot be written as convex linear combinations of some other states. In other words ρ_{pure} is a pure state if and only if there do not exist two different states ρ_1 and ρ_2 such that $\rho_{\text{pure}} = p\rho_1 + (1-p)\rho_2$ for some $p \in (0, 1)$. It can be proved that such characterization of pure states is equivalent with the definition of pure states as Dirac measures.

For a given observable $A \in C^\infty(M)$ and state μ ($\mathrm{d}\mu = \rho\,\mathrm{d}\Omega$) the expectation value of the observable A in the state μ is defined by

$$\langle A \rangle_\mu = \int_M A\,\mathrm{d}\mu = \int_M A \cdot \rho\,\mathrm{d}\Omega. \tag{2.3.3}$$

Note, that the expectation value of the observable A in a pure state δ_x is equal $A(x)$. Indeed,

$$\langle A \rangle_{\delta_x} = \int_M A(y)\delta(x, y)\,\mathrm{d}\Omega(y) = A(x). \tag{2.3.4}$$

2.4 Time evolution of classical systems

One of the observables in the algebra $\mathcal{A}_C(M)$ has a special purpose, namely a Hamiltonian H. This is some distinguished real valued smooth function on M and it corresponds to the total energy of the system. The phase space (M, ω) together with the Hamiltonian H is called a classical Hamiltonian system.

The Hamiltonian H governs the time evolution of the system. Indeed, H generates a Hamiltonian field ζ_H. Integral curves $x(t)$ of the vector field ζ_H, i.e. curves on M satisfying

$$\dot{x}(t) = \zeta_H(x(t)), \tag{2.4.1}$$

represent positions of points $x \in M$ for every instance of time t, which is interpreted as the time development of pure states. Integral curves of a Hamiltonian field ζ_H generate a map $\Phi_t^H \colon M \to M$ (called a phase flow or a Hamiltonian flow) by a prescription: for each point $x \in M$ a curve

$$x(t) = \Phi_t^H(x) \tag{2.4.2}$$

is an integral curve of ζ_H passing through the point x at time $t = 0$. Equation (2.4.1) is called a Hamilton equation and integral curves of the Hamiltonian field are called classical trajectories. In canonical coordinates (q^i, p_j), using formula (2.2.2b), the Hamilton equation takes a form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{2.4.3}$$

An equation of motion of mixed states can be derived from the probability conservation law. From this law follows that every probabilistic measure μ (mixed state) should be constant along any trajectory in the phase space, i.e.

$$\mu(t)(E) = \mu(t + \Delta t)(\Phi_{\Delta t}^H(E)), \quad E \in \mathfrak{B}(M), \tag{2.4.4}$$

which can be written in terms of the pull-back of a measure

$$\mu(t) = (\Phi_{\Delta t}^H)^* \mu(t + \Delta t). \tag{2.4.5}$$

From the above equation it follows that

$$\begin{aligned} 0 &= \lim_{\Delta t \to 0} \frac{(\Phi_{\Delta t}^H)^* \mu(t + \Delta t) - \mu(t)}{\Delta t} = \frac{\mathrm{d}}{\mathrm{d}s} (\Phi_s^H)^* \mu(t + s) \bigg|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} (\Phi_0^H)^* \mu(t + s) \bigg|_{s=0} + \frac{\mathrm{d}}{\mathrm{d}s} (\Phi_s^H)^* \mu(t) \bigg|_{s=0}, \end{aligned} \tag{2.4.6}$$

which implies that

$$\frac{\partial \mu}{\partial t} + \mathcal{L}_{\zeta_H} \mu = 0, \tag{2.4.7}$$

where $\mathcal{L}_{\zeta_H} \mu$ denotes a Lie derivative of the measure μ in the direction of the vector field ζ_H. Equation (2.4.7) is called a Liouville equation and it describes the time development of the state μ.

Let us check if for a pure state $\delta_{x(t)}$ the Liouville equation (2.4.7) is equivalent to the Hamilton equation (2.4.1). From (2.4.7) it follows that

$$0 = \frac{\partial \delta_{x(t)}}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}s}(\Phi_s^H)^* \delta_{x(t)}\Big|_{s=0} = \frac{\partial \delta_{x(t)}}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}s}\delta_{\Phi_{-s}^H(x(t))}\Big|_{s=0}. \tag{2.4.8}$$

From the above equation we get

$$0 = \dot{x}(t) - \frac{\mathrm{d}}{\mathrm{d}s}\Phi_s^H(x(t))\Big|_{s=0} = \dot{x}(t) - \zeta_H(x(t)), \tag{2.4.9}$$

which is just the Hamilton equation (2.4.1).

If a mixed state μ can be written in a form $\mathrm{d}\mu = \rho\, \mathrm{d}\Omega$ for a smooth function ρ, then the Liouville equation (2.4.7) can be written in a different form. Indeed, from (2.4.7) we get

$$0 = \frac{\partial}{\partial t}(\rho(t)\Omega) + \mathcal{L}_{\zeta_H}(\rho(t)\Omega) = \left(\frac{\partial \rho}{\partial t}(t) + \mathcal{L}_{\zeta_H}\rho(t)\right)\Omega, \tag{2.4.10}$$

where the fact that $\mathcal{L}_{\zeta_H}\Omega = 0$, following from (2.1.6), was used. The above equation implies that

$$0 = \frac{\partial \rho}{\partial t} + \mathcal{L}_{\zeta_H}\rho = \frac{\partial \rho}{\partial t} + \zeta_H \rho = \frac{\partial \rho}{\partial t} + \{\rho, H\}. \tag{2.4.11}$$

Hence, the following time evolution equation for the function ρ corresponding to the state μ was received

$$\frac{\partial \rho}{\partial t} - \{H, \rho\} = 0. \tag{2.4.12}$$

Until now the states undergo the time development whereas the observables do not. This corresponds to the Schrödinger picture in quantum mechanics. There is also a dual point of view (which, in turn, corresponds in quantum mechanics to the Heisenberg picture), in which states remain still whereas the observables undergo the time development. A pull-back of the Hamiltonian flow $U_t^H = (\Phi_t^H)^* = e^{t\mathcal{L}_{\zeta_H}}$ is, for every t, an automorphism of the algebra of observables $\mathcal{A}_C(M)$ (it preserves the linear structure as well as the point-wise product and the Poisson bracket). Its action on an arbitrary observable $A \in \mathcal{A}_C(M)$ is interpreted as the time development of A

$$A(t) = U_t^H A(0) = e^{t\mathcal{L}_{\zeta_H}} A(0) = e^{t\zeta_H} A(0) = e^{-t\{H,\cdot\}} A(0). \tag{2.4.13}$$

Differentiating equation (2.4.13) with respect to t we receive the following time evolution equation for an observable A

$$\frac{\mathrm{d}A}{\mathrm{d}t}(t) - \{A(t), H\} = 0. \tag{2.4.14}$$

Let q^i, p_j be observables of position and momentum corresponding to a canonical coordinate system (q^i, p_j), i.e. $q^i(x,0), p_j(x,0)$ are coordinates of a point $x \in M$. From (2.4.14) we get the following system of equations

$$\frac{\mathrm{d}q^i}{\mathrm{d}t}(t) - \{q^i(t), H\} = 0, \quad \frac{\mathrm{d}p_i}{\mathrm{d}t}(t) - \{p_i(t), H\} = 0, \tag{2.4.15}$$

which are just the Hamilton equations (2.4.3) written in a different form. Indeed, a solution of (2.4.15) is of a form $q^i(x,t) = q^i(t)$ and $p_i(x,t) = p_i(t)$ where $q^i(t)$ and $p_i(t)$ are solutions of the Hamilton equations (2.4.3).

Both presented approaches to the time development yield equal predictions concerning the results of measurements, since

$$\begin{aligned}\langle A(0)\rangle_{\mu(t)} &= \int_M A(0)\,\mathrm{d}\mu(t) = \int_M A(0)\,\mathrm{d}\big((\Phi^H_{-t})^*\mu(0)\big) = \int_M (\Phi^H_t)^* A(0)\,\mathrm{d}\mu(0)\\ &= \int_M A(t)\,\mathrm{d}\mu(0) = \langle A(t)\rangle_{\mu(0)}.\end{aligned} \tag{2.4.16}$$

Chapter 3

Quantization of classical mechanics

3.1 Deformation theory of symplectic manifolds

One of the approaches to quantization is deformation quantization developed by Bayen et al. [48, 49, 50]. In this approach quantum mechanics is formulated as a deformation of classical mechanics. Such procedure results in a quantum theory described in a geometric language similar to that of its classical counterpart. This allows introduction in quantum mechanics many concepts from the classical theory, like coordinate systems. Moreover, the formalism of deformation quantization gives a smooth passage from classical to quantum theory, which makes it easy to investigate the classical limit of quantum mechanics.

The main ingredient of deformation quantization is a formal deformation of a Poisson algebra $C^\infty(M)$ of smooth complex-valued functions defined on a phase space M (symplectic manifold). The procedure of formal deformation is based on the Gerstenhaber's theory of deformations of rings and algebras [51]. For a recent review on a subject of deformation quantization refer to [52]. Let $\mathbb{C}[\![\nu]\!]$ denote the ring of formal power series in the parameter ν with coefficients in $\mathbb{C}$ and let $C^\infty(M)[\![\nu]\!]$ be the space of formal power series in ν with coefficients in $C^\infty(M)$. The space $C^\infty(M)[\![\nu]\!]$ is a $\mathbb{C}[\![\nu]\!]$-module.

Definition 3.1.1. A star-product on a symplectic manifold (M, ω) is a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[\![\nu]\!], \quad (f, g) \mapsto f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \tag{3.1.1}$$

which extends $\mathbb{C}[\![\nu]\!]$-linearly to $C^\infty(M)[\![\nu]\!] \times C^\infty(M)[\![\nu]\!]$, such that

(i) C_k are bidifferential operators,

(ii) $(f \star g) \star h = f \star (g \star h)$ (associativity),

(iii) $C_0(f, g) = fg$, $C_1(f, g) - C_1(g, f) = \{f, g\}$,

(iv) $1 \star f = f \star 1 = f$.

One also defines a deformed Poisson bracket by the formula

$$[\![f, g]\!]_\star = \frac{1}{\nu}[f, g]_\star = \frac{1}{\nu}(f \star g - g \star f), \tag{3.1.2}$$

and a formal involution as an antilinear map

$$C^\infty(M) \to C^\infty(M)[\![\nu]\!], \quad f \mapsto f^* = \sum_{k=0}^{\infty} \nu^k B_k(f), \tag{3.1.3}$$

which extends $\mathbb{C}[\![\nu]\!]$-antilinearly to $C^\infty(M)[\![\nu]\!]$, where

(i) B_k are antilinear operators,

(ii) $(f \star g)^* = g^* \star f^*$,

(iii) $(f^*)^* = f$,

(iv) $B_0(f) = \bar{f}$.

From the above definitions it is clear that the $\star$-product, deformed Poisson bracket $[\![\,\cdot\,,\,\cdot\,]\!]_\star$, and involution $*$ are deformations of the point-wise product of functions $\cdot$, Poisson bracket $\{\,\cdot\,,\,\cdot\,\}$, and complex-conjugation:

$$\begin{aligned} f \star g &= fg + o(\nu), \\ [\![f, g]\!]_\star &= \{f, g\} + o(\nu), \\ f^* &= \bar{f} + o(\nu). \end{aligned} \tag{3.1.4}$$

The associativity of the $\star$-product implies that the bidifferential operators C_k satisfy the equations

$$\sum_{n=0}^{k} \big(C_n(C_{k-n}(f, g), h) - C_n(f, C_{k-n}(g, h))\big) = 0, \quad k = 1, 2, \dots . \tag{3.1.5}$$

The deformation of the Poisson algebra $C^\infty(M)$ can be though of as a deformation of a geometrical structure of the symplectic manifold M. The symplectic manifold M is fully described by the Poisson algebra $C^\infty(M)$. Thus by the deformation of $C^\infty(M)$ to some non-commutative algebra we can think of it as describing a non-commutative symplectic manifold.

The existence of a star-product on any symplectic manifold was first proved in 1983 by De Wilde and Lecomte [53]. Later Fedosov [54] gave a recursive construction of a star-product on a symplectic manifold using the framework of Weyl bundles. Independently, Omori et al. [55] gave an alternative proof of the existence of a star-product on a symplectic manifold, also using the framework of Weyl bundles. Finally, in 1997, Kontsevich [56] proved the existence of a star-product on any Poisson manifold.

Let $\star$ and $\star'$ be two star-products on a symplectic manifold (M, ω). These star-products are said to be equivalent if there exists a series

$$S = \sum_{k=0}^{\infty} \nu^k S_k, \quad S_0 = \mathrm{id}, \tag{3.1.6}$$

where S_k are differential operators on $C^\infty(M)$, such that

$$S(f \star g) = Sf \star' Sg. \tag{3.1.7}$$

Alternatively, having a star-product on (M, ω) and a series (3.1.6) one can define a new star-product on (M, ω) by the formula (3.1.7). It can be easily checked that the new star-product indeed will satisfy conditions (i)–(iv) from the definition of a star-product.

The study of equivalences of star-products is best performed in the language of Hochschild cohomologies [51]. The relation of equivalence of star-products is an equivalence relation, thus the set of all star-products on a given symplectic manifold is divided into disjoint equivalence classes. The following result, first received by Nest and Tsygan [57], Bertelson et al. [58], and Deligne [59], characterizes the equivalence classes of star-products.

Theorem 3.1.1. *The equivalence classes of star-products on a symplectic manifold M are parametrized by formal series of elements in the second de Rham cohomology space of M, $H^2(M;\mathbb{C})[\![\nu]\!]$.*

In particular, on a symplectic manifold M for which the second de Rham cohomology space $H^2(M;\mathbb{C})$ vanishes all star-products are equivalent.

3.2 General theory of quantization

In this section we discuss a general theory of quantization of classical Hamiltonian mechanics. Let (M, ω, H) be a classical Hamiltonian system. Such a system can be quantized in the framework of deformation quantization. According to this framework the classical Poisson algebra $\mathcal{A}_C(M) = (C^\infty(M), \cdot, \{\,\cdot\,,\,\cdot\,\}, ^-)$ is deformed to a quantum Poisson algebra $\mathcal{A}_Q(M) = (C^\infty(M)[\![\hbar]\!], \star, [\![\,\cdot\,,\,\cdot\,]\!], *)$, where as the deformation parameter ν is taken $i\hbar$ ($\hbar$ being the Planck's constant). Elements of $C^\infty(M)[\![\hbar]\!]$, self-adjoint with respect to the involution $*$ from $\mathcal{A}_Q(M)$ are observables of the quantum system. To every measurable quantity corresponds some observable. The correspondence between measurable quantities and self-adjoint elements of $C^\infty(M)[\![\hbar]\!]$ is fixed by the choice of quantization and can vary depending on the chosen quantization. In particular, quantum observables do not have to be the same functions as in the classical case; they will be an $\hbar$-deformations of classical observables. They do not even have to be real valued if the involution from $\mathcal{A}_Q(M)$ is not the complex-conjugation. So an explicit choice of quantization of a classical Hamiltonian system is fixed by a choice of both, the $\star$-product and the form of quantum observables. Note that to each classical observable corresponds the whole family of quantum observables which will reduce to the same classical observable in the classical limit. That is to say, if f_C is a classical observable then quantum observables corresponding to it are of the form

$$f = f_C + \sum_{k=1}^{\infty} \hbar^k f_k \tag{3.2.1}$$

for some functions $f_k \in C^\infty(M)$. In other words, it seems that in the quantum world there are more quantities which can be measured than in the classical world. In the classical limit different measurable quantities will reduce to the same measurable quantity.

It seems that there is no way of telling which assignment of measurable quantities to elements of $C^\infty(M)[\![\hbar]\!]$ is appropriate for a given star-product — this can be only verified through experiment. On the other hand, there is very restrictive number of known physical quantum systems, being counterparts of some classical systems. They are mainly described by so called natural Hamiltonians with flat metrics

$$H(q,p) = \frac{1}{2m} g^{ij}(q) p_i p_j + V(q), \tag{3.2.2}$$

where g^{ij} is a flat metric tensor on a configuration space. The knowledge of quantization of such systems is not enough to fix uniquely the quantization and is the source of ambiguities. In consequence, one meets in literature various versions of quantizations which coincide for the class of natural flat Hamiltonians.

If we consider two quantizations of a classical Hamiltonian system (M, ω, H), given by two star-products $\star$ and $\star'$, and two assignments of measurable quantities to elements of $C^\infty(M)[\![\hbar]\!]$, then we say that these two quantizations are equivalent if there exists a series S (3.1.6) such that (3.1.7) holds and which has the property that if A is an observable from the first quantization scheme, corresponding to a given measurable quantity, then $A' = SA$ is an observable from the second quantization scheme corresponding to the same measurable quantity. Note, that in the limit $\hbar \to 0$ both observables A and A' will reduce to the same classical observable.

In what follows we will focus on star-products of the form

$$f \star g = \sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^k C_k(f,g), \tag{3.2.3}$$

which provided the conditions (i)–(iv) from Definition 3.1.1 satisfy also the following properties:

(i) $C_k(f,g) = (-1)^k C_k(g,f)$,

(ii) $\overline{C_k(f,g)} = C_k(\bar{f}, \bar{g})$,

(iii) $\displaystyle\int_M C_k(f,g)\, \mathrm{d}\Omega = 0$ for $f, g \in C_0^\infty(M)$ and $k = 1, 2, \ldots$,

where $C_0^\infty(M)$ denotes the space of smooth compactly supported functions on M, and $\mathrm{d}\Omega$ is the Liouville measure induced by the Liouville form Ω_ω. Conditions (i) and (ii) imply that the complex-conjugation is an involution for this star-product, and from condition (iii) follows that the $\star$-product under the integral sign reduces to the ordinary point-wise product:

$$\int_M f \star g \,\mathrm{d}\Omega = \int_M fg \,\mathrm{d}\Omega, \quad f, g \in C_0^\infty(M). \tag{3.2.4}$$

However, we will not limit ourselves only to star-products of the form (3.2.3) and we will also consider, as illustrative examples, other star-products, in particular, those for which the complex-conjugation is not an involution.

In what follows let $\mathrm{d}l(x) = \dfrac{\mathrm{d}\Omega(x)}{(2\pi\hbar)^N}$ be the normalized Liouville measure and $L^2(M, \mathrm{d}l)$ a Hilbert space of functions defined on the phase space M and square integrable with respect to the measure $\mathrm{d}l$, with the scalar product given by

$$(f, g) = \int_M \overline{f(x)} g(x) \, \mathrm{d}l(x). \tag{3.2.5}$$

So far we considered a quantum Poisson algebra as a formal algebra. That way we did not had to worry about the convergence of formal series appearing during the process of formal quantization. However, such approach is not entirely physical — observables should be functions on a phase space not formal power series. A complete quantum theory require to investigate the convergence of formal series.

Let us give some remarks about the convergence of formal power series appearing in the definition of star-products. Let $\star$ be a star-product on (M, ω). In general it is not possible to find a topology on $C^\infty(M)$ such that the $\star$-product will be convergent for every pair of smooth functions. Thus we have to search for some subspace $\mathcal{A} \subset C^\infty(M)$ with appropriately chosen topology on which the $\star$-product will be convergent. Note, that functions in $\mathcal{A}$ can depend implicitly on $\hbar$. Moreover, we will require that there exists a subalgebra $\mathcal{F} \subset \mathcal{A}$ such that $\mathcal{F}$ is a dense subset of $L^2(M, \mathrm{d}l)$, and for $f, g \in \mathcal{F}$ there holds

$$\|f \star g\| \leq \|f\| \|g\|. \tag{3.2.6}$$

From (3.2.6) it follows that the $\star$-product is continuous on the subspace $\mathcal{F} \times \mathcal{F}$ with respect to the L^2-topology and consequently uniquely extends to the continuous star-product defined on the whole space $L^2(M, \mathrm{d}l)$ and satisfying (3.2.6) for every $f, g \in L^2(M, \mathrm{d}l)$, which is a direct consequence of the fact that $\mathcal{F}$ is dense in $L^2(M, \mathrm{d}l)$.

In the rest of the thesis we will not be dealing with the problem of finding the subspace $\mathcal{A}$ and its topology. In what follows we will tacitly assume that, wherever it is needed, all formal series are convergent. More on the convergence of deformation quantization the reader can find in [60–62], where the authors study the convergence in the framework of C^*-algebras (this is usually referred to as strict deformation quantization). In addition in [63, 64] is studied a non-formal deformation quantization developed in the framework of Fréchet-Poisson algebras. Worth noting are also papers [65, 66] where the convergence of a Moyal product on suitable spaces of functions is investigated.

Note, that the star-product (3.2.3) treated as a formal deformation of the point-wise product is local, i.e. if we choose some $x \in M$ then $(f \star g)(x) \in \mathbb{C}[\![\hbar]\!]$ is fully specified by the values of functions f and g in an arbitrarily small neighborhood of x. This is a direct consequence of the fact that the bidifferential operators C_k are local. However, if we will consider the convergence of the formal series (3.2.3), in general, we end up with a star-product which is not local. In other words for some $x \in M$ the value $(f \star g)(x) \in \mathbb{C}$ takes into account values of functions f and g in points far away from x. Examples of star-products with such property can be found in Section 3.4.

The Hilbert space $L^2(M, \mathrm{d}l)$ together with the $\star$-product has a structure of an algebra, denoted hereafter by $\mathcal{L}$. It is clear that for the algebra $\mathcal{L} = (L^2(M, \mathrm{d}l), \star)$

the complex-conjugation is an involution in this algebra and that under the integral sign the star-product of two functions from $L^2(M, \mathrm{d}l)$ reduces to the point-wise product. Moreover, there holds

$$(g, f \star h) = (\bar{f} \star g, h), \quad f, g, h \in L^2(M, \mathrm{d}l). \tag{3.2.7}$$

If $f \in \mathcal{A}$ and $D(f)$ is a subspace of $\mathcal{A}$ dense in $L^2(M, \mathrm{d}l)$ such that for every $\rho \in D(f)$, $f \star \rho \in L^2(M, \mathrm{d}l)$ then we can associate to f a densely defined operator $f\star$ on the Hilbert space $L^2(M, \mathrm{d}l)$, which domain is equal $D(f)$ and which satisfies

$$(f\star)^\dagger = \bar{f}\star. \tag{3.2.8}$$

Let us define a trace functional by the formula

$$\operatorname{tr}(f) = \int_M f(x)\, \mathrm{d}l(x) \tag{3.2.9}$$

for $f \in L^1(M, \mathrm{d}l)$. The $\star$-product in the algebra $\mathcal{L}$ obey the following property: the ideal $\mathcal{L}^1 = \mathcal{L} \star \mathcal{L}$ is a subset of $L^1(M, \mathrm{d}l)$ and

$$\operatorname{tr}(\bar{f} \star g) = (f, g) \tag{3.2.10}$$

for any $f, g \in L^2(M, \mathrm{d}l)$.

Remark 3.2.1. In this thesis the star-products were introduced as formal series of bidifferential operators. Then, using an appropriate topology on the space of smooth functions, these series could be made convergent. That way we can introduce a star-product on some subspace of $C^\infty(M)$ and then transfer it to the Hilbert space $L^2(M, \mathrm{d}l)$. There is however other way of introducing star-products [67]. One can first define a star-product on some subspace $\mathcal{F} \subset C^\infty(M)$ of smooth functions, which is at the same time required to be a dense subspace in $L^2(M, \mathrm{d}l)$. The subspace $\mathcal{F}$ should be endowed with some topology. Moreover, the star-product should be continuous in $\mathcal{F}$ as well as in $L^2(M, \mathrm{d}l)$, and it is usually defined by some integral formula. From there it can be easily extended to a continuous star-product on the whole space $L^2(M, \mathrm{d}l)$. Denote by $\mathcal{F}'$ the space of continuous linear functionals on $\mathcal{F}$. The elements of $\mathcal{F}'$ are distributions and the space $\mathcal{F}$ is the space of test functions. We can identify functions $f \in \mathcal{F}$ with distributions given by

$$\langle f, g \rangle = \int_M f(x)g(x)\, \mathrm{d}l(x), \quad \text{for every } g \in \mathcal{F}. \tag{3.2.11}$$

Hence, we can write $\mathcal{F} \subset \mathcal{F}'$. For $f \in \mathcal{F}'$ and $g \in \mathcal{F}$ we can define their $\star$-product by

$$\langle f \star g, h \rangle = \langle f, g \star h \rangle, \quad \langle g \star f, h \rangle = \langle f, h \star g \rangle, \quad \text{for every } h \in \mathcal{F}. \tag{3.2.12}$$

Denote by $\mathcal{F}_\star$ the following subset:

$$\mathcal{F}_\star = \{f \in \mathcal{F}' \mid f \star g \in \mathcal{F} \text{ and } g \star f \in \mathcal{F} \text{ for every } g \in \mathcal{F}\}. \tag{3.2.13}$$

In particular, $\mathcal{F} \subset \mathcal{F}_\star$. If the set $\mathcal{F}_\star$ obeys the property

$$\langle f, h \star g \rangle = \langle g, f \star h \rangle, \quad \text{for every } f, g \in \mathcal{F}_\star \text{ and } h \in \mathcal{F}, \tag{3.2.14}$$

then $\mathcal{F}_\star$ is endowed with the algebra structure

$$\langle f \star g, h \rangle = \langle f, g \star h \rangle, \quad \text{for every } f, g \in \mathcal{F}_\star \text{ and } h \in \mathcal{F}, \tag{3.2.15}$$

which is consistent with the involution $\overline{f \star g} = \bar{g} \star \bar{f}$. In such case $\mathcal{F}$ is called a normal subalgebra.

Note that the unity function 1 does not belong to $\mathcal{F}$ or $L^2(M, \mathrm{d}l)$, but is automatically an element of $\mathcal{F}_\star$ and $1 \star f = f \star 1 = f$, for every $f \in \mathcal{F}_\star$. So, $\mathcal{F}_\star$ is an involutive algebra with unity.

In the case $M = \mathbb{R}^{2N}$ the Schwartz space $\mathcal{S}(\mathbb{R}^{2N})$, i.e. the space of rapidly decreasing functions on $\mathbb{R}^{2N}$, is the normal subalgebra for the Moyal product, cf. Section 3.4 and [65].

3.2.1 Quantum states

From definition, through an analogy with the classical case (cf. Section 2.3), quantum states are those functions $\rho \in L^2(M, \mathrm{d}l)$ which satisfy the following conditions

(i) $\rho = \bar{\rho}$ (self-conjugation),

(ii) $\int_M \rho \, \mathrm{d}l = 1$ (normalization),

(iii) $\int_M \bar{f} \star f \star \rho \, \mathrm{d}l \geq 0$ for $f \in C_0^\infty(M)$ (positive-definiteness),

or equivalently

(i') $\rho = \bar{\rho}$ (self-conjugation),

(ii') $\mathrm{tr}(\rho) = 1$ (normalization),

(iii') $\mathrm{tr}(\bar{f} \star f \star \rho) \geq 0$ for $f \in C_0^\infty(M)$ (positive-definiteness).

Quantum states form a convex subset of the Hilbert space $L^2(M, \mathrm{d}l)$. Pure states are defined as extreme points of the set of states, i.e. as those states which cannot be written as convex linear combinations of some other states. In other words ρ_{pure} is a pure state if and only if there do not exist two different states ρ_1 and ρ_2 such that $\rho_{\mathrm{pure}} = p\rho_1 + (1-p)\rho_2$ for some $p \in (0, 1)$. A state which is not pure is called a mixed state.

For certain symplectic manifolds M (cf. Proposition 4.3.6) pure states can be alternatively characterized as functions $\rho_{\mathrm{pure}} \in L^2(M, \mathrm{d}l)$ which are self-conjugated, normalized, and idempotent:

$$\rho_{\mathrm{pure}} \star \rho_{\mathrm{pure}} = \rho_{\mathrm{pure}}. \tag{3.2.16}$$

Mixed states $\rho_{\text{mix}} \in L^2(M, \mathrm{d}l)$ can be characterized as convex linear combinations, possibly infinite, of some families of pure states $\rho^{(\lambda)}_{\text{pure}}$

$$\rho_{\text{mix}} = \sum_{\lambda} p_\lambda \rho^{(\lambda)}_{\text{pure}}, \tag{3.2.17}$$

where $p_\lambda \geq 0$ and $\sum_{\lambda} p_\lambda = 1$.

The interpretation of pure and mixed states is similar as in classical mechanics. When we have the full knowledge of the state of the system then the system is described by a pure state. If we only know that the system is in some state with some probability then the system must be described by a mixed state.

For a given observable $A \in C^\infty(M)[\![\hbar]\!]$ and state ρ the expectation value of the observable A in the state ρ is defined by

$$\langle A \rangle_\rho = \int_M A \star \rho \, \mathrm{d}l = \int_M A\rho \, \mathrm{d}l = \mathrm{tr}(A \star \rho). \tag{3.2.18}$$

3.2.2 Time evolution of quantum systems

The time evolution of a quantum system is governed by a Hamilton function $H \in C^\infty(M)[\![\hbar]\!]$ which is, similarly as in classical mechanics, some distinguished observable, being a deformation of a classical Hamilton function H_C. As in classical theory there are two dual points of view on the time evolution: Schrödinger picture and Heisenberg picture. In the Schrödinger picture states undergo time development while observables do not. An equation of motion for states, through an analogy to the Liouville equation (2.4.12), takes the form

$$\frac{\partial \rho}{\partial t}(t) - [\![H, \rho(t)]\!] = 0. \tag{3.2.19}$$

The formal solution of (3.2.19) takes the form

$$\rho(t) = U(t) \star \rho(0) \star \overline{U(t)}, \tag{3.2.20}$$

where

$$U(t) = e_\star^{-\frac{i}{\hbar}tH} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{i}{\hbar} t \right)^k \underbrace{H \star \cdots \star H}_{k} \tag{3.2.21}$$

is a unitary function as H is self-conjugated:

$$U(t) \star \overline{U(t)} = \overline{U(t)} \star U(t) = 1. \tag{3.2.22}$$

Hence, the time evolution of states can be alternatively expressed in terms of the one parameter group of unitary functions $U(t)$.

In the Heisenberg picture states remain still whereas observables undergo the time development. The time evolution of an observable $A \in C^\infty(M)[\![\hbar]\!]$ is given by the action of the unitary function $U(t)$ from (3.2.21) on A:

$$A(t) = \overline{U(t)} \star A(0) \star U(t) = e^{-t[\![H, \cdot\,]\!]} A(0), \tag{3.2.23}$$

where

$$e^{-t[\![H,\,\cdot\,]\!]} := \sum_{k=0}^{\infty} \frac{1}{k!}(-t)^k \underbrace{[\![H,[\![H,\ldots[\![H,\,\cdot\,]\!]\ldots]\!]]\!]}_{k}. \tag{3.2.24}$$

Differentiating (3.2.23) with respect to t results in the following evolution equation for A:

$$\frac{\mathrm{d}A}{\mathrm{d}t}(t) - [\![A(t),H]\!] = 0. \tag{3.2.25}$$

Equation (3.2.25) is the quantum analogue of the classical equation (2.4.14).

Both presented approaches to the time development yield equal predictions concerning the results of measurements, since

$$\begin{aligned}
\langle A(0)\rangle_{\rho(t)} &= \int_M A(0) \star \rho(t)\,\mathrm{d}l \\
&= \int_M A(0) \star U(t) \star \rho(0) \star \overline{U(t)}\,\mathrm{d}l \\
&= \int_M \overline{U(t)} \star A(0) \star U(t) \star \rho(0)\,\mathrm{d}l \\
&= \int_M A(t) \star \rho(0)\,\mathrm{d}l = \langle A(t)\rangle_{\rho(0)}.
\end{aligned} \tag{3.2.26}$$

3.3 Coordinate systems

The geometrical language which was used to quantize classical systems allowed for quantization to be performed in a coordinate independent way. However, in full analogy with classical mechanics, it is possible to consider quantum theory in some coordinate system. Let $M \supset U \to V \subset \mathbb{R}^{2N}$, $x \mapsto (x^1(x),\ldots,x^{2N}(x))$ be a coordinate system on a phase space M. In analogy with the classical case this coordinate system is called quantum canonical if there holds

$$[\![x^\alpha, x^\beta]\!] = \mathcal{J}^{\alpha\beta}, \tag{3.3.1}$$

where

$$(\mathcal{J}^{\alpha\beta}) = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}. \tag{3.3.2}$$

We will denote a quantum canonical coordinate system $(x^1,\ldots,x^{2N})$ by

$$(q^1,\ldots,q^N,p_1,\ldots,p_N) \equiv (q^i,p_j). \tag{3.3.3}$$

Then the quantum canonicity condition (3.3.1) takes the form

$$[\![q^i,q^j]\!] = [\![p_i,p_j]\!] = 0, \quad [\![q^i,p_j]\!] = \delta^i_j. \tag{3.3.4}$$

The functions q^i and p_j are observables of position and momentum associated with the coordinate system (q^i,p_j). Note that in the limit $\hbar \to 0$ a quantum canonical coordinate system reduces to a classical canonical coordinate system. If (q^i,p_j) and (q'^i,p'_j) are two quantum canonical coordinate systems then the transformation $(q^i,p_j) \mapsto (q'^i,p'_j)$ between these two coordinate systems is called a quantum canonical transformation [42, 68–71].

Let us derive the condition on a coordinate system $(x^1, \dots, x^{2N})$, which has to be satisfied to make it a classical and quantum canonical coordinate system.

Theorem 3.3.1. *A coordinate system $(x^1, \dots, x^{2N})$ is classical and quantum canonical iff*

$$C_1(x^\alpha, x^\beta) = \mathcal{J}^{\alpha\beta}, \tag{3.3.5a}$$
$$C_k(x^\alpha, x^\beta) = 0, \quad k = 3, 5, \dots, \tag{3.3.5b}$$

for every $\alpha, \beta = 1, \dots, 2N$, where C_k are bidifferential operators in the expansion (3.2.3) *of the $\star$-product.*

Proof. From (2.2.3) and (iii) from Definition 3.1.1 we get (3.3.5a). In accordance with (3.3.1) a coordinate system $(x^1, \dots, x^{2N})$ is a quantum canonical coordinate system iff

$$[x^\alpha, x^\beta] = x^\alpha \star x^\beta - x^\beta \star x^\alpha = i\hbar\mathcal{J}^{\alpha\beta}. \tag{3.3.6}$$

The above condition can be written in the form

$$\sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^k \big(C_k(x^\alpha, x^\beta) - C_k(x^\beta, x^\alpha)\big) = i\hbar\mathcal{J}^{\alpha\beta}. \tag{3.3.7}$$

The above equation is equivalent with the following system of equations

$$\frac{1}{2}\big(C_1(x^\alpha, x^\beta) - C_1(x^\beta, x^\alpha)\big) = \mathcal{J}^{\alpha\beta}, \tag{3.3.8a}$$
$$C_k(x^\alpha, x^\beta) = C_k(x^\beta, x^\alpha), \quad k = 2, 3, \dots . \tag{3.3.8b}$$

Equation (3.3.8a) is satisfied due to classical canonicity of the coordinate system. Equation (3.3.8b) due to property (i) of C_k in the expansion (3.2.3) of the $\star$-product can be rewritten in the form

$$C_k(x^\alpha, x^\beta) = (-1)^k C_k(x^\alpha, x^\beta). \tag{3.3.9}$$

The above formula is automatically satisfied for even k, and for odd k we get the condition (3.3.5b). $\square$

If $(x^1, \dots, x^{2N})$ is a coordinate system on M then we can write elements of $C^\infty(M)[\![\hbar]\!]$ in this coordinates receiving formal power series in $C^\infty(V)[\![\hbar]\!]$ where $V \subset \mathbb{R}^{2N}$. In particular, if $f = \sum_{k=0}^{\infty} \hbar^k f_k$ is an element of $C^\infty(M)[\![\hbar]\!]$ then by writing each $f_k \in C^\infty(M)$ in the coordinates $(x^1, \dots, x^{2N})$ we receive a formal power series in $C^\infty(V)[\![\hbar]\!]$. Analogically, we can write a $\star$-product on M in the coordinates $(x^1, \dots, x^{2N})$ receiving a star-product on a subset $V \subset \mathbb{R}^{2N}$. We will denote such star-product by $\star^{(x)}$.

Note, that if $(x^1, \dots, x^{2N})$ is a purely quantum canonical coordinate system, i.e. it is not at the same time classical canonical, then it must depend on $\hbar$ and, in fact, will be a deformation of some classical canonical coordinate system. The components

$\omega_{\alpha\beta}$ of the symplectic form ω for such purely quantum canonical coordinate system will also depend on $\hbar$ and can be expanded in the following series

$$\omega_{\alpha\beta} = \mathcal{J}_{\alpha\beta} + \hbar\omega^{(1)}_{\alpha\beta} + \hbar^2\omega^{(2)}_{\alpha\beta} + o(\hbar^3). \tag{3.3.10}$$

In consequence, the bidifferential operators C_k from the expansion (3.2.3) of the $\star$-product written in the coordinates $(x^1, \dots, x^{2N})$ will depend on $\hbar$. Expanding C_k in the power series of $\hbar$ allows to write the $\star^{(x)}$-product in the form

$$f \star^{(x)} g = \sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^k C'_k(f, g), \tag{3.3.11}$$

where C'_k are new bidifferential operators which are independent on $\hbar$, satisfy conditions (i)–(iii) on page 22, and moreover, in accordance to (3.3.10)

$$C'_1(f, g) = \mathcal{J}^{\alpha\beta}(\partial_{x^\alpha} f)(\partial_{x^\beta} g). \tag{3.3.12}$$

Thus we can show, similarly as in the proof of Theorem 3.3.1, that

$$C'_1(x^\alpha, x^\beta) = \mathcal{J}^{\alpha\beta}, \tag{3.3.13a}$$

$$C'_k(x^\alpha, x^\beta) = 0, \quad k = 3, 5, \dots . \tag{3.3.13b}$$

As a result the $\star^{(x)}$-product can be considered as a coordinate representation, with respect to a classical and quantum canonical coordinate system, of some star-product on a symplectic manifold different than (M, ω).

Let us make some remarks about domains of coordinate systems. If one is interested only in the investigation of a geometry of a classical Hamiltonian system (M, ω, H), then one can consider coordinate systems defined on arbitrary open subsets U of a phase space M. However, for quantum systems this does not remain true since star-products, considered in a non-formal setting, are not local.

The same thing happens when one wishes to investigate integrals over the phase space, e.g., to calculate expectation values of observables, then one cannot do this in an arbitrary coordinate system. The reason for this is that, in general the values of integrals will change if the integration will be performed over some subset $U \subset M$. This argument applies both to classical and quantum theory. The only coordinate systems in which it is meaningful to consider integration are almost global coordinate systems (cf. Section 2.2).

3.4 Natural star-products on symplectic manifolds

3.4.1 Moyal star-product on $\mathbb{R}^{2N}$

Let us take as a phase space M the symplectic vector space $(\mathbb{R}^{2N}, \omega)$, where ω is a symplectic matrix which components in a canonical basis $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_{2N} = (0, 0, \dots, 1)$ on $\mathbb{R}^{2N}$ are equal

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}. \tag{3.4.1}$$

On such symplectic manifold there exists a natural star-product which in canonical coordinates $x = x^\alpha e_\alpha \mapsto (x^1, \dots, x^{2N})$ is given by the formula

$$
\begin{aligned}
f \star_M g &= f \exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}\overleftarrow{\partial}_{x^\mu}\overrightarrow{\partial}_{x^\nu}\right) g \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}(\partial_{x^{\mu_1}}\cdots\partial_{x^{\mu_k}} f)(\partial_{x^{\nu_1}}\cdots\partial_{x^{\nu_k}} g),
\end{aligned} \tag{3.4.2}
$$

where $\omega^{\mu\nu}$ is an inverse matrix to the symplectic matrix $\omega_{\mu\nu}$. The star-product (3.4.2) was first considered by Groenewold [72], Moyal [73], and Berezin [74] and is usually called a Moyal product.

Proposition 3.4.1. *The Moyal product* (3.4.2) *is associative.*

Proof. The Moyal product can be written in a form

$$
(f \star_M g)(x) = \exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}\partial_{y^\mu}\partial_{z^\nu}\right)(f(y)g(z))\Big|_{y=z=x}. \tag{3.4.3}
$$

Derivatives ∂_{x^α} are derivations for the $\star_M$-product, which can be stated as

$$
\partial_{x^\alpha}(f \star_M g)(x) = (\partial_{y^\alpha} + \partial_{z^\alpha})\exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}\partial_{y^\mu}\partial_{z^\nu}\right)(f(y)g(z))\Big|_{y=z=x}. \tag{3.4.4}
$$

Thus

$$
\begin{aligned}
((f \star_M g) \star_M h)(x) &= \exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}\partial_{v^\mu}\partial_{w^\nu}\right)((f \star_M g)(v)h(w))\Big|_{v=w=x} \\
&= \exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}(\partial_{y^\mu} + \partial_{z^\nu})\partial_{w^\nu}\right)\exp\left(\frac{i\hbar}{2}\omega^{\mu'\nu'}\partial_{y^{\mu'}}\partial_{z^{\nu'}}\right)(f(y)g(z)h(w))\Big|_{y=z=w=x} \\
&= \exp\left(\frac{i\hbar}{2}\omega^{\mu\nu}(\partial_{y^\mu}\partial_{w^\nu} + \partial_{z^\mu}\partial_{w^\nu} + \partial_{y^\mu}\partial_{z^\nu})\right)(f(y)g(z)h(w))\Big|_{y=z=w=x} \\
&= (f \star_M (g \star_M h))(x).
\end{aligned} \tag{3.4.5}
$$

□

Proposition 3.4.2. *For elements f, g of the space $C_0^\infty(\mathbb{R}^{2N})$ of compactly supported functions on $\mathbb{R}^{2N}$*

$$
\int_{\mathbb{R}^{2N}} f \star_M g \, \mathrm{d}x = \int_{\mathbb{R}^{2N}} fg \, \mathrm{d}x. \tag{3.4.6}
$$

Proof. Let

$$
C_k(f,g) = \frac{1}{k!}\omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}(\partial_{x^{\mu_1}}\cdots\partial_{x^{\mu_k}} f)(\partial_{x^{\nu_1}}\cdots\partial_{x^{\nu_k}} g). \tag{3.4.7}
$$

Then, using integration by parts we get for $k = 1, 2, \dots$

$$
\begin{aligned}
\int_{\mathbb{R}^{2N}} C_k(f,g)\,\mathrm{d}x &= -\int_{\mathbb{R}^{2N}} \frac{1}{k!}\omega^{\nu_1\mu_1}\omega^{\mu_2\nu_2}\cdots\omega^{\mu_k\nu_k}(\partial_{x^{\nu_1}}\partial_{x^{\mu_2}}\cdots\partial_{x^{\mu_k}} f) \\
&\qquad \times (\partial_{x^{\mu_1}}\partial_{x^{\nu_2}}\cdots\partial_{x^{\nu_k}} g)\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^{2N}} C_k(f,g)\,\mathrm{d}x.
\end{aligned} \tag{3.4.8}
$$

Thus
$$\int_{\mathbb{R}^{2N}} C_k(f, g)\, \mathrm{d}x = 0, \tag{3.4.9}$$
which proves (3.4.6). □

In the rest of the thesis we will use the following conventions concerning the Fourier transform. We will define the Fourier transform of a function f on $\mathbb{R}^{2N}$ by the formula
$$\begin{aligned}(\mathscr{F} f)(\xi) &= \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} f(x) e^{-\frac{i}{\hbar}\xi_\mu x^\mu}\, \mathrm{d}x \\ &= \int_{\mathbb{R}^{2N}} f(x) e^{-\frac{i}{\hbar}\xi_\mu x^\mu}\, \mathrm{d}l(x)\end{aligned} \tag{3.4.10}$$
and the inverse Fourier transform by
$$\begin{aligned}(\mathscr{F}^{-1} f)(x) &= \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} f(\xi) e^{\frac{i}{\hbar}\xi_\mu x^\mu}\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^{2N}} f(\xi) e^{\frac{i}{\hbar}\xi_\mu x^\mu}\, \mathrm{d}l(\xi).\end{aligned} \tag{3.4.11}$$
The Fourier transform has the following properties
$$\mathscr{F}(\partial_{x^\mu} f)(\xi) = \frac{i}{\hbar}\xi_\mu \mathscr{F} f(\xi), \tag{3.4.12}$$
$$\mathscr{F}(f \cdot g) = \mathscr{F} f * \mathscr{F} g, \tag{3.4.13}$$
where $*$ is a convolution of functions defined by
$$(f * g)(x) = \int_{\mathbb{R}^{2N}} f(y) g(x - y)\, \mathrm{d}l(y) = \int_{\mathbb{R}^{2N}} f(x - y) g(y)\, \mathrm{d}l(y). \tag{3.4.14}$$

As an illustrative remark let us give an example of a subspace $\mathcal{A}$ of $C^\infty(\mathbb{R}^{2N})$, with an appropriate topology on which the Moyal product is convergent. Let $\mathcal{A} = \mathscr{F}(\mathcal{E}')$ be the Fourier image of the space of distributions with compact support $\mathcal{E}'$. The space $\mathcal{E}'$ is a dual space to the Fréchet space $\mathcal{E} = C^\infty(\mathbb{R}^{2N})$ equipped with a standard topology of uniform convergence on compact subsets of $\mathbb{R}^{2N}$, together with all derivatives. $\mathcal{E}'$ carries the strong dual topology, i.e. the topology of uniform convergence on bounded sets in $\mathcal{E}$, whereas $\mathcal{A}$ carries the topology induced by $\mathscr{F}$ from $\mathcal{E}'$. By the Paley-Wiener theorem $\mathcal{A}$ is the space of smooth functions on $\mathbb{R}^{2N}$ for which each derivative is polynomially bounded and which extend to entire functions on $\mathbb{C}^{2N}$ of exponential type. That is $f\colon \mathbb{R}^{2N} \to \mathbb{C}$ is an element of $\mathcal{A}$ if and only if f extends to an entire function $f\colon \mathbb{C}^{2N} \to \mathbb{C}$ satisfying for all $z \in \mathbb{C}^{2N}$
$$|f(z)| \leq C(1 + |z|)^n e^{r|\mathrm{Im}(z)|} \tag{3.4.15}$$
for some constants $C > 0$, $r > 0$ and $n \in \mathbb{N}$. Note that all polynomials belong to $\mathcal{A}$.

Theorem 3.4.1. *For $f, g \in \mathcal{A}$ the series $f \star_M g$ is convergent in $\mathcal{A}$. Thus $\mathcal{A}$ is an algebra with respect to the Moyal product $\star_M$.*

The proof of the above theorem can be found in [65]. Worth noting is paper [66] where author introduces slightly different family of subspaces of smooth functions on which the Moyal product is also convergent.

Let $\mathcal{F} = \mathscr{F}(C_0^\infty(\mathbb{R}^{2N}))$. Then $\mathcal{F}$ is a subspace of $\mathcal{A}$ as well as the Schwartz space $\mathcal{S}(\mathbb{R}^{2N})$, i.e. the space of rapidly decreasing functions on $\mathbb{R}^{2N}$. Moreover, $\mathcal{F}$ is dense in $\mathcal{S}(\mathbb{R}^{2N})$ and $L^2(\mathbb{R}^{2N})$.

Theorem 3.4.2. *For $f, g \in \mathcal{F}$ the Moyal product can be written in the following integral form*

$$(f \star_M g)(x) = \frac{1}{(\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} f(x+u)g(x+v) e^{-\frac{2i}{\hbar}\omega_{\mu\nu}u^\mu v^\nu} \,\mathrm{d}u\,\mathrm{d}v. \tag{3.4.16}$$

Proof. Using the properties (3.4.12) and (3.4.13) of the Fourier transform the Moyal product can be written in the following form

$$\begin{aligned}
(f \star_M g)(x) &= \mathscr{F}^{-1}\mathscr{F}(f \star_M g)(x) = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}(f \star_M g)(\xi) e^{\frac{i}{\hbar}\xi_\mu x^\mu} \,\mathrm{d}\xi \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \omega^{\mu_1\nu_1} \cdots \omega^{\mu_k\nu_k} \int_{\mathbb{R}^{2N}} \mathscr{F}(\partial_{x^{\mu_1}} \cdots \partial_{x^{\mu_k}} f)(\eta) \\
&\quad \times \mathscr{F}(\partial_{x^{\nu_1}} \cdots \partial_{x^{\nu_k}} g)(\xi - \eta) e^{\frac{i}{\hbar}\xi_\mu x^\mu} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \omega^{\mu_1\nu_1} \cdots \omega^{\mu_k\nu_k} \int_{\mathbb{R}^{2N}} \frac{i}{\hbar}\eta_{\mu_1} \cdots \frac{i}{\hbar}\eta_{\mu_k} \mathscr{F} f(\eta) \\
&\quad \times \frac{i}{\hbar}(\xi_{\nu_1} - \eta_{\nu_1}) \cdots \frac{i}{\hbar}(\xi_{\nu_k} - \eta_{\nu_k}) \mathscr{F} g(\xi - \eta) e^{\frac{i}{\hbar}\xi_\mu x^\mu} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{-i}{2\hbar}\right)^k \Big(\omega^{\mu\nu}\eta_\mu(\xi_\nu - \eta_\nu)\Big)^k \\
&\quad \times \mathscr{F} f(\eta) \mathscr{F} g(\xi - \eta) e^{\frac{i}{\hbar}\xi_\mu x^\mu} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} \mathscr{F} f(\eta) \mathscr{F} g(\xi - \eta) e^{-\frac{i}{2\hbar}\omega^{\mu\nu}\eta_\mu(\xi_\nu - \eta_\nu)} e^{\frac{i}{\hbar}\xi_\mu x^\mu} \,\mathrm{d}\eta\,\mathrm{d}\xi.
\end{aligned} \tag{3.4.17}$$

Performing the change of variables

$$\begin{aligned}
\eta_\mu &\to \eta_\mu, \\
\xi_\mu &\to \xi_\mu + \eta_\mu
\end{aligned} \tag{3.4.18}$$

we get

$$\begin{aligned}
(f \star_M g)(x) &= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} \mathscr{F} f(\eta) \mathscr{F} g(\xi) e^{\frac{i}{\hbar}(x^\mu + \frac{1}{2}\omega^{\mu\nu}\eta_\nu)\xi_\mu} e^{\frac{i}{\hbar}\eta_\mu x^\mu} \,\mathrm{d}\eta\,\mathrm{d}\xi \\
&= \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F} f(\eta) g(x + \tfrac{1}{2}\omega\eta) e^{\frac{i}{\hbar}\eta_\mu x^\mu} \,\mathrm{d}\eta \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} f(y) g(x + \tfrac{1}{2}\omega\eta) e^{\frac{i}{\hbar}\eta_\mu x^\mu} e^{-\frac{i}{\hbar}\eta_\mu y^\mu} \,\mathrm{d}y\,\mathrm{d}\eta.
\end{aligned} \tag{3.4.19}$$

After performing another change of variables

$$\begin{aligned} y^\mu &\to x^\mu + u^\mu, \\ \eta_\mu &\to 2\omega_{\mu\nu} v^\nu \end{aligned} \tag{3.4.20}$$

we receive the result. □

The integral form of the Moyal product is also valid for $f, g \in \mathcal{S}(\mathbb{R}^{2N})$. Moreover, it can be shown that $\star_M$ is continuous on $\mathcal{S}(\mathbb{R}^{2N})$ and that for $f, g \in \mathcal{S}(\mathbb{R}^{2N})$, $f \star_M g \in \mathcal{S}(\mathbb{R}^{2N})$ and

$$\|f \star_M g\| \leq \|f\| \|g\|, \tag{3.4.21}$$

see e.g. [40, 65]. The extension of the Moyal product from $\mathcal{F}$ to a continuous star-product on $\mathcal{S}(\mathbb{R}^{2N})$ is unique since $\mathcal{F}$ is dense in $\mathcal{S}(\mathbb{R}^{2N})$. Hence the Schwartz space $\mathcal{S}(\mathbb{R}^{2N})$ is an algebra with respect to the Moyal product. The subspace $\mathcal{F}$ is also an algebra with respect to $\star_M$, which is a direct consequence of the fact that the Fourier transform $\mathscr{F}$ is an automorphism of $\mathcal{S}(\mathbb{R}^{2N})$. From (3.4.21) follows that the Moyal product is continuous with respect to the L^2-topology and so can be uniquely extended to a continuous star-product on $L^2(\mathbb{R}^{2N})$.

It is not difficult to check that the integral form (3.4.16) of the Moyal product can be written in the following way

$$(f \star_M g)(q, p) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}(q + \tfrac{1}{2}u, v) \tilde{g}(q - \tfrac{1}{2}v, u) e^{-\frac{i}{\hbar}(u^i + v^i)p_i} \, \mathrm{d}u \, \mathrm{d}v, \tag{3.4.22}$$

where $\tilde{f}$ denotes the Fourier transform of f in the momentum variable

$$\tilde{f}(q, u) = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^N} f(q, p) e^{\frac{i}{\hbar} p_i u^i} \, \mathrm{d}p. \tag{3.4.23}$$

Note, that the Moyal product on the algebra $\mathcal{A}$ is not local, which can be seen from the integral form (3.4.16) of the Moyal product — for a fixed $x \in \mathbb{R}^{2N}$ the value of the integral in (3.4.16) depends on the values of functions f and g far away from x.

The Moyal product (3.4.2) is also a valid star-product on symplectic manifold $M = T^*U = U \times \mathbb{R}^N$, where U is some open subset of $\mathbb{R}^N$. This is a direct consequence of the fact that the Moyal product is a series of bidifferential operators which are local operators. For $f, g \in C_0^\infty(M)$ the integral form (3.4.16) of the Moyal product still makes sense, since f and g can be uniquely extended to smooth functions defined on the whole space $\mathbb{R}^{2N}$ with the same supports as f and g respectively (just by putting the functions f and g equal 0 outside $U \times \mathbb{R}^N$). In such case the expression (3.4.16) still can be formally expanded to the series (3.4.2). Denote by $\mathcal{F}$ the space of smooth functions which momentum Fourier transforms are smooth functions with compact support. For $f, g \in \mathcal{F}$ formula (3.4.22) makes sense and defines the Moyal product of functions f and g. Taking the Fourier transform of (3.4.22) in the momentum variable we receive

$$(f \star_M g)^\sim(q, u) = \int_{\mathbb{R}^N} \tilde{f}(q + \tfrac{1}{2}v, u - v) \tilde{g}(q - \tfrac{1}{2}(u - v), v) \, \mathrm{d}v. \tag{3.4.24}$$

From (3.4.24) and the fact that the convolution of compactly supported smooth functions is also a compactly supported smooth function follows that $(f \star_M g)^\sim$ has compact support and is smooth. Moreover, it is not difficult to see that $f \star_M g$ is smooth. Hence $f \star_M g \in \mathcal{F}$, i.e. $\mathcal{F}$ is an algebra with respect to $\star_M$. Formula (3.4.24) defines actually a twisted convolution of $\tilde{f}$ and $\tilde{g}$.

Let now (M, ω) be a general symplectic manifold and $\star$ a star-product on M of the form (3.2.3). If we choose on M some coordinate system $M \supset U \to V \subset \mathbb{R}^{2N}$, $x \mapsto (x^1, \ldots, x^{2N}) = (q^1, \ldots, q^N, p_1, \ldots, p_N)$ which is at the same time classical and quantum canonical, then in this coordinates the symplectic form ω takes the form (3.4.1) and we receive on the subset V a star-product $\star^{(x)}$. However, on V we can also define a Moyal product (3.4.2) associated to the same symplectic form ω. It happens that these two star-products are always equivalent.

Theorem 3.4.3. *For any coordinate system $(x^1, \ldots, x^{2N})$ on the symplectic manifold (M, ω), which is at the same time classical and quantum canonical, there exists a unique series $S = \sum_{k=0}^{\infty} \hbar^k S_k$ of the form (3.1.6), such that*

$$S(f \star_M^{(x)} g) = Sf \star^{(x)} Sg, \tag{3.4.25a}$$
$$Sx^\alpha = x^\alpha, \tag{3.4.25b}$$
$$\overline{Sf} = S\bar{f}, \tag{3.4.25c}$$

where $\star_M^{(x)}$ is a star-product which in the coordinates $(x^1, \ldots, x^{2N})$ is of the form of the Moyal product. The series S will satisfy (3.4.25) if and only if

$$[S_{2k}, x^\alpha] = \sum_{l=1}^{k} \left(-\frac{1}{4}\right)^l A_{2l}^\alpha S_{2(k-l)}, \tag{3.4.26a}$$
$$[S_{2k}, \partial^\alpha] = \sum_{l=1}^{k} \left(-\frac{1}{4}\right)^l A_{2l+1}^\alpha S_{2(k-l)}, \tag{3.4.26b}$$

and $S_{2k-1} = 0$ for $k = 1, 2, \ldots$, where $\partial^\alpha = \omega^{\alpha\beta}\partial_{x^\beta}$ and $A_k^\alpha f = C_k(x^\alpha, f)$.

The proof of the above theorem is given in Appendix A. Equations (3.4.26) can be used to recursively calculate the series S order by order in $\hbar$. The general solution of (3.4.26) is of the form

$$S_{2k} = \sum_{n=1}^{\infty} \frac{1}{n!}[x^{\alpha_1}, \ldots, [x^{\alpha_{n-1}}, F^{\alpha_n}]]\partial_{\alpha_1} \cdots \partial_{\alpha_n}, \tag{3.4.27}$$

where $F^\alpha = \sum_{l=1}^{k} \left(-\frac{1}{4}\right)^l A_{2l}^\alpha S_{2(k-l)}$. Indeed, it is enough to solve (3.4.26a), since the solution of (3.4.26a) is specified up to an additive function which has to be equal zero by virtue of (3.4.25b). We have that

$$
\begin{aligned}
[S_{2k}, x^\alpha] &= -\sum_{n=1}^{\infty} \frac{1}{n!}[x^\alpha, [x^{\beta_1}, \ldots, [x^{\beta_{n-1}}, F^{\beta_n}]]]\partial_{\beta_1} \cdots \partial_{\beta_n} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(n-1)!}[x^{\beta_1}, \ldots, [x^{\beta_{n-1}}, F^\alpha]]\partial_{\beta_1} \cdots \partial_{\beta_{n-1}} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n!}[x^{\beta_1}, \ldots, [x^{\beta_n}, F^\alpha]]\partial_{\beta_1} \cdots \partial_{\beta_n} \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{n!}[x^{\beta_1}, \ldots, [x^{\beta_n}, F^\alpha]]\partial_{\beta_1} \cdots \partial_{\beta_n} = F^\alpha. \qquad (3.4.28)
\end{aligned}
$$

Note, that since A_k^α are differential operators of finite order the sum in (3.4.27) will be finite.

From Theorem 3.4.3 follows that a quantization of a classical system given by a star-product of the form (3.2.3) and some assignment of measurable quantities to elements $A \in C^\infty(M)[\![\hbar]\!]$, locally is equivalent with a Moyal quantization given by the Moyal product (3.4.2) and an assignment of measurable quantities to elements $A' = SA$. This fact is of fundamental importance for introducing an operator representation of quantum mechanics.

Remark 3.4.1. Note, that Theorem 3.4.3 is also valid for a purely quantum canonical coordinate system $(x^1, \ldots, x^{2N})$ since, in accordance to Section 3.3, the $\star^{(x)}$-product can be considered as a coordinate representation, with respect to a classical and quantum canonical coordinate system, of some star-product on some other symplectic manifold.

In what follows we will consider only such $\star$-products for which, for every almost global classical and quantum canonical coordinate system $M \supset U \to V \subset \mathbb{R}^{2N}$, $x \mapsto (x^1, \ldots, x^{2N})$, the associated series S giving the equivalence with a Moyal product has the property that for every $f \in C_0^\infty(V)$ the series $S(f)$ is convergent to an element of $L^2(V, \mathrm{d}l)$ and

$$
\int_V Sf \,\mathrm{d}l = \int_V f \,\mathrm{d}l, \quad f \in C_0^\infty(V). \qquad (3.4.29)
$$

From (3.4.29) it follows that

$$
(Sf, Sg) = (f, g), \quad f, g \in C_0^\infty(V). \qquad (3.4.30)
$$

Indeed,

$$
\begin{aligned}
(Sf, Sg) &= \int_V \overline{Sf} Sg \,\mathrm{d}l = \int_V S\bar{f} \star^{(x)} Sg \,\mathrm{d}l \\
&= \int_V S(\bar{f} \star_M^{(x)} g) \,\mathrm{d}l = \int_V \bar{f} \star_M^{(x)} g \,\mathrm{d}l = (f, g). \qquad (3.4.31)
\end{aligned}
$$

The above property imposed on the series S guaranties that S can be uniquely extended to a unitary operator defined on the whole Hilbert space $L^2(V, \mathrm{d}l)$ and satisfying

$$
S(f \star_M^{(x)} g) = Sf \star^{(x)} Sg, \quad f, g \in L^2(V, \mathrm{d}l). \qquad (3.4.32)
$$

3.4.2 Family of star-products on T^*E^N

Let us consider an N-dimensional Euclidean space E^N. The cotangent bundle T^*E^N to this space is a $2N$-dimensional manifold naturally endowed with a symplectic structure ω, as was discussed in Section 2.1. Let us choose some Cartesian coordinate system $(q^1,\dots,q^N)$ on E^N. This coordinate system extends to a Cartesian coordinate system $(q^1,\dots,q^N,p_1,\dots,p_N) = (x^1,\dots,x^{2N})$ on the symplectic manifold T^*E^N (see Section 2.1). In this coordinates the symplectic form ω takes the form $\mathrm{d}p_i \wedge \mathrm{d}q^i$. Also the Poisson tensor $\mathcal{P} = \omega^{-1}$ related to the symplectic form ω can be written in the form

$$\mathcal{P} = \mathcal{J}^{\mu\nu}\partial_{x^\mu} \otimes \partial_{x^\nu} = \partial_{q^i} \wedge \partial_{p_i}. \tag{3.4.33}$$

Equation (3.4.33) shows that the Poisson tensor $\mathcal{P}$ can be decomposed into a wedge product of pair-wise commuting vector fields. However, such decomposition is not unique. There are different sequences of commuting vector fields $D_1,\dots,D_{2N}$ such that

$$\mathcal{P} = \mathcal{J}^{\mu\nu}D_\mu \otimes D_\nu = \sum_{i=1}^{N} X_i \wedge Y_i, \tag{3.4.34}$$

where $X_i = D_i$ and $Y_i = D_{N+i}$ for $i = 1,\dots,N$.

In what follows we will define a family of star-products on the symplectic manifold T^*E^N. Let (D_μ) be a sequence of pair-wise commuting global vector fields from the decomposition (3.4.34) of the Poisson tensor $\mathcal{P}$. Define a star-product by the formula

$$f \star g = f \exp\left(\frac{1}{2} i\hbar \mathcal{J}^{\mu\nu} \overleftarrow{D_\mu} \overrightarrow{D_\nu}\right) g. \tag{3.4.35}$$

From the commutativity of vector fields D_μ follows the associativity of the star-product. The proof of this fact is analogical as the proof of Proposition 3.4.1. As was pointed out earlier the sequence (D_μ) is not uniquely specified by the Poisson tensor, thus we can define the whole family of star-products related to the same Poisson tensor.

In particular, if A_μ^ν is a symplectic matrix with constant coefficients i.e. $A^T\mathcal{J}A = \mathcal{J}$ or equivalently $\mathcal{J}^{\mu\nu}A_\mu^\alpha A_\nu^\beta = \mathcal{J}^{\alpha\beta}$, then vector fields $D'_\mu = A_\mu^\nu D_\nu$ also pair-wise commute and satisfy $\mathcal{P} = \mathcal{J}^{\mu\nu}D'_\mu \otimes D'_\nu$. Both sequences (D_μ) and (D'_μ) define the same star-product, as can be checked by a direct computation. Thus the introduced family of star-products is parametrized by elements of the space of sequences (D_μ) modulo the symplectic group $\mathrm{Sp}(2N)$.

The constructed family of star-products consists of equivalent star-products, which is a direct consequence of Theorem 3.1.1.

Example 3.4.1. Let us consider the Poisson manifold $T^*\mathbb{R} \cong \mathbb{R}^2$ with the standard Poisson tensor $\mathcal{P}$. Assume that (q,p) is a Darboux coordinate system. Consider the following vector fields

$$\begin{gathered} X = \partial_q, \quad Y = \partial_p, \\ X' = q^2\partial_q - 2qp\partial_p, \quad Y' = q^{-2}\partial_p. \end{gathered} \tag{3.4.36}$$

It can be checked that $[X, Y] = 0$, $[X', Y'] = 0$ and

$$\mathcal{P} = X \wedge Y = X' \wedge Y'. \tag{3.4.37}$$

Star-products induced by vector fields X, Y and X', Y' are equivalent and the morphism S giving this equivalence is represented by the formula

$$S = \mathrm{id} + \frac{\hbar^2}{4}\left(2q^{-2}\partial_p^2 + q^{-2}p\partial_p^3 - q^{-1}\partial_q\partial_p^2\right) + o(\hbar^4). \tag{3.4.38}$$

Note that vector fields X, Y and X', Y' are related by a canonical transformation $T\colon (q,p) \mapsto T(q,p) = (-q^{-1}, q^2 p)$:

$$(Xf) \circ T = X'(f \circ T), \quad (Yf) \circ T = Y'(f \circ T), \tag{3.4.39}$$

for $f \in C^\infty(\mathbb{R}^2)$.

For a given sequence of vector fields (D_μ) from the decomposition (3.4.34) of the Poisson tensor $\mathcal{P}$ there exists a global coordinate system $(x^1, \dots, x^{2N})$ in which D_μ are coordinate vector fields, i.e. $D_\mu = \partial_{x^\mu}$. Such coordinate system is of course a Darboux coordinate system associated with the symplectic form ω. In this coordinates the star-product (3.4.35) takes the form of the Moyal product (3.4.2). The coordinate system $(x^1, \dots, x^{2N})$ will be called a natural coordinate system of the $\star$-product.

The structure of the symplectic manifold T^*E^N distinguishes one product from the presented family of star-products, namely the one for which the natural coordinate system is the Cartesian coordinate system. Such star-product is indeed uniquely defined since coordinate vector fields of Cartesian coordinate systems are related to each other by linear symplectic transformations and such transformations do not change the star-product (3.4.35) as pointed out earlier. This distinguished star-product will be called a canonical star-product on T^*E^N.

In what follows let us write the canonical star-product on T^*E^N in a different form. To do this let us first write it in a Darboux coordinate system induced from an arbitrary curvilinear coordinates on E^N. Let $\phi\colon (q'^1, \dots, q'^N) \mapsto (q^1, \dots, q^N)$ be a change of coordinates from arbitrary curvilinear coordinates $(q'^1, \dots, q'^N)$ to Cartesian coordinates $(q^1, \dots, q^N)$. The transformation ϕ on E^N induces a canonical transformation $(q', p') \mapsto T(q', p') = (q, p)$ on the symplectic manifold T^*E^N according to the formula (2.1.17).

The canonical star-product in Cartesian coordinates takes the form of a Moyal product (3.4.2). The Moyal product (3.4.2) under the point transformation T transforms to the following star-product:

$$f \star^{(q',p')} g = f \exp\left(\frac{1}{2} i\hbar \mathcal{J}^{\mu\nu} \overleftarrow{D_{x'^\mu}} \overrightarrow{D_{x'^\nu}}\right) g, \tag{3.4.40}$$

where

$$\begin{aligned} D_{q'^i} &= [(\phi'(q'))^{-1}]_i^j \left(\partial_{q'^j} + \Gamma_{jl}^r(q') p'_r \partial_{p'_l}\right), \\ D_{p'_i} &= [\phi'(q')]_j^i \partial_{p'_j} \end{aligned} \tag{3.4.41}$$

is a transformation of Cartesian coordinate vector fields ∂_{q^i}, ∂_{p_i} to a new coordinate chart, and $\Gamma^i_{jk}(q') = [(\phi'(q'))^{-1}]^i_r [\phi''(q')]^r_{jk}$ ($[\phi''(q')]^i_{jk} = \dfrac{\partial^2 \phi^i}{\partial_{q'^j}\partial_{q'^k}}(q')$ is the Hessian of ϕ). Note that the symbols $\Gamma^i_{jk}(q')$ are the Christoffel symbols for the $(q'^1, \dots, q'^N)$ coordinates, associated to the standard linear connection ∇ on the configuration space E^N. Formula (3.4.40) can be written in the form

$$f \star^{(q',p')} g = \sum_{n,m=0}^{\infty} \frac{1}{n!m!} (-1)^m \left(\frac{i\hbar}{2}\right)^{n+m} (D^{j_1 \dots j_m}_{i_1 \dots i_n} f)(D^{i_1 \dots i_n}_{j_1 \dots j_m} g), \tag{3.4.42}$$

where operators $D^{j_1 \dots j_m}_{i_1 \dots i_n}$ are given recursively by

$$\begin{aligned} D^{j_1 \dots j_m}_{i_1 \dots i_{n+1}} f &= D_{i_{n+1}}(D^{j_1 \dots j_m}_{i_1 \dots i_n} f) - \Gamma^k_{i_1 i_{n+1}} D^{j_1 \dots j_m}_{k \dots i_n} f - \dots - \Gamma^k_{i_n i_{n+1}} D^{j_1 \dots j_m}_{i_1 \dots k} f \\ &\quad + \Gamma^{j_1}_{k i_{n+1}} D^{k \dots j_m}_{i_1 \dots i_n} f + \dots + \Gamma^{j_m}_{k i_{n+1}} D^{j_1 \dots k}_{i_1 \dots i_n} f, &(3.4.43a) \\ D^{j_1 \dots j_{m+1}}_{i_1 \dots i_n} f &= D^{j_{m+1}}(D^{j_1 \dots j_m}_{i_1 \dots i_n} f), &(3.4.43b) \\ D_i f &= \partial_{q'^i} f + \Gamma^k_{ij} p'_k \partial_{p'_j} f, &(3.4.43c) \\ D^j f &= \partial_{p'_j} f, &(3.4.43d) \end{aligned}$$

where $\{D_i, D^j\}$ is a so called adopted frame on T^*E^N [75]. Note that the upper indices in the operator $D^{j_1 \dots j_m}_{i_1 \dots i_n}$ commute with the lower indices, i.e. it does not matter if, when calculating $D^{j_1 \dots j_m}_{i_1 \dots i_n} f$, we first use formula (3.4.43a) and then (3.4.43b) or vice verse.

Equation (3.4.42) takes the form

$$\begin{aligned} f \star^{(q',p')} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} (\underbrace{\tilde\nabla \cdots \tilde\nabla}_{k} f)_{i_1 \dots i_n \bar{j}_1 \dots \bar{j}_{k-n}} \\ \times (\underbrace{\tilde\nabla \cdots \tilde\nabla}_{k} g)_{\bar{i}_1 \dots \bar{i}_n j_1 \dots j_{k-n}}, \end{aligned} \tag{3.4.44}$$

where $\bar{i} = N + i$ and $\tilde\nabla$ is a linear connection on the symplectic manifold T^*E^N, which components in the frame $\{D_i, D^j\}$ are equal

$$\tilde\Gamma^i_{jk} = \Gamma^i_{jk}, \quad \tilde\Gamma^{\bar{i}}_{\bar{j}k} = -\Gamma^j_{ik} \tag{3.4.45}$$

with the remaining components equal zero. Equation (3.4.44) can be written in the form

$$\begin{aligned} f \star^{(q',p')} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{n=0}^{k} \binom{k}{n} A^{\mu_1 \nu_1} \cdots A^{\mu_n \nu_n} B^{\mu_{n+1} \nu_{n+1}} \cdots B^{\mu_k \nu_k} \\ \times (\underbrace{\tilde\nabla \cdots \tilde\nabla}_{k} f)_{\mu_1 \dots \mu_k} (\underbrace{\tilde\nabla \cdots \tilde\nabla}_{k} g)_{\nu_1 \dots \nu_k}, \end{aligned} \tag{3.4.46}$$

where

$$A = \begin{pmatrix} 0_N & I_N \\ 0_N & 0_N \end{pmatrix}, \quad B = \begin{pmatrix} 0_N & 0_N \\ -I_N & 0_N \end{pmatrix}. \tag{3.4.47}$$

Equation (3.4.46) takes the form

$$f \star^{(q',p')} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k (A+B)^{\mu_1\nu_1} \cdots (A+B)^{\mu_k\nu_k} (\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k} f)_{\mu_1\ldots\mu_k} \times (\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k} g)_{\nu_1\ldots\nu_k}. \quad (3.4.48)$$

Introducing

$$\omega = A + B = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix} \quad (3.4.49)$$

we finally receive

$$f \star^{(q',p')} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \omega^{\mu_1\nu_1} \cdots \omega^{\mu_k\nu_k} (\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k} f)_{\mu_1\ldots\mu_k} (\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k} g)_{\nu_1\ldots\nu_k}. \quad (3.4.50)$$

Since $D_i \wedge D^j = \partial_{q'^i} \wedge \partial_{p'_j}$, $\omega^{\mu\nu}$ are components of the Poisson tensor in the Darboux frame $\{\partial_{q'^i}, \partial_{p'_j}\}$ as well as in the adopted frame $\{D_i, D^j\}$.

The Christoffel symbols of the linear connection $\tilde{\nabla}$ in the Darboux coordinate frame take the form

$$\begin{gathered} \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^{\bar{i}}_{\bar{j}k} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^{\bar{i}}_{j\bar{k}} = -\Gamma^k_{ji}, \\ \tilde{\Gamma}^{\bar{i}}_{jk} = p_l(\Gamma^r_{jk}\Gamma^l_{ri} + \Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^l_{ij,k}), \end{gathered} \quad (3.4.51)$$

with the remaining components equal zero. From the construction it follows that $\tilde{\nabla}$ is symplectic, i.e. $\tilde{\nabla}\omega = 0$. Moreover, from flatness of the configuration space E^N follows that $\tilde{\nabla}$ is flat and torsionless.

Thus we wrote the canonical star-product on T^*E^N in a covariant form (3.4.50), where $\tilde{\nabla}$ is a connection induced from a standard Levi-Civita connection on E^N. Other star-products on E^N also can be written in a covariant form (3.4.50). As a linear connection $\tilde{\nabla}$ one has to take a connection which components in a natural coordinate system vanish. However, such connection is not related to a standard Levi-Civita connection on E^N.

Equation (3.4.51) defines a lift of the Levi-Civita connection on E^N to a symplectic connection on T^*E^N. It is possible to define a lift of the Levi-Civita connection Γ^i_{jk} on a general Riemannian manifold $\mathcal{Q}$ to a symplectic and torsionless connection $\tilde{\Gamma}^\alpha_{\beta\gamma}$ on the cotangent bundle $T^*\mathcal{Q}$. The resulting connection in the Darboux coordinate frame is given by the formulas

$$\begin{gathered} \tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^{\bar{i}}_{\bar{j}k} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^{\bar{i}}_{j\bar{k}} = -\Gamma^k_{ji}, \\ \tilde{\Gamma}^{\bar{i}}_{jk} = p_l(\Gamma^r_{jk}\Gamma^l_{ri} + \Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^l_{ij,k} - \tfrac{1}{3}R^l_{ijk} - \tfrac{1}{3}R^l_{jik}), \end{gathered} \quad (3.4.52)$$

with the remaining components equal zero. In the adopted frame $\{D_i, D^j\}$ the connection $\tilde{\Gamma}^\alpha_{\beta\gamma}$ takes the form

$$\tilde{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \tilde{\Gamma}^{\bar{i}}_{\bar{j}k} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^{\bar{i}}_{jk} = -\frac{1}{3}p_l(R^l_{ijk} + R^l_{jik}), \quad (3.4.53)$$

with the remaining components equal zero. Straightforward but tedious calculations lead to the following components $\tilde{R}^{\alpha}_{\beta\gamma\delta}$ for the curvature tensor of the symplectic torsionless connection $\tilde{\nabla}$ given by (3.4.52)

$$\begin{gathered}\tilde{R}^{i}_{jkl} = R^{i}_{jkl}, \quad \tilde{R}^{\bar{i}}_{jk\bar{l}} = \frac{2}{3} R^{l}_{(ij)k}, \\ \tilde{R}^{\bar{i}}_{jkl} = -\frac{1}{3} p_r \left(R^{r}_{jkl;i} + R^{r}_{ikl;j} - 6\Gamma^{r}_{s(i} R^{s}_{j)kl} + 4 R^{s}_{(ij)[k} \Gamma^{r}_{l]s} \right),\end{gathered} \tag{3.4.54}$$

with all remaining independent components equal zero, where $(\,\cdot\,,\,\cdot\,)$ and $[\,\cdot\,,\,\cdot\,]$ stand for the symmetrization and anti-symmetrization, respectively. From (3.4.54) it is possible to calculate the components of the Ricci curvature tensor, $\tilde{R}_{\alpha\beta} = \tilde{R}^{\gamma}_{\alpha\gamma\beta}$, receiving

$$\tilde{R}_{ij} = \frac{2}{3} R_{ij}, \quad \tilde{R}_{i\bar{j}} = \tilde{R}_{\bar{i}j} = \tilde{R}_{\bar{i}\bar{j}} = 0. \tag{3.4.55}$$

As we will see later, on a symplectic manifold endowed with a symplectic torsionless connection it is possible to distinguish a family of star-products. In the majority of physically interesting cases as the symplectic manifold is taken the cotangent bundle to a configuration space being a Riemannian manifold. In such case there exists a distinguished connection and thus a family of star-products which can be used to introduce quantization. More about lifts of connections can be found in [75, 76].

Remark 3.4.2. The star-product (3.4.35) is also a valid star-product on more general symplectic manifolds. Let us consider a symplectic manifold M whose Poisson tensor can be written in the form (3.4.34). In addition, let us assume that the first de Rham cohomology class $H^1(M)$ vanishes. This will guarantee the existence of global natural coordinate systems associated to the star-products (3.4.35). On such symplectic manifold M the product (3.4.35) is a valid star-product, which can also be written in a covariant form (3.4.50) with an appropriate linear connection $\tilde{\nabla}$. However, in this case there is no distinguished star-product from the family of products (3.4.35). To distinguish a star-product we have to distinguish a sequence of commuting vector fields (D_μ) from the decomposition (3.4.34) of the Poisson tensor, or equivalently, by distinguishing a flat torsionless symplectic linear connection $\tilde{\nabla}$ on M.

3.4.3 Canonical star-product on $T^*\mathcal{Q}$ with a flat base manifold $\mathcal{Q}$

We can distinguish a star-product on more general symplectic manifolds. Let $\mathcal{Q}$ be an N-dimensional flat Riemannian manifold, and let us take as a symplectic manifold M the cotangent bundle to $\mathcal{Q}$, $M = T^*\mathcal{Q}$. By virtue of (3.4.51) we can lift a flat Levi-Civita connection ∇ on $\mathcal{Q}$ to a flat torsionless symplectic connection $\tilde{\nabla}$ on M. In analogy to (3.4.50) we can define a canonical star-product on M by the following formula

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \omega^{\mu_1\nu_1} \cdots \omega^{\mu_k\nu_k} (\underbrace{\tilde{\nabla} \cdots \tilde{\nabla}}_{k} f)_{\mu_1\ldots\mu_k} (\underbrace{\tilde{\nabla} \cdots \tilde{\nabla}}_{k} g)_{\nu_1\ldots\nu_k}. \tag{3.4.56}$$

It can be proved that the star-product (3.4.56) is associative (see [49]), thus it is a proper star-product on M. Note, that in a case of a non-flat connection $\tilde{\nabla}$ the star-product (3.4.56) in general fails to be associative.

The star-product (3.4.56) can be written in a different form. Let $\widetilde{\exp}\colon TM \to M$ be an exponential map of the connection $\tilde{\nabla}$. For every $x \in M$ there exists a neighborhood $U \subset M$ of x on which $\widetilde{\exp}_x$ is a diffeomorphism of an open subset V of the tangent space T_xM onto U. $\widetilde{\exp}_x$ can be used to locally represent each function $f \in C^\infty(M)$ as a smooth function defined on the vector space T_xM. On each vector space there exists a canonical star-product, namely the Moyal product $\star_M$, thus it is natural to define on M a star-product by the following formula

$$(f \star g)(x) = (\widetilde{\exp}_x^* f \star_M \widetilde{\exp}_x^* g)(0), \tag{3.4.57}$$

where $\widetilde{\exp}_x^* f = f \circ \widetilde{\exp}_x$. Using the formula

$$\frac{\partial^k}{\partial y^{\mu_1} \cdots \partial y^{\mu_k}} f(\widetilde{\exp}_x(y))\bigg|_{y=0} = (\underbrace{\tilde{\nabla} \cdots \tilde{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k}(x) \tag{3.4.58}$$

one can easily see that the star-product (3.4.57) is equal to (3.4.56).

For certain manifolds $\mathcal{Q}$ the star-product (3.4.56) can be written in an integral form. A Riemannian manifold $(\mathcal{Q}, g)$ will be called geodesically simply connected if every pair of points in $\mathcal{Q}$ is connected by a unique geodesic. A Riemannian manifold $(\mathcal{Q}, g)$ will be called almost geodesically simply connected if for every $q \in \mathcal{Q}$ there exists a neighborhood $U \subset \mathcal{Q}$ of q such that $\mathcal{Q} \setminus U$ is of measure zero with respect to the measure induced by the metric volume form ω_g, and every point in U can be connected with q by a unique geodesic. Similarly we define the notion of an (almost) geodesically simply connected symplectic manifold (M, ω) equipped with a torsionless symplectic connection. In that case we replace in the definition the metric volume form ω_g with the Liouville volume form Ω. If $\mathcal{Q}$ is (almost) geodesically simply connected then $T^*\mathcal{Q}$ has the same property. An example of geodesically simply connected Riemannian manifold is the Euclidean space, and an example of almost geodesically simply connected Riemannian manifold is the sphere.

If $M = T^*\mathcal{Q}$ is almost geodesically simply connected then for every $x \in M$ exists a neighborhood $U \subset M$ such that $M \setminus U$ is of measure zero and $\widetilde{\exp}_x$ is a diffeomorphism of an open subset $V \subset T_xM$ onto U. If $f \in C_0^\infty(M)$ is a smooth function with compact support then $\widetilde{\exp}_x^* f \in C_0^\infty(V)$ is also a smooth function with compact support. The function $\widetilde{\exp}_x^* f$ can be uniquely extended to a smooth function defined on the whole tangent space T_xM with the same support as $\widetilde{\exp}_x^* f$, just by putting the function $\widetilde{\exp}_x^* f$ equal 0 outside V. Thus by virtue of (3.4.57) and the integral form of the Moyal product (3.4.16) it follows that for $f, g \in C_0^\infty(M)$ the $\star$-product can be written in the following integral form

$$(f \star g)(x) = \frac{1}{(\pi\hbar)^{2N}} \int_{T_xM} \int_{T_xM} f(\widetilde{\exp}_x(u)) g(\widetilde{\exp}_x(v)) e^{-\frac{2i}{\hbar}\omega_x(u,v)} \, \mathrm{d}u \, \mathrm{d}v. \tag{3.4.59}$$

Note, that the assumption that $M \setminus U$ is of measure zero guaranties that the above integral form of the $\star$-product indeed expands to the series (3.4.56).

Let $(x^1, \ldots, x^{2N})$ be a coordinate system on M which is at the same time classical and quantum canonical. In accordance with Theorem 3.4.3 the $\star^{(x)}$-product is equivalent with the Moyal product for the coordinates $(x^1, \dots, x^{2N})$. In what follows we will derive the form of the respective equivalence morphism S to the second order in $\hbar$. By virtue of Theorem 3.4.3 it follows that only terms with even powers in $\hbar$ are non-zero, thus we only have to calculate S_2. To find the form of S_2 we have to solve the following system of equations

$$[S_2, x^\alpha] = -\frac{1}{4}A_2^\alpha, \tag{3.4.60a}$$

$$[S_2, \partial^\alpha] = -\frac{1}{4}A_3^\alpha, \tag{3.4.60b}$$

where

$$A_k^\alpha f = \frac{1}{k!}\omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k}x^\alpha)_{\mu_1\dots\mu_k}(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k}f)_{\nu_1\dots\nu_k}. \tag{3.4.61}$$

Theorem 3.4.4. *The solution to* (3.4.60) *is of the form*

$$S_2 = -\frac{1}{24}\tilde{\Gamma}_{\alpha\beta\gamma}\partial^\alpha\partial^\beta\partial^\gamma + \frac{1}{16}\tilde{\Gamma}^\mu_{\nu\alpha}\tilde{\Gamma}^\nu_{\mu\beta}\partial^\alpha\partial^\beta, \tag{3.4.62}$$

where $\tilde{\Gamma}_{\alpha\beta\gamma} = \omega_{\alpha\delta}\tilde{\Gamma}^\delta_{\beta\gamma}$.

The proof of the above theorem is given in Appendix B. Note that the condition that $\tilde{\nabla}$ has vanishing torsion can be restated as

$$\tilde{\Gamma}^\alpha_{\beta\gamma} = \tilde{\Gamma}^\alpha_{\gamma\beta}, \tag{3.4.63}$$

and the condition that $\tilde{\nabla}$ is symplectic ($\omega_{\mu\nu;\alpha} = 0$, $\omega^{\mu\nu}{}_{;\alpha} = 0$) in Darboux coordinates can be restated as

$$\omega^{\delta\beta}\tilde{\Gamma}^\alpha_{\beta\gamma} = \omega^{\alpha\beta}\tilde{\Gamma}^\delta_{\beta\gamma}, \tag{3.4.64a}$$

$$\omega_{\delta\alpha}\tilde{\Gamma}^\alpha_{\beta\gamma} = \omega_{\beta\alpha}\tilde{\Gamma}^\alpha_{\delta\gamma}. \tag{3.4.64b}$$

From conditions (3.4.63) and (3.4.64b) we get that $\tilde{\nabla}$ is symplectic and torsionless iff $\tilde{\Gamma}_{\alpha\beta\gamma}$ is symmetric with respect to indices α, β, γ [76].

Remark 3.4.3. The $\star$-product (3.4.56) can be also defined on a general symplectic manifold (M, ω) equipped with a flat torsionless symplectic connection $\tilde{\nabla}$. In such general case formula (3.4.57), considerations about the integral form of the $\star$-product, and the form of the morphism S (3.4.62) remain the same.

3.4.4 Family of star-products on $T^*\mathcal{Q}$ with a non-flat base manifold $\mathcal{Q}$

In this section we will describe a procedure of introducing star-products on a symplectic manifold $M = T^*\mathcal{Q}$ over a non-flat Riemannian manifold $(\mathcal{Q}, g)$. In such general case we will use a connection $\tilde{\nabla}$ on $T^*\mathcal{Q}$, induced from a Levi-Civita connection ∇ on $\mathcal{Q}$, to define a star-product. However, a star-product in the form

(3.4.56) for a curved linear connection $\tilde{\nabla}$ is not a proper star-product (it is not associative). Thus we have to change the star-product (3.4.56) in such a way that for a curved linear connection $\tilde{\nabla}$ it would remain associative. Moreover, we would like it to be equivalent with the Moyal product for every classical and quantum canonical coordinate system.

The general way of defining on a symplectic manifold M a star-product equivalent with the Moyal product is as follows. As in the general case there is no single global coordinate chart, in order to define a product, which will be equivalent with the Moyal product, it is necessary to do this locally for every classical and quantum canonical coordinate chart. Let us take an atlas of classical and quantum canonical coordinate charts $(x_\alpha^1, \dots, x_\alpha^{2N})$ defined on open subsets U_α of the symplectic manifold M. Moreover, let us take some family of linear automorphisms S_α of $C^\infty(U_\alpha)$ with the property: two morphisms S_α and S_β when acted on the Moyal products $\star_M^{(x_\alpha)}$ and $\star_M^{(x_\beta)}$ give star-products, which on the intersection $U_\alpha \cap U_\beta$, are related to each other by the change of variables $(x_\alpha^1, \dots, x_\alpha^{2N}) \mapsto (x_\beta^1, \dots, x_\beta^{2N})$. Every such automorphism S_α can be used to define a star-product on $C^\infty(U_\alpha)$ by acting on the Moyal product $\star_M^{(x_\alpha)}$. All these star-products are consistent on the intersections $U_\alpha \cap U_\beta$ and hence glue together to give a global star-product on $C^\infty(M)$. The question whether such family of automorphisms S_α always exists is nontrivial. Moreover, in the case when such family exists it is not specified uniquely.

In what follows we will use the above procedure to define on $M = T^*\mathcal{Q}$ a family of star-products. We will present the construction to the third order in $\hbar$. Let us take the admissible morphisms S_α in the similar form as for the flat case (see formula (3.4.62))

$$S = \mathrm{id} + \hbar^2 \left(-\frac{1}{24} \tilde{\Gamma}_{\alpha\beta\gamma} \partial^\alpha \partial^\beta \partial^\gamma + \frac{1}{16} (\tilde{\Gamma}^\mu_{\nu\alpha} \tilde{\Gamma}^\nu_{\mu\beta} + 3a \tilde{R}_{\alpha\beta}) \partial^\alpha \partial^\beta \right) + o(\hbar^4), \tag{3.4.65}$$

where a is some real parameter and $\tilde{R}_{\alpha\beta}$ is the Ricci curvature tensor. Then we will receive the one-parameter family of star-products in the form

$$\begin{aligned} f \star_a g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \omega^{\mu_1 \nu_1} \cdots \omega^{\mu_k \nu_k} \Big((\underbrace{\tilde{\nabla} \cdots \tilde{\nabla}}_{k} f)_{\mu_1 \dots \mu_k} (\underbrace{\tilde{\nabla} \cdots \tilde{\nabla}}_{k} g)_{\nu_1 \dots \nu_k} & \\ + B_{\mu_1 \dots \mu_k \nu_1 \dots \nu_k}(f, g) \Big), & \end{aligned} \tag{3.4.66}$$

where $B_{\mu_1 \dots \mu_k \nu_1 \dots \nu_k}$ are bilinear operators given by

$$\begin{aligned} B_0(f,g) &= 0, \\ B_{\mu_1\nu_1}(f,g) &= 0, \\ B_{\mu_1\mu_2\nu_1\nu_2}(f,g) &= -3a \tilde{R}_{\mu_1\mu_2} (\tilde{\nabla}_{\nu_1} f)(\tilde{\nabla}_{\nu_2} g), \\ B_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(f,g) &= -\tilde{R}_{\nu_1\nu_2\nu_3\alpha} \omega^{\alpha\beta} (\tilde{\nabla}\tilde{\nabla}\tilde{\nabla} f)_{\mu_1\mu_2\mu_3} (\tilde{\nabla}_\beta g) \\ &\quad - \tilde{R}_{\mu_1\mu_2\mu_3\alpha} \omega^{\alpha\beta} (\tilde{\nabla}_\beta f)(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla} g)_{\nu_1\nu_2\nu_3} - \frac{9}{2} a \tilde{R}_{\mu_1\mu_2;\mu_3} (\tilde{\nabla}_{\nu_3} f)(\tilde{\nabla}\tilde{\nabla} g)_{\nu_1\nu_2} \\ &\quad + \frac{9}{2} a \tilde{R}_{\mu_1\mu_2;\mu_3} (\tilde{\nabla}\tilde{\nabla} f)_{\nu_1\nu_2} (\tilde{\nabla}_{\nu_3} g) + 9a \tilde{R}_{\mu_2\nu_3} (\tilde{\nabla}\tilde{\nabla} f)_{\mu_1\mu_3} (\tilde{\nabla}\tilde{\nabla} g)_{\nu_1\nu_2} \\ &\quad + \tilde{R}_{\mu_1\mu_2\mu_3\alpha} \tilde{R}_{\nu_1\nu_2\nu_3\gamma} \omega^{\alpha\beta} \omega^{\gamma\delta} (\tilde{\nabla}_\beta f)(\tilde{\nabla}_\delta g), \end{aligned} \tag{3.4.67}$$

and $\tilde{R}_{\alpha\beta\gamma\delta} = \omega_{\alpha\lambda}\tilde{R}^{\lambda}_{\beta\gamma\delta}$ is the curvature tensor. Analogical considerations as in the previous section (see the proof of Theorem 3.4.4) prove that the star-products (3.4.66) with the four first operators $B_{\mu_1\ldots\mu_k\nu_1\ldots\nu_k}$ given by (3.4.67) are equivalent with the Moyal product, up to third order in $\hbar$. Clearly for the flat linear connection $\tilde{\nabla}$ the products (3.4.66) reduces to (3.4.56).

In a special case $a = 0$ the star-product (3.4.66) reduces to

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}(D_{\mu_1\ldots\mu_k}f)(D_{\nu_1\ldots\nu_k}g), \tag{3.4.68}$$

where $D_{\mu_1\ldots\mu_k}$ are linear operators mapping functions to k-times covariant tensor fields given by

$$D_0 f = f, \tag{3.4.69a}$$
$$D_{\mu_1} f = \tilde{\nabla}_{\mu_1} f, \tag{3.4.69b}$$
$$D_{\mu_1\mu_2} f = (\tilde{\nabla}\tilde{\nabla} f)_{\mu_1\mu_2}, \tag{3.4.69c}$$
$$D_{\mu_1\mu_2\mu_3} f = (\tilde{\nabla}\tilde{\nabla}\tilde{\nabla} f)_{\mu_1\mu_2\mu_3} - \tilde{R}_{\mu_1\mu_2\mu_3\alpha}\omega^{\alpha\beta}\tilde{\nabla}_\beta f. \tag{3.4.69d}$$

A direct calculation, with the help of the Ricci identity

$$\tilde{R}_{\alpha\beta\gamma\delta} + \tilde{R}_{\alpha\gamma\delta\beta} + \tilde{R}_{\alpha\delta\beta\gamma} = 0, \tag{3.4.70}$$

shows that operators (3.4.69) are symmetric with respect to indices $\mu_1, \mu_2, \ldots$. Note, that the star-product (3.4.68) up to at least third order in $\hbar$ is a Fedosov star-product associated with the Weyl curvature form $\Omega = \omega$ [54]. It should be noted that for $a \neq 0$ the star-product (3.4.66) is not a Fedosov star-product.

From the presented construction it is clear that when the configuration space $\mathcal{Q}$ is curved there is no single natural star-product on $T^*\mathcal{Q}$ but the whole family of natural star-products. In the considered case (see formula (3.4.65)) the natural star-products are parametrized by a real number a. Also the Fedosov construction of star-products has freedom in taking different Weyl curvature forms Ω.

Remark 3.4.4. The presented construction of the star-products on a symplectic manifold $T^*\mathcal{Q}$ can be generalized, in a straightforward way, to a general symplectic manifold M endowed with a symplectic torsionless linear connection $\tilde{\nabla}$. Formulas (3.4.65)–(3.4.67) remain the same.

Let us extend the introduced family of star-products on $M = T^*\mathcal{Q}$. Using (3.4.52) and (3.4.55) the formula (3.4.65) can be rewritten in the form

$$\begin{aligned} S = \mathrm{id} + \frac{\hbar^2}{4!}\Big(&3\left(\Gamma^i_{lj}\Gamma^l_{ik} + aR_{jk}\right)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{q^i}\partial_{p_j}\partial_{p_k} \\ &+ \left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right)p_i\partial_{p_j}\partial_{p_k}\partial_{p_l}\Big) + o(\hbar^4). \end{aligned} \tag{3.4.71}$$

Let us generalize the formula (3.4.71) in the following way

$$\begin{aligned} S = \mathrm{id} + \frac{\hbar^2}{4!}\Big(&3\left(\Gamma^i_{lj}\Gamma^l_{ik} + aR_{jk}\right)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{q^i}\partial_{p_j}\partial_{p_k} + \left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right)p_i\partial_{p_j}\partial_{p_k}\partial_{p_l} \\ &- 3b\partial_{p_j}(\partial_{q^j} + \Gamma^i_{jl}p_i\partial_{p_l})\partial_{p_k}(\partial_{q^k} + \Gamma^r_{kn}p_r\partial_{p_n})\Big) + o(\hbar^4), \end{aligned} \tag{3.4.72}$$

where b is some real parameter. For a symplectic manifold T^*E^N and Cartesian coordinates (q^i, p_j) all Christoffel symbols $\Gamma^i_{jk} = 0$ and the morphism S (3.4.72) takes the form

$$S = \mathrm{id} - \frac{\hbar^2}{8} b \partial_{q^j} \partial_{p_j} \partial_{q^k} \partial_{p_k} + o(\hbar^4), \tag{3.4.73}$$

and can be considered as the expansion of the following morphism

$$S = \exp\left(-\frac{\hbar^2}{8} b \partial_{q^j} \partial_{p_j} \partial_{q^k} \partial_{p_k} \right). \tag{3.4.74}$$

The morphism S (3.4.74) induces a star-product which takes the form

$$\begin{aligned} f \star_b g = f \exp\Bigg(&\frac{1}{2} i\hbar \overleftarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} - \frac{1}{2} i\hbar \overleftarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^i} + \frac{1}{8} b\hbar^2 (\overleftarrow{\partial}_{q^i} \overleftarrow{\partial}_{p_i} \overleftarrow{\partial}_{q^j} \overleftarrow{\partial}_{p_j} + \overrightarrow{\partial}_{q^i} \overrightarrow{\partial}_{p_i} \overrightarrow{\partial}_{q^j} \overrightarrow{\partial}_{p_j}) \\ &- \frac{1}{8} b\hbar^2 (\overleftarrow{\partial}_{q^i} + \overrightarrow{\partial}_{q^i})(\overleftarrow{\partial}_{p_i} + \overrightarrow{\partial}_{p_i})(\overleftarrow{\partial}_{q^j} + \overrightarrow{\partial}_{q^j})(\overleftarrow{\partial}_{p_j} + \overrightarrow{\partial}_{p_j}) \Bigg) g. \end{aligned} \tag{3.4.75}$$

In general, the star-product induced by the morphism S (3.4.72) for $a = 1$ and $b = 1$ leads to what was called in the paper [11] a "minimal" quantization. Moreover, the same quantization was used in [77–79] in order to investigate the quantum integrability and quantum separability of classical Stäckel systems.

3.4.5 Example of non-canonical star-products on T^*E^N

In what follows we will present a family of star-products on the symplectic manifold T^*E^N, which are not in the form (3.2.3) and for which the complex-conjugation is not the involution. Let $X_1, \ldots, X_N, Y_1, \ldots, Y_N$ be a sequence of pair-wise commuting global vector fields from the decomposition (3.4.34) of the Poisson tensor $\mathcal{P}$. Define a star-product by the formula

$$\begin{aligned} f \star_{\lambda,\alpha,\beta} g = f \exp\Bigg(&i\hbar\lambda \sum_i \overleftarrow{X}_i \overrightarrow{Y}_i - i\hbar(1-\lambda) \sum_i \overleftarrow{Y}_i \overrightarrow{X}_i \\ &+ \hbar\alpha \sum_i \overleftarrow{X}_i \overrightarrow{X}_i + \hbar\beta \sum_i \overleftarrow{Y}_i \overrightarrow{Y}_i \Bigg) g, \end{aligned} \tag{3.4.76}$$

where $\lambda, \alpha, \beta \in \mathbb{R}$. The star-product (3.4.76) is equivalent with the star-product (3.4.35) corresponding to the same sequence (X_i, Y_j) of vector fields. A morphism (3.1.6) giving this equivalence is of the form

$$S_{\lambda,\alpha,\beta} = \exp\left(-i\hbar\left(\frac{1}{2} - \lambda\right) \sum_i X_i Y_i + \frac{1}{2}\hbar\alpha \sum_i X_i X_i + \frac{1}{2}\hbar\beta \sum_i Y_i Y_i \right). \tag{3.4.77}$$

The involution for the $\star_{\lambda,\alpha,\beta}$-product takes the form

$$f^* = \exp\left(-i\hbar(1-2\lambda) \sum_i X_i Y_i \right) \bar{f}. \tag{3.4.78}$$

Equation (3.4.78) indeed defines a proper involution. To see this first note that the involution (3.4.78) can be written in the form $f^* = S_{\lambda,\alpha,\beta}\overline{S^{-1}_{\lambda,\alpha,\beta}f}$. Then from (3.1.7) and the fact that the complex-conjugation is the involution for the $\star$-product (3.4.35) we get

$$\begin{aligned}
(f \star_{\lambda,\alpha,\beta} g)^* &= S_{\lambda,\alpha,\beta}\overline{S^{-1}_{\lambda,\alpha,\beta}(f \star_{\lambda,\alpha,\beta} g)} = S_{\lambda,\alpha,\beta}(\overline{S^{-1}_{\lambda,\alpha,\beta}f \star S^{-1}_{\lambda,\alpha,\beta}g}) \\
&= S_{\lambda,\alpha,\beta}(\overline{S^{-1}_{\lambda,\alpha,\beta}g} \star \overline{S^{-1}_{\lambda,\alpha,\beta}f}) = (S_{\lambda,\alpha,\beta}\overline{S^{-1}_{\lambda,\alpha,\beta}g}) \star_{\lambda,\alpha,\beta} (S_{\lambda,\alpha,\beta}\overline{S^{-1}_{\lambda,\alpha,\beta}f}) \\
&= g^* \star_{\lambda,\alpha,\beta} f^*. \qquad (3.4.79)
\end{aligned}$$

From (3.4.78) it is evident that for $\lambda \neq \frac{1}{2}$ the involution for the $\star_{\lambda,\alpha,\beta}$-product is different than the complex-conjugation and functions self-adjoint with respect to it can be in general complex.

As an example let us consider a quantization given by the $\star_{\lambda,\alpha,\beta}$-product for a one-dimensional case ($N = 1$) and in a natural coordinate system when $X = \partial_q$ and $Y = \partial_p$. Consider complex function $A(q,p) = qp^2 + \hbar\beta q - i\hbar(1-2\lambda)p$. A direct calculation shows that A represents an observable, as it is self-adjoint with respect to the involution $*$ (3.4.78). Moreover, it is equivalent to an observable $A(q,p) = qp^2$ for the Moyal quantization in the same coordinate system.

Chapter 4

Operator representation of quantum mechanics

4.1 Operator representation over a phase space

4.1.1 The case of a phase space $\mathbb{R}^{2N}$

Let us take as a phase space M the symplectic vector space $(\mathbb{R}^{2N}, \omega)$, where ω is a standard symplectic form. Moreover, let us consider on M a star-product which in canonical coordinates $x = x^\alpha e_\alpha \mapsto (x^1, \dots, x^{2N})$, where $e_1, \dots, e_{2N}$ is a canonical basis on $\mathbb{R}^{2N}$, is in the form of the Moyal product (3.4.2). To elements of $C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ we can associate operators defined on the Hilbert space $L^2(\mathbb{R}^{2N}, \mathrm{d}l)$ by the prescription

$$f \mapsto f \star_M . \tag{4.1.1}$$

Formula (4.1.1) gives us a representation of the quantum Poisson algebra $\mathcal{A}_Q(\mathbb{R}^{2N}) = (C^\infty(\mathbb{R}^{2N})[\![\hbar]\!], \star_M)$ in the Hilbert space $L^2(\mathbb{R}^{2N}, \mathrm{d}l)$. In what follows we will investigate the form of the operators $f \star_M$. We will need a notion of a symplectic Fourier transform. For a function $f \in L^1(\mathbb{R}^{2N})$ we define a symplectic Fourier transform of f by the formula

$$\mathscr{F}_\omega f(x) = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} f(y) e^{-\frac{i}{\hbar}\omega(x,y)} \,\mathrm{d}y, \tag{4.1.2}$$

where ω is a standard symplectic form on $\mathbb{R}^{2N}$ given by $\omega(x, y) = \mathcal{J}_{\alpha\beta} x^\alpha y^\beta$ where

$$(\mathcal{J}_{\alpha\beta}) = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}. \tag{4.1.3}$$

Note that $\mathscr{F}_\omega f(x) = \mathscr{F} f(\mathcal{J}^T x)$.

Theorem 4.1.1. *Let f be an element of the space $\mathcal{S}(\mathbb{R}^{2N})$ of Schwartz functions. The operator $f \star_M$ can be written in the following form*

$$f \star_M = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q, p) e^{\frac{i}{\hbar}(p_i \hat{q}^i_{\star_M} - q^i \hat{p}_{\star_M i})} \,\mathrm{d}q \,\mathrm{d}p, \tag{4.1.4}$$

where $\hat{q}^i_{\star_M} = q^i \star_M = q^i + \frac{1}{2} i\hbar \partial_{p_i}$ and $\hat{p}_{\star_M i} = p_i \star_M = p_i - \frac{1}{2} i\hbar \partial_{q^i}$ are operators of position and momentum.

Proof. Let $\rho \in L^2(\mathbb{R}^{2N})$. Using the identity

$$e^{y^i \partial_{x^i}} \rho(x) = \rho(x+y), \quad (y^1, \ldots, y^{2N}) \in \mathbb{R}^{2N} \tag{4.1.5}$$

and the Baker-Campbell-Hausdorff formula we receive that

$$\begin{aligned}
e^{\frac{i}{\hbar}(p'_i \hat{q}^i_{\star M} - q'^i \hat{p}_{\star M i})} \rho(q,p) &= e^{-\frac{i}{2\hbar} q'^i p'_i} e^{\frac{i}{\hbar} p'_i \hat{q}^i_{\star M}} e^{-\frac{i}{\hbar} q'^i \hat{p}_{\star M i}} \rho(q,p) \\
&= e^{-\frac{i}{2\hbar} q'^i p'_i} e^{\frac{i}{\hbar} p'_i (q^i + \frac{1}{2} i\hbar \partial_{p_i})} e^{-\frac{i}{\hbar} q'^i (p_i - \frac{1}{2} i\hbar \partial_{q^i})} \rho(q,p) \\
&= e^{-\frac{i}{2\hbar} q'^i p'_i} e^{\frac{i}{\hbar} p'_i q^i} e^{-\frac{1}{2} p'_i \partial_{p_i}} e^{-\frac{i}{\hbar} q'^i p_i} e^{-\frac{1}{2} q'^i \partial_{q^i}} \rho(q,p) \\
&= e^{\frac{i}{\hbar}(p'_i q^i - q'^i p_i)} e^{-\frac{1}{2} p'_i \partial_{p_i}} e^{-\frac{1}{2} q'^i \partial_{q^i}} \rho(q,p) \\
&= e^{\frac{i}{\hbar}(p'_i q^i - q'^i p_i)} \rho(q - \tfrac{1}{2} q', p - \tfrac{1}{2} p').
\end{aligned} \tag{4.1.6}$$

From the above result we get that

$$\begin{aligned}
&\frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q',p') e^{\frac{i}{\hbar}(p'_i \hat{q}^i_{\star M} - q'^i \hat{p}_{\star M i})} \, \mathrm{d}q' \, \mathrm{d}p' \rho(q,p) = \\
&\quad = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q',p') \rho(q - \tfrac{1}{2} q', p - \tfrac{1}{2} p') e^{\frac{i}{\hbar}(p'_i q^i - q'^i p_i)} \, \mathrm{d}q' \, \mathrm{d}p' \\
&\quad = \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} f(q'',p'') \rho(q - \tfrac{1}{2} q', p - \tfrac{1}{2} p') \\
&\qquad \times e^{\frac{i}{\hbar}(p'_i (q^i - q''^i) - q'^i (p_i - p''_i))} \, \mathrm{d}q' \, \mathrm{d}p' \, \mathrm{d}q'' \, \mathrm{d}p''.
\end{aligned} \tag{4.1.7}$$

After the following change of variables under the integral sign

$$\begin{aligned}
q'^i &\to -2q'^i, \quad q''^i \to q^i + q''^i, \\
p'_i &\to -2p'_i, \quad p''_i \to p_i + p''_i,
\end{aligned} \tag{4.1.8}$$

the above equation can be written in a form

$$\begin{aligned}
&\frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q',p') e^{\frac{i}{\hbar}(p'_i \hat{q}^i_{\star M} - q'^i \hat{p}_{\star M i})} \, \mathrm{d}q' \, \mathrm{d}p' \rho(q,p) = \\
&= \frac{1}{(\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} f(q+q'',p+p'') \rho(q+q',p+p') e^{-\frac{2i}{\hbar}(q'^i p''_i - p'_i q''^i)} \, \mathrm{d}q' \, \mathrm{d}p' \, \mathrm{d}q'' \, \mathrm{d}p'',
\end{aligned} \tag{4.1.9}$$

which is the integral form (3.4.16) of the star-product $f \star_M \rho$. □

Let us now consider on $(\mathbb{R}^{2N}, \omega)$ the following star-product

$$f \star_\lambda g = f \exp\left(i\hbar\lambda \overleftarrow{\partial_{q^i}} \overrightarrow{\partial_{p_i}} - i\hbar(1-\lambda) \overleftarrow{\partial_{p_i}} \overrightarrow{\partial_{q^i}} \right) g, \tag{4.1.10}$$

for $\lambda \in [0,1]$, which is a particular example of the star-product (3.4.76). Using similar considerations as in the proofs of Theorems 3.4.2 and 4.1.1 the following theorem can be proved.

Theorem 4.1.2. *Let f be an element of the space $\mathcal{S}(\mathbb{R}^{2N})$ of Schwartz functions. The operator $f\star_\lambda$ can be written in the following form*

$$f\star_\lambda = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q,p) e^{\frac{i}{\hbar}(p_i\hat{q}^i_{\star_\lambda} - q^i\hat{p}_{\star_\lambda i}) + (\frac{1}{2}-\lambda)q^ip_i)}\,\mathrm{d}q\,\mathrm{d}p, \tag{4.1.11}$$

where $\hat{q}^i_{\star_\lambda} = q^i\star_\lambda = q^i + i\hbar\lambda\partial_{p_i}$ and $\hat{p}_{\star_\lambda i} = p_i\star_\lambda = p_i - i\hbar(1-\lambda)\partial_{q^i}$ are operators of position and momentum.

The right hand sides of equations (4.1.4) and (4.1.11) are symmetric and λ-ordered functions of operators $\hat{q}^i_{\star_\lambda}, \hat{p}_{\star_\lambda j}$ in accordance to a λ-Weyl correspondence rule.

In general, for a Hilbert space $\mathcal{H}$ and self-adjoint operators $\hat{q}^i, \hat{p}_j$ on $\mathcal{H}$ (where $i,j = 1,2,\ldots,N$) satisfying the following commutation relations

$$[\hat{q}^i,\hat{q}^j] = [\hat{p}_i,\hat{p}_j] = 0, \quad [\hat{q}^i,\hat{p}_j] = i\hbar\delta^i_j, \quad i,j = 1,2,\ldots,N, \tag{4.1.12}$$

the λ-Weyl correspondence rule is a procedure of assigning operators defined on the Hilbert space $\mathcal{H}$, to functions defined on a symplectic manifold $(\mathbb{R}^{2N},\omega)$. It was first proposed by Weyl [80] for the symmetric ordering and formally it works by substituting for the variables q^i, p_j the operators $\hat{q}^i, \hat{p}_j$ and appropriately ordering them. The λ-ordered function f of the operators $\hat{q}^i, \hat{p}_j$ will be denoted by $f_\lambda(\hat{q},\hat{p})$ and is given by the formula

$$f_\lambda(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(x)\hat{T}_\lambda(x)\,\mathrm{d}x, \tag{4.1.13}$$

where

$$\hat{T}_\lambda(q,p) = e^{\frac{i}{\hbar}(p_i\hat{q}^i - q^i\hat{p}_i + (\frac{1}{2}-\lambda)q^ip_i)} = e^{\frac{i}{\hbar}p_i\hat{q}^i} e^{-\frac{i}{\hbar}q^i\hat{p}_i} e^{-\frac{i}{\hbar}\lambda q^ip_i} \tag{4.1.14}$$

is the modified Heisenberg operator and $\lambda \in [0,1]$ is a parameter describing different orderings. $\hat{T}_\lambda(x)$ is a unitary operator for every $x \in \mathbb{R}^{2N}$ and $\lambda \in [0,1]$.

In what follows we will investigate on which class of functions the formula (4.1.13) makes sense. Let

$$\rho_{\varphi\psi}(x) = (\varphi, \hat{T}_\lambda(x)\psi), \quad \varphi,\psi \in \mathcal{H}. \tag{4.1.15}$$

It can be easily checked that

$$|\rho_{\varphi\psi}(x)| \leq \|\varphi\|\|\psi\| \tag{4.1.16}$$

for every $x \in \mathbb{R}^{2N}$. Let $\mathscr{F}f \in L^1(\mathbb{R}^{2N})$ and

$$\Lambda(\varphi,\psi) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(x)\rho_{\varphi\psi}(x)\,\mathrm{d}x \tag{4.1.17}$$

be a bilinear form on $\mathcal{H}$. We calculate that

$$|\Lambda(\varphi,\psi)| \leq \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} |\mathscr{F}_\omega f(x)\rho_{\varphi\psi}(x)|\,\mathrm{d}x \leq \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} |\mathscr{F}f(x)|\,\mathrm{d}x\|\varphi\|\|\psi\|. \tag{4.1.18}$$

Thus Λ is bounded and there exists a unique bounded linear operator $f_\lambda(\hat{q},\hat{p})$ such that [81]

$$(\varphi, f_\lambda(\hat{q},\hat{p})\psi) = \Lambda(\varphi,\psi). \tag{4.1.19}$$

That way we gave sense to formula (4.1.13) for f such that $\mathscr{F}f \in L^1(\mathbb{R}^{2N})$.

Equation (4.1.13) makes sense also for wider class of functions f. In such case it has to be treated distributionally. Let $f \in \mathcal{A} = \mathscr{F}(\mathcal{E}')$ be the Fourier image of a distribution with compact support, and let

$$D = \{\psi \in \mathcal{H} \mid \rho_{\varphi\psi} \in C^\infty(\mathbb{R}^{2N}) \text{ for every } \varphi \in \mathcal{H}\}. \tag{4.1.20}$$

Then we can define a bilinear form $\Lambda \colon \mathcal{H} \times D \to \mathbb{C}$ by the formula

$$\Lambda(\varphi, \psi) = \langle \mathscr{F}_\omega f, \rho_{\varphi\psi} \rangle. \tag{4.1.21}$$

It is not difficult to show that

$$|\partial_x^\alpha \rho_{\varphi\psi}(x)| \leq C_\psi^\alpha(x) \|\varphi\| \tag{4.1.22}$$

for every multi-index $\alpha \in \mathbb{N}^{2N}$, $\varphi \in \mathcal{H}$, $\psi \in D$ and $x \in \mathbb{R}^{2N}$, where $C_\psi^\alpha(x)$ is some finite constant independent on φ. Indeed, using the Baker-Campbell-Hausdorff formula the operator $\hat{T}_\lambda(q,p)$ can be written in a form

$$\hat{T}_\lambda(q,p) = e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} e^{\frac{i}{\hbar} p_i \hat{q}^i} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i}. \tag{4.1.23}$$

Hence, using the Leibniz's formula we get

$$\begin{aligned} \partial_{q^{i_1}} \cdots \partial_{q^{i_n}} \partial_{p_{j_1}} \cdots \partial_{p_{j_m}} \hat{T}_\lambda(q,p) = \left(-\frac{i}{\hbar}\right)^n \left(\frac{i}{\hbar}\right)^m \sum_{k=0}^n \binom{n}{k} \lambda^k (1-\lambda)^{n-k} \hat{p}_{i_1} \cdots \hat{p}_{i_k} \\ \times e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} \hat{q}^{j_1} \cdots \hat{q}^{j_m} e^{\frac{i}{\hbar} p_i \hat{q}^i} \hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i}, \end{aligned} \tag{4.1.24}$$

and consequently

$$\begin{aligned} |\partial_{q^{i_1}} \cdots \partial_{q^{i_n}} \partial_{p_{j_1}} \cdots \partial_{p_{j_m}} \rho_{\varphi\psi}(q,p)| \leq \|\varphi\| \Bigg\| \left(-\frac{i}{\hbar}\right)^n \left(\frac{i}{\hbar}\right)^m \sum_{k=0}^n \binom{n}{k} \lambda^k (1-\lambda)^{n-k} \\ \times \hat{p}_{i_1} \cdots \hat{p}_{i_k} e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} \hat{q}^{j_1} \cdots \hat{q}^{j_m} e^{\frac{i}{\hbar} p_i \hat{q}^i} \hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i} \psi \Bigg\|. \end{aligned} \tag{4.1.25}$$

From (4.1.22) we get that

$$\|\rho_{\varphi\psi}\|_{K,\alpha} = \sup_{x \in K} |\partial_x^\alpha \rho_{\varphi\psi}(x)| \leq \sup_{x \in K} C_\psi^\alpha(x) \|\varphi\| = M_{K,\alpha}(\psi) \|\varphi\| \tag{4.1.26}$$

for every compact subset $K \subset \mathbb{R}^{2N}$, multi-index $\alpha \in \mathbb{N}^{2N}$, $\varphi \in \mathcal{H}$, and $\psi \in D$, where $M_{K,\alpha}(\psi)$ is some finite constant independent on φ. From continuity of $\mathscr{F}f$ in $\mathcal{E} = C^\infty(\mathbb{R}^{2N})$ there exists compact set $K \subset \mathbb{R}^{2N}$, $C > 0$, and multi-index $\alpha \in \mathbb{N}^{2N}$ such that

$$|\Lambda(\varphi, \psi)| = |\langle \mathscr{F}_\omega f, \rho_{\varphi\psi} \rangle| \leq C \|\rho_{\varphi\psi}\|_{K,\alpha} \leq C M_{K,\alpha}(\psi) \|\varphi\|. \tag{4.1.27}$$

Thus there exists a unique linear operator $f_\lambda(\hat{q}, \hat{p})$ with domain D such that [81]

$$(\varphi, f_\lambda(\hat{q}, \hat{p})\psi) = \Lambda(\varphi, \psi). \tag{4.1.28}$$

Note, that if there exist a dense subspace $D \subset \mathcal{H}$ such that $\hat{q}^i \colon D \to D$ and $\hat{p}_j \colon D \to D$ for $i, j = 1, 2, \ldots, N$ then the subspace defined by (4.1.20) is equal to this subspace D. In this case for every $f \in \mathcal{A}$ we get a densely defined operator $f_\lambda(\hat{q}, \hat{p})$ with domain D.

It can be calculated that the adjoint of the operator $\hat{T}_\lambda(x)$ is given by

$$\hat{T}_\lambda(x)^\dagger = \hat{T}_{1-\lambda}(-x). \tag{4.1.29}$$

From this follows that

$$f_\lambda(\hat{q}, \hat{p})^\dagger = \bar{f}_{1-\lambda}(\hat{q}, \hat{p}) \tag{4.1.30}$$

on D. As a consequence, in a spacial case $\lambda = \frac{1}{2}$, to real valued functions correspond self-adjoint operators.

Theorem 4.1.3. *For $f(q, p) = K^{i_1 \ldots i_n}(q) p_{i_1} \cdots p_{i_n}$, where $K^{i_1 \ldots i_n}$ is some symmetric complex tensor field on $\mathbb{R}^N$, we get*

$$f_\lambda(\hat{q}, \hat{p}) = \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} \hat{p}_{i_1} \cdots \hat{p}_{i_k} K^{i_1 \ldots i_n}(\hat{q}) \hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n}. \tag{4.1.31}$$

Proof. From (4.1.13) we get

$$\begin{aligned}
f_\lambda(\hat{q}, \hat{p}) &= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} K^{i_1 \ldots i_n}(q') p'_{i_1} \cdots p'_{i_n} e^{-\frac{i}{\hbar}(p_i q'^i - q^i p'_i)} \hat{T}_\lambda(q, p) \,\mathrm{d}q' \,\mathrm{d}p' \,\mathrm{d}q \,\mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} K^{i_1 \ldots i_n}(q') \Big((-i\hbar)^n \partial_{q^{i_1}} \cdots \partial_{q^{i_n}} e^{-\frac{i}{\hbar}(p_i q'^i - q^i p'_i)} \Big) \\
&\quad \times \hat{T}_\lambda(q, p) \,\mathrm{d}q' \,\mathrm{d}p' \,\mathrm{d}q \,\mathrm{d}p \\
&= \frac{1}{(2\pi\hbar)^{2N}} \int_{\mathbb{R}^{2N}} \int_{\mathbb{R}^{2N}} K^{i_1 \ldots i_n}(q') e^{-\frac{i}{\hbar}(p_i q'^i - q^i p'_i)} \Big((i\hbar)^n \partial_{q^{i_1}} \cdots \partial_{q^{i_n}} \hat{T}_\lambda(q, p) \Big) \\
&\quad \times \,\mathrm{d}q' \,\mathrm{d}p' \,\mathrm{d}q \,\mathrm{d}p.
\end{aligned} \tag{4.1.32}$$

Using the Baker-Campbell-Hausdorff formula the operator $\hat{T}_\lambda(q, p)$ can be written in a form

$$\hat{T}_\lambda(q, p) = e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} e^{\frac{i}{\hbar} p_i \hat{q}^i} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i}. \tag{4.1.33}$$

From (4.1.33) and the Leibniz's formula we get

$$\begin{aligned}
(i\hbar)^n \partial_{q^{i_1}} \cdots \partial_{q^{i_n}} \hat{T}_\lambda(q, p) &= (i\hbar)^n \partial_{q^{i_1}} \cdots \partial_{q^{i_n}} e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} e^{\frac{i}{\hbar} p_i \hat{q}^i} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i} \\
&= \sum_{k=0}^{n} \binom{n}{k} \lambda^k \hat{p}_{i_1} \cdots \hat{p}_{i_k} e^{-\frac{i}{\hbar}\lambda q^i \hat{p}_i} e^{\frac{i}{\hbar} p_i \hat{q}^i} (1-\lambda)^{n-k} \hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n} e^{-\frac{i}{\hbar}(1-\lambda) q^i \hat{p}_i}.
\end{aligned} \tag{4.1.34}$$

Substituting (4.1.34) into (4.1.32) and performing integration we get the result. □

From (4.1.31) we get for monomial $q^j p_j$ the operator

$$(\hat{q}^j \hat{p}_j)_\lambda = (1-\lambda) \hat{q}^j \hat{p}_j + \lambda \hat{p}_j \hat{q}^j. \tag{4.1.35}$$

Thus λ parametrizes different orderings, and so for $\lambda = 0$ we get normal ordering, for $\lambda = 1$ anti-normal ordering, and for $\lambda = \frac{1}{2}$ symmetric ordering. In the rest of the thesis we will mainly focus on the case $\lambda = \frac{1}{2}$. In such case we will often omit the symbol λ in $f_\lambda(\hat{q},\hat{p})$ and simply write $f(\hat{q},\hat{p})$.

Formula (4.1.13), by virtue of (3.4.12), can be written in a form

$$f_\lambda(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega S_\lambda^{-1} f(x)\hat{T}_{1/2}(x)\,\mathrm{d}x = (S_\lambda^{-1}f)(\hat{q},\hat{p}), \tag{4.1.36}$$

where

$$S_\lambda = \exp\left(-i\hbar(\tfrac{1}{2}-\lambda)\partial_{q^i}\partial_{p_i}\right). \tag{4.1.37}$$

Using this result we will generalize the concept of the ordering in the following way. For a series S of differential operators in the form

$$S = \mathrm{id} + \sum_{k=1}^{\infty} \hbar^k S_k \tag{4.1.38}$$

we define an S-ordered function of the operators $\hat{q}^i, \hat{p}_j$ by the formula

$$f_S(\hat{q},\hat{p}) = (S^{-1}f)(\hat{q},\hat{p}) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega S^{-1} f(x)\hat{T}_{1/2}(x)\,\mathrm{d}x. \tag{4.1.39}$$

Note, that if on $(\mathbb{R}^{2N},\omega)$ we have some $\star$-product equivalent with the Moyal product, where the equivalence morphism S satisfies

$$Sq^i = q^i, \quad Sp_i = p_i, \tag{4.1.40}$$

then the operator $f\,\star$ is of the form

$$f\,\star = f_S(\hat{q}_\star,\hat{p}_\star), \tag{4.1.41}$$

where $\hat{q}^i_\star = q^i\,\star$ and $\hat{p}_{\star i} = p_i\,\star$. Indeed, from (4.1.40) we get that

$$S\hat{q}^i_{\star_M}S^{-1} = S(q^i \star_M)S^{-1} = Sq^i\,\star = q^i\,\star = \hat{q}^i_\star \tag{4.1.42}$$

and similarly $S\hat{p}_{\star_M i}S^{-1} = \hat{p}_{\star i}$. Thus,

$$\begin{aligned} f\,\star &= S(S^{-1}f \star_M)S^{-1} = S(S^{-1}f)(\hat{q}_{\star_M},\hat{p}_{\star_M})S^{-1} \\ &= (S^{-1}f)(S\hat{q}_{\star_M}S^{-1}, S\hat{p}_{\star_M}S^{-1}) = f_S(\hat{q}_\star,\hat{p}_\star). \end{aligned} \tag{4.1.43}$$

By virtue of Theorem 3.1.1 every star-product on $\mathbb{R}^{2N}$ is equivalent with the Moyal product. Moreover, the equivalence morphism can always be chosen so that (4.1.40) is satisfied, and this requirement uniquely specifies the morphism. Hence, every star-product on $\mathbb{R}^{2N}$ gives rise to an ordering of operators $\hat{q}^i_\star, \hat{p}_{\star j}$. Consequently a quantization can be fixed either by choosing a star-product on a phase space $\mathbb{R}^{2N}$ or equivalently, on a level of the operator representation, by choosing an ordering. As we will see in Section 4.3 in an operator representation over a configuration space to a given star-product on $\mathbb{R}^{2N}$ corresponds the same ordering, of canonical operators $\hat{q}^i, \hat{p}_j$ of position and momentum defined on the Hilbert space $L^2(\mathbb{R}^N)$, as in the operator representation over a phase space.

Remark 4.1.1. The S-ordering rule (4.1.39) is very general and contains as special cases all ordering rules found in the literature. In particular, for a special case of a series S such that

$$S^{-1} = F(-i\hbar\partial_q, i\hbar\partial_p), \tag{4.1.44}$$

where $F\colon \mathbb{R}^{2N} \to \mathbb{C}$ is some general analytic function such that $F(0) = 1$, the S-ordered function of the operators $\hat{q}^i, \hat{p}_j$ reads

$$f_S(\hat{q}, \hat{p}) = \frac{1}{(2\pi\hbar)^N} \int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(q,p) e^{\frac{i}{\hbar}(p_i\hat{q}^i - q^i\hat{p}_i)} F(q,p) \,\mathrm{d}q\,\mathrm{d}p. \tag{4.1.45}$$

The above formula was first considered by Cohen [82]. Thus, it is clear that the very broad family of orderings considered by Cohen and others is a special case of the introduced family of orderings.

In general the morphisms S are not of the form (4.1.44). As an example in a two-dimensional case ($N = 1$) the following three parameter family of morphisms may serve

$$S = \exp\left(-i\hbar a\partial_q\partial_p + i\hbar b q\partial_p^2 - \hbar^2 c\partial_p^3\right), \tag{4.1.46}$$

where $a, b, c \in \mathbb{R}$. This shows that the family of quantizations considered in the thesis is more general than the broad family of quantizations considered by Cohen and others. To illustrate the S-ordering rule let us consider a function $f(q,p) = \frac{1}{2}p^2 + \frac{1}{6}p^3 + \frac{1}{2}q^2$. Then, one finds that

$$(S^{-1}f)(q,p) = \frac{1}{2}p^2 + \frac{1}{6}p^3 + \frac{1}{2}q^2 - i\hbar bq(1+p) + \hbar^2\left(\frac{1}{2}ab + c\right) \tag{4.1.47}$$

and that

$$\begin{aligned} f_S(\hat{q},\hat{p}) &= \frac{1}{2}\hat{p}^2 + \frac{1}{6}\hat{p}^3 + \frac{1}{2}\hat{q}^2 - i\hbar b\left(\hat{q} + \frac{1}{2}\hat{q}\hat{p} + \frac{1}{2}\hat{p}\hat{q}\right) + \hbar^2\left(\frac{1}{2}ab + c\right) \\ &= \frac{1}{2}\hat{p}^2 + \frac{1}{6}\hat{p}^3 + \frac{1}{2}\hat{q}^2 - b\hat{q}\hat{p}\hat{q} + b\hat{p}\hat{q}^2 - \left(\frac{1}{2}ab + \frac{1}{2}b + c\right)\hat{q}\hat{p}\hat{q}\hat{p} \\ &\quad + \left(\frac{1}{2}ab - \frac{1}{2}b + c\right)\hat{q}\hat{p}^2\hat{q} + \left(\frac{1}{2}ab + \frac{1}{2}b + c\right)\hat{p}\hat{q}^2\hat{p} - \left(\frac{1}{2}ab - \frac{1}{2}b + c\right)\hat{p}\hat{q}\hat{p}\hat{q}. \end{aligned} \tag{4.1.48}$$

Remark 4.1.2. Let us consider a two-dimensional phase space $M = \mathbb{R}^2$ and the following star-product defined on it

$$f \star_\lambda g = f \exp\left(\frac{1}{2}i\hbar\overleftarrow{\partial_q}\overrightarrow{\partial_p} - \frac{1}{2}i\hbar\overleftarrow{\partial_p}\overrightarrow{\partial_q} + \hbar\frac{2\lambda - 1}{2\omega}\overleftarrow{\partial_q}\overrightarrow{\partial_q} + \hbar\omega\frac{2\lambda-1}{2}\overleftarrow{\partial_p}\overrightarrow{\partial_p}\right) g, \tag{4.1.49}$$

for $\lambda \in [0,1]$ and $\omega > 0$, which is a particular example of the star-product (3.4.76). In holomorphic coordinates

$$a(q,p) = \frac{\omega q + ip}{\sqrt{2\omega}}, \quad \bar{a}(q,p) = \frac{\omega q - ip}{\sqrt{2\omega}} \tag{4.1.50}$$

the $\star_\lambda$-product takes the form

$$f \star_\lambda g = f \exp\left(\hbar\lambda \overleftarrow{\partial_a}\overrightarrow{\partial_{\bar{a}}} - \hbar(1-\lambda)\overleftarrow{\partial_{\bar{a}}}\overrightarrow{\partial_a}\right) g. \tag{4.1.51}$$

Moreover, the operators $f \star_\lambda$ can be written in a form

$$f \star_\lambda\, = f_\lambda(\hat{a}, \hat{a}^\dagger) = \frac{1}{\pi\hbar}\int \mathscr{F}f(w,\bar{w}) e^{\hbar^{-1}(w\hat{a}^\dagger - \bar{w}\hat{a} + (\frac{1}{2}-\lambda)|w|^2)}\, \mathrm{d}^2 w, \tag{4.1.52}$$

where $\hat{a} = a \star_\lambda$, $\hat{a}^\dagger = \bar{a} \star_\lambda$ are operators of annihilation and creation, and

$$\mathscr{F}f(w,\bar{w}) = \frac{1}{\pi\hbar}\int f(z,\bar{z}) e^{\hbar^{-1}(z\bar{w} - \bar{z}w)}\, \mathrm{d}^2 z, \tag{4.1.53}$$

where $\mathrm{d}^2 z = \mathrm{d}(\operatorname{Re} z)\, \mathrm{d}(\operatorname{Im} z)$, is the symplectic Fourier transform in holomorphic coordinates. The star-product (4.1.51) and the operator function $f_\lambda(\hat{a}, \hat{a}^\dagger)$ are widely used in quantum optics.

4.1.2 The case of a general phase space

First, let us consider a phase space in the form of a cotangent bundle T^*U to an open subset $U \subset \mathbb{R}^N$. On such phase space we can introduce the Moyal product (3.4.2) or, more generally, the $\star_\lambda$-product (4.1.10). If for self-adjoint operators $\hat{q}^i, \hat{p}_j$ defined on some Hilbert space $\mathcal{H}$ and satisfying the commutation relations (4.1.12), and for a function f on T^*U polynomial in momenta we define the corresponding λ-ordered function of the operators $\hat{q}^i, \hat{p}_j$ by the formula (4.1.31), then it can be proved that the operator $f \star_\lambda$ is of the form

$$f \star_\lambda\, = f_\lambda(\hat{q}_{\star_\lambda}, \hat{p}_{\star_\lambda}), \tag{4.1.54}$$

just like in the case of a phase space $\mathbb{R}^{2N}$.

Let us now consider a general phase space in the form of a cotangent bundle $T^*\mathcal{Q}$ to a Riemannian manifold $\mathcal{Q}$, and a general $\star$-product (3.2.3) defined on it. For any canonical coordinates (q^i, p_j) on $T^*\mathcal{Q}$ the $\star^{(q,p)}$-product is equivalent with the Moyal product, in accordance to Theorem 3.4.3. Using the corresponding equivalence morphism S and analogical considerations as in the proof of (4.1.41) we get for a function f polynomial in momenta

$$f \star^{(q,p)}\, = f_S(\hat{q}_\star, \hat{p}_\star). \tag{4.1.55}$$

Note, that for star-products considered in Section 3.4 the action $S^{-1}f$ of the morphism S on a function f polynomial in momenta is again a function polynomial in momenta. Thus, to a general star-product on $T^*\mathcal{Q}$ written in canonical coordinates corresponds an S-ordering of operators of position and momentum.

4.2 Operator calculus

In the following section we will consider the λ-Weyl correspondence rule for a particular example of the Hilbert space $\mathcal{H}$ and the self-adjoint operators $\hat{q}^i, \hat{p}_j$ defined on it. Moreover, we will present a generalization to a case of a symplectic manifold $T^*\mathcal{Q}$ for a general Riemannian manifold $\mathcal{Q}$. Let the Hilbert space $\mathcal{H}$ be the space $L^2(\mathbb{R}^N)$ of square integrable functions on $\mathbb{R}^N$, and let $\hat{q}^i = q^i$ be the operator of multiplication by variable q^i and $\hat{p}_j = -i\hbar\partial_{q^j}$ be the operator of differentiation. Note that the operators $\hat{q}^i, \hat{p}_j$ are properly defined on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ and $\hat{q}^i\colon \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$, $\hat{p}_j\colon \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N)$. Thus the domain D of operators $f_\lambda(\hat{q},\hat{p})$, defined by (4.1.20), is equal $\mathcal{S}(\mathbb{R}^N)$. In such special case of the Hilbert space $\mathcal{H}$ the formula (4.1.13) defining operators $f_\lambda(\hat{q},\hat{p})$ can be written in a different form.

Theorem 4.2.1. *For $\psi \in \mathcal{S}(\mathbb{R}^N)$ there holds*

$$f_\lambda(\hat{q},\hat{p})\psi(q) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(q+\lambda u, p)e^{-\frac{i}{\hbar}u^i p_i}\psi(q+u)\,\mathrm{d}u\,\mathrm{d}p, \tag{4.2.1}$$

where the integral is to be understood in a distributional sense.

Proof. From (4.1.13) we get

$$f_\lambda(\hat{q},\hat{p})\psi(q) = \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(u,v)e^{\frac{i}{\hbar}v_i q^i}e^{-u^i\partial_{q^i}}\psi(q)e^{-\frac{i}{\hbar}\lambda u^i v_i}\,\mathrm{d}u\,\mathrm{d}v. \tag{4.2.2}$$

From the identity

$$e^{-u^i\partial_{q^i}}\psi(q) = \psi(q-u), \tag{4.2.3}$$

which can be easily proved by expanding in a Taylor series the exponent on the left and ψ on the right hand side, we get

$$\begin{aligned}
f_\lambda(\hat{q},\hat{p})\psi(q) &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}} \mathscr{F}_\omega f(u,v)\psi(q-u)e^{\frac{i}{\hbar}(q^i-\lambda u^i)v_i}\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{(2\pi\hbar)^{2N}}\int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^{2N}} f(q',p')e^{-\frac{i}{\hbar}(v_i q'^i - u^i p'_i)}e^{\frac{i}{\hbar}(q^i-\lambda u^i)v_i} \\
&\quad\times \psi(q-u)\,\mathrm{d}q'\,\mathrm{d}p'\,\mathrm{d}u\,\mathrm{d}v \\
&= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^{2N}}\int_{\mathbb{R}^N} f(q',p')\delta(q'-q+\lambda u)e^{\frac{i}{\hbar}u^i p'_i}\psi(q-u)\,\mathrm{d}q'\,\mathrm{d}p'\,\mathrm{d}u \\
&= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(q-\lambda u,p')e^{\frac{i}{\hbar}u^i p'_i}\psi(q-u)\,\mathrm{d}u\,\mathrm{d}p'.
\end{aligned} \tag{4.2.4}$$

□

Note that (4.2.1) can be written in a form

$$f_\lambda(\hat{q},\hat{p})\psi(q) = \int_{\mathbb{R}^N}\tilde{f}(q-\lambda u, u)\psi(q-u)\,\mathrm{d}u, \tag{4.2.5}$$

where $\tilde{f}$ denotes the Fourier transform (3.4.23) of f in the momentum variable.

Let us now move to the more general case of the symplectic manifold M. Let $M = T^*U = U \times \mathbb{R}^N$ where U is some open subset of $\mathbb{R}^N$. In such case for a general Hilbert space $\mathcal{H}$ and operators $\hat{q}^i, \hat{p}_j$ to function f polynomial in momenta, $f(q,p) = K^{i_1\ldots i_n}(q)p_{i_1}\cdots p_{i_n}$, from definition will correspond operator given by (4.1.31). For general function f it is difficult to assign an operator since formula (4.1.13) is no longer properly defined. However, for certain Hilbert spaces $\mathcal{H}$ it is possible to give a formula for an operator $f_\lambda(\hat{q},\hat{p})$ associated to a general function f.

Endow U with some metric tensor g making from U a Riemannian manifold. Let us take as the Hilbert space $\mathcal{H}$ the space $L^2(U,\mathrm{d}\mu)$ of functions on U square integrable with respect to a measure $\mathrm{d}\mu(q) = g^{1/2}(q)\,\mathrm{d}q$ induced by the metric volume form ω_g on U ($g(q) = |\det[g_{ij}(q)]|$ is the determinant of the metric tensor g). In such case for $\psi \in C_0^\infty(U)$ we can generalize formula (4.2.1) in the following way

$$\begin{aligned} f_\lambda(\hat{q},\hat{p})\psi(q) &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(q+\lambda u,p)e^{-\frac{i}{\hbar}u^i p_i}\psi(q+u)\rho(q,u)\,\mathrm{d}u\,\mathrm{d}p \\ &= \int_{\mathbb{R}^N} \tilde{f}(q-\lambda u,u)\psi(q-u)\rho(q,-u)\,\mathrm{d}u, \end{aligned} \tag{4.2.6}$$

where $\rho(q,u) = g^{1/4}(q+u)g^{-1/4}(q)$. Note that since ψ has compact support it does not matter that f and g are not defined on the whole spaces $\mathbb{R}^{2N}$ and $\mathbb{R}^N$, respectively.

Note that for such Hilbert space operators $\hat{q}^i$ and $\hat{p}_j$ are given by

$$\hat{q}^i = q^i, \quad \hat{p}_j = -i\hbar\left(\partial_{q^j} + \frac{1}{2}\Gamma^k_{jk}\right), \tag{4.2.7}$$

where Γ^i_{jk} are Christoffel symbols of the Levi-Civita connection on U. In fact, for functions f polynomial in momenta (4.1.31) holds.

Theorem 4.2.2. *For $f(q,p) = K^{i_1\dots i_n}(q)p_{i_1}\cdots p_{i_n}$, where $K^{i_1\dots i_n}$ is some symmetric complex tensor field on U, formula (4.1.31) holds for operators $\hat{q}^i$ and $\hat{p}_j$ given by (4.2.7).*

Proof. From (4.2.6) we get

$$\begin{aligned} f_\lambda(\hat{q},\hat{p})\psi(q) &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K^{i_1\dots i_n}(q+\lambda u)p_{i_1}\cdots p_{i_n}e^{-\frac{i}{\hbar}u^i p_i}\psi(q+u)\rho(q,u)\,\mathrm{d}u\,\mathrm{d}p \\ &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} K^{i_1\dots i_n}(q+\lambda u)\left((i\hbar)^n\partial_{u^{i_1}}\cdots\partial_{u^{i_n}}e^{-\frac{i}{\hbar}u^i p_i}\right)\psi(q+u) \\ &\quad\times\rho(q,u)\,\mathrm{d}u\,\mathrm{d}p \\ &= \frac{1}{(2\pi\hbar)^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(-i\hbar)^n\partial_{u^{i_1}}\cdots\partial_{u^{i_n}}\left(K^{i_1\dots i_n}(q+\lambda u)\psi(q+u)\rho(q,u)\right) \\ &\quad\times e^{-\frac{i}{\hbar}u^i p_i}\,\mathrm{d}u\,\mathrm{d}p \\ &= (-i\hbar)^n\partial_{u^{i_1}}\cdots\partial_{u^{i_n}}\left(K^{i_1\dots i_n}(q+\lambda u)\psi(q+u)\rho(q,u)\right)\Big|_{u=0}. \end{aligned} \tag{4.2.8}$$

By virtue of the identity

$$\partial_{u^i}\big(g(q+\lambda u)h(q+u)\big) = (\partial_{v^i}+\partial_{w^i})\big(g(q+\lambda v)h(q+\lambda v+(1-\lambda)w)\big)\Big|_{v=w=u} \tag{4.2.9}$$

valid for any functions g and h, (4.2.8) can be written in a form

$$f_\lambda(\hat{q},\hat{p})\psi(q) = (-i\hbar)^n \sum_{k=0}^{n} \binom{n}{k} \partial_{u^{i_1}} \cdots \partial_{u^{i_k}} \partial_{v^{i_{k+1}}} \cdots \partial_{v^{i_n}} \Big(K^{i_1 \dots i_n}(q+\lambda u) \times \psi(q+\lambda u + (1-\lambda)v) g^{1/4}(q+\lambda u+(1-\lambda)v) g^{-1/4}(q)\Big)\Big|_{u=v=0}. \tag{4.2.10}$$

Using the formula

$$\frac{\partial g}{\partial q^j} = 2g\Gamma^k_{jk} \tag{4.2.11}$$

we calculate that

$$-i\hbar\partial_{q^j}(\psi g^{1/4}) = -i\hbar\left(\frac{\partial\psi}{\partial q^j} g^{1/4} + \frac{1}{2}\psi\Gamma^k_{jk} g^{1/4}\right) = (\hat{p}_j\psi)g^{1/4}. \tag{4.2.12}$$

From this and (4.2.10) we receive

$$\begin{aligned} f_\lambda(\hat{q},\hat{p})\psi(q) &= \sum_{k=0}^{n} \binom{n}{k} (-i\hbar)^k (1-\lambda)^{n-k} \partial_{u^{i_1}} \cdots \partial_{u^{i_k}} \Big(K^{i_1\dots i_n}(q+\lambda u) \\ &\quad \times (\hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n}\psi)(q+\lambda u+(1-\lambda)v) g^{1/4}(q+\lambda u+(1-\lambda)v) \\ &\quad \times g^{-1/4}(q)\Big)\Big|_{u=v=0} \\ &= \sum_{k=0}^{n} \binom{n}{k} \lambda^k (1-\lambda)^{n-k} \hat{p}_{i_1} \cdots \hat{p}_{i_k} K^{i_1\dots i_n}(\hat{q}) \hat{p}_{i_{k+1}} \cdots \hat{p}_{i_n}. \end{aligned} \tag{4.2.13}$$

□

Remark 4.2.1. Formula (4.2.6) can be generalized for symplectic manifolds $M = T^*\mathcal{Q}$, where $\mathcal{Q}$ is an almost geodesically simply connected Riemannian manifold. Let $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$ be a Hilbert space of functions on $\mathcal{Q}$ square integrable with respect to a measure $\mathrm{d}\omega_g$ induced by the metric volume form ω_g. For some function $f\colon T^*\mathcal{Q} \to \mathbb{C}$ we can define an operator on $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$ given, for every $\psi \in C_0^\infty(\mathcal{Q})$, by the formula [8]

$$\hat{f}_\lambda\psi(q) = \frac{1}{(2\pi\hbar)^N} \int_{T_q\mathcal{Q}} \int_{T_q^*\mathcal{Q}} f(\exp_q(\lambda u), A(\lambda u)p) e^{-\frac{i}{\hbar}\langle p,u\rangle} \psi(\exp_q(u)) \rho(q,u) \,\mathrm{d}u\,\mathrm{d}p, \tag{4.2.14}$$

where $\rho(q,u) = g^{1/4}(\exp_q(u)) g^{-1/4}(q) \det(\exp_{q*}(u))$, $A(u) = \exp_q^*(u)^{-1}\colon T_q^*\mathcal{Q} \to T^*_{\exp_q(u)}\mathcal{Q}$, $\exp_q\colon T_q\mathcal{Q} \to \mathcal{Q}$ is an exponential map of the Levi-Civita connection on $\mathcal{Q}$, and $\exp_{q*}(u)\colon T_q\mathcal{Q} \to T_{\exp_q(u)}\mathcal{Q}$ is the derivative of $\exp_q$ at point u. Note, that since ψ has compact support and $\exp_q$ is the diffeomorphism taking values in the almost whole manifold $\mathcal{Q}$, (4.2.14) is properly defined.

If $f(q,p) = K^{ij}(q)p_ip_j$ where K^{ij} is a smooth symmetric complex contravariant tensor field on $\mathcal{Q}$, then

$$\hat{f}_\lambda = -\hbar^2\bigg((1-\lambda)^2 K^{ij}\nabla_i\nabla_j + 2\lambda(1-\lambda)\nabla_i K^{ij}\nabla_j + \lambda^2\nabla_i\nabla_j K^{ij} - \frac{2\lambda^2+1}{6} K^{ij} R_{ij}\bigg), \tag{4.2.15}$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection and R_{ij} is the Ricci curvature tensor. The proof of this formula can be found in [8]. In a special case of the symmetric ordering ($\lambda = \frac{1}{2}$) equation (4.2.15) takes the form

$$\hat{f}_{1/2} = -\hbar^2 \left(\nabla_i K^{ij} \nabla_j + \frac{1}{4} K^{ij}{}_{;ij} - \frac{1}{4} K^{ij} R_{ij} \right), \tag{4.2.16}$$

where $K^{ij}{}_{;kl}$ denotes the second covariant derivative of K^{ij}.

4.3 Operator representation over a configuration space

In this section we will present a construction of a natural operator representation of quantum mechanics, which reproduces the usual Hilbert space approach to quantum mechanics. We will be dealing with quantum systems defined on a phase space M in the form of a cotangent bundle $T^*\mathcal{Q}$ to a Riemannian manifold $\mathcal{Q}$. The manifold $\mathcal{Q}$ plays the role of a configuration space of the system. The representation will be constructed in a Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$ of functions on $\mathcal{Q}$ square integrable with respect to a measure $\mathrm{d}\omega_g$ induced by the metric volume form ω_g. The elements of $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$ are interpreted as wave functions describing the states of the quantum system.

4.3.1 The case of a Moyal quantization

First, let us consider the phase space $M = T^*U$, where U is an open subset of $\mathbb{R}^N$ endowed with some metric tensor g. Moreover, we will consider a classical system defined on M and its quantization by means of the Moyal product on M. The first step in construction of the operator representation for such quantum system is an observation that the Hilbert space $L^2(T^*U, \mathrm{d}l)$ can be written as a tensor product of the Hilbert space $L^2(U, \mathrm{d}\mu)$ and the space dual to it. In what follows we present the construction of this tensor product.

In accordance to the Riesz representation theorem the Hilbert space $(L^2(U, \mathrm{d}\mu))^*$ dual to $L^2(U, \mathrm{d}\mu)$ is anti-isomorphic to the Hilbert space $L^2(U, \mathrm{d}\mu)$ and can be naturally identified with $L^2(U, \mathrm{d}\mu)$ itself [81]. After such identification the anti-linear isomorphism $*\colon L^2(U, \mathrm{d}\mu) \to (L^2(U, \mathrm{d}\mu))^*$ takes the form of the complex-conjugation. Denote by $L^2(TU)$ the Hilbert space of functions defined on the tangent bundle $TU = U \times \mathbb{R}^N$ and square integrable with respect to the Lebesgue measure on $U \times \mathbb{R}^N$. Let us introduce a bilinear map of Hilbert spaces $\tilde{W}\colon (L^2(U, \mathrm{d}\mu))^* \times L^2(U, \mathrm{d}\mu) \to L^2(TU)$, which on vectors $\varphi, \psi \in C_0^\infty(U)$ is defined by

$$\tilde{W}(\varphi^*, \psi)(q, u) = \overline{\varphi(q - \tfrac{1}{2}u)} \psi(q + \tfrac{1}{2}u) \bar{\rho}(q, u), \tag{4.3.1}$$

where $\bar{\rho}(q, u) = g^{1/4}(q - \frac{1}{2}u) g^{1/4}(q + \frac{1}{2}u)$. For $\varphi_1, \psi_1, \varphi_2, \psi_2 \in C_0^\infty(U)$ there holds

$$(\tilde{W}(\varphi_1^*, \psi_1), \tilde{W}(\varphi_2^*, \psi_2)) = (\varphi_1^*, \varphi_2^*)(\psi_1, \psi_2). \tag{4.3.2}$$

Indeed,

$$\begin{aligned}(\tilde{W}(\varphi_1^*,\psi_1),\tilde{W}(\varphi_2^*,\psi_2)) &= \int_{U\times\mathbb{R}^N} \overline{\tilde{W}(\varphi_1^*,\psi_1)(q,u)}\tilde{W}(\varphi_2^*,\psi_2)(q,u)\,\mathrm{d}q\,\mathrm{d}u \\ &= \int_{U\times\mathbb{R}^N} \varphi_1(q-\tfrac{1}{2}u)\overline{\psi_1(q+\tfrac{1}{2}u)\varphi_2(q-\tfrac{1}{2}u)}\psi_2(q+\tfrac{1}{2}u) \\ &\quad\times g^{1/2}(q-\tfrac{1}{2}u)g^{1/2}(q+\tfrac{1}{2}u)\,\mathrm{d}q\,\mathrm{d}u. \end{aligned} \tag{4.3.3}$$

Note that since $\varphi_1,\psi_1,\varphi_2,\psi_2$ have compact support, we can extend integration in (4.3.3) to the whole space $\mathbb{R}^N\times\mathbb{R}^N$. After performing the following change of variables

$$\begin{aligned} q_1 &= q-\tfrac{1}{2}u, \\ q_2 &= q+\tfrac{1}{2}u, \end{aligned} \tag{4.3.4}$$

equation (4.3.3) takes the form

$$\begin{aligned}(\tilde{W}(\varphi_1^*,\psi_1),\tilde{W}(\varphi_2^*,\psi_2)) &= \int_{\mathbb{R}^N} \overline{\varphi_2(q_1)}\varphi_1(q_1)g^{1/2}(q_1)\,\mathrm{d}q_1 \int_{\mathbb{R}^N} \overline{\psi_1(q_2)}\psi_2(q_2)g^{1/2}(q_2)\,\mathrm{d}q_2 \\ &= (\varphi_1^*,\varphi_2^*)(\psi_1,\psi_2). \end{aligned} \tag{4.3.5}$$

From property (4.3.2) follows that $\tilde{W}$ is continuous on $C_0^\infty(U)\times C_0^\infty(U)$. Thus, from the fact that $C_0^\infty(U)$ is dense in $L^2(U,\mathrm{d}\mu)$, it can be uniquely extended to a bilinear map defined on the whole space $(L^2(U,\mathrm{d}\mu))^*\times L^2(U,\mathrm{d}\mu)$ and satisfying (4.3.2).

It can be proved that finite linear combinations of vectors $\tilde{W}(\varphi^*,\psi)$ for $\varphi,\psi\in L^2(U,\mathrm{d}\mu)$ create a dense subset of $L^2(TU)$. Thus $\tilde{W}$ is a tensor product of Hilbert spaces $(L^2(U,\mathrm{d}\mu))^*$ and $L^2(U,\mathrm{d}\mu)$.

Now, let us take the inverse Fourier transform of $\tilde{W}(\varphi^*,\psi)$ in momentum variable. That way we receive a bilinear map of Hilbert spaces $W\colon (L^2(U,\mathrm{d}\mu))^*\times L^2(U,\mathrm{d}\mu)\to L^2(T^*U,\mathrm{d}l)$, which on vectors $\varphi,\psi\in C_0^\infty(U)$ takes the form

$$\begin{aligned} W(\varphi^*,\psi)(q,p) &= \int_{\mathbb{R}^N} \tilde{W}(\varphi^*,\psi)(q,u)e^{-\frac{i}{\hbar}u^ip_i}\,\mathrm{d}u \\ &= \int_{\mathbb{R}^N} \overline{\varphi(q-\tfrac{1}{2}u)}\psi(q+\tfrac{1}{2}u)\bar{\rho}(q,u)e^{-\frac{i}{\hbar}u^ip_i}\,\mathrm{d}u. \end{aligned} \tag{4.3.6}$$

Since the Fourier transform in momentum variable is an isomorphism of the Hilbert space $L^2(T^*U,\mathrm{d}l)$ onto the Hilbert space $L^2(TU)$, W is also a tensor product of Hilbert spaces $(L^2(U,\mathrm{d}\mu))^*$ and $L^2(U,\mathrm{d}\mu)$. We will denote this tensor product by $\otimes_W$. In a case $U=\mathbb{R}^N$ with a standard metric tensor g, (4.3.6) is a well known Wigner transform [83, 84].

In what follows we will prove couple properties of the tensor product $\otimes_W$.

Theorem 4.3.1. *For $\varphi,\psi\in L^2(U,\mathrm{d}\mu)$ there holds*

$$\overline{\varphi^*\otimes_W\psi} = \psi^*\otimes_W\varphi, \tag{4.3.7}$$

$$\int_{T^*U} (\varphi^*\otimes_W\psi)\,\mathrm{d}l = (\varphi,\psi). \tag{4.3.8}$$

Proof. Formula (4.3.7) is an immediate consequence of the definition (4.3.6). To prove (4.3.8) it is enough to consider $\varphi, \psi \in C_0^\infty(U)$ since the general case will follow from the continuity of the tensor product $\otimes_W$ and the integral, and the fact that $C_0^\infty(U)$ is dense in $L^2(U, \mathrm{d}\mu)$. From (4.3.6) we have that

$$\begin{aligned}
\frac{1}{(2\pi\hbar)^N} \int_{U\times\mathbb{R}^N} (\varphi^* \otimes_W \psi)(q,p)\,\mathrm{d}q\,\mathrm{d}p &= \frac{1}{(2\pi\hbar)^N} \int_{U\times\mathbb{R}^N} \int_{\mathbb{R}^N} \overline{\varphi(q-\tfrac{1}{2}u)}\psi(q+\tfrac{1}{2}u) \\
&\quad \times e^{-\frac{i}{\hbar}u^i p_i} g^{1/4}(q-\tfrac{1}{2}u) g^{1/4}(q+\tfrac{1}{2}u)\,\mathrm{d}u\,\mathrm{d}q\,\mathrm{d}p \\
&= \int_U \int_{\mathbb{R}^N} \overline{\varphi(q-\tfrac{1}{2}u)}\psi(q+\tfrac{1}{2}u) g^{1/4}(q-\tfrac{1}{2}u) g^{1/4}(q+\tfrac{1}{2}u)\delta(u)\,\mathrm{d}u\,\mathrm{d}q \\
&= \int_U \overline{\varphi(q)}\psi(q) g^{1/2}(q)\,\mathrm{d}q = (\varphi, \psi). \qquad (4.3.9)
\end{aligned}$$

□

Theorem 4.3.2. *Let* $\rho_1 = \varphi_1^* \otimes_W \psi_1$ *and* $\rho_2 = \varphi_2^* \otimes_W \psi_2$ *for* $\varphi_1, \psi_1, \varphi_2, \psi_2 \in L^2(U, \mathrm{d}\mu)$. *Then*

$$\rho_1 \star_M \rho_2 = (\varphi_1, \psi_2)(\varphi_2^* \otimes_W \psi_1). \qquad (4.3.10)$$

Proof. To prove (4.3.10) it is enough to consider $\varphi_1, \psi_1, \varphi_2, \psi_2 \in C_0^\infty(U)$ since the general case will follow from the continuity of the Moyal product $\star_M$, tensor product $\otimes_W$ and scalar product, and the fact that $C_0^\infty(U)$ is dense in $L^2(U, \mathrm{d}\mu)$. From (3.4.22) we have that

$$\begin{aligned}
(\rho_1 \star_M \rho_2)(q,p) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{\rho}_1(q+\tfrac{1}{2}u, v)\tilde{\rho}_2(q-\tfrac{1}{2}v, u) e^{-\frac{i}{\hbar}(u^i+v^i)p_i}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \overline{\varphi_1(q+\tfrac{1}{2}u-\tfrac{1}{2}v)}\psi_1(q+\tfrac{1}{2}u+\tfrac{1}{2}v) g^{1/4}(q+\tfrac{1}{2}u-\tfrac{1}{2}v) \\
&\quad \times g^{1/4}(q+\tfrac{1}{2}u+\tfrac{1}{2}v)\overline{\varphi_2(q-\tfrac{1}{2}v-\tfrac{1}{2}u)}\psi_2(q-\tfrac{1}{2}v+\tfrac{1}{2}u) \\
&\quad \times g^{1/4}(q-\tfrac{1}{2}v-\tfrac{1}{2}u) g^{1/4}(q-\tfrac{1}{2}v+\tfrac{1}{2}u) e^{-\frac{i}{\hbar}(u^i+v^i)p_i}\,\mathrm{d}u\,\mathrm{d}v. \qquad (4.3.11)
\end{aligned}$$

After performing the change of variables

$$\begin{aligned}
w &= u + v, \\
q' &= q + \tfrac{1}{2}u - \tfrac{1}{2}v,
\end{aligned} \qquad (4.3.12)$$

we get

$$\begin{aligned}
(\rho_1 \star_M \rho_2)(q,p) &= \int_{\mathbb{R}^N} \overline{\varphi_1(q')}\psi_2(q') g^{1/2}(q')\,\mathrm{d}q' \int_{\mathbb{R}^N} \overline{\varphi_2(q-\tfrac{1}{2}w)}\psi_1(q+\tfrac{1}{2}w) e^{-\frac{i}{\hbar}w^i p_i} \\
&\quad \times g^{1/4}(q-\tfrac{1}{2}w) g^{1/4}(q+\tfrac{1}{2}w)\,\mathrm{d}w \\
&= (\varphi_1, \psi_2)(\varphi_2^* \otimes_W \psi_1)(q,p). \qquad (4.3.13)
\end{aligned}$$

□

Let $\{\varphi_i\}$ be an orthonormal basis in $L^2(U, \mathrm{d}\mu)$, then $\{\rho_{ij}\} = \{\varphi_i^* \otimes_W \varphi_j\}$ is an orthonormal basis in $L^2(T^*U, \mathrm{d}l)$. From (4.3.7), (4.3.8), and (4.3.10) we get that the basis functions ρ_{ij} have the following properties:

$$\bar{\rho}_{ij} = \rho_{ji}, \tag{4.3.14a}$$
$$\int_{T^*U} \rho_{ij}\,\mathrm{d}l = \delta_{ij}, \tag{4.3.14b}$$
$$\rho_{ij} \star_M \rho_{kl} = \delta_{il}\rho_{kj}. \tag{4.3.14c}$$

Using the basis $\{\rho_{ij}\}$ the following characterization of quantum states can be proved.

Theorem 4.3.3. *Function $\rho \in L^2(T^*U, \mathrm{d}l)$ is a quantum state, i.e. it satisfies*

(i) $\rho = \bar{\rho}$,

(ii) $\displaystyle\int_{T^*U} \rho\,\mathrm{d}l = 1$,

(iii) $\displaystyle\int_{T^*U} \bar{f} \star_M f \star_M \rho\,\mathrm{d}l \geq 0$ *for* $f \in C_0^\infty(T^*U)$,

if and only if ρ is in the form

$$\rho = \sum_\lambda p_\lambda(\varphi_\lambda^* \otimes_W \varphi_\lambda), \tag{4.3.15}$$

where $\varphi_\lambda \in L^2(U, \mathrm{d}\mu)$, $\|\varphi_\lambda\| = 1$, $p_\lambda \geq 0$, *and* $\displaystyle\sum_\lambda p_\lambda = 1$.

Proof. Function ρ can be written in a form

$$\rho = \sum_{i,j} c_{ij}\rho_{ij}, \tag{4.3.16}$$

where $c_{ij} \in \mathbb{C}$ and $\{\rho_{ij}\} = \{\varphi_i^* \otimes_W \varphi_j\}$ is an induced basis in $L^2(T^*U, \mathrm{d}l)$ by the basis $\{\varphi_i\}$ in $L^2(U, \mathrm{d}\mu)$. Properties (i)–(iii) are equivalent to saying that the matrix $\check{c}$ of the coefficients c_{ij} is hermitian ($\check{c} = \check{c}^\dagger$), normalized ($\operatorname{tr}\check{c} = 1$), and positive define ($c_{ii} \geq 0$). Indeed, hermiticity and normalization easily follow from (4.3.14a) and (4.3.14b). To prove positive definite note that (iii) is valid for every $f \in L^2(T^*U, \mathrm{d}l)$ since $C_0^\infty(T^*U)$ is dense in $L^2(T^*U, \mathrm{d}l)$. Thus in particular for basis functions ρ_{kk} and with the help of (4.3.14) we get

$$\begin{aligned} 0 &\leq \int_{T^*U} \rho_{kk} \star_M \rho_{kk} \star_M \rho\,\mathrm{d}l = \int_{T^*U} \rho_{kk} \star_M \rho\,\mathrm{d}l = \sum_{i,j} c_{ij} \int_{T^*U} \rho_{kk} \star_M \rho_{ij}\,\mathrm{d}l \\ &= \sum_{i,j} c_{ij} \int_{T^*U} \delta_{kj}\rho_{ik}\,\mathrm{d}l = \sum_i c_{ik} \int_{T^*U} \rho_{ik}\,\mathrm{d}l = \sum_i c_{ik}\delta_{ik} = c_{kk} \end{aligned} \tag{4.3.17}$$

for every k.

Since the matrix $\check{c}$ is hermitian it can be diagonalized, i.e. there exist an unitary matrix $\check{T}$ such that $c_{ij} = \displaystyle\sum_{k,l} T_{ik}^\dagger(p_k\delta_{kl})T_{lj} = \sum_k T_{ki}^* p_k T_{kj}$ for some $p_k \in \mathbb{R}$. Hence, ρ

takes the form

$$\begin{aligned}\rho &= \sum_{i,j,k} T^*_{ki} p_k T_{kj} (\varphi^*_i \otimes_W \varphi_j) = \sum_k p_k \left(\Big(\sum_i T_{ki}\varphi_i \Big)^* \otimes_W \Big(\sum_j T_{kj}\varphi_j \Big) \right) \\ &= \sum_k p_k (\psi^*_k \otimes_W \psi_k),\end{aligned} \tag{4.3.18}$$

where $\psi_k = \sum_i T_{ki}\varphi_i$. The conditions that $c_{ii} \geq 0$ and $\operatorname{tr} \check{c} = 1$ give that $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$. □

From the above theorem follows that pure states are in the form

$$\rho_{\text{pure}} = \varphi^* \otimes_W \varphi, \tag{4.3.19}$$

for some normalized $\varphi \in L^2(U, \mathrm{d}\mu)$. Conversely, every function ρ of the form (4.3.19) is a pure state. Moreover, from (4.3.10) follows that every pure state is idempotent:

$$\rho_{\text{pure}} \star_M \rho_{\text{pure}} = \rho_{\text{pure}}. \tag{4.3.20}$$

The following theorem states that the inverse is also true.

Theorem 4.3.4. *Every function $\rho \in L^2(T^*U, \mathrm{d}l)$ which satisfies*

(i) $\rho = \bar{\rho}$,

(ii) $\displaystyle\int_{T^*U} \rho \, \mathrm{d}l = 1$,

(iii) $\rho \star_M \rho = \rho$,

is a pure state.

Proof. Function ρ can be written in a form

$$\rho = \sum_{i,j} c_{ij} \rho_{ij}, \tag{4.3.21}$$

where $c_{ij} \in \mathbb{C}$ and $\{\rho_{ij}\} = \{\varphi^*_i \otimes_W \varphi_j\}$ is an induced basis in $L^2(T^*U, \mathrm{d}l)$ by the basis $\{\varphi_i\}$ in $L^2(U, \mathrm{d}\mu)$. Properties (i)–(iii) are equivalent to saying that the matrix $\check{c}$ of the coefficients c_{ij} is hermitian ($\check{c} = \check{c}^\dagger$), normalized ($\operatorname{tr} \check{c} = 1$), and idempotent ($\check{c}^2 = \check{c}$). Since the matrix $\check{c}$ is hermitian it can be diagonalized, i.e. there exist an unitary matrix $\check{T}$ such that $c_{ij} = \sum_{k,l} T^\dagger_{ik} (a_k \delta_{kl}) T_{lj} = \sum_k T^*_{ki} a_k T_{kj}$ for some $a_k \in \mathbb{R}$. Hence, ρ takes the form

$$\begin{aligned}\rho &= \sum_{i,j,k} T^*_{ki} a_k T_{kj} (\varphi^*_i \otimes_W \varphi_j) = \sum_k a_k \left(\Big(\sum_i T_{ki}\varphi_i \Big)^* \otimes_W \Big(\sum_j T_{kj}\varphi_j \Big) \right) \\ &= \sum_k a_k (\psi^*_k \otimes_W \psi_k),\end{aligned} \tag{4.3.22}$$

where $\psi_k = \sum_i T_{ki}\varphi_i$. The conditions that $\check{c}^2 = \check{c}$ and $\operatorname{tr}\check{c} = 1$ give that $a_k^2 = a_k$ and $\sum_k a_k = 1$. Hence $a_k = \delta_{k_0 k}$ for some k_0, from which follows that $\rho = \psi_{k_0}^* \otimes_W \psi_{k_0}$. Thus ρ is a pure state. □

As was noted above pure states $\rho \in L^2(T^*U, \mathrm{d}l)$ are of the form $\rho = \varphi^* \otimes_W \varphi$ for normalized $\varphi \in L^2(U, \mathrm{d}\mu)$. Thus there is a one to one correspondence between pure states and normalized vectors in $L^2(U, \mathrm{d}\mu)$. In what follows we will show that there is in fact a one to one correspondence between states $\rho \in L^2(T^*U, \mathrm{d}l)$ and density operators $\hat{\rho}$ on $L^2(U, \mathrm{d}\mu)$.

First, note that vectors $f \in L^2(T^*U, \mathrm{d}l)$ can be considered as operators $f \star_M$ on $L^2(T^*U, \mathrm{d}l)$ given by the formula

$$(f \star_M)\rho = f \star_M \rho, \quad \rho \in L^2(T^*U, \mathrm{d}l). \tag{4.3.23}$$

From (3.4.21) follows that operators $f\star_M$ are bounded with the norm $\|f\star_M\| \leq \|f\|$. In what follows we will prove that operators $f \star_M$ can be naturally identified with Hilbert-Schmidt operators on $L^2(U, \mathrm{d}\mu)$.

For a Hilbert space $\mathcal{H}$ a bounded operator $\hat{A} \in B(\mathcal{H})$ is called a Hilbert-Schmidt operator if $\operatorname{tr}(\hat{A}^\dagger \hat{A}) < \infty$. The space of all Hilbert-Schmidt operators will be denoted by $B_2(\mathcal{H})$ and it happens to be a Hilbert space with a scalar product given by [85]

$$(\hat{A}, \hat{B})_2 = \operatorname{tr}(\hat{A}^\dagger \hat{B}), \quad \hat{A}, \hat{B} \in B_2(\mathcal{H}). \tag{4.3.24}$$

From the well known relation between the Hilbert-Schmidt norm and the usual operator norm [85]

$$\|\hat{A}\| \leq \|\hat{A}\|_2, \quad \hat{A} \in B_2(\mathcal{H}) \tag{4.3.25}$$

it follows that the inclusion $B_2(\mathcal{H}) \subset B(\mathcal{H})$ is continuous.

Proposition 4.3.1. *For every $\rho \in L^2(T^*U, \mathrm{d}l)$*

$$\rho \star_M = \hat{1} \otimes_W \hat{\rho}, \tag{4.3.26}$$

where $\hat{\rho} \in B_2(L^2(U, \mathrm{d}\mu))$ is some Hilbert-Schmidt operator defined on the Hilbert space $L^2(U, \mathrm{d}\mu)$. Conversely, for every $\hat{\rho} \in B_2(L^2(U, \mathrm{d}\mu))$ the operator $\hat{1} \otimes_W \hat{\rho}$ is of the form $\rho \star_M$ for some $\rho \in L^2(T^*U, \mathrm{d}l)$.

The following properties are fulfilled:

(i) *for $\rho = \varphi^* \otimes_W \psi$, $\hat{\rho} = (\varphi, \cdot\,)\psi$,*

(ii) $\bar{\rho} \star_M = \hat{1} \otimes_W \hat{\rho}^\dagger$,

(iii) $\operatorname{tr}(\rho) \equiv \int_{T^*U} \rho \, \mathrm{d}l = \operatorname{tr}(\hat{\rho})$,

(iv) *for $\rho_1, \rho_2 \in L^2(T^*U, \mathrm{d}l)$ and $\hat{\rho}_1, \hat{\rho}_2 \in B_2(L^2(U, \mathrm{d}\mu))$ such that $\rho_1 \star_M = \hat{1} \otimes_W \hat{\rho}_1$ and $\rho_2 \star_M = \hat{1} \otimes_W \hat{\rho}_2$*

$$(\rho_1, \rho_2) = (\hat{\rho}_1, \hat{\rho}_2)_2, \tag{4.3.27}$$

(v) $\displaystyle\int_{T^*U} \bar{f} \star_M f \star_M \rho \,\mathrm{d}l \geq 0$ *for* $f \in C_0^\infty(T^*U)$ *if and only if* $(\varphi, \hat{\rho}\varphi) \geq 0$ *for* $\varphi \in L^2(U, \mathrm{d}\mu)$.

Proof. First let us prove (i). From (4.3.10) for basis functions $\rho_{ij} = \varphi_i^* \otimes_W \varphi_j$ it follows that

$$\begin{aligned}\rho \star_M \rho_{ij} &= (\varphi^* \otimes_W \psi) \star_M (\varphi_i^* \otimes_W \varphi_j) = (\varphi, \varphi_j)(\varphi_i^* \otimes_W \psi) = \varphi_i^* \otimes_W (\hat{\rho}\varphi_j) \\ &= (\hat{1} \otimes_W \hat{\rho})\rho_{ij},\end{aligned} \tag{4.3.28}$$

which proves (i).

Now, note that for a basis $\{\varphi_i\}$ in $L^2(U, \mathrm{d}\mu)$ the operators $\hat{\rho}_{ij} = (\varphi_i, \cdot\,)\varphi_j$ form a basis in the Hilbert space $B_2(L^2(U, \mathrm{d}\mu))$ of Hilbert-Schmidt operators. From (i) for basis functions $\rho_{ij} = \varphi_i^* \otimes_W \varphi_j$ we have that

$$\rho_{ij} \star_M \ = \hat{1} \otimes_W \hat{\rho}_{ij}. \tag{4.3.29}$$

The general $\rho \in L^2(T^*U, \mathrm{d}l)$ can be written in the form $\rho = \sum_{i,j} c_{ij}\rho_{ij}$ for some $c_{ij} \in \mathbb{C}$. In accordance to (4.3.29) the corresponding Hilbert-Schmidt operator $\hat{\rho}$ is of the form $\hat{\rho} = \sum_{i,j} c_{ij}\hat{\rho}_{ij}$. This proves the first part of the theorem.

It is enough to prove properties (ii)–(iv) for basis functions ρ_{ij}. Property (ii) follows from (4.3.14a) and the fact that $\hat{\rho}_{ij}^\dagger = \hat{\rho}_{ji}$. Property (iii) is a consequence of (4.3.14b) and the identity $\operatorname{tr}(\hat{\rho}_{ij}) = \delta_{ij}$. Property (iv) follows from the equality

$$\begin{aligned}(\rho_{ij}, \rho_{kl}) &= \int_{T^*U} \bar{\rho}_{ij}\rho_{kl}\,\mathrm{d}l = \int_{T^*U} \rho_{ji} \star_M \rho_{kl}\,\mathrm{d}l = \int_{T^*U} \delta_{jl}\rho_{ki}\,\mathrm{d}l = \delta_{jl}\delta_{ik} \\ &= \operatorname{tr}(\hat{\rho}_{ij}^\dagger\hat{\rho}_{kl}) = (\hat{\rho}_{ij}, \hat{\rho}_{kl})_2.\end{aligned} \tag{4.3.30}$$

To prove (v) let us expand ρ and $\hat{\rho}$ in the corresponding basis: $\rho = \sum_{i,j} c_{ij}\rho_{ij}$ and $\hat{\rho} = \sum_{i,j} c_{ij}\hat{\rho}_{ij}$. The property follows from the observation that the positive-definiteness of ρ and $\hat{\rho}$ is equivalent with the inequality $c_{kk} \geq 0$ for every k. □

From Proposition 4.3.1 immediately follows that the Hilbert spaces $L^2(T^*U, \mathrm{d}l)$ and $B_2(L^2(U, \mathrm{d}\mu))$ are naturally isomorphic. The natural isomorphism $\rho \mapsto \hat{\rho}$ is given by $\rho \star_M \ = \hat{1} \otimes_W \hat{\rho}$. The isomorphism $\rho \mapsto \hat{\rho}$ is in fact a representation of the algebra $\mathcal{L} = (L^2(T^*U, \mathrm{d}l), \star_M)$ in the Hilbert space $L^2(U, \mathrm{d}\mu)$ since it satisfies

$$\widehat{f \star_M g} = \hat{f}\hat{g}, \quad \hat{\bar{f}} = \hat{f}^\dagger, \quad \operatorname{tr}(f) = \operatorname{tr}(\hat{f}). \tag{4.3.31}$$

The last property is restricted to the subspace $\mathcal{L}^1 = \mathcal{L} \star_M \mathcal{L}$.

Moreover, from Proposition 4.3.1 follows that there is a one to one correspondence between quantum states $\rho \in L^2(T^*U, \mathrm{d}l)$ and density operators on $L^2(U, \mathrm{d}\mu)$, i.e. trace class operators $\hat{\rho}$ satisfying

(i) $\hat{\rho}^\dagger = \hat{\rho}$,

(ii) $\operatorname{tr}(\hat{\rho}) = 1$,

(iii) $(\varphi, \hat{\rho}\varphi) \geq 0$ for every $\varphi \in L^2(U, \mathrm{d}\mu)$.

The density operators represent quantum states in the operator representation of quantum mechanics.

In what follows we will show that observables $f \in C^\infty(T^*U)[\![\hbar]\!]$ can be naturally identified with operators defined on the Hilbert space $L^2(U, \mathrm{d}\mu)$. Moreover, the presented identification will be in agreement with the Weyl correspondence rule.

Proposition 4.3.2. *Let $f \in C^\infty(T^*U)[\![\hbar]\!]$ and $\rho = \varphi^* \otimes_W \psi$ for $\varphi, \psi \in C_0^\infty(U)$. Then*

$$f \star_M \rho = \varphi^* \otimes_W f(\hat{q}, \hat{p})\psi, \tag{4.3.32a}$$

$$\rho \star_M f = (f(\hat{q}, \hat{p})^\dagger \varphi)^* \otimes_W \psi, \tag{4.3.32b}$$

where $f(\hat{q}, \hat{p})$ is a symmetrically ordered function of canonical operators of position $\hat{q}^i = q^i$ and momentum $\hat{p}_j = -i\hbar(\partial_{q^j} + \frac{1}{2}\Gamma^k_{jk})$, acting in the Hilbert space $L^2(U, \mathrm{d}\mu)$.

Proof. From (3.4.22) we get that

$$\begin{aligned}
(f \star_M \rho)(q, p) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}(q + \tfrac{1}{2}u, v)\tilde{\rho}(q - \tfrac{1}{2}v, u) e^{-\frac{i}{\hbar}(u^i + v^i)p_i} \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}(q + \tfrac{1}{2}u, v)\overline{\varphi(q - \tfrac{1}{2}v - \tfrac{1}{2}u)}\psi(q - \tfrac{1}{2}v + \tfrac{1}{2}u) \\
&\quad \times g^{1/4}(q - \tfrac{1}{2}v - \tfrac{1}{2}u) g^{1/4}(q - \tfrac{1}{2}v + \tfrac{1}{2}u) e^{-\frac{i}{\hbar}(u^i + v^i)p_i} \,\mathrm{d}u\,\mathrm{d}v.
\end{aligned} \tag{4.3.33}$$

After the following change of variables

$$\begin{aligned}
u &\to u - v, \\
v &\to v
\end{aligned} \tag{4.3.34}$$

and using (4.2.6) we receive

$$\begin{aligned}
(f \star_M \rho)(q, p) &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \tilde{f}(q + \tfrac{1}{2}u - \tfrac{1}{2}v, v)\overline{\varphi(q - \tfrac{1}{2}u)}\psi(q + \tfrac{1}{2}u - v) g^{1/4}(q - \tfrac{1}{2}u) \\
&\quad \times g^{1/4}(q + \tfrac{1}{2}u - v) g^{1/4}(q + \tfrac{1}{2}u) g^{-1/4}(q + \tfrac{1}{2}u) e^{-\frac{i}{\hbar}u^i p_i} \,\mathrm{d}u\,\mathrm{d}v \\
&= \int_{\mathbb{R}^N} \overline{\varphi(q - \tfrac{1}{2}u)}(f(\hat{q}, \hat{p})\psi)(q + \tfrac{1}{2}u) g^{1/4}(q - \tfrac{1}{2}u) g^{1/4}(q + \tfrac{1}{2}u) e^{-\frac{i}{\hbar}u^i p_i} \,\mathrm{d}u \\
&= (\varphi^* \otimes_W f(\hat{q}, \hat{p})\psi)(q, p)
\end{aligned} \tag{4.3.35}$$

which proves (4.3.32a).

To prove (4.3.32b) we can use (4.3.7) and (4.3.32a) receiving

$$\rho \star_M f = \overline{\bar{f} \star_M \bar{\rho}} = \overline{\psi^* \otimes_W \bar{f}(\hat{q}, \hat{p})\varphi} = (\bar{f}(\hat{q}, \hat{p})\varphi)^* \otimes_W \psi. \tag{4.3.36}$$

$\square$

Note, that the star-products $f \star_M \rho$ and $\rho \star_M f$ are properly defined by (3.4.22), even though $\tilde{f}$ has no compact support and is not defined on the whole space $\mathbb{R}^N \times \mathbb{R}^N$, since ρ is in the form $\varphi^* \otimes_W \psi$ for φ and ψ with compact support.

From Proposition 4.3.2 follows that operators $f \star_M$ can be written as

$$f \star_M \; = \hat{1} \otimes_W f(\hat{q}, \hat{p}). \tag{4.3.37}$$

Equation (4.3.37) is an analog of (4.3.26) for functions $f \in C^\infty(T^*U)[\![\hbar]\!]$ and it allows to naturally identify functions f with operators $f(\hat{q}, \hat{p})$. That way the Weyl correspondence rule naturally appears in the operator representation of quantum mechanics.

The map $f \mapsto \hat{f} = f(\hat{q}, \hat{p})$ have the following properties

$$\widehat{f \star_M g} = \hat{f}\hat{g}, \quad \hat{\bar{f}} = \hat{f}^\dagger \tag{4.3.38}$$

for functions $f, g \in C^\infty(T^*U)[\![\hbar]\!]$, thus it is a representation of the quantum Poisson algebra $\mathcal{A}_Q(T^*U) = (C^\infty(T^*U)[\![\hbar]\!], \star_M)$ in the Hilbert space $L^2(U, \mathrm{d}\mu)$.

Theorem 4.3.5. *Let $f \in C^\infty(T^*U)[\![\hbar]\!]$ and $\rho \in L^2(T^*U, \mathrm{d}l)$. If $f \star_M \rho \in L^1(T^*U, \mathrm{d}l)$ then*

$$\int_{T^*U} f \star_M \rho \, \mathrm{d}l = \operatorname{tr}(f(\hat{q}, \hat{p})\hat{\rho}). \tag{4.3.39}$$

In particular, if $\rho = \varphi^* \otimes_W \psi$ for $\varphi, \psi \in C_0^\infty(U)$ then

$$\int_{T^*U} f \star_M \rho \, \mathrm{d}l = (\varphi, f(\hat{q}, \hat{p})\psi). \tag{4.3.40}$$

Proof. Let $\{\varphi_i\}$ be a basis in $L^2(U, \mathrm{d}\mu)$ such that φ_i have compact support. From Proposition 4.3.2 and (4.3.8) we have that

$$\int_{T^*U} f \rho_{ij} \, \mathrm{d}l = \int_{T^*U} f \star_M \rho_{ij} \, \mathrm{d}l = (\varphi_i, f(\hat{q}, \hat{p})\varphi_j) \tag{4.3.41}$$

for $\rho_{ij} = \varphi_i^* \otimes_W \varphi_j$. The function ρ can be expanded in the basis ρ_{ij}, $\rho = \sum_{i,j} c_{ij}\rho_{ij}$.

Using this expansion we get that

$$\begin{aligned} \int_{T^*U} f \star_M \rho \, \mathrm{d}l &= \int_{T^*U} f \rho \, \mathrm{d}l = \sum_{i,j} c_{ij} \int_{T^*U} f \rho_{ij} \, \mathrm{d}l = \sum_{i,j} c_{ij} (\varphi_i, f(\hat{q}, \hat{p})\varphi_j) \\ &= \sum_{i,j} c_{ij} \operatorname{tr}(f(\hat{q}, \hat{p})\hat{\rho}_{ij}) = \operatorname{tr}(f(\hat{q}, \hat{p})\hat{\rho}). \end{aligned} \tag{4.3.42}$$

□

From Theorem 4.3.5 follows that the expectation value of an observable $f \in C^\infty(T^*U)[\![\hbar]\!]$ in a state $\rho \in L^2(T^*U, \mathrm{d}l)$ in the operator representation of quantum mechanics is expressed by the formula

$$\langle f \rangle_\rho = \operatorname{tr}(f(\hat{q}, \hat{p})\hat{\rho}). \tag{4.3.43}$$

Moreover, in the operator representation to time evolution equation (3.2.19) corresponds the following equation

$$i\hbar \frac{\partial \hat{\rho}}{\partial t}(t) - [H(\hat{q}, \hat{p}), \hat{\rho}(t)] = 0 \tag{4.3.44}$$

called the von Neumann equation.

4.3.2 The case of a general quantization

Let us consider a configuration space $\mathcal{Q}$ in the form of an N-dimensional almost geodesically simply connected Riemannian manifold, and a phase space $M = T^*\mathcal{Q}$. Moreover, we will consider a classical system defined on M and its quantization by means of a $\star$-product on M. We will begin with constructing the operator representation for some coordinate system on $\mathcal{Q}$. Let $\mathcal{Q} \supset U \to V \subset \mathbb{R}^N$, $q \mapsto (q^1, \dots, q^N)$ be an almost global coordinate system on $\mathcal{Q}$. From the assumption that $\mathcal{Q}$ is almost geodesically simply connected such coordinate system always exists. The coordinate system $(q^1, \dots, q^N)$ induces on M an almost global classical canonical coordinate system $T^*U \to T^*V = V \times \mathbb{R}^N$, $x \mapsto (q^1, \dots, q^N, p_1, \dots, p_N)$. We will assume that this coordinate system is at the same time quantum canonical. For star-products from Section 3.4 this is the case. The quantum system can be written in the coordinates (q^i, p_j). Note, that although $\star$-product is not local it still can be written in the coordinates (q^i, p_j) since this coordinate system is almost globally defined.

The idea behind introducing the operator representation lies in the observation that the quantum system in coordinates (q^i, p_j) is equivalent with a system quantized by the Moyal product, cf. Theorem 3.4.3. If S is a morphism giving this equivalence then S is a unitary operator on the Hilbert space $L^2(T^*V, \mathrm{d}l)$. Let us introduce a tensor product $\otimes_S \colon (L^2(V, \mathrm{d}\mu))^* \times L^2(V, \mathrm{d}\mu) \to L^2(T^*V, \mathrm{d}l)$ by the formula

$$\varphi^* \otimes_S \psi = S(\varphi^* \otimes_W \psi), \quad \varphi, \psi \in L^2(V, \mathrm{d}\mu) \tag{4.3.45}$$

and a function f of S-ordered operators $\hat{q}^i$, $\hat{p}_j$

$$f_S(\hat{q}, \hat{p}) = (S^{-1} f)(\hat{q}, \hat{p}). \tag{4.3.46}$$

Using Theorem 3.4.3 and property (3.4.29) it can be easily proved that all previous formulas and theorems for the case of a Moyal quantization also hold true for a general quantum system in (q^i, p_j) coordinates, provided that the tensor product $\otimes_W$ will be replaced by $\otimes_S$ and operators $f(\hat{q}, \hat{p})$ by $f_S(\hat{q}, \hat{p})$. In particular, there holds.

Proposition 4.3.3. *Let $f \in C^\infty(T^*V)[\![\hbar]\!]$ and $\rho = \varphi^* \otimes_S \psi$ for $\varphi, \psi \in C_0^\infty(V)$. Then*

$$f \star^{(q,p)} \rho = \varphi^* \otimes_S f_S(\hat{q}, \hat{p})\psi, \tag{4.3.47a}$$

$$\rho \star^{(q,p)} f = (f_S(\hat{q}, \hat{p})^\dagger \varphi)^* \otimes_S \psi, \tag{4.3.47b}$$

where $f_S(\hat{q}, \hat{p})$ is an S-ordered function of canonical operators of position $\hat{q}^i = q^i$ and momentum $\hat{p}_j = -i\hbar(\partial_{q^j} + \frac{1}{2}\Gamma^k_{jk})$, acting in the Hilbert space $L^2(V, \mathrm{d}\mu)$.

Proof. From Theorem 3.4.3 and Proposition 4.3.2 we get that

$$\begin{aligned} f \star^{(q,p)} \rho &= SS^{-1}(f \star^{(q,p)} \rho) = S(S^{-1} f \star_M^{(q,p)} S^{-1}\rho) = S\left(\varphi^* \otimes_W (S^{-1} f)(\hat{q}, \hat{p})\psi\right) \\ &= \varphi^* \otimes_S f_S(\hat{q}, \hat{p})\psi \end{aligned} \tag{4.3.48}$$

which proves (4.3.47a). Equation (4.3.47b) can be proved analogically. □

From Proposition 4.3.3 follows that operators $f \star^{(q,p)}$ can be written as

$$f \star^{(q,p)} = \hat{1} \otimes_S f_S(\hat{q}, \hat{p}). \tag{4.3.49}$$

Equation (4.3.49) allows to naturally identify functions $f \in C^\infty(T^*V)[\![\hbar]\!]$ with operators $f_S(\hat{q}, \hat{p})$. Moreover, the map $f \mapsto \hat{f} = f_S(\hat{q}, \hat{p})$ is a representation of the algebra $\mathcal{A}_Q(T^*V) = (C^\infty(T^*V)[\![\hbar]\!], \star^{(q,p)})$ in the Hilbert space $L^2(V, \mathrm{d}\mu)$.

Similarly, the analog of Proposition 4.3.1 holds true, which gives us a representation $\rho \mapsto \hat{\rho}$ of the algebra $\mathcal{L} = (L^2(T^*V, \mathrm{d}l), \star^{(q,p)})$ in the Hilbert space $L^2(V, \mathrm{d}\mu)$ given by $\rho \star^{(q,p)} = \hat{1} \otimes_S \hat{\rho}$.

Furthermore, the following analog of Theorem 4.3.5 can be proved.

Theorem 4.3.6. *Let $f \in C^\infty(T^*V)[\![\hbar]\!]$ and $\rho \in L^2(T^*V, \mathrm{d}l)$. If $f \star^{(q,p)} \rho \in L^1(T^*V, \mathrm{d}l)$ then*

$$\int_{T^*V} f \star^{(q,p)} \rho \, \mathrm{d}l = \operatorname{tr}(f_S(\hat{q}, \hat{p})\hat{\rho}). \tag{4.3.50}$$

In particular, if $\rho = \varphi^* \otimes_S \psi$ for $\varphi, \psi \in C_0^\infty(V)$ then

$$\int_{T^*V} f \star^{(q,p)} \rho \, \mathrm{d}l = (\varphi, f_S(\hat{q}, \hat{p})\psi). \tag{4.3.51}$$

Proof. Let $\{\varphi_i\}$ be a basis in $L^2(V, \mathrm{d}\mu)$ and $\{\rho_{ij}\} = \{\varphi_i^* \otimes_S \varphi_j\}$ an induced basis in $L^2(T^*V, \mathrm{d}l)$. Function ρ can be expanded in the basis $\{\rho_{ij}\}$ resulting in $\rho = \sum_{i,j} c_{ij}\rho_{ij}$.

From Theorems 3.4.3 and 4.3.5, and property (3.4.29) we get that

$$\begin{aligned}
\int_{T^*V} f \star^{(q,p)} \rho \, \mathrm{d}l &= \int_{T^*V} SS^{-1}(f \star^{(q,p)} \rho) \, \mathrm{d}l = \int_{T^*V} S^{-1} f \star_M^{(q,p)} S^{-1}\rho \, \mathrm{d}l \\
&= \sum_{i,j} c_{ij} \int_{T^*V} S^{-1} f \star_M^{(q,p)} S^{-1}\rho_{ij} \, \mathrm{d}l = \sum_{i,j} c_{ij}(\varphi_i, f_S(\hat{q}, \hat{p})\varphi_j) \\
&= \sum_{i,j} c_{ij} \operatorname{tr}(f_S(\hat{q}, \hat{p})\hat{\rho}_{ij}) = \operatorname{tr}(f_S(\hat{q}, \hat{p})\hat{\rho}).
\end{aligned} \tag{4.3.52}$$

□

Note, that for a general quantization the operator representation corresponding to some coordinate system gives us the correspondence rule $f \mapsto f_S(\hat{q}, \hat{p})$ which in general is different than the Weyl correspondence rule. The Weyl correspondence rule is associated only with the Moyal quantization and the Cartesian coordinate system. To create an operator representation of the general quantum system in general coordinates in a consistent way it is needed to use different orderings of position and momentum operators (instead of using Weyl ordering for any quantization and coordinate system we have to use S-orderings).

Let $\mathcal{Q} \supset U \to V \subset \mathbb{R}^N$, $q \mapsto (q^1, \dots, q^N)$ and $\mathcal{Q} \supset U' \to V' \subset \mathbb{R}^N$, $q \mapsto (q'^1, \dots, q'^N)$ be two almost global coordinate systems on $\mathcal{Q}$, and $T^*U \to T^*V = V \times \mathbb{R}^N$, $x \mapsto (q^1, \dots, q^N, p_1, \dots, p_N)$ and $T^*U' \to T^*V' = V' \times \mathbb{R}^N$, $x \mapsto (q'^1, \dots, q'^N, p'_1, \dots, p'_N)$ induced canonical coordinate systems on $T^*\mathcal{Q}$. A map $\phi\colon (q'^1, \dots, q'^N) \mapsto (q^1, \dots, q^N)$ is then a transformation of coordinates on the configuration space $\mathcal{Q}$ and a map $T\colon (q'^1, \dots, q'^N, p'_1, \dots, p'_N) \mapsto (q^1, \dots, q^N, p_1, \dots, p_N)$

is a canonical transformation of coordinates on the phase space $T^*\mathcal{Q}$. The transformation T is given by the formula (2.1.17).

In what follows we will investigate how the operator representation of quantum mechanics behaves after changing the coordinates. First, note that a map $\hat{U}_T\colon L^2(V,\mathrm{d}\mu) \to L^2(V',\mathrm{d}\mu')$ given by

$$(\hat{U}_T\psi)(q') = \psi(\phi(q')) \tag{4.3.53}$$

is an isomorphism of Hilbert spaces. Moreover, a map $L^2(T^*V,\mathrm{d}l) \to L^2(T^*V',\mathrm{d}l)$ given by

$$f \mapsto f \circ T \tag{4.3.54}$$

is also an isomorphism of Hilbert spaces. Let $\otimes_S$ and $\otimes_{S'}$ be tensor products corresponding to star-products $\star^{(q,p)}$ and $\star^{(q',p')}$ respectively. The following theorem can be proved.

Theorem 4.3.7. *For $\varphi,\psi \in L^2(V,\mathrm{d}\mu)$ there holds*

$$(\varphi^* \otimes_S \psi) \circ T = (\hat{U}_T\varphi)^* \otimes_{S'} \hat{U}_T\psi. \tag{4.3.55}$$

From Theorem 4.3.7 follows that operator representations of quantum mechanics corresponding to different coordinate systems are unitarily equivalent. In particular, we get that operators, corresponding to a function $f \in C^\infty(T^*\mathcal{Q})[\![\hbar]\!]$ written in different coordinate systems, are unitarily equivalent:

Theorem 4.3.8. *For $f \in C^\infty(T^*V)[\![\hbar]\!]$ there holds*

$$f'_{S'}(\hat{q}',\hat{p}') = \hat{U}_T f_S(\hat{q},\hat{p})\hat{U}_T^{-1}, \tag{4.3.56}$$

where $f' = f\circ T$ and $\hat{q}^i = q^i$, $\hat{p}_j = -i\hbar(\partial_{q^j} + \frac{1}{2}\Gamma^k_{jk})$ and $\hat{q}'^i = q'^i$, $\hat{p}'_j = -i\hbar(\partial_{q'^j} + \frac{1}{2}\Gamma'^k_{jk})$ are operators of position and momentum corresponding to the coordinates (q^i,p_j) and (q'^i,p'_j) respectively.

Proof. Let $\rho = \varphi^* \otimes_S \psi$. From Proposition 4.3.3 and Theorem 4.3.7 we get from one side

$$(f \star^{(q,p)} \rho) \circ T = f' \star^{(q',p')} (\rho \circ T) = (\hat{U}_T\varphi)^* \otimes_{S'} f'_{S'}(\hat{q}',\hat{p}')\hat{U}_T\psi \tag{4.3.57}$$

and from the other side

$$\begin{aligned}(f \star^{(q,p)} \rho) \circ T &= (\varphi^* \otimes_S f_S(\hat{q},\hat{p})\psi) \circ T = (\hat{U}_T\varphi)^* \otimes_{S'} \hat{U}_T f_S(\hat{q},\hat{p})\psi \\ &= (\hat{U}_T\varphi)^* \otimes_{S'} \hat{U}_T f_S(\hat{q},\hat{p})\hat{U}_T^{-1}\hat{U}_T\psi.\end{aligned} \tag{4.3.58}$$

Comparison of the above two formulas implies the result. □

Remark 4.3.1. If $\hat{q}^i,\hat{p}_j$ are operators of position and momentum defined on the Hilbert space $L^2(V,\mathrm{d}\mu)$ and corresponding to the coordinate system (q^i,p_j), and if $T^{-1}(q,p) = (Q^1(q,p),\ldots,Q^N(q,p),P_1(q,p),\ldots,P_N(q,p))$ is a transformation to the coordinate system (q'^i,p'_j), then the maps Q^i,P_j are observables of position and momentum on the phase space T^*V corresponding to the coordinate system (q'^i,p'_j). To the maps Q^i,P_j we can relate operators $Q^i_S(\hat{q},\hat{p})$, $(P_j)_S(\hat{q},\hat{p})$ which are operators of position and momentum defined on the Hilbert space $L^2(V, \mathrm{d}\mu)$ and corresponding to the coordinate system (q'^i, p'_j). From Theorem 4.3.8 we get that

$$\hat{q}'^i = \hat{U}_T Q^i_S(\hat{q}, \hat{p}) \hat{U}_T^{-1}, \tag{4.3.59a}$$

$$\hat{p}'_j = \hat{U}_T (P_j)_S(\hat{q}, \hat{p}) \hat{U}_T^{-1}. \tag{4.3.59b}$$

Equation (4.3.59a) is a statement of the fact that the unitary operator $\hat{U}_T$ gives a position representation of the quantum system for the operators $Q^i_S(\hat{q}, \hat{p})$, i.e. the unitary operator $\hat{U}_T$ writes the operators $Q^i_S(\hat{q}, \hat{p})$ as operators of multiplication by a coordinate variable.

4.3.3 Invariant form of the operator representation

So far we introduced the operator representation of quantum mechanics corresponding to some coordinate system on the configuration space. In what follows we will use the developed formalism to introduce an operator representation in a coordinate independent way. Let $\phi\colon \mathcal{Q} \supset U \to V \subset \mathbb{R}^N$ be an almost global coordinate system on the configuration space $\mathcal{Q}$ and $\Phi\colon T^*U \to T^*V = V \times \mathbb{R}^N$ a related almost global canonical coordinate system on the phase space $T^*\mathcal{Q}$. Since the coordinate system ϕ is almost globally defined it defines an isomorphism $\hat{U}\colon L^2(\mathcal{Q}, \mathrm{d}\omega_g) \to L^2(V, \mathrm{d}\mu)$ of the Hilbert spaces given by

$$\hat{U}\psi = \psi|_U \circ \phi^{-1}. \tag{4.3.60}$$

Indeed, the restriction $|_U$ is a natural isomorphism of $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$ onto $L^2(U, \mathrm{d}\mu)$ since for $\psi \in L^2(\mathcal{Q}, \mathrm{d}\omega_g)$, ψ and $\psi|_U$ are equal almost everywhere and Hilbert spaces of square integrable functions are constituted of equivalence classes of functions equal almost everywhere. Similarly, the coordinate system Φ defines an isomorphism of the Hilbert space $L^2(T^*\mathcal{Q}, \mathrm{d}l)$ onto the Hilbert space $L^2(T^*V, \mathrm{d}l)$. We can now define a tensor product $\otimes\colon (L^2(\mathcal{Q}, \mathrm{d}\omega_g))^* \times L^2(\mathcal{Q}, \mathrm{d}\omega_g) \to L^2(T^*\mathcal{Q}, \mathrm{d}l)$ by the formula

$$\varphi^* \otimes \psi = ((\hat{U}\varphi)^* \otimes_S \hat{U}\psi) \circ \Phi, \quad \varphi, \psi \in L^2(\mathcal{Q}, \mathrm{d}\omega_g), \tag{4.3.61}$$

where $\otimes_S\colon (L^2(V, \mathrm{d}\mu))^* \times L^2(V, \mathrm{d}\mu) \to L^2(T^*V, \mathrm{d}l)$ is a tensor product corresponding to the coordinate system Φ. The definition of the tensor product $\otimes$ is independent on the choice of a coordinate system. Indeed, if $\phi'\colon \mathcal{Q} \supset U' \to V' \subset \mathbb{R}^N$ is an another almost global coordinate system on $\mathcal{Q}$, $\Phi'\colon T^*U' \to T^*V' = V' \times \mathbb{R}^N$ a related almost global canonical coordinate system on $T^*\mathcal{Q}$ and $\hat{U}'\colon L^2(\mathcal{Q}, \mathrm{d}\omega_g) \to L^2(V', \mathrm{d}\mu')$ a Hilbert space isomorphism induced by ϕ' then $T = \Phi \circ \Phi'^{-1}$ is a canonical transformation of coordinates and $\hat{U}_T = \hat{U}'\hat{U}^{-1}$ a related unitary operator (4.3.53). Then from Theorem 4.3.7 follows that

$$\begin{aligned}\varphi^* \otimes \psi &= ((\hat{U}\varphi)^* \otimes_S \hat{U}\psi) \circ T \circ \Phi' = ((\hat{U}_T\hat{U}\varphi)^* \otimes_{S'} \hat{U}_T\hat{U}\psi) \circ \Phi' \\ &= ((\hat{U}'\varphi)^* \otimes_{S'} \hat{U}'\psi) \circ \Phi'.\end{aligned} \tag{4.3.62}$$

The tensor product $\otimes$ inherits all properties of the tensor products $\otimes_S$. In particular, the following theorems hold.

Proposition 4.3.4. *For every $\rho \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$*

$$\rho\,\star = \hat{1} \otimes \hat{\rho}, \tag{4.3.63}$$

where $\hat{\rho} \in B_2(L^2(\mathcal{Q}, \mathrm{d}\omega_g))$ is some Hilbert-Schmidt operator defined on the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. Conversely, for every $\hat{\rho} \in B_2(L^2(\mathcal{Q}, \mathrm{d}\omega_g))$ the operator $\hat{1} \otimes \hat{\rho}$ is of the form $\rho\,\star$ for some $\rho \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$.

The following properties are fulfilled:

(i) *for $\rho = \varphi^* \otimes \psi$, $\hat{\rho} = (\varphi, \cdot\,)\psi$,*

(ii) $\bar{\rho}\,\star = \hat{1} \otimes \hat{\rho}^\dagger$*,*

(iii) $\operatorname{tr}(\rho) \equiv \int_{T^*\mathcal{Q}} \rho \,\mathrm{d}l = \operatorname{tr}(\hat{\rho})$*,*

(iv) *for $\rho_1, \rho_2 \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$ and $\hat{\rho}_1, \hat{\rho}_2 \in B_2(L^2(\mathcal{Q}, \mathrm{d}\omega_g))$ such that $\rho_1\,\star = \hat{1} \otimes \hat{\rho}_1$ and $\rho_2\,\star = \hat{1} \otimes \hat{\rho}_2$*

$$(\rho_1, \rho_2) = (\hat{\rho}_1, \hat{\rho}_2)_2, \tag{4.3.64}$$

(v) $\int_{T^*\mathcal{Q}} \bar{f} \star f \star \rho \,\mathrm{d}l \geq 0$ *for $f \in C_0^\infty(T^*\mathcal{Q})$ if and only if $(\varphi, \hat{\rho}\varphi) \geq 0$ for $\varphi \in L^2(\mathcal{Q}, \mathrm{d}\omega_g)$.*

Proposition 4.3.5. *Let $f \in C^\infty(T^*\mathcal{Q})[\![\hbar]\!]$ and $\rho = \varphi^* \otimes \psi$ for $\varphi, \psi \in C_0^\infty(\mathcal{Q})$. Then*

$$f \star \rho = \varphi^* \otimes \hat{f}\psi, \tag{4.3.65a}$$

$$\rho \star f = (\hat{f}^\dagger \varphi)^* \otimes \psi, \tag{4.3.65b}$$

where $\hat{f}$ is some operator acting in the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. Furthermore, if $(q^1, \dots, q^N)$ is some almost global coordinate system on $\mathcal{Q}$, $(q^1, \dots, q^N, p_1, \dots, p_N)$ a related canonical coordinate system on $T^*\mathcal{Q}$, and $\hat{U}$ a corresponding unitary operator given by (4.3.60)*, then*

$$\hat{U}\hat{f}\hat{U}^{-1} = f_S(\hat{q}, \hat{p}). \tag{4.3.66}$$

Theorem 4.3.9. *Let $f \in C^\infty(T^*\mathcal{Q})[\![\hbar]\!]$ and $\rho \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$. If $f \star \rho \in L^1(T^*\mathcal{Q}, \mathrm{d}l)$ then*

$$\int_{T^*\mathcal{Q}} f \star \rho \,\mathrm{d}l = \operatorname{tr}(\hat{f}\hat{\rho}). \tag{4.3.67}$$

In particular, if $\rho = \varphi^* \otimes \psi$ for $\varphi, \psi \in C_0^\infty(\mathcal{Q})$ then

$$\int_{T^*\mathcal{Q}} f \star \rho \,\mathrm{d}l = (\varphi, \hat{f}\psi). \tag{4.3.68}$$

From Proposition 4.3.4 follows that the map $\rho \mapsto \hat{\rho}$ is a representation of the algebra $\mathcal{L} = (L^2(T^*\mathcal{Q}, \mathrm{d}l), \star)$ in the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. Furthermore, from Proposition 4.3.5 follows that functions $f \in C^\infty(T^*\mathcal{Q})[\![\hbar]\!]$ can be naturally identified with operators $\hat{f}$ through the formula

$$f\,\star = \hat{1} \otimes \hat{f}. \tag{4.3.69}$$

Moreover, the map $f \mapsto \hat{f}$ is a representation of the quantum Poisson algebra $\mathcal{A}_Q(T^*\mathcal{Q}) = (C^\infty(T^*\mathcal{Q})[\![\hbar]\!], \star)$ in the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. By virtue of Theorems 4.3.3 and 4.3.4 we get the following characterization of quantum states.

Proposition 4.3.6. *Pure states can be alternatively characterized as functions $\rho_{\mathrm{pure}} \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$ which are self-conjugated, normalized, and idempotent:*

$$\rho_{\mathrm{pure}} = \bar{\rho}_{\mathrm{pure}}, \tag{4.3.70a}$$

$$\int_{T^*\mathcal{Q}} \rho_{\mathrm{pure}} \, \mathrm{d}l = 1, \tag{4.3.70b}$$

$$\rho_{\mathrm{pure}} \star \rho_{\mathrm{pure}} = \rho_{\mathrm{pure}}. \tag{4.3.70c}$$

Mixed states $\rho_{\mathrm{mix}} \in L^2(T^*\mathcal{Q}, \mathrm{d}l)$ can be characterized as convex linear combinations, possibly infinite, of some families of pure states $\rho^{(\lambda)}_{\mathrm{pure}}$

$$\rho_{\mathrm{mix}} = \sum_\lambda p_\lambda \rho^{(\lambda)}_{\mathrm{pure}}, \tag{4.3.71}$$

where $p_\lambda \geq 0$ and $\sum_\lambda p_\lambda = 1$.

Remark 4.3.2. If $\phi\colon \mathcal{Q} \supset U \to V \subset \mathbb{R}^N$ is an almost global coordinate system on the configuration space $\mathcal{Q}$, $\Phi\colon T^*U \to T^*V = V \times \mathbb{R}^N$ a related almost global canonical coordinate system on the phase space $T^*\mathcal{Q}$, and $\hat{U}\colon L^2(\mathcal{Q}, \mathrm{d}\omega_g) \to L^2(V, \mathrm{d}\mu)$ an isomorphism of the Hilbert spaces given by (4.3.60), then the maps $q^i = \Phi^i$ and $p_j = \Phi^{j+N}$ $(i, j = 1, 2, \ldots, N)$ are observables of position and momentum corresponding to the coordinate system Φ. To the maps q^i, p_j we can relate operators $\hat{q}^i, \hat{p}_j$ defined on the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. The operators $\hat{q}^i = \phi^i$ are of the form of multiplication operators by coordinate functions ϕ^i and they constitute a complete set of commuting observables. Thus they can be used to create a representation corresponding to the coordinate system ϕ. In this representation operators $\hat{q}^i$ take the form of the multiplication operators by a coordinate variable, which are defined on the Hilbert space $L^2(V, \mathrm{d}\mu)$. In accordance to (4.3.66) the unitary operator giving this representation is equal $\hat{U}$.

4.3.4 Examples of quantum mechanical operators

In what follows we will consider such quantization of a classical system on $T^*\mathcal{Q}$ for which the morphism S giving the equivalence with the Moyal quantization is in the form (3.4.72) for any classical and quantum canonical coordinate system. We will derive the form of operators corresponding to functions linear, quadratic and cubic in momenta. Note, that the terms of order higher or equal to $\hbar^4$ in (3.4.72) are at least of the fourth order in ∂_{p_j} so the formula (3.4.72) is enough to calculate the action of S on functions that are up to cubic in momenta. First, let us consider a function H on $T^*\mathcal{Q}$ linear in momenta, which in some canonical coordinates on $T^*\mathcal{Q}$ takes the form

$$H(q, p) = K^i(q) p_i, \tag{4.3.72}$$

where K^i are components of some vector field K defined on $\mathcal{Q}$. The action of the morphism S on H leaves the function H unchanged:

$$(S^{-1}H)(q,p) = K^i(q)p_i. \tag{4.3.73}$$

From this and (4.1.31) to H will correspond the following self-adjoint operator

$$H_S(\hat{q},\hat{p}) = \frac{1}{2}K^i(\hat{q})\hat{p}_i + \frac{1}{2}\hat{p}_i K^i(\hat{q}). \tag{4.3.74}$$

By virtue of (4.2.7) the above equation can be written in the form

$$H_S(\hat{q},\hat{p}) = -\frac{i\hbar}{2}\left(2K^i\partial_{q^i} + K^i{}_{,i} + \Gamma^k_{ik}K^i\right) = -\frac{i\hbar}{2}\left(2K^i\partial_{q^i} + K^i{}_{;i}\right). \tag{4.3.75}$$

Finally, we can write the above equation in the following invariant form

$$H_S(\hat{q},\hat{p}) = -\frac{i\hbar}{2}\left(2K^i\nabla_i + K^i{}_{;i}\right) = -\frac{i\hbar}{2}\left(K^i\nabla_i + \nabla_i K^i\right). \tag{4.3.76}$$

Now, let us consider a function H on $T^*\mathcal{Q}$ quadratic in momenta, which in some canonical coordinates on $T^*\mathcal{Q}$ takes the form

$$H(q,p) = \frac{1}{2}K^{ij}(q)p_i p_j + V(q), \tag{4.3.77}$$

where K^{ij} are components of some symmetric tensor field K defined on $\mathcal{Q}$ and V is a smooth function on $\mathcal{Q}$. The action of the morphism S on H results in the following function

$$\begin{aligned}(S^{-1}H)(q,p) = \frac{1}{2}K^{ij}(q)p_i p_j + V(q) - \frac{\hbar^2}{2}\Bigg(&\frac{1}{4}K^{ij}{}_{,k}(q)\Gamma^k_{ij}(q) + \frac{1}{4}K^{ij}(q)\Gamma^k_{li}(q)\Gamma^l_{kj}(q) \\ &- \frac{1}{4}bK^{ij}{}_{;ij}(q) + \frac{1}{4}aK^{ij}(q)R_{ij}(q)\Bigg).\end{aligned} \tag{4.3.78}$$

From this and (4.1.31) to H will correspond the following self-adjoint operator

$$\begin{aligned}H_S(\hat{q},\hat{p}) &= \frac{1}{2}\left(\frac{1}{4}K^{ij}(\hat{q})\hat{p}_i\hat{p}_j + \frac{1}{2}\hat{p}_i K^{ij}(\hat{q})\hat{p}_j + \frac{1}{4}\hat{p}_i\hat{p}_j K^{ij}(\hat{q})\right) + V(\hat{q}) \\ &- \frac{\hbar^2}{2}\left(\frac{1}{4}K^{ij}{}_{,k}(\hat{q})\Gamma^k_{ij}(\hat{q}) + \frac{1}{4}K^{ij}(\hat{q})\Gamma^k_{li}(\hat{q})\Gamma^l_{kj}(\hat{q}) - \frac{1}{4}bK^{ij}{}_{;ij}(\hat{q}) + \frac{1}{4}aK^{ij}(\hat{q})R_{ij}(\hat{q})\right).\end{aligned} \tag{4.3.79}$$

By virtue of (4.2.7) the above equation can be written in the form

$$\begin{aligned}H_S(\hat{q},\hat{p}) = -\frac{\hbar^2}{2}\Bigg(&K^{ij}\partial_{q^i}\partial_{q^j} + K^{ij}\Gamma^l_{jl}\partial_{q^i} + K^{ij}{}_{,i}\partial_{q^j} + \frac{1}{2}K^{ij}\Gamma^l_{jl,i} + \frac{1}{4}K^{ij}\Gamma^k_{ik}\Gamma^l_{jl} + \frac{1}{2}K^{ij}{}_{,i}\Gamma^l_{jl} \\ &+ \frac{1}{4}K^{ij}{}_{,ij} + \frac{1}{4}K^{ij}{}_{,k}\Gamma^k_{ij} + \frac{1}{4}K^{ij}\Gamma^k_{li}\Gamma^l_{kj} - \frac{1}{4}bK^{ij}{}_{;ij} + \frac{1}{4}aK^{ij}R_{ij}\Bigg) + V.\end{aligned} \tag{4.3.80}$$

Using the equality ${K^{ij}}_{,k} = -K^{rj}\Gamma^i_{rk} - K^{ri}\Gamma^j_{rk} + {K^{ij}}_{;k}$ the above equation simplifies to

$$H_S(\hat{q},\hat{p}) = -\frac{\hbar^2}{2}\left(K^{ij}\partial_{q^i}\partial_{q^j} + K^{ij}\Gamma^l_{jl}\partial_{q^i} + {K^{ij}}_{,i}\partial_{q^j} + \frac{1}{4}(1-b){K^{ij}}_{;ij} - \frac{1}{4}(1-a)K^{ij}R_{ij}\right) + V. \tag{4.3.81}$$

Note, that (4.3.81) can be written in the following invariant form

$$H_S(\hat{q},\hat{p}) = -\frac{\hbar^2}{2}\left(\nabla_i K^{ij}\nabla_j + \frac{1}{4}(1-b){K^{ij}}_{;ij} - \frac{1}{4}(1-a)K^{ij}R_{ij}\right) + V, \tag{4.3.82}$$

where $\nabla_i K^{ij}\nabla_j = \Delta_K$ is the pseudo-Laplace operator. For a special case when K is the standard metric tensor g on the configuration space, the function H has the form of a natural Hamiltonian (3.2.2), and equation (4.3.82) reduces to

$$H_S(\hat{q},\hat{p}) = -\frac{\hbar^2}{2}\left(g^{ij}\nabla_i\nabla_j - \frac{1}{4}(1-a)R\right) + V. \tag{4.3.83}$$

Observe, that $\nabla_i g^{ij}\nabla_j = g^{ij}\nabla_i\nabla_j = \Delta$ is the Laplace-Beltrami operator. Note, that for a flat metric tensor g the family of morphisms S depends only on the parameter b and consequently we have one-parameter family of quantizations which, according to (4.3.83), coincide for a class of natural Hamiltonians (3.2.2).

Finally, Let us consider a function H on $T^*\mathcal{Q}$, which in some canonical coordinates on $T^*\mathcal{Q}$ is cubic in momenta (we skip the lower terms in momenta):

$$H(q,p) = K^{ijk}(q)p_ip_jp_k, \tag{4.3.84}$$

where K^{ijk} are components of some symmetric tensor field K defined on $\mathcal{Q}$. Similarly as in the previous case we can derive the form of the corresponding self-adjoint operator:

$$\begin{aligned} H_S(\hat{q},\hat{p}) = \frac{1}{2}i\hbar^3\Big(&\nabla_i K^{ijk}\nabla_j\nabla_k + \nabla_i\nabla_j K^{ijk}\nabla_k + \frac{1}{4}(1-b)\nabla_k {K^{ijk}}_{;ij} \\ &+ \frac{1}{4}(1-b){K^{ijk}}_{;ij}\nabla_k - \frac{3}{4}(1-a)\nabla_i K^{ijk}R_{jk} - \frac{3}{4}(1-a)K^{ijk}R_{jk}\nabla_i\Big). \end{aligned} \tag{4.3.85}$$

Note, that the received operators are defined on the Hilbert space $L^2(V, \mathrm{d}\mu)$ and correspond to a given canonical coordinate system (q^i, p_j). These operators are written in an invariant form and consequently they can be treated as operators defined on the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$. Indeed, using the unitary operator $\hat{U}$ related to the coordinate system (q^i, p_j) and given by (4.3.60) we can receive, in accordance to the formula (4.3.66), operators defined on the Hilbert space $L^2(\mathcal{Q}, \mathrm{d}\omega_g)$.

4.3.5 Example of the hydrogen atom

Let us consider a quantum system of the hydrogen atom. A configuration space of such system is the 3-dimensional Euclidean space E^3. It represents the position in space of an electron of the hydrogen atom. A phase space of the system is T^*E^3 and a Hamiltonian H in Cartesian coordinates takes a form

$$H(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{4\pi\epsilon_0} \frac{e^2}{\sqrt{x^2 + y^2 + z^2}}. \tag{4.3.86}$$

As a star-product on T^*E^3 is taken the canonical $\star$-product which in the Cartesian coordinates takes a form of the Moyal product. In the operator representation in the Cartesian coordinate system the Hilbert space of states takes the form of the space $L^2(\mathbb{R}^3)$ of functions on $\mathbb{R}^3$ square integrable with respect to the Lebesgue measure. The canonical operators of position and momentum take the standard form

$$\begin{aligned} \hat{q}_x &= x, & \hat{q}_y &= y, & \hat{q}_z &= z, \\ \hat{p}_x &= -i\hbar\partial_x, & \hat{p}_y &= -i\hbar\partial_y, & \hat{p}_z &= -i\hbar\partial_z, \end{aligned} \tag{4.3.87}$$

and the Hamilton operator, being a symmetrically ordered function H of the operators of position and momentum, takes a form

$$H(\hat{q}_x, \hat{q}_y, \hat{q}_z, \hat{p}_x, \hat{p}_y, \hat{p}_z) = -\frac{\hbar^2}{2m}\Delta - \frac{1}{4\pi\epsilon_0} \frac{e^2}{\sqrt{x^2 + y^2 + z^2}}, \tag{4.3.88}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator in the Cartesian coordinates.

Now, let us consider the quantum system and its operator representation in the spherical polar coordinates. The Moyal product in the Cartesian coordinates, under the point transformation to spherical polar coordinates (2.2.8), transforms to a star-product of the form (3.4.40). In accordance to Theorem 3.4.3 this star-product is equivalent to the Moyal product, where the equivalence morphism S, by virtue of (3.4.71), is equal

$$\begin{aligned} S = \mathrm{id} + \frac{\hbar^2}{4} \Bigg(& \frac{1}{r^2}\partial_{p_r}^2 + \left(\frac{1}{2\tan^2\theta} - 1\right)\partial_{p_\theta}^2 - \partial_{p_\phi}^2 + \frac{1}{r\tan\theta}\partial_{p_r}\partial_{p_\theta} + \frac{1}{r^2}p_\theta\partial_{p_r}^2\partial_{p_\theta} \\ & - \frac{1}{2}p_r\partial_{p_r}\partial_{p_\theta}^2 + \frac{2}{r\tan\theta}p_\phi\partial_{p_r}\partial_{p_\theta}\partial_{p_\phi} - \left(\frac{1}{2}p_r\sin^2\theta + \frac{1}{r}p_\theta\sin\theta\cos\theta\right)\partial_{p_r}\partial_{p_\phi}^2 - \frac{1}{3}p_\theta\partial_{p_\theta}^3 \\ & + \frac{1}{\tan^2\theta}p_\phi\partial_{p_\theta}^2\partial_{p_\phi} - \frac{1}{2}p_\theta\partial_{p_\theta}\partial_{p_\phi}^2 - \frac{1}{3}p_\phi\partial_{p_\phi}^3 + \frac{1}{r^2}p_\phi\partial_r^2\partial_{p_\phi} - \frac{1}{2}r\partial_r\partial_{p_\theta}^2 - \frac{1}{2}r\sin^2\theta\partial_r\partial_{p_\phi}^2 \\ & + \frac{1}{r}\partial_\theta\partial_{p_r}\partial_{p_\theta} - \frac{1}{2}\sin\theta\cos\theta\partial_\theta\partial_{p_\phi}^2 + \frac{1}{r}\partial_\phi\partial_{p_r}\partial_{p_\phi} + \frac{1}{\tan\theta}\partial_\phi\partial_{p_\theta}\partial_{p_\phi}\Bigg) + o(\hbar^4). \end{aligned} \tag{4.3.89}$$

The Hamilton function (4.3.86) in the spherical polar coordinates takes a form

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}, \tag{4.3.90}$$

and the action of the morphism S on the transformed Hamilton function (4.3.90) results in the following function

$$\begin{aligned} (S^{-1}H)(r, \theta, \phi, p_r, p_\theta, p_\phi) = {} & \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\ & - \frac{\hbar^2}{8mr^2}\left(\frac{1}{\sin^2\theta} + 1\right). \end{aligned} \tag{4.3.91}$$

Note the extra term in (4.3.91) dependent on $\hbar$. Thus, the quantum system in the spherical polar coordinates can be described by the Hamiltonian (4.3.90) and the star-product in the form (3.4.40), or equivalently by the Hamiltonian (4.3.91) and the Moyal star-product.

In the operator representation in the spherical polar coordinates the Hilbert space of states is equal $L^2(V, \mathrm{d}\mu)$, where $V = (0,\infty)\times(0,\pi)\times(0,2\pi)$ and $\mathrm{d}\mu(r,\theta,\phi) = r^2 \sin\theta \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi$, and the operators of position and momentum take a form

$$\begin{aligned} &\hat{q}_r = r, && \hat{q}_\theta = \theta, && \hat{q}_\phi = \phi, \\ &\hat{p}_r = -i\hbar\left(\partial_r + \frac{1}{r}\right), && \hat{p}_\theta = -i\hbar\left(\partial_\theta + \frac{1}{2\tan\theta}\right), && \hat{p}_\phi = -i\hbar\partial_\phi. \end{aligned} \tag{4.3.92}$$

The Hamilton operator is calculated as an S-ordered Hamilton function (4.3.90) of the operators of position and momentum (4.3.92), or equivalently as a symmetrically ordered function (4.3.91) of these operators:

$$\begin{aligned} H_S(\hat{q}_r, \hat{q}_\theta, \hat{q}_\phi, \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi) &= (S^{-1}H)(\hat{q}_r, \hat{q}_\theta, \hat{q}_\phi, \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi) \\ &= -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \frac{1}{\tan\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\right] - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}. \end{aligned} \tag{4.3.93}$$

Note, that the expression in square brackets is just the Laplace operator written in spherical coordinates. A direct computation shows that the operators (4.3.88) and (4.3.93) are unitarily equivalent, where a unitary operator giving this equivalence is equal

$$\hat{U}_T \colon L^2(\mathbb{R}^3) \to L^2(V, \mathrm{d}\mu), \quad (\hat{U}_T\psi)(r,\theta,\phi) = \psi(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta). \tag{4.3.94}$$

Note, that the property that the spherical polar coordinates are almost global is crucial to get the unitarity of $\hat{U}_T$. Since the operators (4.3.88) and (4.3.93) are unitarily equivalent they have the same spectra, and solving the eigenvalue problem of one of these operators gives the solution for the other operator.

Chapter 5

Quantum trajectories

5.1 Preliminaries

We will consider the Moyal quantization of a classical Hamiltonian system (M, ω, H), where a phase space $M = \mathbb{R}^{2N}$, symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$, and Hamiltonian $H \in C^\infty(M)$ is an arbitrary real function.

The solution of quantum Hamiltonian equations

$$\dot{Q}^i(t) = [\![Q^i(t), H]\!], \quad \dot{P}_j(t) = [\![P_j(t), H]\!], \tag{5.1.1}$$

where $Q^i(q,p,0) = q^i$ and $P_j(q,p,0) = p_j$, i.e., the Heisenberg representation (3.2.25) for observables of position and momentum, generates a quantum flow Φ_t in phase space according to an equation

$$\Phi_t(q,p;\hbar) = (Q(q,p,t;\hbar), P(q,p,t;\hbar)). \tag{5.1.2}$$

For every instance of time t the map Φ_t is a quantum canonical transformation (quantum symplectomorphism) from the coordinates q^i, p_j to new coordinates $q'^i = Q^i(q,p,t;\hbar)$, $p'_j = P_j(q,p,t;\hbar)$. In other words Φ_t preserves the quantum Poisson bracket: $[\![Q^i(t), P_j(t)]\!] = \delta^i_j$ (this can be easily seen from (5.1.8) and the fact that $[\![Q^i(0), P_j(0)]\!] = [\![q^i, p_j]\!] = \delta^i_j$).

The flow Φ_t, as every other quantum canonical transformation, can act on observables and states as a simple composition of maps. Such classical action can also be used to transform the algebraic structure of the quantum Poisson algebra so that the action will be an isomorphism of the initial algebra and its transformation. A star-product $\star_t$ being the Moyal $\star$-product transformed by Φ_t is defined by the formula

$$(f \star g) \circ \Phi_t^{-1} = (f \circ \Phi_t^{-1}) \star_t (g \circ \Phi_t^{-1}), \quad f, g \in C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]. \tag{5.1.3}$$

The $\star_t$-product takes the form of the Moyal product but with derivatives ∂_{q^i}, ∂_{p_i} replaced by some other derivations D_{q^i}, D_{p_i} of the algebra $C^\infty(\mathbb{R}^{2N})$:

$$f \star_t g = f \exp\left(\frac{1}{2} i\hbar \overleftarrow{D_{q^i}} \overrightarrow{D_{p_i}} - \frac{1}{2} i\hbar \overleftarrow{D_{p_i}} \overrightarrow{D_{q^i}}\right) g, \tag{5.1.4}$$

where derivations D_{q^i}, D_{p_i} are transformations of the derivatives ∂_{q^i}, ∂_{p_i}:

$$(\partial_{q^i} f) \circ \Phi_t^{-1} = D_{q^i}(f \circ \Phi_t^{-1}), \quad (\partial_{p_i} f) \circ \Phi_t^{-1} = D_{p_i}(f \circ \Phi_t^{-1}). \tag{5.1.5}$$

The crucial point of our construction is the observation that for quantum flows the $\star_t$-product is equivalent to the Moyal product (see Theorem 3.4.3 and Remark 3.4.1). Strictly speaking, to a quantum flow Φ_t there corresponds a unique isomorphism S_t of the form (3.1.6) satisfying

$$S_t(f \star g) = S_t f \star_t S_t g, \tag{5.1.6a}$$

$$S_t q^i = q^i, \quad S_t p_j = p_j, \tag{5.1.6b}$$

$$S_t \bar{f} = \overline{S_t f}. \tag{5.1.6c}$$

A formal solution of the time evolution equation (3.2.25) for an observable $A \in C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ can be expressed by the formula

$$A(t) = e^{-t[\![H,\,\cdot\,]\!]}A(0) = e_\star^{\frac{i}{\hbar}tH} \star A(0) \star e_\star^{-\frac{i}{\hbar}tH}. \tag{5.1.7}$$

In particular, the solution of (5.1.1) takes the form

$$Q^i(t) = e^{-t[\![H,\,\cdot\,]\!]}Q^i(0) = e_\star^{\frac{i}{\hbar}tH} \star Q^i(0) \star e_\star^{-\frac{i}{\hbar}tH}, \tag{5.1.8a}$$

$$P_j(t) = e^{-t[\![H,\,\cdot\,]\!]}P_j(0) = e_\star^{\frac{i}{\hbar}tH} \star P_j(0) \star e_\star^{-\frac{i}{\hbar}tH}, \tag{5.1.8b}$$

which for a fixed initial condition $Q^i(q,p,0) = q^i$ and $P_j(q,p,0) = p_j$ represents a particular quantum trajectory.

A time evolution of an observable $A \in C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ should be alternatively expressed by an action of the quantum flow Φ_t on A. The composition of Φ_t with observables (the classical action of Φ_t on observables) does not result in a proper time evolution of observables. Thus it is necessary to deform this classical action. We will prove that a proper action of the quantum flow Φ_t on functions from $C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ (a pull-back of Φ_t) is given by the formula

$$\Phi_t^* A = (S_t A) \circ \Phi_t, \tag{5.1.9}$$

where S_t is an isomorphism associated to the quantum canonical transformation Φ_t^{-1}. Indeed, (5.1.9) can be proved first by noting that

$$\Phi_t^* Q^i(0) = (S_t Q^i(0)) \circ \Phi_t = Q^i(0) \circ \Phi_t = Q^i(t) = e^{-t[\![H,\,\cdot\,]\!]}Q^i(0) \tag{5.1.10}$$

and similarly

$$\Phi_t^* P_j(0) = e^{-t[\![H,\,\cdot\,]\!]}P_j(0), \tag{5.1.11}$$

where the fact that $S_t q^i = q^i$ and $S_t p_j = p_j$ was used, which on the other hand was a consequence of the quantum canonicity of Φ_t. Secondly, Φ_t^* given by (5.1.9) is an automorphism of $\mathcal{A}_Q(\mathbb{R}^{2N})$ as

$$\begin{aligned}\Phi_t^*(A \star B) &= (S_t(A \star B)) \circ \Phi_t = (S_t A \star_t S_t B) \circ \Phi_t = ((S_t A) \circ \Phi_t) \star ((S_t B) \circ \Phi_t) \\ &= \Phi_t^* A \star \Phi_t^* B,\end{aligned} \tag{5.1.12}$$

where $\star_t$ denotes a star-product transformed by Φ_t^{-1}. Thus

$$\Phi_t^* = e^{-t[\![H,\,\cdot\,]\!]} \tag{5.1.13}$$

holds true since every function in $C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]$ can be expressed as a $\star$-power series.

In a complete analogy with classical theory one can define a quantum Hamiltonian vector field by $\zeta_H = [\![\,\cdot\,, H]\!]$. Then (5.1.13) states that Φ_t is a flow of the quantum Hamiltonian vector field ζ_H. Also in an analogy with classical mechanics $\{\Phi_t\}$ is a one-parameter group of quantum canonical transformations with respect to a multiplication defined by

$$\Phi_{t_1}\Phi_{t_2} = (S_{t_2}\Phi_{t_1}) \circ \Phi_{t_2}, \tag{5.1.14}$$

where $S_{t_2}\Phi_{t_1}$ denotes a map $\mathbb{R}^{2N} \to \mathbb{R}^{2N}$ given by the formula

$$S_{t_2}\Phi_{t_1} = (S_{t_2}Q^1(t_1), \dots, S_{t_2}P_N(t_1)), \tag{5.1.15}$$

where $\Phi_{t_1} = (Q^1(t_1), \dots, Q^N(t_1), P_1(t_1), \dots, P_N(t_1))$. Multiplication defined in such a way satisfies properties similar to their classical counterparts:

$$\Phi_0 = \mathrm{id}, \quad \Phi_{t_1}\Phi_{t_2} = \Phi_{t_1+t_2}, \tag{5.1.16}$$

proving that $\{\Phi_t\}$ is a group. Further on we will call it a quantum composition. The quantum composition rule given by (5.1.14) is properly defined since it respects the quantum pull-back of flows:

$$(\Phi_{t_1}\Phi_{t_2})^* = \Phi_{t_2}^* \circ \Phi_{t_1}^*. \tag{5.1.17}$$

Indeed, it is enough to show (5.1.17) for an arbitrary $\star$-monomial. For simplicity we will present the proof for a two-dimensional case and for a $\star$-monomial $q \star p$. Using the fact that $S_t q = q$ and $S_t p = p$ for every t, following from quantum canonicity of the flow Φ_t, one calculates that

$$\begin{aligned}
(\Phi_{t_2}^* \circ \Phi_{t_1}^*)(q \star p) &= \Phi_{t_2}^*\big((S_{t_1}(q \star p)) \circ \Phi_{t_1}\big) = \Phi_{t_2}^*\big((q \star_{t_1} p) \circ \Phi_{t_1}\big) \\
&= \Phi_{t_2}^*\big(Q(t_1) \star P(t_1)\big) = \big(S_{t_2}(Q(t_1) \star P(t_1))\big) \circ \Phi_{t_2} \\
&= \big(S_{t_2}Q(t_1) \star_{t_2} S_{t_2}P(t_1)\big) \circ \Phi_{t_2} = (q \star_{t_2,t_1} p) \circ S_{t_2}\Phi_{t_1} \circ \Phi_{t_2},
\end{aligned} \tag{5.1.18}$$

where $\star_{t_1}$, $\star_{t_2}$, denote Moyal products transformed, respectively, by transformations $\Phi_{t_1}^{-1}$, $\Phi_{t_2}^{-1}$, and $\star_{t_2,t_1}$ denotes the $\star_{t_2}$-product transformed by $(S_{t_2}\Phi_{t_1})^{-1}$. Now, from the relation $S_{T_1 \circ T_2} = S_{T_1,T_2}S_{T_1}$ valid for any quantum canonical transformations T_1, T_2 defined on the whole phase space ($S_{T_1\circ T_2}$ is an isomorphism intertwining star-products $\star$ and $\star_{T_1\circ T_2}$, S_{T_1,T_2} intertwines $\star_{T_1}$ with $\star_{T_1\circ T_2}$, and S_{T_1} intertwines $\star$ with $\star_{T_1}$, where $\star_{T_1}$ and $\star_{T_1\circ T_2}$ are Moyal products transformed, respectively, by transformations T_1 and $T_1 \circ T_2$), one receives that

$$S_{(\Phi_{t_1}\Phi_{t_2})^{-1}}(q \star p) = S_{\Phi_{t_2}^{-1},(S_{t_2}\Phi_{t_1})^{-1}}S_{t_2}(q \star p) = S_{\Phi_{t_2}^{-1},(S_{t_2}\Phi_{t_1})^{-1}}(q \star_{t_2} p) = q \star_{t_2,t_1} p. \tag{5.1.19}$$

Hence

$$(\Phi_{t_2}^* \circ \Phi_{t_1}^*)(q \star p) = S_{(\Phi_{t_1}\Phi_{t_2})^{-1}}(q \star p) \circ S_{t_2}\Phi_{t_1} \circ \Phi_{t_2} = (\Phi_{t_1}\Phi_{t_2})^*(q \star p). \tag{5.1.20}$$

In the limit $\hbar \to 0$, (5.1.8) reduces to classical phase space trajectories

$$\begin{gathered}
Q^i(t) = e^{-t\{H,\,\cdot\,\}}Q^i(0), \quad P_j(t) = e^{-t\{H,\,\cdot\,\}}P_j(0), \\
Q^i(q,p,0) = q^i, \quad P_j(q,p,0) = p_j,
\end{gathered} \tag{5.1.21}$$

which are formal solutions of classical Hamilton equations

$$\dot{Q}^i(t) = \{Q^i(t), H\}, \quad \dot{P}_j(t) = \{P_j(t), H\}. \tag{5.1.22}$$

In more explicit form classical trajectories are represented by a flow (diffeomorphism)

$$\Phi_t(x, p) = (Q(x, p, t), P(x, p, t)), \tag{5.1.23}$$

which is an $\hbar \to 0$ limit of the quantum flow (5.1.2). Diffeomorphism (5.1.23) is a classical symplectomorphism. An action of the classical flow Φ_t on functions from $\mathcal{A}_C(\mathbb{R}^{2N})$ (a pull-back of Φ_t) is just a simple composition of functions with Φ_t, being an $\hbar \to 0$ limit of (5.1.9)

$$\Phi_t^* A = A \circ \Phi_t. \tag{5.1.24}$$

$\{\Phi_t\}$ forms a one-parameter group of canonical transformations, preserving a classical Poisson bracket: $\{Q^i(t), P_j(t)\} = \delta^i_j$, with a multiplication being an ordinary composition of maps

$$\Phi_{t_1}\Phi_{t_2} = \Phi_{t_1} \circ \Phi_{t_2}, \tag{5.1.25}$$

which is the $\hbar \to 0$ limit of (5.1.14).

5.2 Examples

5.2.1 Example 1: Harmonic oscillator

In this example we will consider quantum trajectories of the harmonic oscillator. The Hamiltonian of the harmonic oscillator is given by the equation

$$H(q, p) = \frac{1}{2}\left(p^2 + \omega^2 q^2\right). \tag{5.2.1}$$

It happens that in such case the quantum trajectory coincides with the classical one. Indeed, one can show that

$$Q(t) = e^{-t[\![H,\,\cdot\,]\!]}Q(0) = e^{-t\{H,\,\cdot\,\}}Q(0), \tag{5.2.2a}$$

$$P(t) = e^{-t[\![H,\,\cdot\,]\!]}P(0) = e^{-t\{H,\,\cdot\,\}}P(0) \tag{5.2.2b}$$

and in explicit form classical/quantum trajectory $\Phi_t = (Q(t), P(t))$ of the harmonic oscillator is

$$Q(q, p, t) = q\cos\omega t + \omega^{-1}p\sin\omega t, \tag{5.2.3a}$$

$$P(q, p, t) = p\cos\omega t - \omega q\sin\omega t. \tag{5.2.3b}$$

Observe that the classical action (composition) of Φ_t on the algebra of observables preserves the Moyal product, i.e.,

$$(f \star g) \circ \Phi_t = (f \circ \Phi_t) \star (g \circ \Phi_t), \quad f, g \in C^\infty(\mathbb{R}^{2N})[\![\hbar]\!]. \tag{5.2.4}$$

Thus in accordance with (5.1.6) the unique isomorphism S_t associated with Φ_t is equal $S_t = \mathrm{id}$. This means that the action of the flow Φ_t on observables (5.1.9) as well as the quantum composition rule (5.1.14) for the flow is equal to the classical composition. In other words the time evolution of observables is the same as in classical case. The difference between the classical and quantum system is in the admissible states which evolve along the flow. In classical case states are probabilistic distribution functions, whereas in quantum case states are quasi-probabilistic distribution functions. In particular, classical pure states are Dirac distribution functions, however, quantum pure states will no longer be of such form due to the Heisenberg uncertainty principle.

5.2.2 Example 2

In this example let us consider a two particle system described by the Hamiltonian

$$H(q,p) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + kq^1p_2^2, \tag{5.2.5}$$

where m_1 and m_2 are masses of particles and k is a coupling constant. The solution of quantum Hamilton equations (5.1.1) reads [71]

$$Q^1(t) = q^1 + \frac{1}{m_1}p_1t - \frac{k}{2m_1}p_2^2t^2, \tag{5.2.6a}$$

$$P_1(t) = p_1 - kp_2^2t, \tag{5.2.6b}$$

$$Q^2(t) = q^2 + \left(\frac{1}{m_2}p_2 + 2kq^1p_2\right)t + \frac{k}{m_1}p_1p_2t^2 - \frac{k^2}{3m_1}p_2^3t^3, \tag{5.2.6c}$$

$$P_2(t) = p_2, \tag{5.2.6d}$$

which coincides again with a solution of classical Hamilton equations. However, in accordance with (5.1.3) the received quantum flow Φ_t transforms the Moyal product to the following product

$$f \star_t g = f \exp\left(\frac{1}{2}i\hbar\overleftarrow{D_{q^i}}\overrightarrow{D_{p_i}} - \frac{1}{2}i\hbar\overleftarrow{D_{p_i}}\overrightarrow{D_{q^i}}\right) g, \tag{5.2.7}$$

where

$$D_{q^1} = \partial_{q^1} + 2ktp_2\partial_{q^2}, \tag{5.2.8a}$$

$$D_{p_1} = \partial_{p_1} + \frac{1}{m_1}t\partial_{q^1} + \frac{k}{m_1}t^2p_2\partial_{q^2}, \tag{5.2.8b}$$

$$D_{q^2} = \partial_{q^2}, \tag{5.2.8c}$$

$$D_{p_2} = \partial_{p_2} - 2ktp_2\partial_{p_1} - \frac{k}{m_1}t^2p_2\partial_{q^1} + \left(\frac{1}{m_2}t + 2ktq^1 - \frac{k}{m_1}t^2p_1 - \frac{k^2}{m_1}t^3p_2^2\right)\partial_{q^2}. \tag{5.2.8d}$$

Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product with the $\star_t$-product takes the form

$$S_t = \exp\left(\frac{1}{8}\hbar^2\frac{k}{m_1}t^2\partial_{q^1}\partial_{q^2}^2 + \frac{1}{4}\hbar^2kt\partial_{p_1}\partial_{q^2}^2 + \frac{1}{12}\hbar^2\frac{k^2}{m_1}t^3p_2\partial_{q^2}^3\right). \tag{5.2.9}$$

Indeed, a direct calculations show that the relations (5.1.6) are satisfied. More details of the construction of S_t the reader can find in [42].

As in this case $S_{t_2}\Phi_{t_1} = \Phi_{t_1}$, the group multiplication for $\{\Phi_t\}$ is just a composition of maps, as one could expect since Φ_t is simultaneously the classical and quantum trajectory. However, the action of Φ_t on observables and states does not reduce in general to a composition of maps (5.1.24). This shows that the time evolution of quantum observables differs in general from the time evolution of classical observables.

One can check by direct calculations that the action of the quantum flow Φ_t on an observable A, given by (5.1.9), indeed describes the quantum time evolution of A. As an example let us take $A(q,p) = q_1 q_2^2$. Then

$$(S_t A)(q,p) = q_1 q_2^2 + \frac{1}{4}\hbar^2 \frac{k}{m_1} t^2 \tag{5.2.10}$$

and it can be checked that

$$A(t) = (S_t A) \circ \Phi_t = Q^1(t)(Q^2(t))^2 + \frac{1}{4}\hbar^2 \frac{k}{m_1} t^2 \tag{5.2.11}$$

satisfies the time evolution equation (3.2.25).

5.2.3 Example 3

In this example we will consider a system described by a Hamiltonian

$$H(q,p) = q^2 p^2. \tag{5.2.12}$$

The solution of quantum Hamilton equations (5.1.1) reads [37]

$$Q(q,p,t;\hbar) = \sec^2(\hbar t) q \exp\left(\frac{2}{\hbar}\tan(\hbar t) qp\right), \tag{5.2.13a}$$

$$P(q,p,t;\hbar) = \sec^2(\hbar t) p \exp\left(-\frac{2}{\hbar}\tan(\hbar t) qp\right), \tag{5.2.13b}$$

for $|t| < \frac{\pi}{2\hbar}$. This solution is a deformation of a classical one given by the limit $\hbar \to 0$

$$Q_C(q,p,t) = q e^{2tqp}, \quad P_C(q,p,t) = p e^{-2tqp}. \tag{5.2.14}$$

The induced quantum flow Φ_t is an example of a flow for which Φ_t, for every $t \in (-\frac{\pi}{2\hbar}, \frac{\pi}{2\hbar}) \setminus \{0\}$, is not a classical symplectomorphism, since

$$\{Q(t), P(t)\} = \sec^4(\hbar t) \neq 1. \tag{5.2.15}$$

In accordance with (5.1.3) the quantum flow Φ_t transforms the Moyal product to the following product

$$f \star_t g = f \exp\left(\frac{1}{2} i\hbar \overleftarrow{D_q}\overrightarrow{D_p} - \frac{1}{2} i\hbar \overleftarrow{D_p}\overrightarrow{D_q}\right) g, \tag{5.2.16}$$

where

$$
\begin{aligned}
D_q &= \sec^2(\hbar t)\big(1+2ta(\hbar t)qp\big)\exp\big(2ta(\hbar t)qp\big)\partial_q \\
&\quad - 2t\sec^2(\hbar t)a(\hbar t)p^2\exp\big(2ta(\hbar t)qp\big)\partial_p, \qquad (5.2.17a)\\
D_p &= 2t\sec^2(\hbar t)a(\hbar t)q^2\exp\big(-2ta(\hbar t)qp\big)\partial_q \\
&\quad + \sec^2(\hbar t)\big(1-2ta(\hbar t)qp\big)\exp\big(-2ta(\hbar t)qp\big)\partial_p, \qquad (5.2.17b)
\end{aligned}
$$

and $a(x) = \dfrac{\tan(x)}{x\sec^4(x)}$. Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product with the $\star_t$-product, up to the second order in $\hbar$, takes the form

$$
\begin{aligned}
S_t = \mathrm{id} + \hbar^2\bigg(&\frac{1}{6}(3t^2q^3 + 4t^3q^4p)\partial_q^3 + \frac{1}{6}(3t^2p^3 - 4t^3qp^4)\partial_p^3 \\
&+ \frac{1}{2}(-tp - t^2qp^2 + 4t^3q^2p^3)\partial_q\partial_p^2 + \frac{1}{2}(tq - t^2q^2p - 4t^3q^3p^2)\partial_q^2\partial_p \\
&+ (2t^2q^2 + 2t^3q^3p)\partial_q^2 + (2t^2p^2 - 2t^3qp^3)\partial_p^2 + (-2t^2qp)\partial_q\partial_p\bigg) + o(\hbar^4). \qquad (5.2.18)
\end{aligned}
$$

Indeed, expanding relations (5.1.6) with respect to $\hbar$ one can prove that S_t in the above form satisfies these relations up to $o(\hbar^2)$.

From the fact that Φ_t is a purely quantum trajectory, we deal with the quantum group multiplication (5.1.14) for $\{\Phi_t\}$ as well as the quantum action (5.1.9) of Φ_t on observables and states. Indeed, expanding (5.2.13) with respect to $\hbar$:

$$
Q(q,p,t;\hbar) = Q_C\left(1 + \hbar^2\left(t^2 + \frac{2}{3}t^3qp\right)\right) + o(\hbar^4), \qquad (5.2.19a)
$$

$$
P(q,p,t;\hbar) = P_C\left(1 + \hbar^2\left(t^2 - \frac{2}{3}t^3qp\right)\right) + o(\hbar^4) \qquad (5.2.19b)
$$

and applying isomorphism S_t (5.2.18), it can be calculated that the quantum composition law

$$
Q(t_1+t_2) = S_{t_2}Q(t_1)\circ\Phi_{t_2} = S_{t_1}Q(t_2)\circ\Phi_{t_1}, \qquad (5.2.20a)
$$

$$
P(t_1+t_2) = S_{t_2}P(t_1)\circ\Phi_{t_2} = S_{t_1}P(t_2)\circ\Phi_{t_1} \qquad (5.2.20b)
$$

holds up to $o(\hbar^2)$. Note also, that the flow Φ_t is not defined for all $t \in \mathbb{R}$ but only on an interval $(-\frac{\pi}{2\hbar}, \frac{\pi}{2\hbar})$, contrary to classical flows which are always globally defined. This is an interesting result showing that in general the quantum time evolution do not have to be defined for all instances of time t.

Chapter 6

Summary

In the thesis was developed an invariant quantization procedure of classical Hamiltonian mechanics. The main results include:

- use of deformation approach to quantization for developing an invariant description of quantum mechanics,
- construction of the two-parameter family of star-products on a cotangent bundle to a general Riemannian manifold, which reproduces most of the results received by different approaches to quantization found in the literature,
- construction of the operator representation of quantum mechanics for any coordinates on the configuration space and in a coordinate independent way, this includes generalization of the concept of ordering of operators of position and momentum,
- development of the theory of quantum trajectories on a phase space.

The presented theory is a promising starting point for a further development. Especially interesting would be to create a quantum analog of classical theories of integrable systems such as bi-Hamiltonian systems. The received geometrical approach to quantum mechanics, which includes coordinate transformations and quantum trajectories, gives a potential possibility of developing such theories. Some preliminary results were received in our papers [79, 86].

Another interesting development of the presented formalism would be an incorporation of spin degrees of freedom. Some results in this direction can be found in the literature [87, 88], where the authors use a Grassmann variant of classical mechanics.

The received theory allowed for quantizing systems defined on curved spaces. This can be used to introduce quantization of systems with constrains. Moreover, it is possible to formulate a relativistic version of the theory, which together with the possibility of quantizing systems on curved spaces allows for introducing quantum mechanics coupled with a classical gravitational field [89].

The theory of quantum trajectories can be used to investigate quantum geometry of a phase space. In particular, it might be possible to develop a version of non-commutative geometry in which quantum mechanics could be described. Moreover, the phase space formalism of quantum theory may be adopted to describe a non-commutative quantum mechanics in which a non-commutativity of observables of position is introduced [90].

The presented formalism could also be used as a starting point in developing a theory of quantum fields in the language of deformation quantization [91].

Appendix

A Proof of Theorem 3.4.3

First let us endow $C^\infty(M)[\![\hbar]\!]$ with a topology. The space $C^\infty(M)$ can be considered as a Fréchet space with a standard topology of uniform convergence on compact subsets in all derivatives. The space of formal power series $C^\infty(M)[\![\hbar]\!]$ can be treated as the Cartesian product of countable family of copies of the spaces $C^\infty(M)$, i.e. formal series $\sum_{k=0}^{\infty} \hbar^k f_k$ can be identified with sequences $(f_0, f_1, f_2, \dots)$. We can hence endow the space $C^\infty(M)[\![\hbar]\!]$ with the product topology.

We will prove Theorem 3.4.3 by directly constructing the morphism S. The proof will constitute with a series of lemmas.

Lemma A.1. *Equations* (3.4.25) *are equivalent with the following equations*

$$S(x^\alpha \star_M^{(x)} f) = x^\alpha \star^{(x)} Sf, \tag{A.1a}$$

$$Sx^\alpha = x^\alpha, \tag{A.1b}$$

$$\overline{Sf} = S\bar{f}. \tag{A.1c}$$

Proof. Indeed, if the conditions (3.4.25) are fulfilled then trivially the conditions (A.1) are fulfilled. Assume now, that the conditions (A.1) are fulfilled. From (A.1) it follows that (3.4.25a) will be satisfied for every f in the form of a $\star_M^{(x)}$-polynomial. For example when $f = x^\alpha \star_M^{(x)} x^\beta$ then

$$\begin{aligned} S(f \star_M^{(x)} g) &= S(x^\alpha \star_M^{(x)} x^\beta \star_M^{(x)} g) = x^\alpha \star^{(x)} S(x^\beta \star_M^{(x)} g) = x^\alpha \star^{(x)} x^\beta \star^{(x)} Sg \\ &= Sx^\alpha \star^{(x)} Sx^\beta \star^{(x)} Sg = S(x^\alpha \star_M^{(x)} x^\beta) \star^{(x)} Sg = Sf \star^{(x)} Sg. \end{aligned} \tag{A.2}$$

Every polynomial with coefficients from $\mathbb{C}[\![\hbar]\!]$ can be written as a $\star_M^{(x)}$-polynomial. Since the space of polynomials is a dense subspace of $C^\infty(M)$ the space of polynomials with coefficients from $\mathbb{C}[\![\hbar]\!]$ is dense in $C^\infty(M)[\![\hbar]\!]$. The morphism S as well as $\star_M^{(x)}$ and $\star^{(x)}$-products are continuous as formal series of differential operators, hence (3.4.25a) will be satisfied for every $f \in C^\infty(M)[\![\hbar]\!]$. □

The operator $x^\alpha \star_M^{(x)}$ takes the form

$$x^\alpha \star_M^{(x)} = x^\alpha + \frac{1}{2} i\hbar \partial^\alpha, \tag{A.3}$$

and the operator $x^\alpha \star^{(x)}$ can be written in the form

$$x^\alpha \star^{(x)} = x^\alpha + \frac{1}{2} i\hbar \partial^\alpha + \sum_{k=2}^{\infty} \left(\frac{i\hbar}{2}\right)^k A_k^\alpha, \tag{A.4}$$

where $\partial^\alpha = \mathcal{J}^{\alpha\beta}\partial_{x^\beta}$ and $A_k^\alpha f = C_k(x^\alpha, f)$.

Lemma A.2. *Let* $S = \sum_{k=0}^{\infty} \hbar^k S_k$, *where* $S_0 = \mathrm{id}$. *Then* S *will satisfy* (A.1) *iff*

$$[S_{2k}, x^\alpha] = \sum_{l=1}^{k} \left(-\frac{1}{4}\right)^l A_{2l}^\alpha S_{2(k-l)}, \tag{A.5a}$$

$$[S_{2k}, \partial^\alpha] = \sum_{l=1}^{k} \left(-\frac{1}{4}\right)^l A_{2l+1}^\alpha S_{2(k-l)}, \tag{A.5b}$$

and $S_{2k-1} = 0$ *for* $k = 1, 2, \ldots$.

Proof. Equation (A.1a) takes the form

$$\begin{aligned}\sum_{k=0}^{\infty} \hbar^k S_k x^\alpha + \sum_{k=0}^{\infty} \frac{1}{2} i\hbar^{k+1} S_k \partial^\alpha = \sum_{k=0}^{\infty} \hbar^k x^\alpha S_k + \sum_{k=0}^{\infty} \frac{1}{2} i\hbar^{k+1} \partial^\alpha S_k \\ + \sum_{k=0}^{\infty}\sum_{l=2}^{\infty} \left(\frac{i}{2}\right)^l \hbar^{k+l} A_l^\alpha S_k.\end{aligned} \tag{A.6}$$

Regrouping terms with even and odd k and l in the last term in the above equation we get

$$\begin{aligned}\sum_{k=0}^{\infty} \hbar^k [S_k, x^\alpha] + \frac{1}{2} i \sum_{k=0}^{\infty} \hbar^{k+1} [S_k, \partial^\alpha] = \sum_{n=0}^{\infty}\sum_{l=1}^{\infty} \hbar^{2n+2l} \left(-\frac{1}{4}\right)^l A_{2l}^\alpha S_{2n} \\ + \sum_{n=0}^{\infty}\sum_{l=1}^{\infty} \hbar^{2n+2l+1} \left(-\frac{1}{4}\right)^l A_{2l}^\alpha S_{2n+1} + \frac{1}{2} i \sum_{n=0}^{\infty}\sum_{l=1}^{\infty} \hbar^{2n+2l+1} \left(-\frac{1}{4}\right)^l A_{2l+1}^\alpha S_{2n} \\ + \frac{1}{2} i \sum_{n=0}^{\infty}\sum_{l=1}^{\infty} \hbar^{2n+2l+2} \left(-\frac{1}{4}\right)^l A_{2l+1}^\alpha S_{2n+1}.\end{aligned} \tag{A.7}$$

Regrouping terms with even and odd k in the left hand side of the above formula and replacing the summation over n and l by a summation over $k = n + l$ and l we receive

$$\sum_{k=0}^{\infty}\hbar^{2k}[S_{2k},x^\alpha]+\sum_{k=0}^{\infty}\hbar^{2k+1}[S_{2k+1},x^\alpha]+\frac{1}{2}i\sum_{k=0}^{\infty}\hbar^{2k+1}[S_{2k},\partial^\alpha]+$$
$$+\frac{1}{2}i\sum_{k=0}^{\infty}\hbar^{2k+2}[S_{2k+1},\partial^\alpha]=\sum_{k=1}^{\infty}\hbar^{2k}\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l}S_{2(k-l)}$$
$$+\sum_{k=1}^{\infty}\hbar^{2k+1}\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l}S_{2(k-l)+1}+\frac{1}{2}i\sum_{k=1}^{\infty}\hbar^{2k+1}\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l+1}S_{2(k-l)}$$
$$+\frac{1}{2}i\sum_{k=1}^{\infty}\hbar^{2k+2}\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l+1}S_{2(k-l)+1}. \quad (A.8)$$

Comparing terms with the same order in $\hbar$ and using (A.1c) we get the following recursive equations for S_k

$$[S_{2k},x^\alpha]=\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l}S_{2(k-l)}, \quad (A.9a)$$
$$[S_{2k},\partial^\alpha]=\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l+1}S_{2(k-l)}, \quad (A.9b)$$

and

$$[S_1,x^\alpha]=0, \quad (A.10a)$$
$$[S_1,\partial^\alpha]=0 \quad (A.10b)$$
$$[S_{2k+1},x^\alpha]=\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l}S_{2(k-l)+1}, \quad (A.10c)$$
$$[S_{2k+1},\partial^\alpha]=\sum_{l=1}^{k}\left(-\frac{1}{4}\right)^l A^\alpha_{2l+1}S_{2(k-l)+1}, \quad (A.10d)$$

for $k=1,2,\ldots$. From (A.10a) and (A.10b) we get that $S_1=\text{const}$, and by virtue of (A.1b) this implies that $S_1=0$. Thus, from (A.10c) and (A.10d), we get that $S_{2k+1}=0,\ k=1,2,\ldots$. □

To prove Theorem 3.4.3 we have to prove that (A.5) have a solution. Before doing this let us prove the following lemmas.

Lemma A.3. *A system of equations*

$$[B,x^\alpha]=F^\alpha,\quad \alpha=1,2,\ldots,2N, \quad (A.11a)$$
$$[B,\partial^\alpha]=G^\alpha,\quad \alpha=1,2,\ldots,2N, \quad (A.11b)$$

where $F^\alpha = \sum_{n\geq 0} f^\alpha_{\mu_1\ldots\mu_n}(x)\partial^{\mu_1}\cdots\partial^{\mu_n}$ *and* $G^\alpha = \sum_{n\geq 0} g^\alpha_{\mu_1\ldots\mu_n}(x)\partial^{\mu_1}\cdots\partial^{\mu_n}$ *are some differential operators, have a solution* B *iff*

$$[F^\alpha, x^\beta] = [F^\beta, x^\alpha], \tag{A.12a}$$

$$[F^\alpha, \partial^\beta] = [G^\beta, x^\alpha], \tag{A.12b}$$

$$[G^\alpha, \partial^\beta] = [G^\beta, \partial^\alpha] \tag{A.12c}$$

for all $\alpha, \beta = 1, 2, \ldots, 2N$.

Proof. First let us assume that (A.11) have a solution. From Jacobi's identity we have that

$$[[B, x^\alpha], \partial^\beta] + [[x^\alpha, \partial^\beta], B] + [[\partial^\beta, B], x^\alpha] = 0, \tag{A.13}$$

from which follows that

$$[[B, x^\alpha], \partial^\beta] = [[B, \partial^\beta], x^\alpha]. \tag{A.14}$$

Using (A.11) from this we receive (A.12b). Equations (A.12a) and (A.12c) can be received analogically.

Now, let us assume that (A.12a) is satisfied. From the form of F^α it can be easily seen that (A.11a) for $\alpha = 1$ have a solution. Assume that for some $\gamma \geq 1$ (A.11a) have a solution for all $\alpha \leq \gamma$. This solution is not unique but there exists a family of solutions such that if B and B' are solutions of (A.11a) for all $\alpha \leq \gamma$ then there exists an operator $H^{(\gamma)}$ such that $B' = B + H^{(\gamma)}$ and $[H^{(\gamma)}, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, \gamma$. From (A.12a) and (A.11a) we have that

$$[[B, x^\alpha], x^{\gamma+1}] = [F^{\gamma+1}, x^\alpha], \quad \alpha = 1, 2, \ldots, \gamma. \tag{A.15}$$

Using Jacobi's identity the above equation takes the form

$$[[B, x^{\gamma+1}], x^\alpha] = [F^{\gamma+1}, x^\alpha], \quad \alpha = 1, 2, \ldots, \gamma. \tag{A.16}$$

From this follows that

$$[B, x^{\gamma+1}] = F^{\gamma+1} + H^{(\gamma)}, \quad \alpha = 1, 2, \ldots, \gamma \tag{A.17}$$

for some operator $H^{(\gamma)}$ such that $[H^{(\gamma)}, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, \gamma$. From the freedom of the solution B there exists B for which $H^{(\gamma)} = 0$. Hence for all $\alpha \leq \gamma + 1$ (A.11a) have a solution. Thus we inductively proved that (A.11a) have a solution for all $\alpha = 1, 2, \ldots, 2N$.

Now, let us assume that (A.12) is satisfied. As was shown above (A.11a) have a solution. This solution is not unique but there exists a family of solutions such that if B and B' are solutions of (A.11a) then there exists an operator H such that $B' = B + H$ and $[H, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, 2N$. From (A.12b) and (A.11a) we have that

$$[[B, x^\alpha], \partial^1] = [G^1, x^\alpha], \quad \alpha = 1, 2, \ldots, 2N. \tag{A.18}$$

Using Jacobi's identity the above equation takes the form

$$[[B, \partial^1], x^\alpha] = [G^1, x^\alpha], \quad \alpha = 1, 2, \ldots, 2N. \tag{A.19}$$

From this follows that

$$[B, \partial^1] = G^1 + H^{(1)}, \tag{A.20}$$

for some operator $H^{(1)}$ such that $[H^{(1)}, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, 2N$. From the freedom of the solution B there exists B for which $H^{(1)} = 0$. Hence we have shown that there exists a solution to the system of equations

$$[B, x^\alpha] = F^\alpha, \quad \alpha = 1, 2, \ldots, 2N, \tag{A.21a}$$

$$[B, \partial^1] = G^1. \tag{A.21b}$$

This solution is specified up to an operator H such that $[H, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, 2N$ and $[H, \partial^1] = 0$. Assume now that for $\gamma \geq 1$ there exists a solution B to the system of equations

$$[B, x^\alpha] = F^\alpha, \quad \alpha = 1, 2, \ldots, 2N, \tag{A.22a}$$

$$[B, \partial^\beta] = G^\beta, \quad \beta = 1, 2, \ldots, \gamma, \tag{A.22b}$$

specified up to an operator H such that $[H, x^\alpha] = 0$ ($\alpha = 1, 2, \ldots, 2N$) and $[H, \partial^\beta] = 0$ ($\beta = 1, 2, \ldots, \gamma$). From (A.12b) and (A.11a) we have that

$$[[B, x^\alpha], \partial^{\gamma+1}] = [G^{\gamma+1}, x^\alpha], \quad \alpha = 1, 2, \ldots, 2N. \tag{A.23}$$

Using Jacobi's identity the above equation takes the form

$$[[B, \partial^{\gamma+1}], x^\alpha] = [G^{\gamma+1}, x^\alpha], \quad \alpha = 1, 2, \ldots, 2N. \tag{A.24}$$

From this follows that

$$[B, \partial^{\gamma+1}] = G^{\gamma+1} + H^{(\gamma)}, \tag{A.25}$$

for some operator $H^{(\gamma)}$ such that $[H^{(\gamma)}, x^\alpha] = 0$ for all $\alpha = 1, 2, \ldots, 2N$. Moreover, $H^{(\gamma)}$ satisfies: $[H^{(\gamma)}, \partial^\beta] = 0$ for all $\beta = 1, 2, \ldots, \gamma$. Indeed,

$$[[B, \partial^{\gamma+1}], \partial^\beta] = [G^{\gamma+1}, \partial^\beta] + [H^{(\gamma)}, \partial^\beta], \quad \beta = 1, 2, \ldots, \gamma, \tag{A.26}$$

from which follows, by virtue of Jacobi's identity and (A.12c), that

$$[[B, \partial^\beta], \partial^{\gamma+1}] = [G^\beta, \partial^{\gamma+1}] + [H^{(\gamma)}, \partial^\beta], \quad \beta = 1, 2, \ldots, \gamma. \tag{A.27}$$

Since B satisfies (A.22b) we receive that

$$[H^{(\gamma)}, \partial^\beta] = 0, \quad \beta = 1, 2, \ldots, \gamma. \tag{A.28}$$

From the freedom of the solution B there exists B for which $H^{(\gamma)} = 0$. Hence (A.22) have a solution for $\beta \leq \gamma + 1$. Thus we inductively proved that (A.11) have a solution for all $\alpha = 1, 2, \ldots, 2N$. □

From Lemma A.3 we get:

Lemma A.4. *The system of equations* (A.5) *for* $k = 1, 2, \dots$ *have a solution iff*

$$\sum_{l=0}^{k} [A^{\alpha}_{2l}, A^{\beta}_{2(k-l)}] = 0, \tag{A.29a}$$

$$\sum_{l=0}^{k} [A^{\alpha}_{2l+1}, A^{\beta}_{2(k-l)}] = 0, \tag{A.29b}$$

$$\sum_{l=0}^{k} [A^{\alpha}_{2l+1}, A^{\beta}_{2(k-l)+1}] = 0, \tag{A.29c}$$

for all $\alpha, \beta = 1, 2, \dots, 2N$.

Proof. We will prove the lemma by induction. Directly from Lemma A.3 follows that for $k = 1$ the assumption of the lemma is true. Assume that for $k = 1, 2, \dots, K$ where $K \geq 1$ the assumption of the lemma holds. From Lemma A.3 the system of equations (A.5) for $k = K + 1$ have a solution iff

$$\left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\alpha}_{2l} S_{2(K+1-l)}, A^{\beta}_{0}\right] = \left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\beta}_{2l} S_{2(K+1-l)}, A^{\alpha}_{0}\right], \tag{A.30a}$$

$$\left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\alpha}_{2l} S_{2(K+1-l)}, A^{\beta}_{1}\right] = \left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\beta}_{2l+1} S_{2(K+1-l)}, A^{\alpha}_{0}\right], \tag{A.30b}$$

$$\left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\alpha}_{2l+1} S_{2(K+1-l)}, A^{\beta}_{1}\right] = \left[\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\beta}_{2l+1} S_{2(K+1-l)}, A^{\alpha}_{1}\right]. \tag{A.30c}$$

Equation (A.30a), by virtue of the Leibniz's rule, is equivalent with the following equation

$$\begin{aligned}\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} [A^{\alpha}_{2l}, A^{\beta}_{0}] S_{2(K+1-l)} + \sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\alpha}_{2l} [S_{2(K+1-l)}, A^{\beta}_{0}] = \\ = \sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} [A^{\beta}_{2l}, A^{\alpha}_{0}] S_{2(K+1-l)} + \sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\beta}_{2l} [S_{2(K+1-l)}, A^{\alpha}_{0}].\end{aligned} \tag{A.31}$$

Using (A.5a) we have that

$$\begin{aligned}\sum_{l=1}^{K+1} \left(-\frac{1}{4}\right)^{l} A^{\alpha}_{2l} [S_{2(K+1-l)}, A^{\beta}_{0}] &= \sum_{l=1}^{K} \sum_{r=1}^{K+1-l} \left(-\frac{1}{4}\right)^{l+r} A^{\alpha}_{2l} A^{\beta}_{2r} S_{2(K+1-l-r)} \\ &= \sum_{n=2}^{K+1} \sum_{r=1}^{n-1} \left(-\frac{1}{4}\right)^{n} A^{\alpha}_{2(n-r)} A^{\beta}_{2r} S_{2(K+1-n)}.\end{aligned} \tag{A.32}$$

Using (A.32) and (A.29a) for $k = 1$, (A.31) can be rewritten in the form

$$\sum_{l=2}^{K+1} \left(-\frac{1}{4}\right)^{l} \left([A^{\beta}_{0}, A^{\alpha}_{2l}] + \sum_{r=1}^{l-1} [A^{\beta}_{2r}, A^{\alpha}_{2(l-r)}] + [A^{\beta}_{2l}, A^{\alpha}_{0}] \right) S_{2(K+1-l)} = 0, \tag{A.33}$$

which in turn can be written as

$$\sum_{l=2}^{K+1}\left(-\frac{1}{4}\right)^{l}\sum_{r=0}^{l}[A^{\beta}_{2r},A^{\alpha}_{2(l-r)}]S_{2(K+1-l)}=0. \tag{A.34}$$

Using the inductive assumption the above equation reduces to

$$\sum_{r=0}^{K+1}[A^{\beta}_{2r},A^{\alpha}_{2(K+1-r)}]=0, \tag{A.35}$$

which proves that (A.30a) is equivalent with (A.29a) for $k=K+1$. Analogically we prove that (A.30b) and (A.30c) are equivalent with (A.29b) and (A.29c). This ends the induction. □

Now we are ready to prove Theorem 3.4.3.

Proof of Theorem 3.4.3. We have to show that the system of equations (A.5) have a solution. From Lemma A.4 it is enough to show that (A.29) holds. From (3.1.5) we get

$$\begin{aligned}\sum_{l=0}^{2k}A^{\alpha}_{l}(A^{\beta}_{2k-l}f)&=\sum_{l=0}^{2k}C_l(x^{\alpha},C_{2k-l}(x^{\beta},f))=\sum_{l=0}^{2k}C_l(C_{2k-l}(x^{\alpha},x^{\beta}),f)\\&=\sum_{l=0}^{k}C_{2l}(C_{2(k-l)}(x^{\alpha},x^{\beta}),f)+\sum_{l=0}^{k-1}C_{2l+1}(C_{2(k-l)-1}(x^{\alpha},x^{\beta}),f).\end{aligned} \tag{A.36}$$

The second term in the last equality in (A.36) vanishes because of the classical and quantum canonicity condition (Theorem 3.3.1). Hence, with the use of property (i) on page 22 equation (A.36) reduces to

$$\begin{aligned}\sum_{l=0}^{2k}A^{\alpha}_{l}(A^{\beta}_{2k-l}f)&=\sum_{l=0}^{k}C_{2l}(C_{2(k-l)}(x^{\alpha},x^{\beta}),f)=\sum_{l=0}^{k}C_{2(k-l)}(C_{2l}(x^{\beta},x^{\alpha}),f)\\&=\sum_{l=0}^{2k}A^{\beta}_{2k-l}(A^{\alpha}_{l}f).\end{aligned} \tag{A.37}$$

Thus we get that

$$\sum_{l=0}^{2k}[A^{\alpha}_{l},A^{\beta}_{2k-l}]=0. \tag{A.38}$$

Analogically we get that

$$\sum_{l=0}^{2k+1}[A^{\alpha}_{l},A^{\beta}_{2k-l+1}]=0. \tag{A.39}$$

On the other hand from (3.1.5) we have that

$$\sum_{l=0}^{k}C_l(C_{k-l}(x^{\beta},f),x^{\alpha})=\sum_{l=0}^{k}C_l(x^{\beta},C_{k-l}(f,x^{\alpha})), \tag{A.40}$$

which can be rewritten in the form

$$\sum_{l=0}^{k}(-1)^l A_l^\alpha(A_{k-l}^\beta f) = \sum_{l=0}^{k}(-1)^{k-l} A_l^\beta(A_{k-l}^\alpha f) = \sum_{l=0}^{k}(-1)^l A_{k-l}^\beta(A_l^\alpha f). \tag{A.41}$$

Thus we get that

$$\sum_{l=0}^{2k}(-1)^l[A_l^\alpha, A_{2k-l}^\beta] = 0, \tag{A.42a}$$

$$\sum_{l=0}^{2k+1}(-1)^l[A_l^\alpha, A_{2k-l+1}^\beta] = 0. \tag{A.42b}$$

By adding (A.38) to (A.42a) we receive (A.29a) and by subtracting them we get (A.29c). By adding or subtracting (A.39) to (A.42b) we receive (A.29b). $\square$

B Proof of Theorem 3.4.4

From (3.4.61) and (3.4.64a) we get that

$$\begin{aligned} A_2^\alpha &= -\frac{1}{2}\omega^{\mu_1\nu_1}\omega^{\mu_2\nu_2}\tilde{\Gamma}^\alpha_{\mu_1\mu_2}(\partial_{\nu_1}\partial_{\nu_2} - \tilde{\Gamma}^\beta_{\nu_1\nu_2}\partial_\beta) \\ &= -\frac{1}{2}\tilde{\Gamma}^\alpha_{\mu_1\mu_2}\partial^{\mu_1}\partial^{\mu_2} - \frac{1}{2}\omega^{\mu_1\alpha}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_2}\tilde{\Gamma}^{\mu_2}_{\nu_1\nu_2}\partial^{\nu_2}. \end{aligned} \tag{B.1}$$

On the other hand

$$\begin{aligned} [S_2, x^\alpha] &= -\frac{1}{24}\omega^{\delta\alpha}\tilde{\Gamma}_{\delta\beta\gamma}\partial^\beta\partial^\gamma - \frac{1}{24}\omega^{\beta\alpha}\tilde{\Gamma}_{\delta\beta\gamma}\partial^\delta\partial^\gamma - \frac{1}{24}\omega^{\gamma\alpha}\tilde{\Gamma}_{\delta\beta\gamma}\partial^\delta\partial^\beta \\ &\quad + \frac{1}{16}\omega^{\gamma\alpha}\tilde{\Gamma}^\mu_{\nu\gamma}\tilde{\Gamma}^\nu_{\mu\beta}\partial^\beta + \frac{1}{16}\omega^{\beta\alpha}\tilde{\Gamma}^\mu_{\nu\gamma}\tilde{\Gamma}^\nu_{\mu\beta}\partial^\gamma \\ &= \frac{1}{8}\tilde{\Gamma}^\alpha_{\beta\gamma}\partial^\beta\partial^\gamma + \frac{1}{8}\omega^{\gamma\alpha}\tilde{\Gamma}^\mu_{\nu\gamma}\tilde{\Gamma}^\nu_{\mu\beta}\partial^\beta, \end{aligned} \tag{B.2}$$

which proves (3.4.60a). From (3.4.61) we can calculate that

$$\begin{aligned} A_3^\alpha = \frac{1}{6}\omega^{\mu_1\nu_1}\omega^{\mu_2\nu_2}\omega^{\mu_3\nu_3}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}x^\alpha)_{\mu_1\mu_2\mu_3}\Big(&\partial_{\nu_1}\partial_{\nu_2}\partial_{\nu_3} - \tilde{\Gamma}^\beta_{\nu_1\nu_2}\partial_{\nu_3}\partial_\beta - \tilde{\Gamma}^\beta_{\nu_3\nu_1}\partial_{\nu_2}\partial_\beta \\ &- \tilde{\Gamma}^\beta_{\nu_2\nu_3}\partial_{\nu_1}\partial_\beta + (\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}x^\beta)_{\nu_1\nu_2\nu_3}\partial_\beta\Big). \end{aligned} \tag{B.3}$$

The above equation can be rewritten in a different form. To do this first let us prove that

$$\omega^{\mu_1\nu_1}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}x^\alpha)_{\mu_1\mu_2\mu_3} = \omega^{\alpha\mu_1}\tilde{\Gamma}^{\nu_1}_{\mu_2\mu_3,\mu_1} + \omega^{\alpha\mu_1}\tilde{R}^{\nu_1}_{\mu_2\mu_3\mu_1}, \tag{B.4a}$$

$$\omega^{\mu_2\nu_2}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}x^\alpha)_{\mu_1\mu_2\mu_3} = \omega^{\alpha\mu_2}\tilde{\Gamma}^{\nu_2}_{\mu_1\mu_3,\mu_2} + \omega^{\alpha\mu_2}\tilde{R}^{\nu_2}_{\mu_1\mu_3\mu_2}. \tag{B.4b}$$

Indeed, with the help of (3.4.64) we can calculate that

$$\begin{aligned} \omega^{\mu_1\nu_1}(\tilde{\nabla}\tilde{\nabla}\tilde{\nabla}x^\alpha)_{\mu_1\mu_2\mu_3} &= \omega^{\mu_1\nu_1}(-\tilde{\Gamma}^\alpha_{\mu_2\mu_1,\mu_3} + \tilde{\Gamma}^\beta_{\mu_1\mu_3}\tilde{\Gamma}^\alpha_{\beta\mu_2} + \tilde{\Gamma}^\beta_{\mu_2\mu_3}\tilde{\Gamma}^\alpha_{\beta\mu_1}) \\ &= \omega^{\mu_1\alpha}(-\tilde{\Gamma}^{\nu_1}_{\mu_2\mu_1,\mu_3} + \tilde{\Gamma}^\beta_{\mu_2\mu_3}\tilde{\Gamma}^{\nu_1}_{\beta\mu_1}) + \omega^{\mu_1\beta}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^\alpha_{\beta\mu_2} \\ &= \omega^{\mu_1\alpha}(R^{\nu_1}_{\mu_2\mu_1\mu_3} - \tilde{\Gamma}^{\nu_1}_{\mu_2\mu_1,\mu_3} + \tilde{\Gamma}^\beta_{\mu_2\mu_3}\tilde{\Gamma}^{\nu_1}_{\beta\mu_1}) + \omega^{\mu_1\beta}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^\alpha_{\beta\mu_2}, \end{aligned} \tag{B.5}$$

and that

$$\omega^{\mu_1\beta}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\alpha}_{\beta\mu_2} = \omega^{\mu_1\beta}\delta^{\alpha}_{\gamma}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\gamma}_{\beta\mu_2} = \omega^{\mu_1\beta}\omega^{\alpha\delta}\omega_{\delta\gamma}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\gamma}_{\beta\mu_2} = \omega^{\mu_1\beta}\omega^{\alpha\delta}\omega_{\beta\gamma}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\gamma}_{\delta\mu_2}$$
$$= \delta^{\mu_1}_{\gamma}\omega^{\alpha\delta}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\gamma}_{\delta\mu_2} = -\omega^{\delta\alpha}\tilde{\Gamma}^{\nu_1}_{\mu_1\mu_3}\tilde{\Gamma}^{\mu_1}_{\delta\mu_2}, \tag{B.6}$$

from which follows (B.4a). (B.4b) can be proved analogically. Hence using (3.4.64a), (B.4) and the condition

$$\omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}(\tilde{\nabla}\cdots\tilde{\nabla}x^{\alpha})_{\mu_1\ldots\mu_k}(\tilde{\nabla}\cdots\tilde{\nabla}x^{\beta})_{\nu_1\ldots\nu_k} = 0, \quad k = 3, 5, \ldots \tag{B.7}$$

following from the quantum canonicity condition (3.3.5b) of the coordinate system $(x^1, \ldots, x^{2N})$ we get

$$A^{\alpha}_3 = \frac{1}{6}\omega^{\alpha\mu_1}\left(\tilde{\Gamma}^{\nu_1}_{\mu_2\mu_3,\mu_1} + \tilde{R}^{\nu_1}_{\mu_2\mu_3\mu_1}\right)\partial_{\nu_1}\partial^{\mu_2}\partial^{\mu_3}$$
$$+ \frac{1}{2}\omega^{\alpha\mu_1}\left(\tilde{\Gamma}^{\nu_1}_{\mu_2\mu_3,\mu_1} + \frac{1}{3}\tilde{R}^{\nu_1}_{\mu_2\mu_3\mu_1} + \frac{2}{3}\tilde{R}^{\nu_1}_{\mu_3\mu_2\mu_1}\right)\tilde{\Gamma}^{\mu_2}_{\nu_1\nu_2}\partial^{\mu_3}\partial^{\nu_2}. \tag{B.8}$$

On the other hand

$$[S_2, \partial^{\alpha}] = -\frac{1}{24}\omega^{\alpha\delta}\tilde{\Gamma}^{\lambda}_{\beta\gamma,\delta}\partial_{\lambda}\partial^{\beta}\partial^{\gamma} - \frac{1}{8}\omega^{\alpha\delta}\tilde{\Gamma}^{\nu}_{\mu\beta,\delta}\tilde{\Gamma}^{\mu}_{\nu\lambda}\partial^{\lambda}\partial^{\beta}, \tag{B.9}$$

which shows that S_2 in the form (3.4.62) will satisfy (3.4.60b) since from the flatness assumption $\tilde{R}^{\alpha}_{\beta\gamma\delta} = 0$.

Bibliography

[1] B. Podolsky. *Quantum-mechanically correct form of Hamiltonian function for conservative systems.* Phys. Rev. **32**(5), (1928) pp. 812–816

[2] B. S. DeWitt. *Point transformations in quantum mechanics.* Phys. Rev. **85**(4), (1952) pp. 653–661

[3] B. S. DeWitt. *Dynamical theory in curved spaces. I. A review of the classical and quantum action principles.* Rev. Mod. Phys. **29**(3), (1957) pp. 377–397

[4] J.-L. Gervais and A. Jevicki. *Point canonical transformations in the path integral.* Nucl. Phys. **110B**(1), (1976) pp. 93–112

[5] B. Carter. *Killing tensor quantum numbers and conserved currents in curved space.* Phys. Rev. D **16**(12), (1977) pp. 3395–3414

[6] H. Essén. *Quantization and independent coordinates.* Am. J. Phys. **46**(10), (1978) p. 983

[7] H. Dekker. *On the path integral for diffusion in curved spaces.* Physica (Utrecht) **103A**(3), (1980) pp. 586–596

[8] Z. J. Liu and M. Quian. *Gauge invariant quantization on Riemannian manifolds.* Trans. Amer. Math. Soc. **331**(1), (1992) pp. 321–333

[9] C. Duval and V. Ovsienko. *Conformally equivariant quantum Hamiltonians.* Selecta Math. New Ser. **7**(3), (2001) pp. 291–320

[10] S. E. Loubon Djounga. *Conformally invariant quantization at order three.* Lett. Math. Phys. **64**(3), (2003) pp. 203–212

[11] C. Duval and G. Valent. *Quantum integrability of quadratic Killing tensors.* J. Math. Phys. **46**(5), (2005) p. 053516

[12] M. Born, P. Jordan, and W. Heisenberg. *Zur Quantenmechanik. II.* Z. Phys. **35**(8–9), (1926) pp. 557–615

[13] P. Jordan. *Über kanonische Transformationen in der Quantenmechanik.* Z. Phys. **37**(4–5), (1926) pp. 383–386

[14] P. Jordan. *Über kanonische Transformationen in der Quantenmechanik. II.* Z. Phys. **38**(6–7), (1926) pp. 513–517

[15] F. London. *Über die Jacobischen Transformationen der Quantenmechanik.* Z. Phys. **37**(12), (1926) pp. 915–925

[16] F. London. *Winkelvariable und kanonische Transformationen in der Undulationsmechanik.* Z. Phys. **40**(3–4), (1927) pp. 193–210

[17] P. A. M. Dirac. *The physical interpretation of the quantum dynamics.* Proc. R. Soc. A **113**(765), (1927) pp. 621–641

[18] P. A. M. Dirac. *The Principles of Quantum Mechanics.* Oxford University Press, New York, fourth edition (1958)

[19] P. A. Mello and M. Moshinsky. *Nonlinear canonical transformations and their representations in quantum mechanics.* J. Math. Phys. **16**(10), (1975) pp. 2017–2028

[20] M. Moshinsky and T. H. Seligman. *Canonical transformations to action and angle variables and their representations in quantum mechanics.* Ann. Phys. **114**(1–2), (1978) pp. 243–272

[21] M. Moshinsky and T. H. Seligman. *Canonical transformations to action and angle variables and their representations in quantum mechanics II. The Coulomb problem.* Ann. Phys. **120**(2), (1979) pp. 402–422

[22] G. García-Calderón and M. Moshinsky. *Wigner distribution functions and the representation of canonical transformations in quantum mechanics.* J. Phys. A **13**(6), (1980) pp. L185–L188

[23] R. Dirl, P. Kasperkovitz, and M. Moshinsky. *Wigner distribution functions and the representation of a nonbijective canonical transformation in quantum mechanics.* J. Phys. A **21**(8), (1988) pp. 1835–1846

[24] H. Lee and W. S. l'Yi. *Non-hermitian techniques of canonical transformations in quantum mechanics.* Phys. Rev. A **51**(2), (1995) pp. 982–988

[25] J.-H. Kim and H.-W. Lee. *Canonical transformations and the Hamilton-Jacobi theory in quantum mechanics.* Can. J. Phys. **77**(6), (1999) pp. 411–425

[26] Y. Ohnuki and S. Watanabe. *Characterization of point transformations in quantum mechanics.* J. Anal. Appl. **1**, (2003) p. 193

[27] A. Anderson. *Quantum canonical transformations and integrability. Beyond unitary transformations.* Phys. Lett. B **319**(1–3), (1993) pp. 157–162. `arXiv:hep-th/9302061v2`

[28] A. Anderson. *Canonical transformations in quantum mechanics.* Ann. Phys. **232**(2), (1994) pp. 292–331. `arXiv:hep-th/9305054v1`

[29] L. de Broglie. *Remarques sur la nouvelle mécanique ondulatoire.* C. R. Acad. Sci. **183**, (1926) p. 272

[30] L. de Broglie. *La mécanique ondulatoire et la structure atomique de la matière et du rayonnement.* J. Phys. Radium **8**(5), (1927) p. 225

[31] D. Bohm. *A suggested interpretation of the quantum theory in terms of "hidden" variables. I.* Phys. Rev. **85**(2), (1952) pp. 166–179

[32] D. Bohm. *A suggested interpretation of the quantum theory in terms of "hidden" variables. II.* Phys. Rev. **85**(2), (1952) pp. 180–193

[33] P. R. Holland. *The Quantum Theory of Motion: An Account of the de Broglie-Bohm Causal Interpretation of Quantum Mechanics.* Cambridge University Press, Cambridge (1993)

[34] R. E. Wyatt. *Quantum Dynamics with Trajectories: Introduction to Quantum Hydrodynamics.* Springer, Berlin (2005)

[35] R. G. Littlejohn. *The semiclassical evolution of wave packets.* Phys. Rep. **138**(4–5), (1986) pp. 193–291

[36] M. A. Rieffel. *The classical limit of dynamics for spaces quantized by an action of* $\mathbb{R}^d$. Can. J. Math. **49**(1), (1997) pp. 160–174

[37] N. C. Dias and J. N. Prata. *Features of Moyal trajectories.* J. Math. Phys. **48**(1), (2007) p. 012109. `arXiv:quant-ph/0604167v1`

[38] M. I. Krivoruchenko and A. Faessler. *Weyl's symbols of Heisenberg operators of canonical coordinates and momenta as quantum characteristics.* J. Math. Phys. **48**(5), (2007) p. 052107. `arXiv:quant-ph/0604075`

[39] T. A. Osborn and F. H. Molzahn. *Moyal quantum mechanics: the semiclassical Heisenberg dynamics.* Ann. Phys. **241**(1), (1995) pp. 79–127

[40] M. Błaszak and Z. Domański. *Phase space quantum mechanics.* Ann. Phys. **327**(2), (2012) pp. 167–211. `arXiv:1009.0150[math-ph]`

[41] M. Błaszak and Z. Domański. *Quantum trajectories.* Phys. Lett. A **376**(47–48), (2012) pp. 3593–3598. `arXiv:1208.0720[math-ph]`

[42] M. Błaszak and Z. Domański. *Canonical transformations in quantum mechanics.* Ann. Phys. **331**, (2013) pp. 70–96. `arXiv:1208.2835[math-ph]`

[43] M. Błaszak and Z. Domański. *Canonical quantization of classical mechanics in curvilinear coordinates. Invariant quantization procedure.* Ann. Phys. **339**, (2013) pp. 89–108. `arXiv:1305.4518[math-ph]`

[44] M. Błaszak and Z. Domański. *Natural star-products on symplectic manifolds and related quantum mechanical operators.* Ann. Phys. **344**, (2014) pp. 29–42. `arXiv:1311.3115[math-ph]`

[45] Z. Domański and M. Błaszak. *On equivalence of star-products in arbitrary canonical coordinates* (2013). `arXiv:1305.4026[math-ph]`

[46] P. Libermann and C. M. Marle. *Symplectic Geometry and Analytical Mechanics.* D. Reidel Publishing Company, Dordrecht, Holland (1987)

[47] M. Fecko. *Differential Geometry and Lie Groups for Physicists.* Cambridge University Press, New York (2006)

[48] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer. *Quantum mechanics as a deformation of classical mechanics.* Lett. Math. Phys. **1**(6), (1975–1977) pp. 521–530

[49] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer. *Deformation theory and quantization. I. deformations of symplectic structures.* Ann. Phys. **111**(1), (1978) pp. 61–110

[50] F. Bayen, M. Flato, C. Frønsdal, A. Lichnerowicz, and D. Sternheimer. *Deformation theory and quantization. II. physical applications.* Ann. Phys. **111**(1), (1978) pp. 111–151

[51] M. Gerstenhaber. *On the deformation of rings and algebras.* Ann. Math. **79**(1), (1964) pp. 59–103

[52] S. Gutt. *Deformation quantization of Poisson manifolds.* Geom. Topol. Mon. **17**, (2011) pp. 171–220

[53] M. De Wilde and P. B. A. Lecomte. *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds.* Lett. Math. Phys. **7**(6), (1983) pp. 487–496

[54] B. V. Fedosov. *A simple geometrical construction of deformation quantization.* J. Diff. Geom. **40**(2), (1994) pp. 213–238

[55] H. Omori, Y. Maeda, and A. Yoshioka. *Weyl manifolds and deformation quantization.* Adv. Math. **85**(2), (1991) pp. 224–255

[56] M. Kontsevich. *Deformation quantization of Poisson manifolds.* Lett. Math. Phys. **66**(3), (2003) pp. 157–216. `arXiv:q-alg/9709040v1`

[57] R. Nest and B. Tsygan. *Algebraic index theorem for families.* Adv. Math. **113**, (1995) pp. 151–205

[58] M. Bertelson, M. Cahen, and S. Gutt. *Equivalence of star products.* Class. Quant. Grav. **14**, (1997) pp. A93–A107

[59] P. Deligne. *Déformations de l'algèbre des fonctions d'une variété symplectique: Comparaison entre Fedosov et De Wilde, Lecomte.* Selecta Math. New Ser. **1**(4), (1995) pp. 667–697

[60] M. Rieffel. *Deformation quantization for actions of* $\mathbb{R}^n$. Mem. Am. Math. Soc. **506**

[61] T. Natsume. *C^*-algebraic deformation quantization and the index theorem.* Math. Phys. Stud. **23**, (2001) pp. 142–150

[62] T. Natsume, R. Nest, and P. Ingo. *Strict quantizations of symplectic manifolds.* Lett. Math. Phys. **66**, (2003) pp. 73–89

[63] H. Omori, Y. Maeda, N. Miyazaki, and A. Yoshioka. *Deformation quantization of Fréchet-Poisson algebras — convergence of the Moyal product.* Math. Phys. Stud. **22**, (2000) pp. 233–246

[64] H. Omori, Y. Maeda, N. Miyazaki, and A. Yoshioka. *Orderings and non-formal deformation quantization.* Lett. Math. Phys. **82**(2–3), (2007) pp. 153–175

[65] J. M. Gracia-Bondía and J. C. Várilly. *Algebras of distributions suitable for phase-space quantum mechanics.* J. Math. Phys. **29**(4), (1988) pp. 869–879

[66] M. A. Soloviev. *Algebras with convergent star products and their representations in Hilbert spaces.* J. Math. Phys. **54**, (2013) p. 073517

[67] M. V. Karasev and T. A. Osborn. *Cotangent bundle quantization: entangling of metric and magnetic field.* J. Phys. A **38**, (2005) pp. 8549–8578

[68] J. Hietarinta. *Quantum canonical transformations as integral transformations.* Phys. Rev. D **25**(8), (1982) pp. 2103–2117

[69] T. Dereli, T. Hakioğlu, and A. Teğmen. *Quantum canonical transformations in Weyl-Wigner-Groenewold-Moyal formalism.* Int. J. Mod. Phys. A **24**(24), (2009) pp. 4573–4587. `arXiv:0904.3416v2[math-ph]`

[70] T. Curtright, D. B. Fairlie, and C. Zachos. *Features of time-independent Wigner functions.* Phys. Rev. D **58**(2), (1998) pp. 25002–25015. `arXiv:hep-th/9711183v3`

[71] N. C. Dias and J. N. Prata. *Time dependent transformations in deformation quantization.* J. Math. Phys. **45**(3), (2004) pp. 887–901. `arXiv:hep-th/0311218v1`

[72] H. J. Groenewold. *On the principles of elementary quantum mechanics.* Physica (Utrecht) **12**(7), (1946) pp. 405–460

[73] J. E. Moyal. *Quantum mechanics as a statistical theory.* Proc. Cambridge Philos. Soc. **45**(1), (1949) pp. 99–124

[74] F. A. Berezin. *Quantization.* Math. USSR Izv. **8**(5), (1975) pp. 1109–1165

[75] K.-P. Mok. *Metrics and connections on the cotangent bundle.* Kodai Math. Sem. Rep. **28**(2–3), (1977) pp. 226–238

[76] J. F. Plebański, M. Przanowski, and F. J. Turrubiates. *Induced symplectic connection on the phase space.* Acta Phys. Pol. B **32**(1), (2001) pp. 3–16

[77] S. Benenti, C. Chanu, and G. Rastelli. *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. I. The completeness and Robertson conditions.* J. Math. Phys. **43**(11), (2002) pp. 5183–5222

[78] S. Benenti, C. Chanu, and G. Rastelli. *Remarks on the connection between the additive separation of the Hamilton-Jacobi equation and the multiplicative separation of the Schrödinger equation. II. First integrals and symmetry operators.* J. Math. Phys. **43**(11), (2002) pp. 5223–5253

[79] M. Błaszak, Z. Domański, A. Sergyeyev, and B. Szablikowski. *Integrable quantum Stäckel systems.* Phys. Lett. A **377**(38), (2013) pp. 2564–2572. `arXiv:1305.5686[nlin.SI]`

[80] H. Weyl. *Quantenmechanik und Gruppentheorie.* Z. Phys. **46**(1–2), (1927) pp. 1–46

[81] W. Mlak. *Wstep do Teorii Przestrzeni Hilberta.* Państwowe Wydawnictwo Naukowe, Warszawa (1970)

[82] L. Cohen. *Generalized phase-space distribution functions.* J. Math. Phys. **7**(5), (1966) pp. 781–786

[83] E. P. Wigner. *On the quantum correction for thermodynamic equilibrium.* Phys. Rev. **40**(5), (1932) pp. 749–759

[84] G. S. Agarwal and E. Wolf. *Calculus for functions of noncommuting operators and general phase-space method in quantum mechanics I.* Phys. Rev. D **2**(10), (1970) pp. 2161–2186

[85] M. Grabowski and R. S. Ingarden. *Mechanika Kwantowa. Ujecie w Przestrzeni Hilberta.* Państwowe Wydawnictwo Naukowe, Warszawa (1989)

[86] M. Błaszak, Z. Domański, and B. Silindir. *Flat minimal quantizations of Stäckel systems and quantum separability.* Ann. Phys. **351**, (2014) pp. 152–165. `arXiv:1401.0434[nlin.SI]`

[87] A. C. Hirshfeld and P. D. Henselder. *Deformation quantization for systems with fermions.* Ann. Phys. **302**(1), (2002) pp. 59–77

[88] I. Galaviz, H. Garcia-Compean, M. Przanowski, and F. J. Turrubiates. *Weyl-Wigner-Moyal formalism for Fermi classical systems.* Ann. Phys. **323**(2), (2008) pp. 267–290. `arXiv:hep-th/0612245v1`

[89] F. Antonsen. *Quantum theory in curved spacetime using the Wigner function.* Phys. Rev. D **56**(2), (1997) pp. 920–935

[90] N. C. Dias, M. de Gosson, F. Luef, and J. N. Prata. *A deformation quantization theory for noncommutative quantum mechanics.* J. Math. Phys. **51**(7), (2010) pp. 072101–072112

[91] T. Curtright and C. Zachos. *Phase-space quantization of field theory.* Prog. Theor. Phys. **135**, (1999) pp. 244–258. `arXiv:hep-th/9903254`

Publications of the author

[1] M. Błaszak and **Z. Domański**. *Phase space quantum mechanics.* Ann. Phys. **327**(2), (2012) pp. 167–211. `arXiv:1009.0150[math-ph]`

[2] M. Błaszak and **Z. Domański**. *Quantum trajectories.* Phys. Lett. A **376**(47–48), (2012) pp. 3593–3598. `arXiv:1208.0720[math-ph]`

[3] M. Błaszak and **Z. Domański**. *Canonical transformations in quantum mechanics.* Ann. Phys. **331**, (2013) pp. 70–96. `arXiv:1208.2835[math-ph]`

[4] M. Błaszak, **Z. Domański**, A. Sergyeyev, and B. Szablikowski. *Integrable quantum Stäckel systems.* Phys. Lett. A **377**(38), (2013) pp. 2564–2572. `arXiv:1305.5686[nlin.SI]`

[5] M. Błaszak and **Z. Domański**. *Canonical quantization of classical mechanics in curvilinear coordinates. Invariant quantization procedure.* Ann. Phys. **339**, (2013) pp. 89–108. `arXiv:1305.4518[math-ph]`

[6] M. Błaszak and **Z. Domański**. *Natural star-products on symplectic manifolds and related quantum mechanical operators.* Ann. Phys. **344**, (2014) pp. 29–42. `arXiv:1311.3115[math-ph]`

[7] M. Błaszak, **Z. Domański**, and B. Silindir. *Flat minimal quantizations of Stäckel systems and quantum separability.* Ann. Phys. **351**, (2014) pp. 152–165. `arXiv:1401.0434[nlin.SI]`

[8] **Z. Domański** and M. Błaszak. *On equivalence of star-products in arbitrary canonical coordinates* (2013). `arXiv:1305.4026[math-ph]`

Oświadczenie

Ja, niżej podpisany

Ziemowit Domański,
doktorant w Zakładzie Fizyki Matematycznej Wydziału Fizyki
Uniwersytetu im. Adama Mickiewicza w Poznaniu

oświadczam, że przedkładaną rozprawę doktorską pt:

Admissible invariant canonical quantizations of classical mechanics

napisałem samodzielnie. Oznacza to, że przy pisaniu rozprawy, poza niezbędnymi konsultacjami, nie korzystałem z pomocy innych osób, a w szczególności nie zlecałem opracowania rozprawy lub jej części innym osobom, ani nie odpisywałem tej rozprawy lub jej części od innych osób.

Oświadczam również, że egzemplarz rozprawy doktorskiej w formie wydruku komputerowego jest zgodny z egzemplarzem rozprawy doktorskiej w formie elektronicznej.

Jednocześnie przyjmuję do wiadomości, że gdyby powyższe oświadczenie okazało się nieprawdziwe, decyzja o nadaniu mi stopnia naukowego doktora zostanie cofnięta.

....
Poznań, dnia 15 grudnia 2014