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Decompositions of Complete Uniform Multipartite Hypergraphs

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Abstract

In recent years, researchers have studied the existence of complete uniform hypergraphs into small-order hypergraphs. In particular, results on small 3-uniform graphs including loose 3, 4, and 5 cycles have been studied, as well as 4-uniform loose cycles of length 3. As part of these studies, decompositions of multipartite hypergraphs were constructed. In this paper, we extend this work to higher uniformity and order as well as expand the class of hypergraphs.

1 Introduction

A common problem in combinatorial designs is graph decompositions. Most results on graph decompositions are for simple graphs. Results on hypergraph decompositions are limited to small hypergraphs of small uniformity or to a limited class of hypergraphs. The author contributed this research in [2] and [3]. The goal of this research is to extend the study of hypergraph decompositions to a wider range of graph classes and for graphs of any size and uniformity.

1.1 Main Results

In our paper we present a new technique for decomposing complete multipartite graphs. This technique applies to hypergraphs of any size and uniformity, and we present three graph classes to which the decomposition technique applies. Finally, we introduce a property of hypergraphs which relates coloring of the hypergraphs to decomposition of complete multipartite graphs and give some neccesary conditions for this property.

1.2 Graph Decompositions

The study of combinatorial designs concerns the arrangements of finite sets into subsets so that certain properties are met. A famous problem in combinatorial designs is the existence of a *Steiner triple system.* A Steiner triple system is a set of points S and a set of three element subsets of S, T, such that every pair of points in S occurs in exactly one triple in T. The order of the system is $|S|$. A Steiner triple system can be rephrased as a graph decomposition. Suppose each triple in a Steiner triple system of order n is a graph where the vertices are the points of the triple and the edges form a triangle between the vertices. Then the Steiner triple system is equivalent to a decomposition of K_n into triangles.

Example 1.1. $\{\{1,2,3,4,5,6,7\},\{\{1,2,4\},\{2,3,5\},\{3,4,7\},\{4,5,1\},\{5,6,2\},\{6,7,3\},\{7,1,4\}\}\}$ *is a Steiner triple system of order* 7*. The corresponding graph decomposition is shown in Figure 1.*

We now present an overview of graph decompositions. A *decomposition* of a graph K is a set $\Delta = \{G_1, G_2, \ldots, G_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_s) = E(K)$. If each element of Δ is isomorphic to a fixed graph G, then Δ is called a *G-decomposition* of K. A G-decomposition of K_v is also known as a *G-design of order v.* A K_k-design of order v is an *$S(2,k,v)$-design* or a *Steiner system.* An $S(2,k,v)$-design is also known as a *balanced incomplete block design of index 1* or a *$(v,k,1)$-BIBD.* The problem of determining all v for which there exists a G-design of order v is of special interest (see [1] for a survey).

The notion of decompositions of graphs naturally extends to decompositions of uniform hypergraphs. A *hypergraph* H consists of a finite nonempty set V of *vertices* and a set $E = \{e_1, e_2, \ldots, e_m\}$ of nonempty subsets of V called *hyperedges.* If for each $e \in E$ we have $|e| = \lambda$, then H is said to be *λ-uniform.* Thus

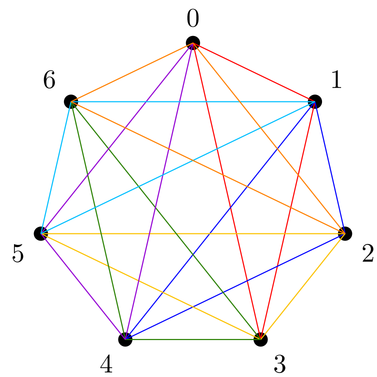

Figure 1: A decomposition of K_7 into triangles, or a Steiner triple system of order 7

graphs are 2-uniform hypergraphs. To indicate that a hypergraph H has uniformity λ, we write $H^{(\lambda)}$. The complete λ-uniform hypergraph on the vertex set V has the set of all λ-element subsets of V as its edge set and is denoted by $K_V^{(\lambda)}$. If $v = |V|$, then $K_v^{(\lambda)}$ is called the *complete λ-uniform hypergraph of order v* and is used to denote any hypergraph isomorphic to $K_V^{(\lambda)}$.

Example 1.2. *The four-uniform complete graph of order five is*
$K_5^{(4)} = \Big(\mathbb{Z}_5, \big\{\{0,1,2,3\},\{1,2,3,4\},\{0,2,3,4\},\{0,1,3,4\},\{0,1,2,4\}\big\}\Big)$.

A *decomposition* of a hypergraph K is a set $\Delta = \{H_1, H_2, \ldots, H_s\}$ of pairwise edge-disjoint subgraphs of K such that $E(H_1) \cup E(H_2) \cup \cdots \cup E(H_s) = E(K)$. If each element H_i of Δ is isomorphic to a fixed hypergraph H, then H_i is called an *H-block*, and Δ is called an *H-decomposition* of K. If there exists an H-decomposition of K, then we may simply state that H *decomposes* K. A common problem in hypergraph decompositions is to find the necessary and sufficient conditions on n for the existence of a decomposition of $K_n^{(\lambda)}$ into isomorphic copies of H. The authors of [4] found these conditions for each 3-uniform hypergraph with at most 3 edges and at most 6 vertices. To that end, the authors found decompositions of complete tripartite hypergraphs. Similar results for 3-uniform loose 5-cycles and 4-uniform loose 3-cycles were found in [2] and [3]. Decompositions of λ-partite, λ-uniform hypergraphs into Hamiltonian cycles were studied in [7] and [9].

1.3 Additional Notation and Terminology

If a and b are integers, we define $[a, b]$ to be $\{r \in \mathbb{Z} : a \leq r \leq b\}$. Let $\mathbb{Z}_n$ denote the group of integers modulo n. We next define some notation for certain types of hypergraphs.

Let $V(H)$ denote the set of vertices for a hypergraph H and let $E(H)$ denote the set of edges for a hypergraph H.

In this paper we are interested in decompositions of complete uniform multipartite hypergraphs. Let $U_1, U_2, \ldots U_\lambda$ be pairwise disjoint sets. The hypergraph with vertex set $\bigcup_{i=1}^{\lambda} U_i$ and edge set consisting of all λ-element sets having exactly one vertex in each of $U_1, U_2, \ldots U_\lambda$ is denoted by $K^{(\lambda)}_{U_1,U_2,\ldots U_\lambda}$. If $|U_i| = u_i$ for $1 \leq i \leq \lambda$, we may use $K^{(\lambda)}_{u_1,u_2,\ldots u_\lambda}$ to denote any hypergraph that is isomorphic to $K^{(\lambda)}_{U_1,U_2,\ldots U_\lambda}$. A complete multipartite hypergraph is *balanced* if each set has the same cardinality. We denote a complete λ-uniform, λ-partite hypergraph where each set has n vertices by $K^{(\lambda)}_{\lambda\times n}$.

Example 1.3. *The three-uniform complete multipartite graph for three parts of size four is*
$K^{(3)}_{3\times 4} = \{\{\mathbb{Z}_3 \times \mathbb{Z}_3\}, \{\{(i,0),(j,1),(k,2)\} : i,j,k \in \mathbb{Z}_4\}\}$.

We also refer to multipartite-like hypergraphs. Let $U_1, U_2, \ldots U_m$ be pairwise disjoint sets. The hypergraphs with vertex set $\bigcup_{i=1}^{m} U_i$ and edge set consisting of all λ-element sets having at least one vertex in each of $U_1, U_2, \ldots U_m$ is denoted by $L^{(\lambda)}_{U_1,U_2,\ldots U_m}$. If $|U_i| = u_i$ for $1 \le i \le m$, we may use $L^{(\lambda)}_{u_1,u_2,\ldots u_m}$ to denoted a hypergraph isomorphic to $L^{(\lambda)}_{U_1,U_2,\ldots U_m}$.

In this paper we are interested in three particular classes of λ-uniform hypergraphs. They are

- The λ-*uniform loose n cycle*, denoted $LC_n^{(\lambda)}$ which has vertex set $\{v_1, v_2, \ldots, v_{n(\lambda-1)}\}$ and edge set $\{\{v_{i(\lambda-1)+1}, v_{i(\lambda-1)+2}, \ldots, v_{i(\lambda-1)+\lambda}\} : 0 \le i \le n-2\} \cup \{v_1, v_{n(\lambda-1)}, v_{n(\lambda-1)-1}, \ldots, v_{n(\lambda-1)-(\lambda-2)}\}$. Notice that $d(v) = 2$ for $v \in \{v_{i(\lambda-1)+1} : 0 \le n-2\}$. Denote this set of vertices S_2. Denote the complement of S_2 in $V(LC_n^{(\lambda)}$ by S_1 and note that each $v \in S_1$ has $d(v) = 1$. An example is shown in Figure 2.

- The λ-*uniform loose n path*, denoted $LP_n^{(\lambda)}$ which has vertex set $\{v_1, v_2, \ldots, v_{n(\lambda-1)+1}\}$ and edge set $\{\{v_{i(\lambda-1)+1}, v_{i(\lambda-1)+2}, \ldots, v_{i(\lambda-1)+\lambda}\} : 0 \le i \le n-1\}$. An example is shown in Figure 3.

- A star forest, where a λ-uniform star denoted $S_n^{(\lambda)}$ is a hypergraph with vertex set $\{v_1, \ldots v_{n(\lambda-1)+1}\}$ and edge set $\{\{v_1, v_{i(\lambda-1)+2}, v_{i(\lambda-1)+3}, \ldots, v_{i(\lambda-1)+\lambda}\} : 0 \le i \le n-1\}$ and a star forest is a hypergraph in which the connected components are stars. In this paper we assume that all stars in a star forest have the same uniformity. An example of a star is shown in Figure 4.

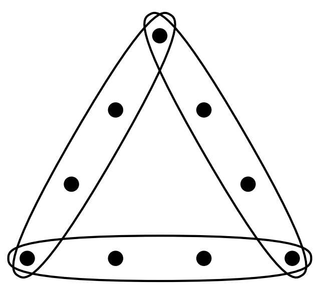

Figure 2: The 4-uniform loose 3-cycle, $LC_3^{(4)}$

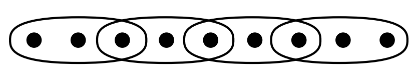

Figure 3: The 3-uniform loose 4-path, $LP_4^{(3)}$

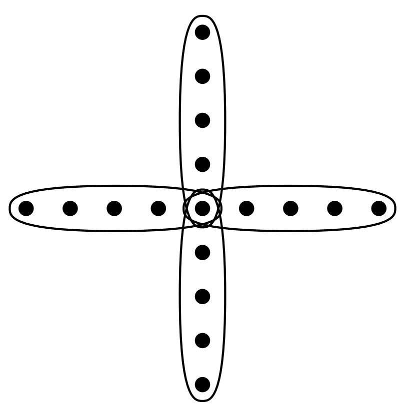

Figure 4: A 5-uniform 4-star

We define the *degree of a vertex v* to mean the number of edges which contain v. We denote the degree of a vertex as $d(v)$. Similarly, we define the *degree of a set of vertices* $\{v_1, \ldots, v_k\}$ to mean the number of edges which contain the entire set of vertices.

We define the *order* of the hypergraph H to mean the cardinality of the vertex set of H and the *size* of the hypergraph H to mean the cardinality of the edge set of H.

2 A technique for decomposing $K^{(\lambda)}_{\lambda \times n}$

The main result of our research is a technique for decomposing $K^{(\lambda)}_{\lambda\times n}$ which can be applied to hypergraphs of any size and uniformity. The two uniform version of this problem is the existence of a decomposition is studied in [5],[6], [8], [10] and [11]. These results are discussed more in Section 4. The study of decompositions of $K^{(\lambda)}_{\lambda\times n}$ is also motivated by the spectrum problem for λ uniform hypergraphs. In [3] and [4], the authors used the fact that a decomposition of $K^{(3)}_{mx}$ is equivalent to a decomposition of $xK^{(3))}_{m} \cup \binom{x}{2} L^{(3)}_{m,m} \cup \binom{x}{3} K^{(3)}_{m,m,m}$. Similarly in [2] the authors used the fact that a decomposition of $K^{(4)}_{mx}$ is equivalent to a decomposition of $xK^{(4))}_{m} \cup \binom{x}{2} L^{(4)}_{m,m} \cup \binom{x}{3} L^{(3)}_{m,m,m} \cup \binom{x}{4} K^{(4)}_{m,m,m,m}$. This led the author to consider the decompositions of $K^{(\lambda)}_{\lambda\times n}$. The methods used in [3] and [4] involve labelling a set of copies of a hypergraph H. The cardinality of this set increases as the size and uniformity of H increases. This led the authors of [2] to create a method which only involves labelling one copy of H. Our technique is a generalization of this method for hypergraphs of any size, uniformity and for a wide range of graph classes.

If it is possible to label the vertices H with λ colors such that no two vertices of an edge are the same color, then we say the graph is λ-colorable. From this point, any reference to a coloring of H will mean a coloring in which no two vertices of an edge are the same color. For any coloring $\mathcal{C}$ of H, define the color class $\mathcal{C}_x$ to mean the set of all vertices of H colored x. Define the $\mathcal{C}_{x,y}$-induced subgraph of H to be the graph with vertex set $\mathcal{C}_x \cup \mathcal{C}_y$ and edges $\{u, v\} \subseteq e$ for some $e \in E(H)$ where $u \in \mathcal{C}_x$ and $v \in \mathcal{C}_y$. A coloring and a $\mathcal{C}_{0,1}$ induced subgraph are shown in Figure 5. For a λ-colorable hypergraph H, we often refer to the edge set of H as a set of ordered pairs in $\mathbb{Z}_\lambda \times \mathbb{Z}_n$. Then $(v_{i,j}, j)$ gives the coordinate of the vertex in the jth column of the ith edge of H.

Theorem 1. *Suppose H is a λ-uniform, λ-colorable hypergraph of size n. If for some coloring $\mathcal{C}$ of H with λ colors, there exist two colors x and y such that the $\mathcal{C}_{x,y}$-induced subgraph of H decomposes $K^{(2)}_{n,n}$ then H decomposes $K^{(\lambda)}_{\lambda\times n}$.*

Proof. Assume that for a coloring $\mathcal{C}$ of H with the colors $\{0, \dots, \lambda-1\}$ there exists a $\mathcal{C}_{0,1}$-induced subgraph of H which decomposes $K^{(2)}_{n,n}$. Call this graph G and call Δ a G-decomposition of $K^{(2)}_{n,n}$. Let $V(K^{(\lambda)}_{\lambda\times n}) = \mathbb{Z}_\lambda \times \mathbb{Z}_n$. For each $G \in \Delta$, let $\{\{(g_{0,i}, 0), (g_{1,i}, 1)\} : 1 \le i \le n\}$ be an embedding of G in $K^{(2)}_{n,n}$. Let $\{\{(v_{0,i}, 0), (v_{1,i}, 1), \dots, (v_{\lambda-1,i}, \lambda-1)\} : 1 \le i \le n\}$ be an embedding of H in $K^{(\lambda)}_{\lambda\times n}$. Let

$$H_{G,j_2,j_3\dots,j_{\lambda-1}} = \bigcup_{i=1}^{n} \{(g_{0,i}, 0), (g_{1,i}, 1), (v_{2,i} + j_2, 2), (v_{3,i} + j_3, 3), \dots, (v_{\lambda-1,i} + j_{\lambda+1}, \lambda-1)\}.$$

We show that an H decomposition of $K^{(\lambda)}_{\lambda\times n}$ is $\Gamma = \bigcup\limits_{G\in\Delta} \bigcup\limits_{0\le j_2,j_3,\dots,j_{\lambda-1}\le n-1} H_{G,j_2,j_3\dots,j_{\lambda-1}}$.

There are n^λ edges in $K^{(\lambda)}_{\lambda\times n}$. There are n copies of $G \in \Delta$, so there are $n^{\lambda-1}$ copies of $H \in \Gamma$. Therefore the the number of edges in the copies of $H \in \Gamma$ is $nn^{\lambda-1} = n^\lambda$ which is equal to the number of edges in $K^{(\lambda)}_{\lambda\times n}$.

Next we will show that for each $e \in E(K^{(\lambda)}_{\lambda\times n})$, $e \in E(H)$ for some $H \in \Gamma$. Consider the edge $\{(w_0, 0), (w_1, 1), \dots, (w_{\lambda-1}, \lambda-1)\} \in E(K^{(\lambda)}_{\lambda\times n})$. Since G decomposes $K^{(2)}_{n,n}$, there is a $G^* \in \Delta$ such that $\{(w_0, 0), (w_1, 1)\} \in E(G^*)$. Suppose this is the ith edge in G^*. For each for each $k > 1$, there exists a $j^*_k \in \mathbb{Z}_n$ such that $v_{k,i} + j^*_k = w_k$. Then $\{(w_0, 0), (w_1, 1), \dots, (w_{\lambda-1}, \lambda-1)\} \in E(H_{G^*,j^*_2,j^*_3\dots,j^*_{\lambda-1}})$.

Since for each $e \in E(K^{(\lambda)}_{\lambda\times n})$, $e \in E(H)$ for some $H \in \Gamma$ and there are exactly $n^{\lambda-1}$ copies of $H \in \Gamma$, each edge in $E(K^{(\lambda)}_{\lambda\times n})$ is in $E(H)$ for exactly one copy of H. Then Γ is an H-decomposition of $K^{(\lambda)}_{\lambda\times n}$. $\square$

3 Graph classes which decompose $K^{(\lambda)}_{\lambda\times n}$

In this section, we present three corollaries to Theorem 1 which illustrate the decomposition technique. The decomposition technique was inspired when the author was studying decompositions of $K_n^{(\lambda)}$ into loose cycles. In [2] the authors found a decomposition of $K^{(4)}_{3,3,3,3}$ into $LC_3^{(4)}$ and in [3] the authors found a decomposition of $K^{(n)}_{5,5,5}$ into $LC_5^{(3)}$. While studying a finding a decomposition of $K^{(4)}_{3,3,3,3}$ into $LC_3^{(4)}$, the author was able to generalize the technique to any loose cycle of any uniformity and to other graph classes.

Corollary 2. *$LC_n^{(\lambda)}$ decomposes $K^{(\lambda)}_{\lambda\times n}$.*

Proof. Consider $LC_n^{(\lambda)}$ as defined in subsection 1.3. Let G be the graph with vertex set $\{v_2, v_{n(\lambda-1)}\} \cup \{v_{i(\lambda-1)+1} : 1 \leq i \leq n-1\}$ and edge set $\{\{v_2, v_\lambda\}, \{v_{n(\lambda-1)}, v_{n(\lambda-1)-(\lambda-2)}\}\} \cup \{\{v_{i(\lambda-1)+1}, v_{i(\lambda-1)+\lambda}\} : 1 \leq i \leq n-2\}$. Then G is a path formed by two vertex sets of different edges in $E(LC_n^{(\lambda)})$. A path is bipartite and is shown to decompose $K^{(2)}_{n,n}$ in [11]. It follows from Theorem 1 that $LC_n^{(\lambda)}$ decomposes $K^{(\lambda)}_{\lambda\times n}$. $\square$

Corollary 3. *$LP_n^{(\lambda)}$ decomposes $K^{(\lambda)}_{\lambda\times n}$.*

Proof. Consider $LP_n^{(\lambda)}$ as defined in subsection 1.3. Let G be the graph with vertex set $\{v_{i(\lambda-1)+1} : 0 \leq i \leq n\}$ and edge set $\{\{v_{i(\lambda-1)+1}, v_{i(\lambda-1)+\lambda}\} : 0 \leq i \leq n-1\}$. Then G is a path formed by two vertex sets of different edges in $E(LP_n^{(\lambda)})$. A path is bipartite and is shown to decompose $K^{(2)}_{n,n}$ in [11]. It follows from Theorem 1 that $LP_n^{(\lambda)}$ decomposes $K^{(\lambda)}_{\lambda\times n}$. $\square$

Corollary 4. *A λ-uniform star forest of size n decomposes $K^{(\lambda)}_{\lambda\times n}$.*

Proof. Let $\{S_1, \ldots, S_m\}$ be a forest of λ uniform stars as defined in subsection 1.3 where S_i has size s_i and $\sum i = 1^m s_i = n$. Then let G be the graph with vertex set $\bigcup_{i=1}^{m} \{v_1\} \cup \{v_{j(\lambda-1)+2} : 0 \leq j \leq s_i - 1\}$ and edge set $\bigcup_{i=1}^{m} \{\{v_1, v_{j(\lambda-1)+2}\} : 0 \leq j \leq s_i - 1\}$. Then G is a star forest of size n which is shown to decompose $K^{(2)}_{n,n}$ in [5]. It follows from Theorem 1 that $\{S_1, \ldots, S_m\}$ decomposes $K^{(\lambda)}_{\lambda\times n}$. $\square$

4 Property P

To apply Theorem 1 for a hypergraph H, there must exist a simple 2-uniform graph formed by two vertex sets of edges, and this graph must decompose $K^{(2)}_{n,n}$. The problem of decomposing $K^{(2)}_{n,n}$ is well studied. It is conjectured by Graham and Häggkvist that every tree of size n decomposes $K^{(2)}_{n,n}$ (see [6]). The authors of [8] prove the conjecture for some families of trees. Sotteau proved in [10] that cycles of length $2k$ decompose $K^{(2)}_{2k,2k}$

We instead focus on the existence of a simple 2-uniform graph formed by two vertex sets of edges.

Definition 4.1. *Let H be a λ-uniform hypergraph on n-vertices. We say H has property P if there exists a coloring $\mathcal{C}$ with the colors $\{0, 1, \ldots, \lambda-1\}$ such that no $\{u, v\} \subset V(H)$ with $d(\{u, v\}) > 1$ is an edge in the $\mathcal{C}_{0,1}$-induced subgraph.*

We now present a lemma which connects property P to Theorem 1.

Lemma 5. *Assume H is a λ-uniform, λ-colorable hypergraph of size n. If for some λ-coloring $\mathcal{C}$ of H there exist two colors x and y such that the $\mathcal{C}_{x,y}$-induced subgraph of H decomposes $K^{(2)}_{n,n}$, then H has property P.*

Proof. Assume that for a coloring of H with color set $\{0, \ldots, \lambda-1\}$ there exists a $\mathcal{C}_{0,1}$-induced subgraph of H which decomposes $K^{(2)}_{n,n}$. Call this graph G. Since G decomposes $K^{(2)}_{n,n}$, then it is a simple graph. Therefore, $d(u, v) = 1$ if $\{u, v\} \in E(G)$. It follows that H has property P. $\square$

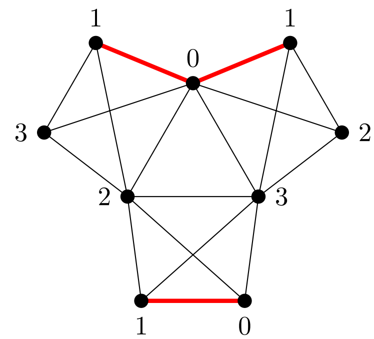

Figure 5: A hypergraph colored with the colors $\{0, 1, 2\}$. The $\mathcal{C}_{0,1}$-induced subgraph is highlighted in red

Lemma 5 implies that if a hypergraph has property P, then finding a decomposition of $K^{(2)}_{n,n}$ into the $\mathcal{C}_{0,1}$-induced subgraph is all that is required to find a decomposition of $K^{(\lambda)}_{\lambda\times n}$. In other words, the problem of finding a decomposition of a λ-uniform hypergraph is reduced to finding a decomposition of a simple graph. In the final section of the paper, we focus on necessary conditions on a hypergraph to have property P. This gives insight on hypergraphs for which we can apply Theorem 1.

5 Some necessary conditions for P

We now present some necessary conditions on a hypergraph so that the hypergraph has property P. In particular, we present one necessary condition for general λ based on the *chromatic number* of a graph formed by subsets of edges.

Definition 5.1. *Let H be a λ-uniform hypergraph. Let H^2 be the set graph with edge set, $\{\{u, v\} \subset V(H) : d(\{u, v\}) > 1\}$.*

Definition 5.2. *For a graph G, let the chromatic number, denoted $\chi(G)$, be the fewest colors needed to color a graph such that no two adjacent vertices share a color.*

Theorem 6. *Let H be a λ-uniform, λ-colorable hypergraph. If $\chi(H^2) = \lambda$, then H does not have property P.*

Proof. Suppose H is a λ-uniform, λ-colorable hypergraph and $\chi(H^2) = \lambda$. For the sake of contradiction, assume H has property P. Then there exists a coloring $\mathcal{C}$ with the colors $\{0, \ldots, \lambda - 1\}$ such that for every $v \in V(H^2) \cap \mathcal{C}_1$ and $u \in V(H^2) \cap \mathcal{C}_0$, $\{u, v\} \notin E(H^2)$. Then $\mathcal{C}^*$ defined by $\mathcal{C}^*_i = \mathcal{C}_i$ for $2 \leq i \leq \lambda - 1$ and $\mathcal{C}^*_1 = \mathcal{C}_1 \cup \mathcal{C}_0$ is a coloring of H^2 with $\lambda - 1$ colors so $\chi(H^2) < \lambda$ which is a contradiction, so H does not have property P. $\square$

5.1 Some necessary conditions for P when H is 3-uniform

We now present some necessary conditions for P when H is 3-uniform. In particular, we take advantage of the fact that for any two vertices $\{u, v\}$ with $d(\{u, v\}) > 1$, either u or v is in $\mathcal{C}_2$ for some coloring $\mathcal{C}$ with the colors $\{0, 1, 2\}$. This leads us to a subgraph we call the special path.

Definition 5.3. *Let the Special Path, denoted SP_n, for odd $n > 1$, be the 3-uniform hypergraph with $V(SP_n) = \{0, 1, \ldots, (5n+1)/2\}$ and $E(SP_n) = \{\{5i, 5i+1, 5i+3\}, \{5i, 5i+2, 5i+3\}, \{5i+3, 5i+4, 5i+5\} : 0 \leq i \leq \mathbb{Z}_{(n-3)/2}\} \cup \{\{(5n+1)/2, (5n-1)/2, (5n-5)/2\}, \{(5n+1)/2, (5n-3)/2, (5n-5)/2\}\}$. For $n = 1$ let $V(SP_1) = \mathbb{Z}_4$ and $E(SP_1) = \big\{\{0, 1, 3\}, \{0, 2, 3\}\big\}$. Call vertex 0 and vertex $(5n+1)/2$ the ends of the special path. A special path is shown in Figure 6.*

Lemma 7. *If H is a 3-uniform, 3-colorable hypergraph with property P which contains a special path, then for some coloring of H with the colors $\{0, 1, 2\}$ at least one of the ends of the special path must be colored 2.*

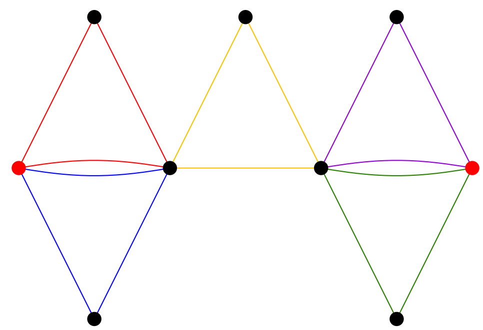

Figure 6: A SP_3

Proof. (Induction) Let H be a 3-uniform, 3-colorable hypergraph with property P which contains a special path.

Base Case: Call the ends of SP_1 a and b. Clearly in some coloring $\mathcal{C}$ which has the colors $\{0,1,2\}$, either a or b must be in $\mathcal{C}_2$ or else $d(\{a,b\}) > 1$ and $\{a,b\}$ is in the $\mathcal{C}_{0,1}$-induced subgraph.

Inductive Step: Assume that for some coloring $\mathcal{C}$ with the colors $\{0,1,2\}$, one end of the SP_{n-2} must be in $\mathcal{C}_2$. Notice that $V(SP_n) = V(SP_{n-2}) \cup \{(5n+1)/2, (5n-1)/2, (5n-3)/2, (5n-5)/2, (5n-7)/2\}$ and $E(SP_n) = E(SP_{n-2}) \cup \{\{(5n-9)/2, (5n-7)/2, (5n-5)/2\}, \{(5n-5)/2, (5n-3)/2, (5n+1)/2\}, \{(5n-5)/2, (5n-1)/2, (5n+1)/2\}\}$. From the induction hypothesis, vertex 0 or vertex $(5n-9)/2$ is in $\mathcal{C}_2$. If vertex 0 is in $\mathcal{C}_2$, then one end of SP_n is in $\mathcal{C}_2$. Otherwise vertex $(5n-9)/2$ is in $\mathcal{C}_2$. Since vertex $(5n-5)/2$ is adjacent to vertex $(5n-9)/2$, vertex $(5n-5)/2$ is not in $\mathcal{C}_2$. Since $d((5n-5)/2, (5n+1)/2) > 1$, vertex $(5n+1)/2$ must be in $\mathcal{C}_2$. Therefore, the assumption that one end of the SP_{n-2} must be in $\mathcal{C}_2$ implies that one end of SP_n must be in $\mathcal{C}_2$. From this, one end of SP_n must be in $\mathcal{C}_2$ for every odd n. ☐

Corollary 8. *Suppose H is a 3-uniform, 3-colorable hypergraph with property P that contains a special path, let a and b be the ends of the special path and let c be adjacent to a and b. Then for any coloring $\mathcal{C}$ in which a or b is in $\mathcal{C}_2$, $c \notin \mathcal{C}_2$*

Proof. Let H be a 3-uniform, 3-colorable hypergraph with property P that contains a special path with ends a and b. From Lemma 7, either a or b is in $\mathcal{C}_2$ for some coloring $\mathcal{C}$ of H with the colors $\{0,1,2\}$. Since c is adjacent to a and b, $c \notin \mathcal{C}_2$. ☐

It is well known that a 2-uniform graph is bipartite if and only if it does not contain an odd cycle. We obtain a similar result for coloring loose odd cycles with $\{0,1,2\}$.

Lemma 9. *Suppose H is a 3-uniform, 3-colorable hypergraph which contains a loose n cycle for some odd n. Then for any coloring $\mathcal{C}$ of H with the colors $\{0,1,2\}$, there are between 1 and $(n-1)/2$ vertices in $S_2 \cap \mathcal{C}_2$.*

Proof. Let H be a 3-uniform, 3-colorable hypergraph which contains a loose n cycle for some odd n. If more than $(n-1)/2$ vertices are in $S_2 \cap \mathcal{C}_2$, then two of those vertices are adjacent by the pigeonhole principle. Then there are at most $(n-1)/2$ vertices in $S_2 \cap \mathcal{C}_2$. Similarly, there are at most $(n-1)/2$ vertices in $S_2 \cap \mathcal{C}_0$ and at most $(n-1)/2$ vertices in $S_2 \cap \mathcal{C}_1$ so there are at most $n-1$ vertices in S_2 which are not in $\mathcal{C}_2$. From this, there is at least one vertex in $S_2 \cap \mathcal{C}_2$. ☐

Lemma 7, Corollary 8 and Lemma 9 give restrictions on which vertices can or cannot be in $\mathcal{C}_2$. We now present three ways in which these restrictions intersect so that a hypergraph cannot have property P.

Theorem 10. *If H is a 3-uniform, 3-colorable hypergraph and H contains a special path whose ends are adjacent to both ends of other special paths, then H does not have property P.*

Proof. Assume for the sake of contradiction that H is a 3-uniform, 3-colorable hypergraph and H contains a special path whose ends are adjacent to both ends of another special path and H has property P. Consider the special path whose ends are adjacent to both ends of other special paths. By Lemma 7, at least one end of the special path is in $\mathcal{C}_2$ for some coloring $\mathcal{C}$ with the colors $\{0,1,2\}$. However since this vertex is adjacent to both ends of a special path, it cannot be in $\mathcal{C}_2$ by Corollary 8, so a contradiction arises. From this, H does not have property P. $\square$

Theorem 11. *Suppose H is a 3-uniform, 3-colorable hypergraph which contains a loose n cycle for some odd n. If each vertex in S_1 or if each vertex in S_2 is adjacent to both ends of some special path, then H does not have property P.*

Proof. Assume that H is a 3-uniform, 3-colorable hypergraph which contains a loose n cycle for some odd n. Assume for the sake of contradiction that H has property P. We divide the proof into two cases.

Case 1: Assume that each vertex in S_1 is adjacent to both ends of some special path. Then for some coloring C with the colors $\{0, 1, 2\}$, each vertex in S_1 is in $\mathcal{C}_0$ or $\mathcal{C}_1$ by Corollary 8. Then each of these vertices is adjacent to a vertex in S_2 which is also in $\mathcal{C}_2$. Since there are n vertices in S_1, there must be at least $n/2$ vertices in S_2 which are in in $\mathcal{C}_2$. This is a contradiction to Lemma 9.

Case 2: Assume that each vertex in S_2 is adjacent to both ends of some special path. Then each vertex in S_2 is in $\mathcal{C}_0$ or $\mathcal{C}_1$ for some coloring $\mathcal{C}$ of H with the colors $\{0, 1, 2\}$ by Corollary 8. Then no vertex in S_2 is in $\mathcal{C}_2$, which contradicts Lemma 9.

In both cases, a contradiction arises so H does not have property P. $\square$

Theorem 12. *Suppose H is a 3-uniform, 3-colorable hypergraph which contains n special paths for some odd n. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be the sets of ends of the specials paths. Let S_2 and T_2 be the sets of vertices with degree 2 for two loose n cycles in contained in H. If $\{a_1, \ldots, a_n\} = S_2$ and $\{b_1, \ldots, b_n\} = T_2$, then H does not have property P.*

Proof. Assume that H is a 3-uniform, 3-colorable hypergraph which contains n special path for some odd n. Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be the sets of ends of the specials paths with $\{a_1, \ldots, a_n\} = S_2$ and $\{b_1, \ldots, b_n\} = T_2$. Assume for the sake of contradiction that H has property P. By Lemma 7 at least n of the vertices in $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ are in $\mathcal{C}_2$ for some coloring $\mathcal{C}$ of H with the colors $\{0, 1, 2\}$. However since $\{a_1, \ldots, a_n\} = S_2$ and $\{b_1, \ldots, b_n\} = T_2$, Lemma 9 implies that at most $n-1$ of $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ are in $\mathcal{C}_2$. This is a contradiction, so H does not have property P. $\square$

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