Properties of random coverings of graphs

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“*When you are a Bear of Very Little Brain, and you Think of Things, you find sometimes that a Thing which seemed very Thingish inside you is quite different when it gets out into the open and has other people looking at it.*

Kiedy się jest Misiem o Bardzo Małym Rozumku i myśli się o Rozmaitych Rzeczach, to okazuje się czasami, że Rzecz, która zdawała się bardzo Prosta, gdy miało się ją w głowie, staje się całkiem inna, gdy wychodzi z głowy na świat i inni na nią patrzą.”

A.A. Milne, *The House at Pooh Corner (Chatka Puchatka)*, 1928

Dla wszystkich, którzy przyczynili się do jej powstania
i tych, zainteresowanych na tyle, by ją przeczytać.
For those who helped make it possible
and those who are reading it.

“ I can no other answer make but thanks, and thanks. ”

William Shakespeare, *Twelfth Night*, 1602

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Abstract

In the thesis we study selected properties of random coverings of graphs introduced by Amit and Linial in 2002. A random n-covering of a graph G, denoted by $\tilde{G}$, is obtained by replacing each vertex v of G by an n-element set $\tilde{G}_v$ and then choosing, independently for every edge $e = \{x, y\} \in E(G)$, uniformly at random a perfect matching between $\tilde{G}_x$ and $\tilde{G}_y$.

The first problem we consider is the typical size of the largest topological clique in a random covering of given graph G. We show that asymptotically almost surely a random n-covering $\tilde{G}$ of a graph G contains the largest topological clique which is allowed by the structure of G.

The second property we examine is the existence of a Hamilton cycle in $\tilde{G}$. We show that if G has minimum degree at least 5 and contains two edge disjoint Hamilton cycles whose union is not a bipartite graph, then asymptotically almost surely $\tilde{G}$ is Hamiltonian.

Streszczenie

Przedmiotem rozprawy doktorskiej są asymptotyczne własności losowych nakryć grafów zdefiniowanych przez Amita i Liniala w 2002 roku, jako nowy model grafu losowego. Dla zadanego grafu bazowego G losowe nakrycie stopnia n, oznaczane jako $\tilde{G}$, otrzymujemy poprzez zastąpienie każdego wierzchołka v przez n-elementowy zbiór $\tilde{G}_v$ oraz wybór, dla każdej krawędzi $\{x, y\} \in E(G)$, z równym prawdopodobieństwem, losowego skojarzenia pomiędzy zbiorami $\tilde{G}_x$ i $\tilde{G}_y$.

Pierwszym zagadnieniem poruszanym w pracy jest oszacowanie wielkości największej topologicznej kliki zawartej (jako podgraf) w losowym nakryciu danego grafu G. Udało się pokazać, że asymptotycznie prawie na pewno losowe nakrycie $\tilde{G}$ grafu G zawiera największą dopuszczalną przez strukturę grafu bazowego topologiczną klikę.

Drugim badanym zagadnieniem jest pytanie o istnienie w podniesieniu grafu cyklu Hamiltona. W pracy pokazujemy, że jeżeli graf G ma minimalny stopień co najmniej 5 i zawiera dwa krawędziowo rozłączne cykle Hamiltona, których suma nie jest grafem dwudzielnym, to asymptotycznie prawie na pewno $\tilde{G}$ jest grafem hamiltonowskim.

Contents

What is there that confers the noblest delight? What is that which swells a man's breast with pride above that which any other experience can bring to him? Discovery! (...) To give birth to an idea, to discovery a great thought-an intellectual nugget, right under the dust of a field that many a brain-plough had gone over before.

Mark Twain, *The Innocents Abroad*, 1869

1 Introduction

The main object of this thesis is to study selected properties of random coverings of graphs. This model has been introduced by Amit and Linial in order to transfer the topological notion of covering maps to the case of graphs. Then they defined a probabilistic structure on the set of all graphs that cover a fixed base graph.

Let us recall first one of the simplest and most frequently used model of random graphs: the binomial random graph $G(n, p)$. In this model, a graph is generated by drawing n vertices and adding edges between them with probability p, independently for each pair of vertices. $G(n, p)$ has been proved useful in many constructions of graphs with certain unusual properties, such as graphs with large chromatic number and large girth, graphs with some special extremal properties as well as in modelling various processes in statistical physics [19].

Nevertheless, the binomial model has some serious limitations. For instance, it poorly reflects the properties of so called Internet graphs. Moreover, since, roughly speaking, we cannot force $G(n, p)$ to have some local properties of certain types, there are some problems when it is applied to constructing error correcting codes, random maps, or to provide tight estimates for Ramsey numbers.

Random coverings of graphs meet some of these needs. The model of random coverings we are interested in was introduced by Amit and Linial [2]. The concept comes from the topological notion of covering maps. A graph is a topological object (e.g. it can be viewed as a one dimensional simplicial complex), so covering maps can be defined and studied also for graphs. Later however to distinguish this model from the other existing concepts of coverings in graph theory, as edge coverings or vertex coverings, it was proposed [3] to use the name "lift" instead of "covering". From this point on we will be mainly using the second name.

For graphs G and H, a map $\pi : V(H) \to V(G)$ is a *covering map* from H to G if for every $v \in V(H)$ the restriction of π to the neighbourhood of v is a bijection onto the neighbourhood of $\pi(v) \in V(G)$. If such a mapping exists, we say that H is a lift of G and G is the base graph for H. It is easy to see that for connected graphs the number of vertices which are mapped to one vertex of the base graph is the same for all vertices $v \in G$. We denote this common value by n and call it the degree of covering. The set of all graphs that are n-lifts of G is denoted $L_n(G)$. The random n-lift of a graph G is obtained by choosing uniformly at random one graph from the set $L_n(G)$. More formal definition of the model can be found in Chapter 2.

Our interest lies in the asymptotic properties of lifts of graphs, when the parameter n goes to infinity. In particular, we say that a property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity. Sometimes, instead of saying that the random lift of G has aas a property $\mathcal{A}$, we write that almost every random lift of a graph G has $\mathcal{A}$, or, briefly, just that the random lift of G has property $\mathcal{A}$.

Random lifts of graphs are interesting mathematical objects by their own and there are several papers which study how typical properties of random lifts reflect properties of the their base graphs [2, 3, 4, 26, 28]. Nonetheless, the main motivation to introduce this model has been its applications, so let us mention some of them. The first one is to solve problems in extremal graph theory and construct graphs with good expanding properties [1, 12, 27, 29]. Amit and Linial also suggested that random lifts can be found useful in some algorithmic problems, in particular, they were able to reformulate the Unique Game Conjecture in terms of random lifts [25]. Recently the idea of random coverings has been pushed further to study a random higher-dimensional complexes [5]. One can notice that every covering map is also a homomorphism of graphs, but the converse is not true. Thus, one can consider coverings as the special class of homomorphisms of graphs, and study whether conjectures concerning homomorphism of graphs holds for the subclass of coverings.

For the applications the main challenge is to turn lifts and random lifts into tools in the study of important questions in computational complexity and discrete mathematics. The most spectacular result obtained with random lifts of graphs concern spectral properties of graphs. Lifts can be used to construct regular graphs with large spectral gap. Currently we know how to construct Ramanujan graphs (i.e. d-regular graphs with second eigenvalue $\lambda_2 \leq 2\sqrt{d-1}$) only for $d = p^\alpha + 1$, with p being a prime number [31]. Bilu and Linial [6] presented a new explicit construction for expander graphs with nearly optimal spectral gap, namely having second eigenvalue of order $O(\sqrt{d \log^3 d})$. The construction is based on a series of 2-lift operations. Recently Marcus, Spielman and Srivastava [30] extended this result showing that there exist infinite families of regular bipartite Ramanujan graphs of every degree bigger than 2.

As we have already mentioned there are only a handful of papers concerning asymptotic properties of random lifts. In the paper in which they introduced the model Amit and Linial [2] proved that random lifts are highly connected. In the second paper on random lifts the authors proved that random lifts have good expanding properties [3]. The infinite d-regular tree is an ideal expander. The main challenge is to find a finite graph with similar combinatorial and spectral properties. One idea is to look at the minors of a graph. An infinite tree has no non-trivial minors. The question is which of the minors M of a graph G are persistent, meaning they are minors of almost every lift of G. Drier and Linial [13] studied existence of minors in random lifts of complete graphs, proving existence of topological cliques of certain sizes in lifts of small degree. We continue the study of existence of topological cliques in random lifts of graphs [35], showing that almost every random lift of a given graph contains a topological clique as large as it is permitted by the structure of G (see Theorem 18 below).

In Chapter 3 of this thesis we discuss basic properties of random lifts focusing especially on their connectivity properties. In the next part of the thesis, we prove the existence of large topological cliques in random lifts. Using basically the same argument, we will argue that asymptotically almost surely a random lift of a graph G with minimum degree $\delta \geq 2k - 1$ is k-linked. This result is a substantial strengthening of the theorem by Amit and Linial [2] from their first paper on random coverings.

We say that a theorem is a *zero-one law* if it specifies that an event of a certain type either happens asymptotically almost surely or asymptotically almost surely does not happen. This will mean to us that the probability of an occurrence of such event tends either to zero, or to one, as $n \to \infty$. Some part of research in the area of random lifts is connected with such theorems. Linial and Rozenman [28] showed that for any graph G its random lifts either almost surely has a perfect matching or almost surely does not have such a matching. Similar question has been raised regarding existence of Hamiltonian cycles [26]:

Problem 1. Is it true that asymptotically almost surely for every G almost every or almost none of the graphs in $L_n(G)$ have a Hamilton cycle?

Problem 2. Let G be a d-regular graph with $d \geq 3$. Is it true that random n-lift of G is asymptotically almost surely Hamiltonian?

In fact, the question about existence of Hamiltonian cycle is one of the most studied in the topic of random lifts. Chebolu and Frieze [9] proved that random lifts of appropriately large complete directed graphs asymptotically almost surely contains a Hamiltonian cycle. Burgin, Chebolu, Cooper and Frieze [8] proved that there exists a constant c such that almost every lift of complete graphs on more than c vertices contains a Hamiltonian cycle. Together with Łuczak and Ł.Witkowski we were able to show that almost every random lift of a graph G with minimum degree at least 5 and two edge disjoint Hamiltonian cycles whose union is not a bipartite graph is Hamiltonian [33]. The proof of this fact can be found in Chapter 5.

Let us also mention similar concentration questions raised for the chromatic number of lifts of graphs. We do not know whether for every graph G the chromatic number of almost every lift of G tends to concentrate in one value [4]. The simplest case for which this question remains open is the complete graph on five vertices K_5. It is easy to show that chromatic number of random lift of K_5 is a.a.s. either 3 or 4, but we do not know whether both these values are obtained with probability bounded away from zero, or the chromatic number of a random lift of K_5 almost surely takes only one of them. Farzad and Theis tried to solve this problem but they were able to prove only that random lifts of K_5 minus one edge are almost surely 3-colourable [15].

In the Chapter 2 we will recall basic definitions and notions that we use in this thesis. Here we also define the model of random coverings of graph we shall be dealt with. In Chapter 3 we presents known properties of random lifts in a more thorough way. Moreover some useful facts concerning asymptotic properties of lifts, which are used in next chapters, are proven in this part of the thesis.

> *I do not carry such information in my mind since it is readily available in books... The value of a college education is not the learning of many facts but the training of the mind to think.*

Albert Einstein, *In response to question about the speed of sound*, *NYT* 1921

2 Preliminaries

We start with basic definitions of terms and notions that are used throughout the thesis. A *simple graph* or shortly a *graph*, is a pair of sets $G = (V, E)$, where $E \subset V^{(2)} = \{\{x, y\} \subset V : x \neq y\}$. The set V, also denoted as $V(G)$, is called the set of *vertices* of G. The set E (sometimes denoted $E(G)$) is called the set of *edges* of G. The number of vertices $|G| = |V(G)|$ is called the *order* of G and $e(G) = |E(G)|$ is the *size* of G. If H is a graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$, then we say that H is a *subgraph* of G.

The most common questions that is asked about graphs is about their connectivity properties. The set of *neighbours* of a vertex v is denoted

$$N(v) = \{w \in V(G) : \{v, w\} \in E(G)\}.$$

For $\{v, w\} \in E(G)$ we say that a vertex $w \in N(v)$ is *adjacent* to v and an edge $\{v, w\}$ is *incidence* to v and w. The number of neighbours of a given vertex $d(v) = |N(v)|$ is called the *degree* of the vertex. The *minimum degree* over all vertices of G is denoted

$$\delta(G) = \min_{v \in V(G)} d(v),$$

while for the *maximum degree* over all vertices of G we write

$$\Delta(G) = \max_{v \in V(G)} d(v).$$

A *walk* is an alternating sequence of vertices and distinct edges, beginning and ending at vertices, where each vertex is incident to the edges that precede and follow it in the sequence.

If all the vertices in a walk are different we call it a *path*. The length of a path is the number of edges which belong to it. The path together with the edge joining its ends forms a *cycle*. A cycle containing all the vertices of a graph is called *Hamiltonian* or a *Hamilton cycle*. A graph which contains a Hamiltonian cycle as a subgraph is called *Hamiltonian*. Finding a Hamilton cycle is one of the most important problems in graph theory and has many applications in clustering of data arrays, route assignments, analysis of the structure of crystals and others [24].

For vertices u and v the *distance* $dist(u, v)$ is the length of the shortest path connecting u to v. The set of all vertices at distance at most d to vertex set S is called a *closed d-neighbourhood* and denoted

$$\hat{N}_d(S) = \{u \in V(G) : \min_{v \in S} dist(u, v) \leq d\}.$$

The set $N_d(S) = \hat{N}_d(S) \backslash S$ will be called an *open d-neighbourhood* or shortly a *d-neighbourhood* of S.

A graph is *connected* if for every pair of vertices $u, v \in V(G)$ there is a path in G from u to v (called uv path). A graph is *k-connected* if for every pair of vertices $u, v \in V(G)$ there are k vertex-disjoint paths in G from u to v. Equivalently, by Menger's theorem [11], graph is k-connected if and only if it stays connected after removing any set of $k - 1$ vertices.

A graph H is called a *minor* of a graph G if it can be obtained from G by a series of edge contradiction and deletions, and possibly omitting some vertices and edges. A graph that is obtained by replacing the edges of H with vertex disjoint paths is called a *subdivision* of H. If X is isomorphic to a subgraph of G, and X is a subdivision of a graph H, we say that H is a *topological minor* of G. Clearly, each topological minor is a minor as well, but it is easy to see that converse is not true.

We distinguish several special classes of graphs. By K_n we denote a graph in which each pair of vertices is an edge (i.e. $E(K_n) = V^{(2)}$), and called it the *complete graph*, or *clique* of order n. A graph whose vertices can be divided into two disjoint sets U and V such that every edge connects a vertex in U to a vertex in V is called a *bipartite graph*. If all vertices in G have the same degree equal d, then G is called *d-regular*. A set of disjoint edges of a graph is called a *matching*; a matching covering all vertices from $V(G)$ is called a *perfect matching*. A connected graph with no cycles is called *a tree*. Vertices of degree one in a tree are called *leaves*.

2.1 Coverings

The notion of covering maps between graphs is a restriction of more general topological notion of covering maps to the case of graphs (notice that graphs can be viewed as one dimensional simplicial complexes). A *covering map* of topological spaces $f : X \to Y$ is an open surjective map that is locally homeomorphism. We will define a covering map of graphs in terms of homomorphism of graphs.

Definition. Let G and H be graphs. A *homomorphism* of G to H is a function $f : V(G) \to V(H)$ such that

$$\{x, y\} \in E(G) \Rightarrow \{f(x), f(y)\} \in E(H).$$

By $H \to G$ we denote the existence of a homomorphism of H onto G. Notice that the smallest k for which there is a homomorphism of G onto K_k is the chromatic number of G. A covering map between graphs is a "locally bijective" homomorphism.

Definition. For graphs G and H a homomorphism $\Gamma : V(H) \to V(G)$ is a *covering map* if for every $x \in V(H)$, the neighbourhood $N(x)$ can be mapped 1-to-1 onto $N(\Gamma(x))$.

We denote the covering of a graph G as $\tilde{G}$, and call the graph G the *base graph* of the covering while $\tilde{G}$ is called a *lift* of G. For each vertex $v \in G$ the inverse image $\Gamma^{-1}(v)$ is called the *fiber above* v and denoted $\tilde{G}_v$. For simplicity we sometimes say that u lies above v when $\Gamma(u) = v$.

The best way to visualize a covering is to put vertices of fibers as vertical stacks above the vertices of the base graph G as in Figure 2.1. It is easy to see that the condition of covering map being locally homomorphic forces all fibers to have the same size, provided G is connected. This common cardinality is called the *degree* of covering. If degree of a covering Γ equals to n we call it an n-covering.

We will mostly use the term *lift* rather than covering, to distinguished it from other concept of coverings in graph theory e.g. vertex covering, edge covering. That is why $\tilde{G}$ will often be called an n-lift of G, or simple a lift of G.

2.2 Random coverings

Let $\mathcal{G}$ be a family of graphs. A graph chosen from G according to some random experiment is called a *random graph*. A *random n-covering* of a graph G will be obtained by choosing a graph $\tilde{G}$ at random from the set $L_n(G)$ of all n-lifts of G. Notice that an edge $\{u, v\} \in E(G)$ results in a random matching between vertices from fibers $\tilde{G}_u$ and $\tilde{G}_v$. It is easy to see that,

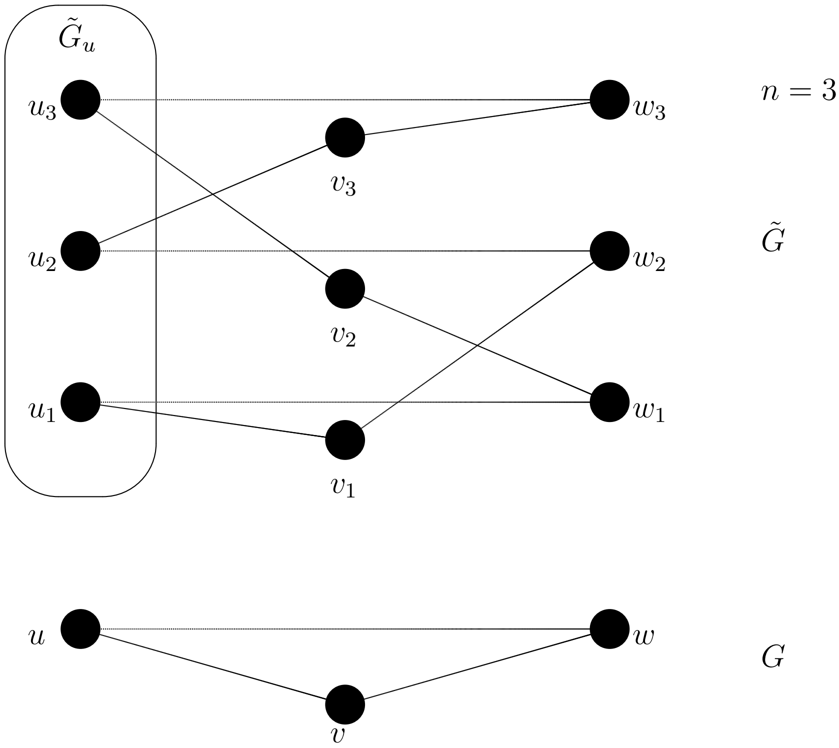

Figure 2.1: Example of a 3-covering (3-lift) $\tilde{G}$ of the graph $G = K_3$. Covering assigns u_i's to u, v_i's to v and w_i's to w. Vertices $\{u_1, u_2, u_3\}$ creates a fiber $\tilde{G}_u$ above vertex u.

equivalently, a random covering of a graph G can be generated by choosing independently and uniformly at random, for every edge $\{u, v\} \in E(G)$, a perfect matching between $\tilde{G}_u$ and $\tilde{G}_v$.

Nevertheless most of the time we would use yet another approach to choose a random lift. Let G be a base graph and $\tilde{G}$ be its lift. For every edge $\{u, v\} \in E(G)$ we choose its orientation and label all vertices in every fiber in $\tilde{G}$ from 1 to n. Then the matching between two fibers is determined by a single permutation on n elements. Whenever $u_i \in \tilde{G}_u$ is connected with $v_j \in \tilde{G}_v$, we put j on i-th position of the permutation. For example in the Figure 2.1 the permutation for edge $\{u, v\}$ equals (132). Changing the permutation results in obtaining different matching and consequently different covering. In forthcoming chapters we will call vertices of a fiber labelled $1, ..., k$ as *k lexicographically first vertices of the fiber*.

Notice that a chosen orientation of the edges has no real effect on possible outcome, since reversing the edges and inverting the permutation yield the same covering. Nevertheless if we want to precisely describe a covering we need to orient each edge e in order to know how to attach single permutation to it. It is also easy to see that indeed all coverings of G can be obtained in this manner.

Thus, formally we can define a *random n-covering* in a following way: choose a permutation $\sigma_e \in S_n$ uniformly and independently for every (oriented) edge $e = \{u, v\}$ in G and then connect u_i to $v_{\sigma_e(i)}$. One can also think about choosing the permutations non-uniformly or not independently, but none of those variations is a subject of this thesis. The following definition by Linial and Amit gives a formal description of the model.

Definition. Given a graph G, a *random labelled n-covering of G* is obtained by arbitrarily orienting the edges of G, choosing permutations σ_e in S_n for each edge e uniformly and independently, and constructing the graph $\tilde{G}$ with n vertices $u_1, ..., u_n$ for each vertex u of G and edges $e_i = \{u_i, v_{\sigma_e(i)}\}$ whenever $e = \{u, v\}$ is an oriented edge. A covering $\Gamma : \tilde{G} \to G$ is then defined by $\Gamma(u_i) = u$.

Note that analogously to the case of the binomial random graph model, the standard model is defined for labelled graphs, where vertices of each fiber are equipped with a labelling $\{1, ..., n\}$. However, it was shown in [2] that asymptotic properties of coverings are the same in the labelled and unlabelled model.

2.3 Probability

In this work we shall deal only with finite probability spaces. Typically our probability space would be the set of all random n-lift of a given graph G. Each graph has the same probability to be drawn. Thus properties of graphs become events in this probability space and usually we will consider random variables which count the number of specific structures in such random graph.

Our interest lies in the asymptotic properties of random lifts, that is when $n \to \infty$. In particular, we say that a graph property holds *asymptotically almost surely*, or, briefly, *aas*, if its probability tends to 1 as n tends to infinity. In other words a graph H drawn at random from $L_n(G)$ has this property with probability $1 - \epsilon_n$, where $\epsilon_n \to 0$ as $n \to \infty$.

Throughout the paper we will use standard probabilistic inequalities to estimate the probabilities of events. The first one, the union bound, says that for any set of events $X_1, ..., X_n$, we have

$$\mathbf{Pr} \bigcup_{i=1}^{n} X_i \leq \sum_{i=1}^{n} \mathbf{Pr}[X_i].$$

Markov's inequality states that for any random variable $X \geq 0$,

$$\mathbf{Pr}[X \geq \lambda] \leq \frac{\mathbf{E}[X]}{\lambda}.$$

Note that if X is a random variable with non-negative integer values, then Markov's inequality with $\lambda = 1$ implies that

$$\mathbf{Pr}[X > 0] \leq \mathbf{E}[X],$$

In particular if $\mathbf{E}[X] \to 0$, then $\mathbf{Pr}[X = 0] \to 1$.

The last inequality is particularly useful if X counts the occurrence of some structure we want to avoid. In the setup of lifts we look at the behaviour of expected value of X as degree n tends to infinity, arguing that almost every random lift does not have the desired structure or property.

Another frequently used tool in the theory of random structures is Chebyshev's inequality. It says that for any random variable X with finite expected value $\mathbf{E}[X]$, a finite non-zero variance $\mathbf{Var}[X]$ and for any $t > 0$ we have

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq t] \leq \frac{\mathbf{Var}[X]}{t^2}.$$

A common feature in many probabilistic arguments is the need to show that a random variable with large probability is not too far from its mean. A better estimate for the tails of X than the one given by Chebyshev's bound is the result of Chernoff's. Chernoff's inequality states that if $X \in B(r, p)$ (i.e. if X has the binomial distribution with parameters r and p), then for every ϵ, $0 < \epsilon \leq 3/2$,

$$\mathbf{Pr}[|X - \mathbf{E}[X]| \geq \epsilon\, \mathbf{E}[X]] \leq 2 \exp\left(-\frac{\epsilon^2}{3}\, \mathbf{E}[X]\right).$$

In the thesis we also use some results from the theory of branching processes. Let X be an integer-valued non-negative random variable with probability mass function for each $k = 0, 1, ...$ given by $p_k = \mathbf{Pr}[X = k]$. We say that a sequence of random variables Y_n, $n = 0, 1, 2, ...$, is a *branching process* if

1. $Y_0 = 1$

2. $Y_{n+1} = X_1^{(n)} + X_2^{(n)} + ... + X_{Y_n}^{(n)}$,

Where $X_j^{(n)}$ is the number of descendants produces by the j^{th} ancestor of the n^{th} generation and the $X_j^{(n)}$ are i.i.d. random variables with the same distribution as X. We say that distribution of X is the generating distribution of the branching process.

This definition describes one of the simplest models for population growth. The process starts at time 0 with one ancestor: $Y_0 = 1$. At time $n = 1$ this ancestor dies producing a random number of descendants $Y_1 = X_1^{(0)}$. This process continues while $Y_n > 0$. If $Y_n = 0$, for some n, the branching process stops and we say that it dies out. Thus, Y_{n+1} is the number of descendants in the $(n+1)^{th}$ generation produced by Y_n individuals of generation n.

The random variable X defined above specifies the probability distribution on the number of offspring. We denote $\mathbf{E}[X] = \mu$ and $\mathbf{Var}[X] = \sigma^2$. Let $f : [0, 1] \to \mathbb{R}$ denote the probability generating function of X, defined as

$$f_X(x) = f(x) = \sum_{i \geq 0} x^i \, \mathbf{Pr}[X = i],$$

and

$$\rho_n = \mathbf{Pr}[Y_n = 0]$$

be the probability that the population is extinct by generation n. The probability π_0 that the branching process dies out is then the limit of those probabilities.

$$\pi_0 = \mathbf{Pr}[\text{the process dies out}] = \mathbf{Pr}[Y_n = 0 \text{ for some } n] = \lim_{n\to\infty} \mathbf{Pr}[Y_n = 0] = \lim_{n\to\infty} \rho_n.$$

The basic result in the theory of branching processes is the following (see e.g. [17]).

Theorem 1. *If $\mu > 1$ and $\mathbf{Pr}[X = 0] > 0$, then π_0 is equal to the smallest solution of the equation $f(x) = x$ which belongs to the interval $(0, 1)$.*

Note that this means that whenever $\mu > 1$ the probability that the process survives is strictly positive. We will be particularly interested in branching processes where the number of descendants is given by a binomially distributed random variable. Let $X \in B(r, p)$. Then the probability generating function of X equals

$$f_X(x) = \sum_{i=0}^{n} \binom{r}{i} x^i p^i (1-p)^{r-i} = (1 - p + xp)^r.$$

Thus the probability of extinction ρ_n of the branching process defined by X is uniquely determined by the solution of the equation

$$(1 - p + xp)^r = x \tag{2.1}$$

In the Chapter 5 of this thesis we construct a branching process with generating distribution given by $X \in B(3, 0.49)$. As $\mathbf{E}[X] = 1.47 > 1$, from above paragraph we know that with probability greater than 0.61 such a process will never die out. At this point we will need to estimate the grow of such a process, namely we want to know what is the expected number of individuals in the n-th generation.

Lemma 2. *Let X be a random variable with binomial distribution $B(r, p)$, where $rp > 1$. Let Y_n denote the number of individuals in n-th generation of the branching process with generating distribution given by distribution of X. For a given m let us choose the smallest n such that $\sum_{i=0}^{n-1} Y_i \geq 2m/(rp - 1)$. Then with probability at least $2\exp(-m\frac{rp-1}{6rp})$ we have $Y_n \geq m$.*

Proof. Note that the probability that there are fewer than m ancestors in the last generation is bounded from above by the probability that random variable $Z = \sum_{i=1}^{t} X_i$ defined as the sum of t, $t \geq 2m/(rp-1)$ independent random variables $X_i \in B(r,p)$ is less than $m+t-1$. Observe that Z has the binomial distribution $B(tr,p)$; in particular $\mathbf{E}\, Z = trp$. Thus, from Chernoff's inequality, we get

$$\begin{aligned}
\mathbf{Pr}[Y_n \leq m] &\leq \mathbf{Pr}[Z \leq m+t] \\
&\leq \mathbf{Pr}\left[\mathbf{E}[Z] - Z \geq \mathbf{E}[Z] - (m+t)\right] \\
&\leq \mathbf{Pr}\left[\mathbf{E}[Z] - Z \geq \frac{\mathbf{E}[Z] - (m+t)}{\mathbf{E}[Z]}\,\mathbf{E}[Z]\right] \\
&\leq 2\exp\left(-\frac{(trp - m - t)^2}{3trp}\right) \\
&\leq 2\exp\left(-\frac{(t(rp-1) - \frac{t(rp-1)}{2})^2}{3trp}\right) \\
&\leq 2\exp\left(-\frac{t(rp-1)^2}{12rp}\right) \\
&\leq 2\exp\left(-\frac{m(rp-1)}{6rp}\right). \qquad (2.2)
\end{aligned}$$

This concludes the proof of Lemma 2. □

Young man, in mathematics you don't understand things. You just get used to them.

Johann von Neumann, 1921

3

Properties of random lifts of graphs

In this chapter we survey results concerning properties of random lifts and show some of their properties which will be useful in the upcoming chapters. At the end of the chapter we also mention results on matchings and chromatic number of random lifts which strictly speaking are not related to the issues we are concerned in this thesis, but since, in general, not much is known about the properties of random coverings we like to present a current picture of the whole area.

Let us recall that we shall be only interested in the asymptotic properties of random lifts, when $n \to \infty$. Thus in every proof in this and following chapters we claim that all inequalities we state holds only for sufficiently large n.

It is easy to see that some properties of the base graph are in a way preserved by the covering graph. For example the degrees of the vertices in the fibers are the same as the degree of a vertex they are mapped to, and so the lift of a d-regular graph is d-regular. Since the covering map is a homomorphism, the chromatic number of the lift is not greater than the chromatic number of the base graph. On the other hand lifts of graphs can have much better connectivity properties than base graphs. Our main interest lies in a question how the family of lifts preserves and reflects the local and global structure of the base graph. The simplest case is when the base graph is a tree. An easy argument proves that a lift of a tree T is a collection of disjoint trees isomorphic to T.

Fact 1. *Let $\Gamma : \tilde{G} \to G$ be an n-covering. Every tree T in G is covered in $\tilde{G}$ by n disjoint trees isomorphic to T.*

Proof. We will prove this fact by induction on the size of a tree. The base case is a single vertex t. A covering of one vertex is simply a sum of n disjoint vertices. For the induction hypothesis, suppose that the statement of the fact is true for every connected tree on $k-1$ vertices. Now consider a tree T on k vertices with a vertex u of degree one in T. Let v be a vertex adjacent to u in T. A covering of $T \backslash u$ is a sum of n disjoint trees $T_1, ..., T_n$ isomorphic to T. Consider an edge $e = \{u, v\}$, its lift match trees $T_1, ..., T_n$ with n vertices that covers u.

□

For T being a path the above property is sometimes called the unique path-lifting property of random lifts [2]. Since relabelling vertices on fiber typically does not change properties of the covering we may always assume that copies of a path in the lift are contained in different „layers" on fibers. In particular if E is a set of edges that does not contain a cycle, then the probability of any property of the covering is unchanged if we condition on all the permutations assigned to edges in E being the identity.

3.1 General properties of random lifts

Adding one edge to a tree results in creating a cycle in a graph. A random lift of a cycle is the first non-trivial case we have to review. One can easily check that the lift of a cycle is a set of disjoint cycles, but in this case lengths of those cycles varies.

Lemma 3. *Let $h \geq 3$. Asymptotically almost surely a random lift of a cycle C_h on h vertices consists of a collection of at most $2\log n$ disjoint cycles.*

Proof. If we remove one edge e from a cycle, then a lift of the path obtained in this way is a collection of n disjoint paths (see Fact 1 above). Lifting the missing edge e is the same as matching at random the two sets of ends of those paths or connecting those ends according to some random permutation. The number of cycles created after joining those paths is then the same as the number of cycles in a random permutation on set $[n] = \{1, 2, \ldots, n\}$. The precise distribution of the number of cycles in random permutation is well known [16], but here we estimate it for the completeness of the argument.

Let $X_d = X_d(n)$ denote the number of d-cycles in the random permutation on $[n]$. There are $(d-1)!$ ways of arranging given d symbols in a cycle, $(n-d)!$ permutations of the remaining symbols and $n!$ permutations in total, so the probability for d given symbols to form a cycle in a permutation chosen uniformly at random from the set of all permutations of n symbols is

$$\frac{(d-1)!(n-d)!}{n!}$$

There are $\binom{n}{d}$ selections of d out of n symbols, so for expected number of d-cycles $\mathbf{E}\, X_d$ we have

$$\mathbf{E}\, X_d = \binom{n}{d}(d-1)!\frac{(n-d)!}{n!} = \frac{1}{d}. \tag{3.1}$$

Thus, if $X = X(n) = \sum_{d=1}^{n} X_d$ denotes the total number of cycles, then

$$\mathbf{E}\, X = \sum_{d=1}^{n} \mathbf{E}\, X_d = \sum_{d=1}^{n} \frac{1}{d} = \log n + O(1). \tag{3.2}$$

In order to compute the variance note that if we fix a cycle in a random permutation, then each permutation on the remaining vertices is equally likely. Hence

$$\mathbf{E}\, X(n)[(X(n)-1)] = \sum_{d=1}^{n}\sum_{\ell=1}^{n-d} \mathbf{E}\, X_d(n)\, \mathbf{E}\, X_\ell(n-d) = \sum_{d=1}^{n}\sum_{\ell=1}^{n-d} \frac{1}{d\ell}\,. \tag{3.3}$$

Let $s = n\exp(-\sqrt{\log n})$. Then,

$$\begin{aligned}\mathbf{E}\, X[(X-1)] &= \sum_{d=1}^{s}\sum_{\ell=1}^{n-d}\frac{1}{d\ell} + \sum_{d=s+1}^{n}\sum_{\ell=1}^{n-d}\frac{1}{d\ell}\\ &= (\log s + O(1))(\log n + O(1)) + (\log(n/s) + O(1))O(\log n)\\ &= (\log n)^2 + O\big(\log^{3/2} n\big),\end{aligned} \tag{3.4}$$

and

$$\mathbf{Var}\, X = \mathbf{E}\, X(X-1) + \mathbf{E}\, X - (\mathbf{E}\, X)^2 = O\big(\log^{3/2} n\big)\,.$$

Hence, from Chebyshev's inequality we get

$$\mathbf{Pr}[X \geq 2\log n] \leq \mathbf{Pr}[|X - \mathbf{E}\, X| \geq 0.5\log n] \leq \frac{4\,\mathbf{Var}\, X}{\log^2 n} = o(1)\,.$$

□

The lifts of more complex graphs are much harder to describe. That is why from this point on we focus on selected graph properties that are preserved in lifts. In the case of general graphs it can be proven that all short cycles are typically "sparsely distributed" in the lifts.

Lemma 4. *Let G be a simple graph, then asymptotically almost surely no two cycles of $\tilde{G}$ of length smaller than $(\log\log n)^2$ lie within distance less than $(\log\log n)^2$ from each other.*

Proof. Let G be a simple graph of order k. Let Z be a random variable which counts the number of pairs of cycles in $\tilde{G} \in L_n(G)$, which are shorter than $(\log\log n)^2$ and either intersect each other, or are connected by a path of length at most $(\log\log n)^2$. We bound from above the expected value of Z by counting the number of paths P of length at most $3(\log\log n)^2$ such that both ends of P are adjacent to some element of P (we denote this new random variable as Z').

For a given ordered set of m vertices $\{u_1, ..., u_m\}$ and two selected vertices u_i and u_j, the probability that there is a path $u_1...u_m$ with additional edges between u_1 and u_i, and u_m and u_j in $\tilde{G}$ is less than $\left(\frac{1}{n-m}\right)^{m+1}$ (since for every edge $\{u_x, u_{x+1}\}$ at most m places in the fiber $\tilde{G}_{u_{x+1}}$ can be occupied by edges of P). For every m there are $\binom{kn}{m}$ ways to choose m vertices out of kn vertices of $\tilde{G}$. On the given set of m vertices we can build $m!$ different paths and furthermore there are at most m^2 ways to choose u_i and u_j, thus

$$\begin{aligned}\mathbf{E}\, Z' &\leq \sum_{m=1}^{3(\log\log n)^2} \binom{kn}{m} m!\, m^2 \left(\frac{1}{n-m}\right)^{m+1} \\ &\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(kn)^m}{m!} \frac{m!\, m^2}{(n-m)^{m+1}} \\ &\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(kn)^m m^2}{(n-m)^{m+1}}\end{aligned}$$

Since $m < n/2$, we have

$$\begin{aligned}\mathbf{E}\, Z' &\leq \sum_{m=1}^{3(\log\log n)^2} \frac{(2k)^{m+1} m^2}{n} \\ &\leq \frac{9(2k)^{3(\log\log n)^2}(\log\log n)^6}{n} \leq \frac{\exp\left((\log\log n)^3\right)}{n}.\end{aligned}$$

Consequently, from Markov's inequality,

$$\mathbf{Pr}[Z > 0] \leq \mathbf{E}\, Z \leq \mathbf{E}\, Z' = o(1)\,,$$

and the assertion follows. □

Our next result states that for almost every lift of a graph G, $d = d(n) \leq 5\log\log n$ and any constant C, the d-neighbourhoods of the C lexicographically first vertices of every fiber are mutually disjoint and have a structure of a tree.

Lemma 5. *Let G be a simple graph, with $\delta(G) \geq 2$, and $C > 0$ be a constant. Asymptotically almost surely $\tilde{G}$ has the following property. For any vertex $v \in G$, the C lexicographically first vertices from the fiber above vertex v are at distance at least $11\log\log n$ from each other and each such vertex is at distance at least $11\log\log n$ from any cycle shorter than $10\log\log n$.*

Proof. Let G be a simple graph of order k. Let $\mathcal{C}_v$ denote the set of C lexicographically first vertices from fiber $\tilde{G}_v$ (vertices with labels between 1 and C), and take $x, y \in \mathcal{C}_v$. Let $Z_{x,y}$ be a random variable that counts the number of paths connecting x with y which are shorter than $11 \log n \log n$. For a given ordered set of m vertices $\{u_1, ..., u_m\}$, the probability that there is a path $xu_1...u_m y$ in $\tilde{G}$ is less than $\left(\frac{1}{n-m}\right)^{m+1}$. Hence, similarly as in the proof of Lemma 4, we have

$$\begin{aligned}\mathbf{E}\, Z_{x,y} &\leq \sum_{m=1}^{11 \log\log n} \binom{kn}{m} (m!) \left(\frac{1}{n-m}\right)^{m+1} \\ &\leq \sum_{m=1}^{11 \log\log n} \frac{(kn)^m}{m!} \frac{m!}{(n-m)^{m+1}} \\ &\leq \sum_{m=1}^{11 \log\log n} \frac{(kn)^m}{(n-m)^{m+1}}.\end{aligned}$$

Since $m < n/2$, we have

$$\begin{aligned}\mathbf{E}\, Z_{x,y} &\leq \sum_{m=1}^{11 \log\log n} \frac{(2k)^{m+1}}{n} \\ &\leq 11 \log\log n \frac{(2k)^{11 \log\log n}}{n} \leq \frac{\exp\left((\log\log n)^2\right)}{n}.\end{aligned}$$

Let Z be a random variable that counts, for every fiber $\tilde{G}_v$ in $\tilde{G}$, the number of paths shorter than $11 \log\log n$ connecting any pair of vertices in $\mathcal{C}_v$. There are k different fibers and $\binom{C}{2}$ different pairs on each fiber, so using the union bound we get

$$\mathbf{E}\, Z \leq k \binom{C}{2} \mathbf{E}\, Z_{a,b} \leq k \binom{C}{2} \frac{\exp\left((\log\log n)^2\right)}{n} \longrightarrow 0.$$

Thus $\mathbf{Pr}[Z > 0] = o(1)$ which proves the first part of the statement.

Now we would like to count the expected number of cycles shorter than $10 \log\log n$ which are at distance smaller than $11 \log\log n$ to any vertex in $\mathcal{C}_v$. Let Q be a random variable that counts the number of paths starting at vertex of $\mathcal{C}_v$ that are shorter than $22 \log\log n$ and for which there is an edge connecting the last vertex with some of the first $n-2$ vertices of this path. Then similar estimates as in the proof of Lemma 4 together with the union bound gives

$$
\begin{aligned}
\mathbf{E}\, Q &\leq \sum_{m=1}^{22\log\log n} (Ck)\binom{kn}{m}\, m!\, m \left(\frac{1}{n-m}\right)^{m+1} \\
&\leq \sum_{m=1}^{22\log\log n} (Ck)\frac{(kn)^m m}{(n-m)^{m+1}} \\
&\leq \sum_{m=1}^{22\log\log n} C\frac{(2k)^{m+1} m}{n} \\
&\leq C\frac{(2k)^{22\log\log n} 22(\log\log n)}{n} \longrightarrow 0.
\end{aligned}
$$

Hence, asymptotically almost surely such a cycle does not appear in $\tilde{G}$ and the assertion follows. □

3.2 Connectivity

In this section we focus on the connectivity properties of random lifts. More precisely, we study the expected number of vertex disjoint paths connecting any two vertices in a random lift. Let δ denote the minimum degree of a graph G. Then for an ℓ-connected graph G we obviously have $\ell \leq \delta$. Notice that the lift $\tilde{G}$ of a graph G with minimum degree δ contains vertices of degree δ and therefore it is at most δ-connected. We already know that there exist examples of graphs (e.g. cycles) with $\delta \leq 2$ such that their random lifts are aas not connected. Amit and Linial [2] proved that if $\delta \geq 3$, then almost every random lift is in fact δ-connected. We present here a simple proof of this fact, much shorter than the original argument of Amit and Linial.

Theorem 6 ([2])**.** *Let G be a connected simple graph with minimal degree $\delta \geq 3$. Then asymptotically almost surely an n-lift $\tilde{G}$ is δ-connected.*

Proof. Let G be a connected graph with $\delta(G) \geq 3$. Let us recall that for $X \subset G$ by $N(X)$ we denote the set of vertices from $V(\tilde{G})\backslash X$ that are adjacent to some vertex in X. By Menger's theorem [11] to show that a covering $\tilde{G}$ is δ-connected, we need to show that for every subset X of vertices of G with $|X| < |\tilde{G}|/2$, we have $|N(X)| \geq \delta$. Notice that it is enough to show that this property is true for connected subsets X of $\tilde{G}$.

Whenever we take a connected set $X \subseteq V(\tilde{G})$ of a size $x = |X| \leq \log\log n$ we have $N(X) \leq c\log\log n$, for some constant $c > 0$ which does not depend on n. By Lemma 4 aas there is at most one cycle in a subgraph of $\tilde{G}$ induced on $N(X) \cup X$. Therefore either there are $|X|$ edges inside the set X and all vertices in X are connected to different

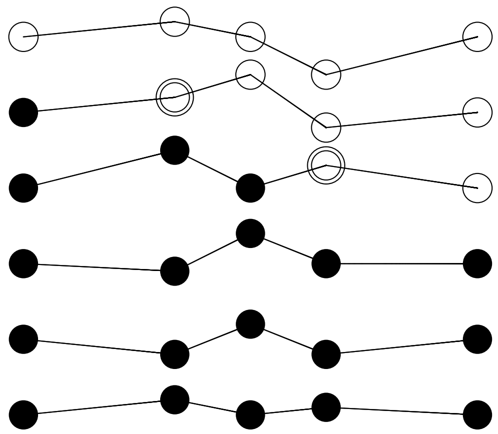

Figure 3.1: An example of the set X which is not 2-outside with a lift of path. The black vertices are in X, the white ones are outside X. Double-circled points are elements from the set $N(X)$.

vertices outside X, or there are $|X| - 1$ edges inside the set X and there is at most one vertex in $N(X)$ which is connected to at most two different vertices from X. Thus $|N(X)|$ equals to the number of all edges coming from X minus the edges which are inside X i.e. $\delta|X| - 2|X| = (\delta - 2)|X|$. The inequality $(\delta - 2)|X| \geq \delta$ holds for all $|X| \geq \frac{\delta}{\delta - 2}$, while for $|X| \leq 2$ the statement is trivial. Thus the assertion of the lemma holds for $|X| \leq \log\log n$.

In order to deal with the case when $|X| = x > \log\log n$ we lift G in two stages. Let T be a spanning tree of a graph G. First we lift edges of T; then we lift the rest of the graph. Let us recall that, by the Fact 1, the lift $\tilde{T}$ of T consists of n disjoint copies of T; we denote them by $T_1, \ldots, T_n$.

We say that a set of vertices $X \subset V(\tilde{G})$ is *α-outside of $\tilde{T}$* if all except at most $\alpha - 1$ trees from the family $T_1, \ldots, T_n$ which intersect X are entirely contained in X. We show first that each subset of X which is not α-outside of $\tilde{T}$ has neighbourhood at least α. Indeed, it is enough to note that if X properly intersects some tree T_i, then T_i contributes at least one vertex to $N(X)$ (see Figure 3.1, where we illustrate it for the case when T is a path). Consequently, whenever X intersects properly at least α trees from $\tilde{T}$, we have $|N(X)| \geq \alpha$.

Thus, to conclude the proof, we can restrict our attention to the sets X of size x, where

$$\log\log n \leq x \leq |\tilde{G}| = nk/2$$

and X is δ-outside of $\tilde{T}$ and show that aas for all of them we have $|N(X)| \geq \delta$. Let choose a set X with the above property. Then at least $\frac{x}{|G|} - \delta$ trees from $\tilde{T}$ are entirely contained in X. Let v be a vertex of degree one in the tree T and let $X_i = \tilde{G}_i \cap X$. There are $\delta - 1$ edges connecting v with other vertices $d_1, ..., d_{\delta-1}$ from G, where, let us recall, $\delta \geq 3$. We prove that the probability that vertices from X_v are connected to fewer than δ vertices from $\tilde{G}_{d_1} \backslash X_{d_1} \cup \cdots \cup \tilde{G}_{d_{\delta-1}} \backslash X_{d_{\delta-1}}$ tends to zero as n goes to infinity. More specifically, let $B(x)$ denote the expected number of sets X of size x such that $|N(X_v) \cap (V(\tilde{G}) \backslash X)| \leq \delta$. We shall show that

$$\sum_{y=\log\log n}^{nk/2} B(x) = o(1). \tag{3.5}$$

Let us divide X into two parts: X_1 that contains trees from $\tilde{T}$ which are entirely contained in X, and X_2 containing trees from $\tilde{T}$ which intersect properly with X. Let $x_1 = |X_1|$ and $x_2 = |X_2|$, where $x_1 + x_2 = x$. Thus we have

$$B(x) = \sum_{x_2=0}^{(\delta-1)(k-1)} \sum_{x_1=\log\log n - x_2}^{nk/2-x_2} B'(x_1, x_2),$$

where $B'(x_1, x_2)$ is the expected number of sets X with partition into $|X_1| = x_1$ and $|X_2| = x_2$ such that $|N(X_v) \cap (V(\tilde{G}) \backslash X)| \leq \delta$. Now we try to estimate the probability of $B'(x_1, x_2)$.

In order to pick the set X we have to choose $q = x_1/k$ trees that are contained in X, and then select possible additional $z \leq \delta - 1$ trees that are not entirely contained in X. Next we have to decide which x_2 out of $z(k-1)$ vertices of the second type trees we want to include in X. Finally the set X can also contains up to $\delta - 1$ vertices that are not elements of previously chosen trees (otherwise, from previous analysis, the neighbourhood $N(X)$ would be greater than δ).

Let v be a vertex of degree one in T. Vertex v has $\delta - 1$ neighbours outside T. For every edge $e = \{v, d_i\}$ there is a matching between sets $\tilde{G}_v$ and $\tilde{G}_{d_i}$, so the probability that $|N(X_v) \cap (\tilde{G}_{d_i} \backslash X_{d_i})| < \delta$ is bounded from above by the probability that the chosen random set of q elements would be a subset of $q + z + \delta - 1$ vertices (q vertices from X_1, z vertices from X_2 and up to $\delta - 1$ vertices of $N(X)$ that are not in X). Thus

$$\begin{aligned} B'(x_1, x_2) &\le \binom{n}{q}\binom{n}{z}\binom{nk}{\delta-1}\binom{(k-1)z}{x_2}\left(\frac{\binom{q+z+\delta-1}{q}}{\binom{n}{q}}\right)^{\delta-1} \\ &\le \binom{n}{q}\left(\binom{nk}{\delta-1}\right)^2\binom{(k-1)(\delta-1)}{x_2}\left(\frac{\binom{q+2\delta-2}{q}}{\binom{n}{q}}\right)^{\delta-1} \\ &\le \binom{n}{q}\left(\binom{nk}{\delta-1}\right)^2\binom{(k-1)(\delta-1)}{x_2}\left(\frac{(q+2\delta-2)\cdot \ldots \cdot(2\delta-1)}{n\cdot \ldots \cdot(n-q+1)}\right)^{\delta-1}. \end{aligned}$$

Note that $c = \binom{(k-1)(\delta-1)}{x_2}$ is a constant that does not depend on n. Moreover,

$$\begin{aligned} B'(x_1, x_2) &\le c(nk)^{2\delta-2}\frac{n!}{(q)!(n-q)!}\left(\frac{(q+2\delta-2)\cdot \ldots \cdot(2\delta-1)}{n\cdot \ldots \cdot(n-q+1)}\right)^{\delta-1} \\ &\le c(nk)^{2\delta-2}\left(\frac{n\cdot \ldots \cdot(n-q+1)}{q!}\right)\left(\frac{(q+2\delta-2)\cdot \ldots \cdot(2\delta-1)}{n\cdot \ldots \cdot(n-q+1)}\right)^{\delta-1} \\ &\le c(nk)^{2\delta-2}(2q)^{2\delta}\left(\frac{(q+2\delta-2)\cdot \ldots \cdot(2\delta-1)}{n\cdot \ldots \cdot(n-q+1)}\right)^{\delta-2} \\ &= c(nk)^{2\delta-2}\left(\frac{2x_1}{k}\right)^{2\delta}\left(\frac{(\frac{x_1}{k}+2\delta-2)\cdot \ldots \cdot(2\delta-1)}{n\cdot \ldots \cdot(n-\frac{x_1}{k}+1)}\right)^{\delta-2}. \end{aligned}$$

Now for $\log\log n \le x_1 \le \log^2 n$, we have

$$B'(x_1, x_2) \le c(nk)^{4\delta}\left(\frac{\log^2 n}{n-\log^2 n}\right)^{(\delta-2)\log\log n} = o(1/n^2)$$

while for $\log^2 n \le x_1 \le n/2$, we get

$$B'(x_1, x_2) \le c(nk)^{4\delta}\left(\frac{1}{2}\right)^{(\delta-2)\log^2 n} = o(1/n^2)$$

Since $\delta \ge 3$, equation (3.5) holds and so the assertion follows. □

In the next Chapter 4 we prove that almost every random lift of minimal degree at least $2k-1$ has much stronger connectivity property, namely it is k-linked (see Chapter 4 for definition). Furthermore the lengths of the paths connecting every pair of vertices in the definition of k-linked graph can be chosen to have order $O(\log n)$.

As we have seen in order for a graph to be α-connected, for any subset $|S| \le |V(G)|/2$ of vertices of a graph, we need its neighbourhood to be greater than α. A natural question to ask is whether it is possible to obtain a stronger property, i.e. the size of neighbourhood of S to be some function of the size of S. This question may be asked in terms of number of edges connecting set S with the rest of the graph or the number of vertices adjacent to some vertex from S. A parameter that measure this property in the case of the size of edge-cut between S and $G \backslash S$ is the edge expansion.

Definition. Let G be a graph with v vertices. For $S \subset V(G)$, let ∂S be the set of edges with one vertex in S and one outside S. The *edge expansion* $\xi(S)$ is defined to be $|\partial S|/|S|$, and the edge expansion of G is

$$\xi(G) = \min\{\xi(S) : S \subset V(G), |S| \le |V(G)|/2\}.$$

Graphs which have a large edge expansion are called *expanders*. These graphs have the property that it is easy to get from one point to any other in the graph. Notice that a lift $\tilde{G}$ cannot have higher edge expansion than G. Given $S \subset V(G)$ with some small $\xi(S)$, take $\tilde{S}$ to be union of the fibers $\tilde{G}_u = \{u\} \times [n]$, for $u \in S$. Then $\xi(\tilde{S}) = \xi(S)$ and $|\tilde{S}| \le |V(\tilde{G})|/2$ iff $|S| \le |V(G)|/2$. Amit and Linial proved that edge expansions of lifts are asymptotically almost surely bounded away from 0.

Theorem 7 ([3]). *Let $G = (V, E)$ be a connected graph with $|E| > |V|$. Then there is a positive constant $\xi_0 = \xi_0(G)$ such that aas lift $\tilde{G}$ has edge expansion at least ξ_0.*

Note that the constant in the theorem of Amit and Linial is a function of the order of the graph G. If we put a restriction on the size of the set S, we can show a bound for the size of of the set $N(S)$ (where $N(S)$ is the set of vertices from $V(\tilde{G}) \backslash S$ that are adjacent to some vertex in S) as a function of the minimum degree of a graph G.

Lemma 8. *Let $\delta \ge 12$, for every simple graph G of order k with minimum degree δ aas every subset $|S|$ of vertices in $\tilde{G}$ with*

$$|S| \le \frac{n}{1000k^4\delta}$$

satisfy

$$|S \cup N(S)| > \frac{\delta}{3}|S|.$$

Proof. Let G be a graph of order k. Let S be any subset of vertices of $\tilde{G}$ and denote its size by s. We estimate the probability of an event $B(s)$ that any of the sets of size $s \le \alpha n$, for $\alpha = \frac{1}{1000k^4\delta}$, has a neighbourhood smaller than $\frac{\delta}{3}|S|$ and show that this probability tends to zero as $n \to \infty$. For a given set of vertices $T \in \tilde{G}$, $|T| \leq (\delta/3 - 1)s$ the probability that $\hat{N}(S) \subset S \cup T$ is bounded from above by

$$\left(\frac{|S| \cup |T|}{n}\right)^{\frac{\delta}{2} \cdot s},$$

since for each edge of each vertex $v \in S$ we have to choose its end in $S \cup T$. All together there are at least $s \cdot (\delta/2)$ neighbours to be chosen (we use such bound in respect to the worst case when all edges lies inside the set S), where each neighbour can be chosen from all vertices of appropriate fibers. There are $\binom{nk}{s}$ sets S with $|S| = s$ and $\binom{nk}{(\delta/3-1)s}$ choices for T, so we need to show that

$$\sum_{s=1}^{\alpha n} B(s) = o(1), \tag{3.6}$$

where

$$B(s) = \binom{nk}{s} \binom{nk}{(\frac{\delta}{3} - 1)s} \left(\frac{\delta s}{3n}\right)^{\delta s/2}.$$

Using the fact that

$$\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$$

we get

$$\begin{aligned}
B(s) &\leq \left(\frac{ekn}{s}\right)^s \left(\frac{ekn}{(\delta/3 - 1)s}\right)^{(\delta/3-1)s} \left(\frac{e\delta s}{3n}\right)^{\delta s/2} \\
&\leq \left(\left(\frac{e\delta}{3}\right)^{\delta/2} \left(\frac{3ek}{\delta - 3}\right)^{\delta/3} \left(\frac{s}{n}\right)^{\delta/6}\right)^s \\
&\leq \left(\left(2ek\right)^{2\delta/3} \left(\frac{\delta s}{n}\right)^{\delta/6}\right)^s.
\end{aligned}$$

Hence, if $s \leq \alpha n$, where $\alpha = \frac{1}{1000k^4\delta}$, we have

$$\left((2ek)^4 \delta\alpha\right)^{\delta/6} < 0.99,$$

therefore $B(s) = o(1/n)$ and the assertion follows. □

There have been extensive studies of lifts in terms of their expanding features, and this topic has brought a lot of attention because of their important applications. However, most of them concentrate on lift of special classes of graphs (so called Ramanujan graphs) or a construction of finite lifts, and so have different flavour than other results presented in this chapter. Since covering of this topic would require a commodious introduction and analysis we do not present those results in this thesis; more information on this topic can be found in [1, 6, 29].

3.3 Minors

Drier and Linial [13] discussed the existence of minors and topological minors in the lifts of graphs. They used slightly different approach and consider the behaviour of the n-lifts of complete graph of order ℓ, when $n = n(\ell)$. They proved that for $n \leq O(\log \ell)$ almost every n-lift of the complete graph K_ℓ contains a clique minor of size $\Theta(\ell)$, and for $n > \log \ell$ it contains a clique minor of size at least $\Omega\left(\frac{\ell\sqrt{n}}{\sqrt{\log \ell}}\right)$. The last result was shown to be tight as long as $\log \ell < n < \ell^{1/3-\epsilon}$.

Denote by $\sigma(L)$ the size of the largest clique which topological minor can be found in a lift L. The following bound holds for every lift of complete graph K_ℓ.

Lemma 9 ([13])**.** *Let $\tilde{K}_\ell$ be a lift of K_ℓ, then*

$$\Omega(\sqrt{\ell}) \leq \sigma(\tilde{K}_\ell) \leq \ell$$

Indeed since every vertex in $L \in \tilde{K}_\ell$ has only $n-1$ neighbours it is easy to notice that $\sigma(L) \leq n$. Lower bounds comes from theorem of Komlós and Szemerédi [22] that says that every graph of average degree d contains a subdivision of $K^{\Omega(\sqrt{d})}$. For n sufficiently large Drier and Linial proved the following results for random lifts.

Theorem 10 ([13])**.** *Aas for $L \in L_n(K_\ell)$ we have $\sigma(L) \leq O(\sqrt{\ell n})$.*

Theorem 11 ([13])**.** *If $\ell \geq \Omega(n)$, then aas for $L \in L_n(K_\ell)$, we have $\sigma(L) \geq \Omega(n)$.*

Authors left the problem of finding topological minors in lifts of complete graphs when $n \geq \Omega(\ell)$ and for lifts of general base graphs as an open question. The main question in this area is to understand, for a given graph, which of its minors M is persistent, i.e. M is a minor of almost every lift of G; and which are not.

In the Chapter 4 we show that in almost every lift of any graph G we can find a topological clique of size equal to the maximal degree in the $core(G)$ plus one (see definitions in Chapter 4). In particular it implies that for fixed d and $n \to \infty$, we have $\sigma(\tilde{K}_d) = d$. This results is best possible.

3.4 Other properties

There are only a handful papers on random lifts, therefore only few properties of those graphs has been studied. Thus for the completion of the picture we briefly present here also results on matching and chromatic number of random coverings, even though they are not the topic of research presented in upcoming chapters.

3.4.1 Matchings in random lifts

Some part of the research in the area of random lifts is dedicated to analyse which properties of random lifts are aas preserved by almost all or almost none of the lifts regardless of the choice of the base graph. Let us consider the property that a graph contains a perfect matching. It is easy to see that a lift of the perfect matching in G is a perfect matching in $\tilde{G}$. However, it is possible that G does not have a perfect matching while aas every lift does. The main role in determining whether the lift of a graph contains a perfect matching plays a concept of *fractional matching*.

Definition. A *fractional matching* in a graph $G = (V, E)$ is mapping $f : E \to \mathbb{R}^+$ such that $\sum_{e=\{v,x\}} f(e) \leq 1$ for every vertex $v \in V$. If the equality holds at every vertex, f is called a *perfect fractional matching*.

Since a covering graph can have an odd number of vertices we define an *almost-perfect matching*, as a matching that misses at most one vertex. A perfect matching in $\tilde{G}$ determines a fractional perfect matching in G. Indeed for each edge of G the $f(e)$ is the proportion of edges which belongs to the matching in the lift of e. It turned out that this condition is also sufficient for lift to admit a perfect matching.

Theorem 12 ([28])**.** *Let G be a graph that satisfies the following conditions:*

1 *G is connected.*

2 |*E(G)*| > |*V(G)*|.

3 *G has a perfect fractional matching.*

Then asymptotically almost surely a lift $\tilde{G}$ *has an almost-perfect matching.*

Linial and Rozenman were able to prove even more tight classification result.

Theorem 13 ([28])**.** *Let G be finite connected graph. Exactly one of the following situations occurs:*

1 *Every lift* $\tilde{G}$ *of G has a perfect matching. This occurs when G has a perfect matching.*

2 *Asymptotically almost surely a lift* $\tilde{G}$ *of G has an almost-perfect matching.*

3 *Asymptotically almost surely in a lift* $\tilde{G}$*, the largest matching misses* $\Theta(\log n)$ *vertices. This happens e.g. when G is an odd cycle.*

4 *Asymptotically almost surely every matching in an* n*-lift* $\tilde{G}$ *misses* $\Omega(n)$ *vertices. This happens if* $\sum f(e) \leq (1/2 - \epsilon)|V|$ *for every fractional matching in G.*

The implicit constants on the Θ *and* Ω *terms depend only on G.*

3.4.2 Chromatic number

We say that G is *k-colourable* if one can assign the colors $\{1, .., k\}$ to the vertices in $V(G)$, in such a way that every vertex gets exactly one color and no edge in $E(G)$ has both of its endpoints coloured the same color. The smallest k such that G is k-colourable is called the *chromatic number of G*. It turns out that finding the distribution of the chromatic number $\chi(\tilde{G})$ of random lifts of G is an interesting and challenging problem. We will focused on two parameters which are in a sense upper and lower bound on the chromatic number of lifts.

Definition.

$$\tilde{\chi}_h(G) = \min\{k \mid \chi(\tilde{G}) \leq k \text{ for almost every lift } \tilde{G} \text{ of } G\}$$

$$\tilde{\chi}_l(G) = \max\{k \mid \chi(\tilde{G}) \geq k \text{ for almost every lift } \tilde{G} \text{ of } G\}$$

Obviously $\tilde{\chi}_l(G) \leq \tilde{\chi}_h(G) \leq \chi(G)$. Linial, Amit and Matousek [4] conjecture that the chromatic number of random lifts concentrates essentially in a single value.

Conjecture 1. *For every graph G, $\tilde{\chi}_l(G) = \tilde{\chi}_h(G)$.*

Conjecture has been settled in the affirmative for bipartite graphs, cubic graphs and certain "blow-ups" of graphs (see Proposition 1 below). For paths and trees the chromatic number of their lift is aas equal 2. A lift of a graph with at least one odd cycle has chromatic number at least 3, since with high probability such lift contain an odd cycle. The smallest graph for which we do not know if this conjecture is true is K_5, the complete graph on 5 vertices. The chromatic number of its n-lift is a.a.s. either 3 or 4, but so far we do not know the probability distribution of $\chi(\tilde{K_5})$. For the complete graph on 5 vertices minus one edge the chromatic number of the random lift was found by Farzad and Theis.

Theorem 14 ([15])**.** *Asymptotically almost surely a random lift of $K_5 \backslash e$, (i.e. the complete graph of order 5, minus one edge) is 3-colourable.*

As it comes to determining the values of $\tilde{\chi}_l(G)$ and $\tilde{\chi}_h(G)$ in general case the following was proven by Amit, Linial and Matousek [4].

Theorem 15 ([4])**.** *For every graph G,*

$$\tilde{\chi}_l(G) \geq \sqrt{\frac{\chi(G)}{3 \log \chi(G)}}$$

As a matter of fact, the authors of this result conjectured that it can be substantially improved.

Conjecture 2 ([4]). *For each graph G,*

$$\tilde{\chi}_l(G) \geq C\frac{\chi(G)}{\log \chi(G)}$$

A better estimate can be obtained if instead of the chromatic number $\chi(G)$ we use the fractional chromatic number $\chi_f(G)$, defined as the minimum total weight of linear combination of independent sets, such that the weight at each vertex is at least 1.

Theorem 16 ([4]). *For each graph G,*

$$\tilde{\chi}_l(G) \geq \Omega\left(\frac{\chi_f(G)}{\log^2 \chi_f(G)}\right)$$

On the other hand, a theorem of Kim [21] on the chromatic number of graphs with high girth (the length of shortest cycle in a graph), yields an upper bound on $\tilde{\chi}_h(G)$. This bound can be proven to be tight for some classes of graphs.

Theorem 17 ([21]). *Let G be a graph with minimal degree $\delta = \delta(G)$. Then*

$$\tilde{\chi}_h(G) \leq \frac{\delta}{\ln \delta}(1 + o(1))$$

For complete graphs, we have then the following estimates

Corollary 1. *There exist constants $A > B > 0$ such that*

$$A\frac{r}{\log r} \geq \tilde{\chi}_h(K_r) \geq \tilde{\chi}_l(K_r) \geq B\frac{r}{\log r}$$

The above means that if we randomly lift complete graphs, its chromatic number drops from r to $r/\log r$. On the other hand there exist graphs whose chromatic numbers are preserved for all theirs lifts.

Proposition 1 ([4]). *For any graph G with $\chi(G) \geq 2$, put $r = 3\chi(G)\log\chi(G)$, and let H be constructed from G by replacing each vertex by an independent set of size r and every edge by a complete bipartite graph $K_{r,r}$. Then aas a lift $\tilde{H}$ of H has chromatic number $\chi(\tilde{H}) = \chi(H) = \chi(G)$.*

> “ *Oh, he seems like an okay person, except for being a little strange in some ways. All day he sits at his desk and scribbles, scribbles, scribbles. Then, at the end of the day, he takes the sheets of paper he's scribbled on, scrunches them all up, and throws them in the trash can.* ”
>
> John von Neumann's housekeeper, describing her employer.

4 Topological cliques in random lifts

In this part of the work we give a more detailed insight into the size of the largest topological clique in random lifts of graphs and some of the properties related to it.

Let us recall that a graph obtained by replacing edges of H with vertex disjoint paths is called a *subdivision* of H. If X is isomorphic to a subgraph of G, and X is a subdivision of a clique K_ℓ, we say that there is a *topological clique* of order ℓ in G. The vertices in G corresponding to the vertices in K_ℓ are then called branch vertices.

Observe that a vertex v of degree d can be a branch vertex in a topological clique of size at most $d+1$. Moreover no vertex of degree one can be a vertex connecting two branch vertices. That is why the concept of the core of a graph is crucial for the analysis of topological cliques.

Definition. The *core* of a connected graph G, denoted as $core(G)$, is the unique maximal subgraph of G with minimum degree at least two.

The $core(G)$ can be obtained from G by an algorithm that repeatedly removes vertices of degree one [11]. Therefore the core of the lift $\tilde{G}$ is the same as the lift of the $core(G)$. Consequently the maximum size of the topological clique contained in the lift of the graph G is bounded from above by $\Delta(core(G))+1$.

The main theorem of this chapter is that this bound is tight. That is, for any graph G, a random lift $\tilde{G}$ aas contains a topological clique of size $\Delta(core(G))+1$.

Theorem 18. *For a given graph G asymptotically almost surely $\tilde{G}$ contains a topological clique of size $\Delta(core(G)) + 1$. Moreover, the clique can be chosen in such a way that each path joining two branch vertices is shorter than $c \log n$, for some constant $c = c(|G|)$.*

4.1 Idea of the proof

In this section we present the main idea and describe some obstacles which we shall have to overcome in the proof of Theorem 18. We also introduce some of the notation that is used throughout this chapter. Since the proof uses significant amount of symbols to distinguish the case when we talk about the base graph, we will type all the symbols corresponding to the base graph in bold.

The idea behind the proof is roughly the following. Let **G** be a simple, connected graph and let $\tilde{G}$ be a graph chosen randomly from the set $L_n(\mathbf{G})$. Denote by **H** the core of **G** and let $\tilde{H}$ be its lift. Our goal is to find a topological clique of size $\Delta(\tilde{H}) + 1$ in $\tilde{H}$. Therefore the branch vertices of such clique must have degree at least $\Delta(\tilde{H})$. Let **v** be a vertex of the maximum degree in **H**. Since vertex **v** could be the only vertex of degree $\Delta(H)$ in **G** we focus our attention on vertices from the fiber $\tilde{H}_v$. We will show that if we take the lexicographically first $\Delta(\tilde{H}) + 1$ vertices from fiber $\tilde{H}_v$ (vertices of $\tilde{H}_v$ labelled from 1 to $\Delta(\tilde{H}) + 1$), then asymptotically almost surely they are branch vertices of a topological clique. Denote the set of $\Delta(\tilde{H}) + 1$ lexicographically first vertices of $\tilde{H}_v$ as $U = \{u_1, u_2, \ldots, u_{\Delta(\tilde{H})+1}\}$.

Let **W** be a family of directed closed walks in **H** which start and end in **v** and let $\tilde{W}$ denote the lift of those walks. In order to find a topological clique in $\tilde{H}$ we perform a breadth-first search type procedure. Starting from u_i we follow the lifts of walks from the family **W**. Since walks in **W** are closed, for every walk its lift is a path which starts at u_i ends at some vertex $q \in \tilde{H}_v$. Next, for every end we continue the same expansion operation. The set of vertices reached after z iterations of this process will be denoted by $R_z(u_i)$.

The proof consists of two parts. First, using general structural properties of random lifts, we show that for $\ell = \log\log^4 n$ aas sets $R_\ell(u_i)$ are of the size $O(\log^4 n)$. Next, we prove that aas those sets can be further extended to the size of $O(\sqrt{n}\log n)$. Finally we show that with probability tending to one, for every pair $u_i, u_j \in U$ there would be a common vertex $x \in R_{\ell'}(u_i) \cap R_{\ell'}(u_j)$. Thus along the path we used to get to x we can find a path connecting u_i to u_j. We repeat this reasoning for every pair of vertices in U (they are designed to be the branch vertices of our topological clique).

The main technical obstacle in the argument is that paths which connect the branch vertices should be vertex disjoint. Thus, in the process of generating $R_{\ell'}(u_i)$ we want to avoid the vertices which have been added to the sets $R_{\ell'}(u_j)$ generated earlier. Hence, whenever we reach already "visited" vertex we will not use this vertex to expand $R_{\ell'}(u_i)$. Consequently sets $\hat{R}_{\ell'}(u_i)$ modified in such a way will be slightly smaller than in the case in which they would be generated independently from each other. We argue that this difference is not substantial and would not affect the probability that the random sets $\hat{R}_{\ell'}(u_j)$ and $\hat{R}_{\ell'}(u_i)$ have a non-empty intersection.

4.2 Preliminaries

As mentioned above an important part in our argument is played by the family of directed closed walks $\mathbf{W} = \{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{\Delta(\tilde{H})}\}$ in **H** which start and end at **v**. We choose those walks in such a way that their first edges are different. It is easy to show that such a family always exists. Assume we start a walk choosing an edge $\mathbf{e} = \{\mathbf{v}, \mathbf{x}\}$. There are two cases, either **e** lies on a cycle and we choose this cycle as our walk, or we continue with choosing the next edge **e'** $= \{\mathbf{x}, \mathbf{y}\}$ adjacent to **e**. Since **H** has minimum degree greater than 2 at some point of this procedure we will reach an edge which lies on a cycle **C** in **G**. The path from **v** to this edge together with **C** and way back to vertex **v** will be our closed walk $\mathbf{W}_i$.

We will use walks $\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{\Delta(\tilde{H})}\}$ to recursively build sets of vertices of the graph $\tilde{H}$ which can be reached from u_i. Let $T_0(u_i) = R_0(u_i) = u_i$. Denote by $\tilde{W}_j(u_1)$ the lift of a closed path $\mathbf{W}_j$ that starts at u_i. Then we set $T_1(u_i) = (\bigcup_j \tilde{W}_j(u_i))\backslash\{u_i\}$, and by $R_1(u_i) = T_1(u_i) \cap \tilde{H}_v$ denote the set of all vertices of the fiber above vertex **v** in which those walks end. Next we continue to use the lifts $\tilde{W}_i$, for $i \in \{1, 2, \ldots, \Delta(\tilde{H})\}$, to travel from $R_1(u_i)$ back to the fiber $\tilde{H}_v$, then $T_2(u_i) = (\bigcup_j \bigcup_{u' \in R_1(u_1)} \tilde{W}_j(u'))\backslash(T_1(u_1) \cup T_0(u_i))$ and analogously $R_2(u_i) = T_2(u_i) \cap \tilde{H}_v$. In general we set $R_\ell(u_i) = T_\ell(u_i) \cap H_v$ and call it *ℓ-vicinity* of u_i. The set of vertices $T_\ell(u_i)$ is defined recursively, we take vertices of all paths from $\tilde{W}$ which start at vertices from $R_{\ell-1}(u_i)$ and are not part of any $T_j(u_i)$, for $j = 0, ..., \ell - 1$.

For a single path the probability that the lift $\tilde{W}_j(u_i)$ of a walk $\mathbf{W}_j$ ends at given vertex $z \in \tilde{H}_v$ equals $\frac{1}{n}$. But when we generate two different paths $\tilde{W}_j(u_i)$ and $\tilde{W}_k(u_i)$, since those two paths can cross, this estimate is no longer true. Nonetheless, as we will see shortly, we can treat them as almost independent from each other.

As mentioned before in each step of the branching through graph $\tilde{H}$ we are avoiding vertices visited in previous steps. The reason is that we do not want to have an intersection between generated paths, moreover we want the neighbourhoods to be generated randomly and (roughly) independently. To this end, during our procedure we will generate the random lift $\tilde{G}$ on the way, i.e. if we visit a vertex from the lift we reveal its incident edges as a result of the random experiment by choosing one out of the possible edges. Let us call a vertex $v \in \tilde{H}$ as *active* if we did not generate any edge incidence to it, and call all vertices that are not active as *inactive*. Our object is to avoid inactive vertices since if at any point of the procedure we reach an inactive vertex, then at least some of edges incident to it are already chosen, which interfere with our probabilistic analysis. Let D be the set of inactive vertices in the graph $\tilde{H}$. Let $D_v = D \cap \tilde{H}_v$ and $D_{W_k} = D_v \cap \{P \in \tilde{W}_k : |P \cap D| \geq 1\}$ be the set of ends of those walks from $\tilde{W}_k$ which contains at least one inactive vertex.

We allow two walks $\mathbf{W}_i$ and $\mathbf{W}_j$ to intersect in **G** at vertices other than **v**. Notice that for every common point $\mathbf{a} \in \mathbf{W}_i \cap \mathbf{W}_j$ in the lifted graph, every path $P \in \tilde{W}_i$ intersect with exactly one $P' \in \tilde{W}_j$. Therefore, whenever we use path P to expand $R_\ell(u_i)$ it prevents us from using exactly one $P' \in \tilde{W}_j$ it intersects with. Thus, in this case, in order to prevent P' from being a part of $R_\ell(u_i)$, for any prospective vertex, we generate it and add its vertices to the set of inactive vertices. This implies that we would not branch from those vertices in the future. Let **c** denote the total number of intersections between walks $\{\mathbf{W}_1, \mathbf{W}_2, ..., \mathbf{W}_{\Delta(\tilde{H})}\}$ apart from at vertex **v**. Note that **c** is bounded from above by the square of the number of vertices of **G** which, let us recall, is a constant which does not depend on n.

Note that whenever we expand the $T_\ell(u_i)$ there is no point to use edges by which we arrived to the points of $R_{\ell-1}(u_i)$ from $R_{\ell-2}(u_i)$. Otherwise it would contradict the assumption that we want to avoid branching from vertices that we have visited in previous steps. Moreover, for any set $T_\ell(u_i)$ we exclude the vertices which were elements of $T_k(u_i)$, for every $k < \ell$. Similarly we exclude vertices from intersections between $T_k(u_j)$ and $T_\ell(u_i)$, for any $u_i, u_j \in U$ and respectively $i < j$ and $k < \ell$. The modified sets obtained by applying this rule are denoted as $\hat{R}_\ell(u_i)$ and $\hat{T}_\ell(u_i)$.

Additionally let us point out that the set $\hat{R}_\ell(u)$ has a structure of a tree T rooted at u_i, which has all vertices placed on the fiber $\tilde{H}_v$. We can order the vertices of this tree from the root to the leaves. Observe that because $\delta(T) \geq 3$ the sizes of $\hat{R}_\ell(u)$ are expected to grow exponentially with ℓ, at least for small ℓ.

4.3 Proof

Proof of Theorem 18. Let $\tilde{H}$ be the core of the graph $\tilde{G}$ and **v** be a vertex of maximal degree in **H**. If $\Delta(\mathbf{H}) = 2$, then **H** is a cycle. The lift of a cycle is a sum of disjoint cycles, so the lift of **G** contains a topological clique of size 3. Therefore we may assume $\Delta(\mathbf{H}) > 2$ and, since we are considering the core of **G**, we have also $\delta(\mathbf{H}) \geq 2$. For the remainder of this section, we condition on the event that a graph $\tilde{H}$ satisfies conditions of Lemma 5 (i.e.

that the $\Delta(\tilde{H})+1$ lexicographically first vertices of the fiber $\tilde{H}_v$ are at distance $11\log\log n$ from each other and short cycles in $\tilde{H}$).

Let $U = \{u_1, u_2, \ldots, \ldots, u_{\Delta(\tilde{H})+1}\}$ be the set of the $\Delta(\tilde{H})+1$ lexicographically first vertices from $\tilde{H}_v$. As we have assumed, due to Lemma 5, vertices in U are at distance at least $11\log\log n$ from each other and any cycle of length at most $10\log\log n$. Therefore for $q \leq 5\log\log n$ and all the i's we can choose neighbourhoods $N_q(u_i)$ which form a tree and are disjoint from each other. Let q be the smallest number such that for each $i = 1, ..., \Delta(\tilde{H})+1$ the size of the q-neighbourhood $N_q(u_i)$ in the graph $\tilde{H}$ is at least $\log^4 n$. Note that in these neighbourhoods the distance between two vertices from the fiber $\tilde{H}_v$ is bounded by order of **G**, which is a constant number that does not grow with n.

For all trees $N_q(u_i)$ we restrict our attention only to vertices from $\tilde{H}_v$. Let $M(u_i)$ denote a graph whose set of vertices is $N_q(u_i) \cap \tilde{H}_v$. Two vertices x, y are connected in $M(u_i)$ if and only if they are the closest neighbours in the $N_q(u_i)$, i.e. there is no $z \in N_q(u_i)$ such that $d(x,y) = d(x,z) + d(z,y)$. Notice that, since $N_q(u_i)$'s are trees, the $M(u_i)$ is a tree which is a topological minor of $N_q(u_i)$. Moreover due to the fact that the distance (in the $N_q(u_i)$) between two vertices connected by an edge in $M(u_i)$ can be bounded by a constant number, the number of vertices of each $M(u_i)$ is of order $\Theta(\log^4 n)$. We subdivide those trees into disjoint subtrees. For each u_i, we choose a subset of vertices $U_i = \{u_i^1, ..., u_i^{\Delta(H)}\} \in M(u_i)$ and divide $M(u_i)$ into disjoint connected subtrees $M(u_i^1) \cup ... \cup M(u_i^{\Delta(H)})$ such that they are all rooted at u_i^j's and each one is of order $\Theta(\log^4 n)$.

After choosing the u_i's and generating the M_{u_i}'s the set D of inactive vertices is the sum of $\{u_1, u_2, \ldots, u_{\Delta(H)+1}\}$ together with vertices of $N_q(u_i)$'s and vertices of walks which cross those neighbourhoods. Our ultimate goal is to expand the vicinities $\hat{R}_\ell(u_i^j)$ to the size of $\sqrt{n}\log n$. We show that we can obtain it aas deactivating at most $O(\sqrt{n}\log n)$ vertices. We will use this value in the proof as the bound for the size of D.

Consider the first pair of vertices (u_1^2, u_2^1). Our first goal is to expand the set $\hat{R}_q(u_1^2)$. At the beginning $\hat{R}_q(u_1^2)$ is equivalent to the set of leaves of the tree $M(u_1^2)$ and it size is of order $\Theta(\log^4 n)$. We expand this vicinity in the following manner: We generate consecutively $\hat{R}_{q+1}(u_1^2), \hat{R}_{q+2}(u_1^2), ..., \hat{R}_\ell(u_1^2)$. The set D_v contains all ends of those walks from $\tilde{W}$ that contains at least one inactive vertex. The process of expanding the vicinity $\hat{R}_q(u_1^2)$ can be approximated by choosing, for each vertex $w' \in \hat{R}_q(u_1^2)$, and for each walk $\tilde{W}_k(w')$, $k \in \{1, 2, ..., \Delta(\tilde{H})\}$, an element from the set $\tilde{H}_v - D_{\tilde{W}_k}$ at random with uniform distribution. The probability of success is then equivalent to the probability that chosen vertex is not an element of D_v. If we succeed, then we add vertices of given path to the $\hat{R}_{q+1}(u_1^2)$ and to the set D. Furthermore if a walk $\tilde{W}_k(w')$ crosses any other walk $\tilde{W}_j(w'')$ then we set vertices of the walk $\tilde{W}_j(w'')$ as inactive.

We continue the expansion until for some ℓ we have

$$|\hat{R}_\ell(u_1^2)| = \Theta(\sqrt{n}/\log^3 n).$$

Let $\mathcal{A}$ denote the event that at some point of expanding the vicinity of the vertex w' we choose an inactive vertex. The probability of event $\mathcal{A}$ to occur is bounded by

$$\mathbf{Pr}[\mathcal{A}] \leq c\frac{\sqrt{n}}{\log^3 n} \cdot \frac{|D_v|}{n - \max_k |D_{\tilde{W}_k}|} \leq c\frac{\sqrt{n}}{\log^3 n} \cdot \frac{O(\sqrt{n}\log n)}{n - O(\sqrt{n}\log n)} \leq \frac{O(1)}{\log^2 n} \longrightarrow 0. \quad (4.1)$$

We repeat this action for all $c\log^4 n$ leaves in $M(u_1^2)$. As we just showed the probability of failure in expanding the vicinity for a single vertex is bounded by $\frac{O(1)}{\log^2 n}$. Thus the probability of the event $\mathcal{B}$ that we fail to expand one half of the vertices from $M(u_1^2)$, is bounded by

$$\mathbf{Pr}[\mathcal{B}] \leq 2^{c\log^4 n} \left(\frac{O(1)}{\log^2 n}\right)^{c\log^4 n} = o(n^{-3\Delta(H)}) \longrightarrow 0. \quad (4.2)$$

Thus asymptotically almost surely we can expand the vicinities of half of the leaves of $M(u_1^2)$ to the size of $\sqrt{n}/\log^3 n$, avoiding all inactive vertices. Therefore in total we expand the vicinity $\hat{R}_\ell(u_1^2)$ to the size $\Theta(\sqrt{n}\log n)$. Notice that, since each step deactivates a constant number of vertices, the number of vertices we deactivated during the process is also of order $\Theta(\sqrt{n}\log n)$.

As we would like to find a path between u_1^2 and u_2^1, in the next step we repeat the same reasoning in respect to the vertex u_2^1. We proceed in exactly the same manner as with the vertex u_1^2, trying to expand $\hat{R}_\ell(u_2^1)$, starting from $M(u_2^1)$, step by step. The only difference is that the size of the set of inactive vertices grow since we also want to avoid all connections between a vertex $w_2^1 \in \hat{R}_\ell(u_2^1)$ and vertices of $\hat{R}_\ell(u_1^2)$. Thus, as before, the probability of the event $\mathcal{A}'$, that at some point of expanding the vicinity of the leaf $w' \in M(u_2^1)$ we choose a vertex which is inactive is bounded by

$$\mathbf{Pr}[\mathcal{A}'] \leq c\frac{\sqrt{n}}{\log^3 n} \cdot \frac{|D_v|}{n - \max_k |D_{\tilde{W}_k}|} \leq \frac{O(1)}{\log^2 n} \longrightarrow 0.$$

Likewise in the previous case we repeat this action for all $c\log^4 n$ leaves in $M(u_2^1)$. Again, the probability of failure in expanding the vicinity for a single vertex is bounded by $\frac{O(1)}{\log^2 n}$. Thus, the probability of the event $\mathcal{B}'$ that we fail to expand half of the leaves from $M(u_2^1)$, is bounded by $o(n^{-3\Delta_H})$.

Finally we can expand both sets $\hat{R}(u_1^2)$ and $\hat{R}(u_1^2)$ to the size of $\Theta(\sqrt{n}\log n)$. In order to connect vertices u_1^2 and u_2^1 by a path we need to find some vertex $x \in \hat{R}(u_1^2) \cap \hat{R}(u_1^2)$, then the path $u_1^2...x...u_2^1$ would connect u_1 with u_2. The probability that such a vertex does not exist can be bounded above by the probability that a randomly chosen set $\hat{R}(u_2^1) \subseteq H_v$ of size $\sqrt{n}\log n$ avoids $\hat{R}(u_1^2)$. This probability is smaller than

$$\begin{aligned}\frac{\binom{n-|D|-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{n-|D|}{\sqrt{n}\log n}} &\leq \frac{(n-|D|-\sqrt{n}\log n)!}{(\sqrt{n}\log n)!(n-|D|-2\sqrt{n}\log n)!}\cdot\frac{(\sqrt{n}\log n)!(n-|D|-\sqrt{n}\log n)!}{(n-|D|)!}\\ &= \frac{(n-|D|-2\sqrt{n}\log n)(n-|D|-2\sqrt{n}\log n+1)\cdot\ldots\cdot(n-|D|-\sqrt{n}\log n)}{(n-|D|-\sqrt{n}\log n)(n-|D|-\sqrt{n}\log n+1)\cdot\ldots\cdot(n-|D|)}\\ &\leq \left(1-\frac{\log n}{\sqrt{n}-O(\log n)}\right)^{\sqrt{n}\log n} = o(n^{-3\Delta(H)}) \longrightarrow 0. \qquad (4.3)\end{aligned}$$

Our aim is to connect u_i's by disjoint paths so that they create a topological clique. Therefore we will take pairs of vertices (u_i^j, u_j^i) for $i \neq j$ and $(u_i^j, u^i_{\Delta(H)+1})$ for $i = j$ and try to build a set of disjoint paths between them.

The argument for each of the pairs of vertices u_i^j, u_j^i is similar to the one above. Again the only thing that changes is the size of the set of inactive vertices D we have to avoid, but since there are only $(\Delta(H)+1)^2$ pairs it will never grow beyond $O(\sqrt{n}\log n)$. Consequently all the previous calculations carry over to this case. Thus, the probability of choosing some previously visited vertex while expanding the vicinity of any leaf of $M(u_i^j)$ or $M(u_i^j)$ to the size of $\sqrt{n}/\log^3 n$, is bounded by $\frac{O(1)}{\log^2 n}$ as in (4.1). Hence, as before, the probability of the event $\mathcal{B}''$, that we fail to expand one half of the vertices from $M(u_j^i)$ and half from $M(u_i^j)$ is $o(n^{-3\Delta(H)})$ (see (4.2)). This implies that in $O(\log n)$ stages we can expand vicinities of leaves from $M(u_j^i)$ and $M(u_i^j)$ to the size of $\sqrt{n}\log n$. Finally, as in (4.3), the probability that we do not find a vertex which connects these two vicinities can be bounded from above by

$$\frac{\binom{n-|D|-\sqrt{n}\log n}{\sqrt{n}\log n}}{\binom{n-|D|}{\sqrt{n}\log n}} = o(n^{-3\Delta(H)}) \longrightarrow 0, \qquad (4.4)$$

Thus, we have showed that the probability of failure in connecting any pair is of order $o(n^{-3\Delta(H)}) = o(1)$. Because there are only finite number of pairs, the probability that we do not find a topological clique of size $\Delta_H + 1$ can also be bounded by $o(1)$. Note that for each vertex u_i^j we choose some vertex at distance at most $5\log\log n$ from u_i^j and in $O(\log n)$ steps we connected it with some other u_j^i. Thus, the generated paths connecting u_i^j's with u_i^j's are of length $O(\log n)$. □

4.4 Links

In the Section 3.2 we reviewed results on the connectivity properties of random lifts in terms of the number of vertices or edges you have to delete from a graph to separate given subset of vertices from the rest of the graph. Now we consider a related, yet slightly different problem.

Definition. A graph G with at least $2k$ vertices is said to be *k-linked* if for every $2k$ distinct vertices $s_1, s_2, ..., s_k, t_1, t_2, ..., t_k$ it contains k vertex-disjoint paths $P_1, P_2, ..., P_k$ such that P_i connects s_i to t_i, $1 \leq i \leq k$.

Notice that, from Menger's theorem [11], each k-linked graph is k-connected, but the converse is far from being true (for example a cycle is 2-connected but it is not 2-linked).

Now we try to answer the question about maximal k, for which almost every random lift of a given graph is k-linked. Jung [20] and, independently, Larman and Mani [23] proved that every $2k$-connected graph that contains a K_{3k} as a topological minor is k-linked. Combining their result with Theorem 18 and Theorem 6 we get the following corollary.

Corollary 2. *If G is a connected graph with minimum degree δ, then aas $\tilde{G} \in L_n(G)$ is $\min\{\Delta(\mathit{core}(G))/3, \delta/2\}$-linked.*

A slight modification of the argument used in the proof of Theorem 18 together with result from Lemma 4 gives us better result.

Theorem 19. *For a given graph G with $\delta(G) = 2k - 1 \geq 3$ asymptotically almost surely $\tilde{G} \in L_n(G)$ is k-linked.*

Proof. Let $S = s_1, s_2, ..., s_k, t_1, t_2, ..., t_k$ be a set of $2k$ vertices. In the proof we condition on the random lift $\tilde{G}$ to fulfils Lemma 4 (any two short cycles are far away from each other). Our plan is to mimic the proof of Theorem 18.

Let us consider two cases: Let us assume first that the vertices from the set S are at distance $11 \log\log n$ from each other and all short cycles in $\tilde{G}$, then we can choose one fiber $\tilde{G}_u$ and connect them, by paths of length smaller than $|G|$, to vertices $u_1, ..., u_{2k}$ from this fiber. Those vertices will have the same properties as vertices from the statement of Lemma 5. Thus from this point we proceed in the same way as in the proof of Theorem 18. Asymptotically almost surely designated vertices $u_1, ..., u_{2k}$ will be the branch vertices of a topological clique in $\tilde{G}$. Hence we can find a vertex-disjoint paths connecting s_i with t_i.

If any two vertices $x, y \in S$ are at smaller distance to each other than $11 \log\log n$, then we would like to switch them to ones which are far from all the others vertices in S. By Lemma 4 the $(\log\log n)^2$-neighbourhoods of vertices in S have at most one cycle. It means we can find a path connecting x with vertex $\bar{x} \in N_q(x)$, $q = (\log\log n)^2$ which is at distance greater than $11 \log\log n$ from any vertex in S. We can repeat this operation for all vertices in S. As the minimum degree is at least $2k - 1$ branching through the neighbourhood tree we either find a short path connecting particular s_i with t_i inside $N_q(s_i) \cap N_q(t_i)$ or vertex-disjoint paths connecting x's with $\bar{x}$'s.

In this way we can create some short paths connecting some of pairs s_i and t_i inside $N_q(s_i) \cap N_q(t_i)$ and for those among vertices $s_1, s_2, ..., s_k$ and $t_1, t_2, ..., t_k$ which remain unmatched we can find a set of disjoint paths connecting them with vertices $u_1, ..., u_k$ on the fiber of $\tilde{G}_u$. Furthermore, we can assume that u_i's are at distance at least $11 \log \log n$ from each other and all, apart from at most one, short cycles in $\tilde{G}$. It is easy to notice that such a single cycle do not influence the analysis made in the proof of Theorem 18. Then we can mark all vertices of constructed paths and their neighbours as inactive and mimic the argument from the proof of Theorem 18 to construct the topological clique on set $u_1, u_2, ..., u_k$. Then, to find a path from s_i to t_i one needs to go from s_i to the branch vertex u_i, next use edges of the clique to reach the branch vertex u_{k+i} matched to t_i and finally go to t_i.

The probability that we fail in any step of the proof is less than $o(n^{-3\Delta(H)})$ (see the estimates in (4.2)-(4.4)). Since there are at most $\binom{n|G|}{2k} \leq (n|G|)^{2k}$ possibilities to choose $2k$ vertices out of $n|G|$ vertices of the lift of G the probability of failure in connecting any of them tends to 0 as $n \to \infty$.

□

Let us remark that the above statement does not hold for $k = 1$ even if $\delta(G) = 2$, since asymptotically almost surely random lift of a cycle is not connected. On the other hand, for $k \geq 2$ it is clearly best possible, since k-linked graph can not contain a vertex of degree at most $2k - 2$. Indeed, in this case we can put v as s_1, as t_1 take any vertex outside $N(v)$, and separate s_1 from t_1 by $N(v) \subseteq \{s_2, ..., s_k, t_2, ..., t_k\}$.

4.5 k-diameter

In the previous section we showed that for any two sets of k vertices we can connect pairs of vertices from those sets by mutually disjoint paths. In addition the proof of this fact gives us an insight into the length of the paths connecting those vertices. A parameter which is focused on the length of different paths connecting any pair of two vertices in a graph is the *k-diameter of a graph.*

Let G be a k-connected graph and u, v, $u \neq v$, be any pair of vertices of G. Let $\mathcal{P}_k(u, v)$ be a family of k vertex disjoint paths between u and v, i.e.

$$\mathcal{P}_k(u, v) = \{P_1, P_2, ..., P_k\}, \text{ where } |P_1| \leq |P_2| \leq ... \leq |P_k|$$

and $|p_i|$ denotes the number of edges in path p_i. The *k-distance* $d_k(u, v)$ between vertices u and v is the minimum $|p_k|$ among all $\mathcal{P}_k(u, v)$ and the *k-diameter* $d_k(G)$ of G is defined as the maximum k-distance $d_k(u, v)$ over all pairs u, v of vertices of G.

The concept of k-diameter comes from analysis of the performance of routing algorithms [10] but has also drawn some attention as a graph parameter [18]. In the case of random lifts of a given graph G, for all vertices $u, v \in V(G)$, by the proof of Theorem 18, we know that for almost every random lift whenever we choose nearest neighbours of u and v we find a set of disjoint paths connecting vertices from these two sets. Thus, as an immediate consequence of Theorem 18 we get the following result.

Corollary 3. *If G is a connected graph with minimum degree $\delta \geq 3$, then aas δ-diameter of $\tilde{G} \in L_n(G)$ is $O(\log n)$.*

□

Do not keep saying to yourself, if you can possibly avoid it, "But how can it be like that?" because you will get "down the drain," into a blind alley from which nobody has yet escaped. Nobody knows how it can be like that.

Richard Feynman, *The Character of Physical Law, 1965.*

5

Hamilton cycles in random lifts

Finding a Hamiltonian cycle is one of the most celebrated problems in graph theory and theory of random graphs (see [7] for many results in this area). It is no surprise that it also caught the attention of researchers in the case of random lifts. The main question is whether it is true that for every G either almost all or almost none of the random lifts of G contain a Hamilton cycle as in the case for perfect matching (see Theorem 13). In a weaker version of the problem, posed by Linial [26], we ask whether this property is true for a subclass of d-regular graphs.

Problem 1. *Let G be a d-regular connected graph with $d \geq 3$. Is it true that almost every random lift of G is Hamiltonian?*

Burgin, Chebolu, Cooper and Frieze have proven that for sufficiently large complete graphs and complete bipartite graphs almost every lift is Hamiltonian.

Theorem 20 ([8])**.** *There exists a constant t_0, such that if $t \geq t_0$, then asymptotically almost surely $\tilde{K}_t$ is Hamiltonian.*

Theorem 21 ([8])**.** *There exists a constant t_1, such that if $t \geq t_1$, then asymptotically almost surely $\tilde{K}_{t,t}$ is Hamiltonian.*

Chebolu and Frieze [9] were able to expand this result to the random lifts of complete directed graphs (where lifted edges preserve orientation of edges from the base graph). In the previous work [34] on this problem we show that the constant t_0 is less than 30. Here we present the proof of a stronger statement.

Theorem 22. *Let G be a graph with minimum degree at least five which contains at least two edge disjoint Hamilton cycles whose union is not a bipartite graph. Then aas $\tilde{G}$ is Hamiltonian.*

The structure of the proof is the following. First we describe the idea behind the algorithm which finds the Hamilton cycle in $\tilde{G}$. Then we present the algorithm. In the last section we show that asymptotically almost surely it succeeds in finding Hamilton cycle in $\tilde{G}$.

5.1 Preliminaries

Our algorithm will use the path reversal technique of Pósa [32]. Let G be any connected graph and $P = v_0 v_1 ... v_m$ be a path in G. If $1 \leq i \leq m-2$ and $\{v_m, v_i\}$ is an edge of G, then $P' = v_0 v_1 ... v_i v_m v_{m-1} ... v_{i+1}$ is a path in G with the same vertex set as P. We call P' a Pósa rotation of P with the preserved *starting point* v_0 and the *pivot* v_i. Note that used edge $\{v_m, v_i\}$ in path P' is not incidence to its new end. By $\mathcal{P}_q(P, v_0)$ we denote the set of all paths of G which can be obtained from P by at most q rotations preserving the starting point v_0.

For the clarity of argument in this and next two sections symbols that corresponds to elements of base graph will be written in bold. Let $\mathbf{G}$ be a connected graph on k vertices with $\delta(\mathbf{G}) \geq 5$ which contains two edge disjoint Hamilton cycles $\mathbf{H}_1$ and $\mathbf{H}_2$. Choose any vertex $\mathbf{h}_1$ and label each vertex twice according to its appearance in Hamilton cycles i.e. $\mathbf{H}_1 = \mathbf{h}_1 \mathbf{h}_2 \dots \mathbf{h}_k \mathbf{h}_1$ and $\mathbf{H}_2 = \mathbf{h}'_1 \mathbf{h}'_2 \dots \mathbf{h}'_k \mathbf{h}'_1$, where $\mathbf{h}'_1 = \mathbf{h}_1$.

Due to Lemma 3 aas the random lift of $\mathbf{H}_1$ consists of disjoint cycles $C_1, C_2, ..., C_\ell$, where $\ell \leq 2 \log n$. We refer to these as *basic cycles*. We will use the property that cycles in the lift preserve the order of vertices from the cycles in the base graph, i.e. for every basic cycle there exists $r \geq 1$ such that it can be written as

$$h_1^1 h_2^1 \dots h_k^1 h_1^2 h_2^2 \dots h_k^2 \dots h_1^r h_2^r \dots h_k^r h_1^1, \tag{5.1}$$

where h_j^i is an element of the fiber $\tilde{G}_{h_j}$.

Let $\mathbf{G}_1 = \mathbf{G} - \mathbf{H}_1$. The main idea of our argument goes as follows. First we generate the lift $\tilde{H}_1$, next we try to connect cycles $C_1, C_2, ..., C_\ell$ into one long path using edges of $\tilde{G}_1$. Note that $\delta(\tilde{G}_1) \geq 3$. To this end at some point we will have to use the property that $\mathbf{G}_1$ contains the Hamilton cycle $\mathbf{H}_2$ to close the path into a cycle.

Denote the longest cycle in $\tilde{H}_1$ by C. We shall try to connect C to any other basic cycle in the lift using the edges of $\tilde{G}_1$. Once we succeed in finding connecting edge, we break the cycle C and connect it to some other basic cycle. The path created in such a way will be denoted by P. Subsequently we want to increase the length of P by "absorbing" one basic cycle at a time. We shall do it by generating edges of $\tilde{G}_1$ which are incident to one of the ends of the path P. If we connect it to some basic cycle, say C'_s, then we replace P by a longer path adding all vertices from C'_s, otherwise, either we try to connect the ends of P to create a new cycle C, or try to replace P by another path using Pósa transformation (in fact, since the probability that we extend P is small, we use Pósa transformations right away to produce a lot of paths with the same vertex set as P as candidates for further extensions). If the obtained cycle C is still not a Hamilton cycle, then we try to merge C with some of the remaining basic cycles and repeat the procedure.

5.2 The algorithm

We generate a graph $\tilde{G}$ in each step of the algorithm edge by edge. First we generate the lift $\tilde{H}_1$, next at each point randomly, for a given vertex v, we choose its neighbours in $\tilde{G}_1$ from all available candidates. We shall do it vertex by vertex since we want to use the fact that for each vertex the choice of its neighbours we generate at some stage of the algorithm do not depend much on the previous history of the procedure.

Whenever we have already generated an edge from $\tilde{G}_1$ adjacent to a vertex v we call such a vertex *inactive*, vertices that are not inactive are called *active*. We will denote the set of inactive vertices by D.

In the analysis of the algorithm we shall show that asymptotically almost surely in order to merge P with one of the remaining basic cycles we deactivates at most $5n^{4/5}$ vertices. Since the number of basic cycles is a.a.s. less than $2\log n$ (see Section 5.3 below) and at each iteration we connect one basic cycle to the cycle C, it implies that in order to perform the whole procedure, we need to generate edges of $\tilde{G}_1$ incident to not more than $10n^{4/5}\log n < n^{5/6}$ vertices. In the next chapter we show that in fact at the end of the algorithm execution aas we have $|D| < n^{5/6}$.

The algorithm consists of seven phases.

Phase 1 – Cycle Lift

Generate a lift $\tilde{H}_1$ of $\boldsymbol{H}_1$. Assign C to be the longest cycle in $\tilde{H}_1$.

Phase 2 – Cycle Merge

Given a cycle C and a set of basic cycles $C'_1, \dots, C'_s$ disjoint with C do the following:

A. *If $0 < \sum_{i=1}^{s} |V(C'_i)| < n^{9/10}$ take any vertex v which belongs to a basic cycle C'_1 and generate edges of $\tilde{G}_1$ incident to it. If one of these edges e connects C'_1 to C assign* *to P a path whose vertex set is $V(C) \cup V(C'_1)$ and those two parts are joined by e. Otherwise take another vertex $v' \in C'_1$ and repeat the operation.*

B. *If $\sum_{i=1}^{s} |V(C'_i)| \geq n^{9/10}$ choose any $n^{1/3}$ active vertices of C which are at distance at least 2 from the set of all inactive vertices and generate edges of $\tilde{G}_1$ incident to them. If one of these edges e connects C'_i to C assign to P a path whose vertex set is $V(C) \cup V(C'_i)$ and those two parts are joined by e. Otherwise repeat the operation.*

Phase 3 – Path Merge

Given a path P and some basic cycles $C'_1, \ldots, C'_s$, if any end of P is connected to a basic cycle C'_i replace P by a new path with vertex set $V(P) \cup V(C'_i)$.

Phase 4 – Cloning Path

Let us suppose we are given a path P whose both ends are active, and a set of basic cycles $C'_1, \ldots, C'_s$.

Repeat following actions:

Take $P = w_1 w_2 \ldots w_t$ and apply to it repeatedly Pósa transformation preserving starting point w_1. Continue until $\log^2 n$ different paths starting at w_1 and ending at w_{i_j}, $j = 1, 2, \ldots, \log^2 n$ will be found. Now reverse each of these paths and apply to each of them the transformation preserving point w_{i_j}. Continue to perform the operations until one of the following conditions holds:

— there is an edge connecting path $P \in \mathcal{P}_q(P, w_1)$ with some basic cycle C'_{i_0},

— there is an edge connecting path $P \in \mathcal{P}_{q'}(P_{i_j}, w_{i_j})$ with some basic cycle C'_{i_0},

— there are $r = \log^2 n$ paths $P_1, \ldots, P_r$ such that each of them has the same vertex set as P, and all $2r$ vertices which are ends of these paths are pairwise different and active.

In the case that one of the first two conditions is met go back to Phase 3, in the case that the third condition holds continue to Phase 5.

Phase 5 – Multiplying Ends

For every path $P_1, \ldots, P_r$ constructed in the Phase 4 split the vertex set $V(P_j)$ of P_j into two roughly equal disjoint sets $V_1, V_2 \subset V(P_j)$, $|V_1|, |V_2| \geq (|V(P_j)| - 1)/2$. Thus every path $P_j = w_1 w_2 \ldots w_t$ splits into two paths $P'_j = w_1 w_2 \ldots w_{i-1} w_i$ and $P''_j = w_{i+1} w_{i+2} \ldots w_t$, where $i = \lceil t/2 \rceil$.

At any point of the phase if there is:

— an edge closing some path P_j to form a cycle, then go to Phase 2,

— an edge connecting P_j with some basic cycle, then go to Phase 3.

Repeat simultaneously for each path $P_1, \ldots, P_r$:

Apply a series of Pósa transformations to the path P'_j which preserve the starting point w_i, and a series of Pósa transformations to the path P''_j which preserve starting point w_{i+1}.

(We apply a single Pósa transformation to each of the paths in turn before we apply the next Pósa transformation).

Stop if for any path you find two sets $S_1 \subset V_1$, $S_2 \subset V_2$, such that $|S_1|, |S_2| \geq n^{3/5}\log^2 n$ with the following property:

> *For every $x \in S_1$ and $y \in S_2$ there is a path P_{xy} of length $|P_j|$ which starts at x ends at y whose first $|V_1|$ vertices are those from V_1 and last $|V_2|$ vertices are those from V_2.*

Phase 6 – Adjusting

Choose any edge $\{\boldsymbol{x},\boldsymbol{y}\}$ from $\boldsymbol{G} \setminus (\boldsymbol{H}_1 \cup \boldsymbol{H}_2)$. Use at most $|G|(|S_1| + |S_2|)$ Pósa transformations to switch the end w_1 of the path P' and the end w_t of the path P'' to replace the sets S_1, S_2 generated in the previous stage by slightly smaller sets $S_1' \subset V_1$, $S_2' \subset V_2$, $|S_1'|, |S_2'| \geq n^{3/5}$, such that S_1' is contained in the fiber $\tilde{G}_x$ and $S_2' \subset \tilde{G}_y$.

Phase 7 – Closing a cycle

Generate edges between $\tilde{G}_x$ and $\tilde{G}_y$ incident to vertices from S_1'. If one of them has an end in S_2' then STOP if the resulted cycle is a Hamilton cycle, or otherwise go to Phase 2.

5.3 The analysis of the algorithm

In this section we show that aas the algorithm returns a Hamiltonian cycle and, consequently, Theorem 22 follows.

Phase 1. We start the analysis of the algorithm with Phase 1. As already mentioned, Lemma 3 states that the random lift of H_1 asymptotically almost surely consists of disjoint cycles $C_1, C_2, ..., C_\ell$, where $\ell \leq 2\log n$. Note that this means that the length of the longest cycle $C \in \tilde{H}_1$ is at least $n/(2\log n)$. Observe that since the number of basic cycles is bounded from above by $2\log n$, Phases 2 and 3 can be invoked at most $2\log n$ times.

We shall show that with probability at least $1 - o(1/\log n)$ during Phases 2-7 we create a cycle C each time deactivating fewer that $5n^{4/5}$ vertices. Thus, at the end of the execution of the algorithm we will have $|D| \leq 10n^{4/5}\log n \leq n^{5/6}$. This bound, which states that to build a cycle we need only to generate a small portion of the random n-lift, shall be used very often in the analysis of our procedure.

Note that in any step in which we deactivate a vertex either it is already in P, or we have just added it to P. Consequently, all vertices outside P are active. Moreover, since at every point of the algorithm we want the ends of path P to be active vertices, we perform Pósa rotation only in the case when the new end of P is an active vertex. Notice that this happen to be true whenever in the rotation the pivot is at distance at least 2 from any inactive vertex.

Phase 2. In this step we want to connect cycle C with any basic cycle disjoint with it, creating a long path P. As stated above we require that vertices which connect those two cycles are not adjacent to any inactive vertices.

In case A the total number of vertices in the remaining basic cycles which are yet to be joined to C is smaller than $n^{0.9}$. The probability that a vertex from the basic cycle C_1' has a neighbour inside C which is at distance at least 2 from any inactive vertex is larger than

$$\frac{n - n^{9/10} - \Delta(G) \cdot |D|}{n - |D|} \geq 1 - \frac{2n^{9/10}}{0.9n} = 1 - o(1/\log n),$$

since we need to exclude vertices outside C together with all inactive vertices and their neighbours. Hence, aas the merging deactivates only one vertex and we do not need to repeat the procedure more times.

For case B note that since $|C| \geq n/(2\log n)$ and there is always fewer than $n^{5/6}$ inactive vertices, one can greedily select $n^{1/3}$ vertices of C which are at distance at least 2 from any inactive vertex and from each other. Clearly, the probability that some of these vertices is adjacent in $\tilde{G}_1$ to one of the basic cycles is bounded from above by

$$1 - \left(\frac{n - n^{9/10}}{n}\right)^{n^{1/3}} \leq 1 - \left(1 - n^{-1/10}\right)^{n^{1/3}} = 1 - o(1/\log n).$$

Again aas we succeed with first set of $n^{1/3}$ vertices and we do not need to repeat the procedure.

Altogether each time we invoke this phase with probability at least $1 - o(1/\log n)$ we deactivate at most $n^{1/3}$ vertices.

Phase 3.

We do not generate any edges in this step, and so we do not deactivate any vertices.

Phase 4. Let $P = w_1, ..., w_t$. Our aim is either to find an edge of $\tilde{G}_1$ joining one end of a path $P' \in \mathcal{P}_q(P, w_1)$, or $P'' \in \mathcal{P}_{q'}(P_{i_j}, w_{i_j})$, to one of the cycles outside P and go to Phase 3, or to find for $r = \log^2 n$ a set of paths $P_1, \ldots, P_r$ such that each of them has the same vertex set as P, and all $2r$ vertices which are ends of these paths are different and active.

There are two stages in this phase. First we take path P and find a set of r paths which start at w_1 and whose r ends are distinct and active. Notice that after any Pósa transformation we want the new end to be active so we require that the pivot w_i has no inactive neighbours. Thus we estimate the probability that in any of the $\log^2 n$ possibly required Pósa transformations the new end of our transformed path either is connected to a vertex which is at distance less than 2 to any inactive vertex, or is the neighbour of one of the ends of previously generated paths. This probability can be crudely bounded above by

$$\log^2 n \, \frac{(\Delta(G) \cdot |D| + \log^2 n)}{n - |D|} \leq \frac{\Delta(G) \cdot \log^4 n \cdot n^{5/6}}{0.9n} = o(1/\log n)\,.$$

In the second stage we take all paths $P_1, ..., P_r$ and apply to them the Pósa transformations preserving the ends chosen in the first stage. At this time the structure of each path is distinct, so in the process of applying consecutive transformations we might get different results for each path. Moreover we want those new ends to be different from the ends generated in previous stage. Thus we take the first path P_1 and apply transformations in order to generate a set of $\log^2 n$ active ends for it and choose one of them as the end of P_1. Then we take path P_2; if it admits the same transformations as P_1, then we select one of the vertices generated for P_1, which has not already been taken, as the end for P_2. In the opposite case we apply Pósa transformations for P_2 and generate a new set of $\log^2 n$ ends for it. We repeat the same operations for all other paths. Notice that in the worst case scenario we need to make at most $\log^4 n$ single transformations in total. Similarly to previous case the probability that in any of $\log^4 n$ required Pósa transformations the new end of our transformed path is either connected to a vertex which is within distance less than 2 to any inactive vertex, or has been the neighbour of the end of one of the previously generated paths, is bounded from above by

$$\log^4 n \, \frac{(\Delta(G) \cdot |D| + \log^4 n)}{n - |D|} \leq \frac{\Delta(G) \cdot \log^8 n \cdot n^{5/6}}{0.9n} = o(1/\log n)\,.$$

Note also that each time we run this phase we deactivate at most $\log^2 n + \log^4 n \leq 2\log^4 n$ vertices.

Phase 5. Let us recall that, roughly speaking, in this phase we want to take any of the paths $P_j = w_1 w_2 \ldots w_t$ constructed in the previous case, split it into two halves $P' = w_1 w_2 \ldots w_{i-1} w_i$ and $P'' = w_{i+1} w_{i+2} \ldots w_t$, where $i = \lceil t/2 \rceil$, and apply to them transformations preserving respectively w_i and w_{i+1} in order to find at least $n^{3/5} \log^2 n$ new feasible ends for each of them.

We show that the probability that we succeed in doing it for a given path is bounded away from zero, by some constant $\alpha > 0$. Thus if we repeat this for $\log^2 n$ paths, then with probability $1 - o(1/\log n)$ for at least one of them we expand the set of feasible ends to the required size.

The existence of a constant $\alpha > 0$ follows easily from the theory of branching process (see Section 2.3). Indeed, take one path, say P', and first generate all its possible ends using the transformation preserving the end w_i (this will be the first generations of ends), then apply consecutive transformation to obtained ends in order to get the second generations of ends, and so on. In each step we generate at least three new vertices (since the minimum degree of $\tilde{G}_1 = \tilde{G} - \tilde{H}_1$ is at least three) and we fail if we choose in such a trial either a vertex from the other path P'', or a vertex which is adjacent to inactive vertex or one of the ends chosen so far. Since $|P''| \leq n/2 + 1 \leq 0.501n$, the probability of making a bad choice is in each step bounded from above by

$$\frac{0.501n + \Delta(G)|D|}{n - |D|} \leq 0.51.$$

Consequently, the number of successful choices (i.e. the ones which either lead to a new end or allow us to go to Phase 3) in one round is stochastically bounded from below by the binomially distributed random variable $B(3, 0.49)$.

Thus, let us recall, we treat the process of applying consecutive Pósa transformations as a branching process. Since every active vertex v has at least 3 edges in $\tilde{G}_1$ which are still to be revealed, the possible number of descendants for each ancestor is bounded from below by 3. The probability of producing new individual in the next generation equals the probability that generated edge connects v with either a vertex of P' which is not adjacent to an inactive vertex or vertex generated in previous steps or with a vertex outside $P' \cup P''$. Since the number of inactive vertices is at most $n^{5/6} = o(n)$ and clearly $|P''| \leq 0.501n$, the process of generating feasible ends for the path P can be stochastically bounded from below by the branching process defined by a variable with binomial distribution $B(3, 0.49)$.

Since $3 * 0.49 > 1$, by Theorem 1 with probability $\beta > 0.61$ the branching process will not die out. Furthermore, in Section 2.3, we showed that with probability at least $1 - 2\exp(-n^{3/5})$ the first time we get $n^{3/5}\log^2 n$ vertices in one generation the total number of descendants in the whole process is bounded from above by $5n^{3/5}\log^2 n$ (see Lemma 2). Consequently, with probability at least $\beta/2$, after using at most $5n^{3/5}\log^2 n$ vertices we either merge the end of P' with one of basic cycles (and so go to Phase 3) or generate at least $n^{3/5}\log^2 n$ different active ends for this path. Hence, the probability that it happens at the same time for P' and P'' is bounded from below by $\alpha = (\beta/2)^2$.

As we have already mentioned at the beginning the previous phase of the algorithm provided us not one, but $\log^2 n$ paths with different ends. Consequently, with probability

$$1 - (1 - \alpha)^{\log^2 n} = 1 - o(1/\log n)$$

we succeed in expanding the set of feasible ends for at least one of the paths. Hence, with probability at least $1 - o(1/\log n)$ this phase of the algorithm can be completed with the total number of deactivated vertices bounded from above by $5\Delta(G)n^{3/5}\log^2 n \log^2 n \leq n^{4/5}$.

Phase 6. The sets S_1 and S_2 found in the previous phase are such that each edge connecting them creates a cycle. Such a cycle is either a Hamilton cycle or can be merged to some remaining basic cycles (back in the Phase 2).

Note however that vertices in S_1 and S_2 are spread over fibers of $\tilde{G}$. In particular, it might happen that sets S_1 and S_2 are placed in two fibers which corresponds to non-adjacent vertices of $\mathbf{G}$ and so we cannot expect them to be connected by an edge in $\tilde{G}$. In the best case scenario we have $S_1 \subseteq \tilde{G}_x$, $S_2 \subseteq \tilde{G}_y$ and $\{\mathbf{x,y}\} \in E(\mathbf{G})$. Hence, in this phase we want to "switch" elements of the sets S_1 and S_2 (or at least a large portion of them) to the chosen fibers $\tilde{G}_x$ and $\tilde{G}_y$. In order to do that we use the property that $\mathbf{G}_1$ contains Hamilton cycle $\mathbf{H}_2$ disjoint from $\mathbf{H}_1$.

Let $P' = w_1 w_2 \ldots w_i$ be defined as "the half" of the path we have dealt with in the previous phase, and let $w_1 \in S_1$. We would like to argue that, with probability bounded away from zero by some constant $\gamma > 0$, we can deactivate at most $|\mathbf{G}|$ vertices in order to either connect P' to some remaining basic cycle, or turn P' by a sequence of transformations preserving the end w_i into a path with an end on the chosen fiber $\tilde{G}_x$.

Let us recall first that P' has been obtained in the process of merging and transforming basic cycles obtained in the first phase. Each of the basic cycles has a periodic structure (see (5.1)), which implies that they are evenly distributed across the fibers of the lift. Let $k = |\mathbf{G}|$ where, let us recall, k is a constant which does not grow with n. In the case when the length of P' is smaller than $n/3$ the total length of basic cycles outside P' and P'' is $m \geq n/3$ and furthermore each fiber contains precisely m/k vertices which belong to basic cycles outside $V(P') \cup V(P'')$. Consequently, with positive probability (at least $m/(nk) \geq 1/(3k)$) we merge the end of P' with a basic cycle deactivating just one vertex.

Let us consider now the more challenging case, when P is very long and the length of P' is at least $n/3$. We are interested in the structure of the path P', namely to what extent it preserves the structure of basic cycles. Whenever we have joined two cycles or perform a Pósa transformation we have perturbed the cyclic distribution of vertices. More precisely, a single merge or transformation could spoil at most three of sequences $...h_1^i h_2^i \ldots h_{k-1}^i h_k^i h_1^{i+1}...$ which occurred in the path P. See Figure 1 for an example of transformation, note that after transformation in part of the path the order of the vertices in the sequence is reversed.

Observe that the number of joins and transformations made to a path P' is bounded by the number of inactive vertices. Since during the algorithm we deactivate at most $n^{5/6}$ vertices, there are at least $(n/3k) - 3n^{5/6} > 2n/(7k)$ sequences of consecutive vertices which belong to fibers given by the order of vertices in H_1. Some of the sequences could get reversed in the transformations (see Figure 5.1), but at least half of them, i.e. at least $n/(7k)$, are sequences of consecutive vertices appearing in the same order which is either $\mathbf{h}_1 \ldots \mathbf{h}_{k-1}\mathbf{h}_k\mathbf{h}_1$ or $\mathbf{h}_1\mathbf{h}_k\mathbf{h}_{k-1} \ldots \mathbf{h}_2\mathbf{h}_1$. In the former case we say that the orientation of the sequence is positive, in the latter one we say that it is negative.

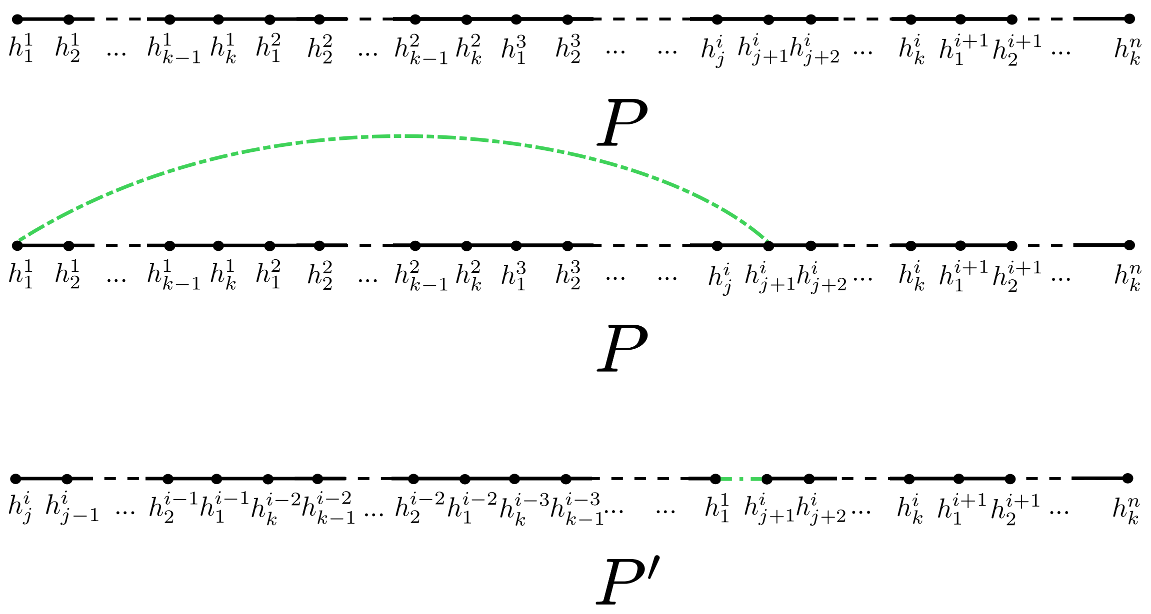

Figure 5.1: The path P above consists of sequences of vertices from a path which is a part of a lift Hamilton cycle $\mathbf{H}_1 = \mathbf{h}_1 ... \mathbf{h}_k$ in the base graph, where by h_i^j we denote a vertex from the fiber above $\mathbf{h}_i$. After Pósa transformation with the pivot h_{j+1}^i we get a new path P'. Notice that in this way we broke down the sequence $h_1^i ... h_k^i h_1^{i+1}$. Moreover sequences $h_1^1 ... h_1^{i-1}$ are reversed in the path P'.

Thus, let us choose $n/(7k)$ sequences with the same orientation. We subdivide P' into $k-1$ connected sections, such that each of them contain at least $z = n/(8k^2)$ sequences of the same orientation and denote those sections as $Q_1, ..., Q_{k-1}$. See Figure 5.2 for an example.

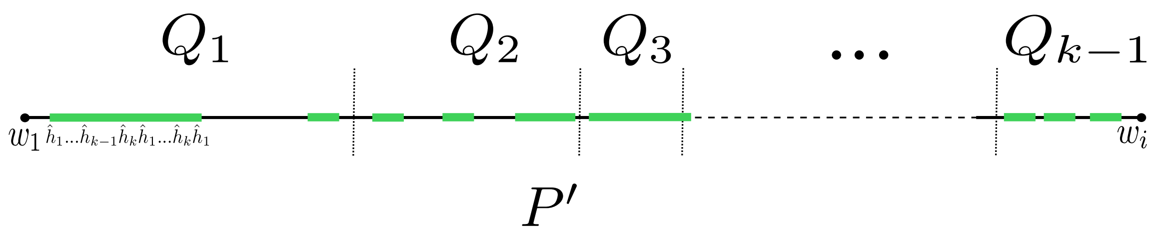

Figure 5.2: The path P' divided into sections $Q_1, ..., Q_{k-1}$. By $\hat{h}_i$ we denote that vertex is an element of the fiber above $\mathbf{h}_i$. Green segments indicate sequences of vertices which belong to fibers given by order of vertices in $\mathbf{H}_1$ (there could be more than one sequence in one segment).

Let $\vec{\mathbf{H}}_1$ be a directed cycle created by orienting edges of $\mathbf{H}_1$ in one direction. Let us recall that $\mathbf{H}_2 = \mathbf{h}'_1\mathbf{h}'_2 \dots \mathbf{h}'_k\mathbf{h}'_1$ is the second Hamiltonian cycle in the base graph $\mathbf{G}$ which does not share any edges with the cycle $\mathbf{H}_1$. In the definition below we treat each undirected edge of $\mathbf{H}_2$ as a pair of edges with opposite orientations.

Definition. A directed path $\mathbf{P}$ in $\mathbf{G}$ is called $\mathbf{H}_2\vec{\mathbf{H}}_1$-alternating if it starts by an edge of $\mathbf{H}_2$, ends with an edge of $\vec{\mathbf{H}}_1$ and its edges belong alternatively to Hamilton cycle $\mathbf{H}_2$ and directed cycle $\vec{\mathbf{H}}_1$.

The proof of the following lemma can be find in the next section of this paper.

Lemma 23. *Let $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ be edge disjoint Hamilton cycles such that $\boldsymbol{H} = \boldsymbol{H}_1 \cup \boldsymbol{H}_2$ is a non-bipartite graph, then for every pair of vertices $v, u \in \boldsymbol{H}$ there exists a $\boldsymbol{H}_2\vec{\boldsymbol{H}}_1$-alternating path from v to u.*

Denote by u the end of P we want to switch and let $\tilde{G}_x$ be a fiber which we want to switch u onto. Without loss of generality we may assume that the end u of P' belongs to the fiber $\tilde{G}_z$ above some vertex $\mathbf{z}$. By Lemma 23 in $\mathbf{G}$ there exist a $\mathbf{H}_2\vec{\mathbf{H}}_1$-alternating path $\mathbf{R} = \mathbf{z}\mathbf{b}_1\mathbf{a}_1\mathbf{b}_2\mathbf{a}_2, .., \mathbf{b}_{\ell-1}\mathbf{a}_\ell\mathbf{b}_\ell\mathbf{x}$ from $\mathbf{z}$ to $\mathbf{x}$.

Now let us try to generate an edge from u to a vertex $b^1 \in \tilde{G}_{b_1}$. The probability that such a vertex exists and belongs to Q_1 equals $1/9k^2$. If b^1 is not a part of Q_1 then we stop, otherwise we use it as the pivot in Pósa transformation that would change P' into a path P'_1 which ends at a vertex $a^1 \in \tilde{G}_{a_1}$. Then we continue the transformations in the same manner as in the first step. We try to connect a^1 to a vertex $b^2 \in \tilde{G}_{b_2}$ which belongs to Q_2. Again we succeed with probability $1/9k^2$. Next we use b^2 as the pivot in Pósa transformation that would change P'_1 into a path P'_2 which ends at a vertex a^2 from fiber above $\mathbf{a}_2$. We apply the same operations until we get to vertex $w \in \tilde{G}_x$. Notice that since pivot for path P_i is closer on a path P_i to the vertex w_i than the pivot used for path P_{i-1} the transformation does not change the orientation of the sequences in the $Q_{i+1}, ..Q_{k-1}$. See Figure 3 for an example.

Note that since the length of a $\mathbf{H}_2\vec{\mathbf{H}}_1$-alternating path is bounded by $2(k-1)$ during the process we have to generate at most $k-1$ edges (those which belongs to $\tilde{\mathbf{H}}_2$). Hence, with probability at least $(9k^2)^{-k}$ we can move a given vertex from S_1, from any fiber to the designated fiber above vertex $\mathbf{x}$. The same analysis can be repeated in respect to the second path P'' and vertices from S_2 which we would like to place on fiber $\tilde{G}_y$. Since $|S_1|, |S_2| \geq n^{3/5}\log^2 n$, with probability at least $1 - \exp(-n^{3/5}) = 1 - o(1/\log n)$ we can successfully switch at least $n^{3/5}$ of them. Note that in this process we deactivated at most $2|\mathbf{G}|n^{3/5}\log^2 n < n^{4/5}$ new vertices.

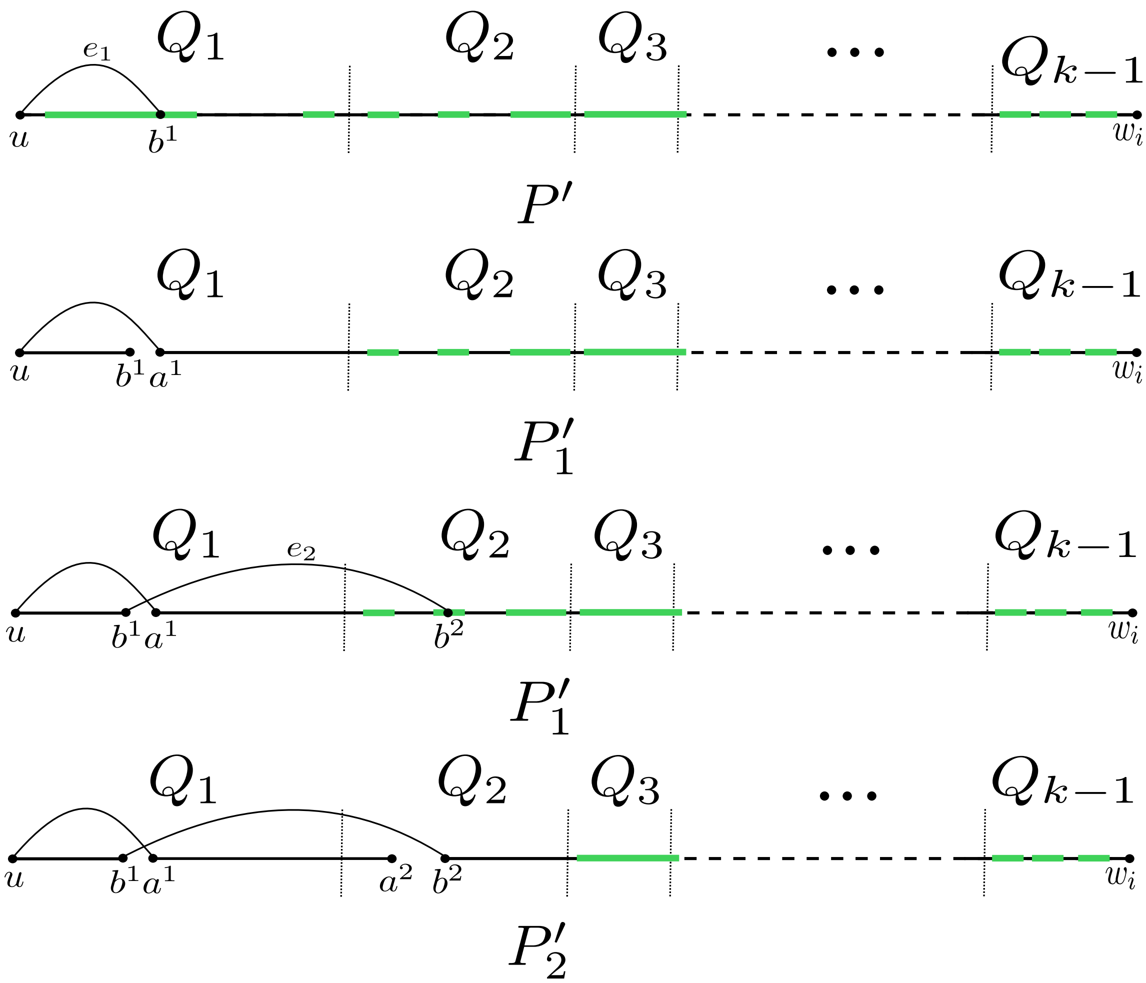

Figure 5.3: Two steps of the process of switching the end of path P' onto desired fiber. Green sections indicates positively oriented sequences of vertices from fibers above consecutive vertices of the cycle $\mathbf{H}_1$. The vertex u is the end of path P'. The edge e_1 connects vertex u with vertex b^1 from fiber $\tilde{G}_{b_1}$. The edge e_2 connects vertex a^1 with vertex b^2 from fiber $\tilde{G}_{b_2}$.

Phase 7. Since S'_1 and S'_2 belong to different fibers which correspond to adjacent vertices from G the probability that there are no edges between S'_1 and S'_2 is bounded from above by

$$\left(\frac{n-|S'_1|-|D|}{n-|D|}\right)^{|S'_2|} \leq \left(\frac{n-|S'_1|}{n}\right)^{|S'_2|} \leq \exp\left(-\frac{|S'_1||S'_2|}{2n}\right)$$
$$\leq \exp(-n^{1/6}/2) = o(1/\log n)\,.$$

Clearly, in the last phase we deactivated at most $|S'_1| \leq n^{4/5}$ vertices.

This completes the analysis of the algorithm and the proof of Theorem 22. □

5.4 Proof of the Lemma

In this section we present the proof of the Lemma 23. In fact we shall show a slightly more general result.

Lemma 24. *Let $\vec{H}_1$ be a directed Hamilton cycle and H_2 be a connected d-regular graph, edge disjoint with H_1, which is such that $H = H_1 \cup H_2$ is a non-bipartite graph. Then for every pair of vertices $v, u \in H$ there exists a $H_2\vec{H}_1$-alternating path from v to u.*

Proof. For the purpose of the proof let color edges of $\vec{H}_1$ *red* and edges of H_2 *blue*. We proceed in following way: starting from vertex v we build $H_2\vec{H}_1$-alternating paths to other vertices. We mark vertices in H *red* if we leave this vertex by a red edge and respectively *blue* we leave this vertex by a blue edge (vertices that admits both colors are denoted as red-blue). Additionally we color vertex v blue. Notice that $H_2\vec{H}_1$-alternating path is now equivalent to a BlueRed-alternating path, starting with blue edge and ending with red edge (that is, ending in a blue or red-blue vertex).

Denote by N the set of vertices that are not reached from vertex v by an $H_2\vec{H}_1$ alternating path (that did not get any color) and by R, B, RB the sets of vertices which are coloured red, blue and by both colors respectively. Let us make the following observations.

(i) There are no directed red edges $\{xy\}$ from $x \in R \cup RB$ to $y \in N$, because that would imply $y \in B$. The same argument shows that there are no blue edges between RB and N.

(ii) There is in total at most one directed red edge $\{xy\}$ between sets N and $B \cup RB$, or sets B and RB, or within the set B because apart from the starting vertex v in order to color vertex blue we have to first reach it by a red edge (come to it from red vertex).

(iii) There are no blue edges $\{xy\}$ between $x \in RB$ and $y \in B$, since in this case we would be able to reach x by red edge, use blue edge to get to y and then leave y by red edge, which results in $y \in RB$. The same argument proves that there are no red edges directed from RB to R, no blue edges inside B, and no red edges inside R.

Figure 4 shows all the possible edges which can occur between sets R, B, RB and N.

Denote by $|X \to Y|$ the number of red edges coming from the set X to the set Y. Since the red edges form the directed Hamilton cycle $\vec{H}_1$ in G there is exactly one red edge coming out and one red edge coming in to every vertex in G. Thus we can estimate the size of R counting red edges incoming to it. Therefore

$$|R| = |N \to R| + |B \to R|.$$

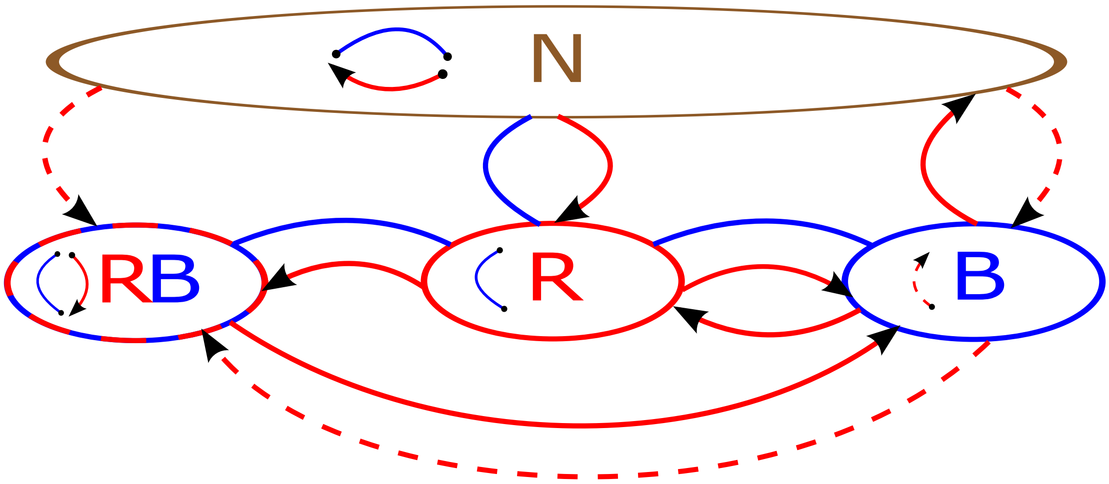

Figure 5.4: Diagram of edges between vertices in $H = H_1 \cup H_2$ created by following $H_2\vec{H}_1$-alternating paths from some vertex $v \in H$. Edges of Hamilton cycle H_1 are directed and coloured red, and edges of Hamilton cycle H_2 are coloured blue.

In a similar way (counting red edges going out from vertices), we have

$$|B| = |B \to N| + |B \to R| + |B \to RB| + |B \to B|$$

The red edges form a directed Hamilton cycle, thus the number of red edges coming in the set N equals to the number of red edges coming out from the set N. Hence, by (ii), we have $|B \to N| + |B \to RB| + |B \to B| \geq |N \to R|$, which implies that $|B| \geq |R|$.

Each blue and red vertex is incident to exactly d blue edges. Notice that all blue edges that are leaving B go to the vertices in R, which, together with $|B| \geq |R|$, implies that $|B| = |R|$. Let us consider now three possible sizes of set $|B| = |R|$.

Case 1. $|B| = |R| = 0$.

Then all vertices of G belong to $RB \cup N$, and since both H_1 and H_2 are connected graphs we must have $N = \emptyset$. Consequently, all vertices are coloured with both colors and the assertion follows.

Case 2. $0 < |B| + |R| < |G|$.

Then the blue edges between R and B induce the d-regular subgraph of H_2, which is clearly a component of H_2. This contradicts the fact that H_2 is connected.

Case 3. $|B| + |R| = |G|$.

In this case G is a bipartite graph in which red vertices form one part of the partition and blue vertices form the other, which contradicts the assumption of the lemma.

This concludes the proof of the lemma. □

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