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SOLVABILITY IN CLASSICAL SENSE OF QUASI-LINEAR NON-COOPERATIVE ELLIPTIC SYSTEMS AND APPLICATION

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In this article is studied the solvability in classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ sense of quasi-linear non-cooperative weakly coupled systems of elliptic second-order PDE. The main tool for the research is the method of sub- and super-solutions. The result is applied to a model example describing two dimensional non-super-conformal minimal surface M^2 in R^4.

1. Introduction

One of the major applications of the comparison principle is the method of sub- and super-solutions. It is applied to a quasi-linear non-cooperative elliptic system and gives some sufficient conditions for it solvability in C^2.

Let $\Omega \in R^n$ be a bounded domain with smooth boundary $\partial\Omega$. In this article are considered quasi-linear weakly-coupled elliptic systems of the type

$$Q^l(u) = -\mathrm{div}\, a^l(x, u^l, Du^l) + F^l(x, u^1, \dots, u^N, Du^l) = f^l(x) \ \text{ in } \ \Omega \tag{1}$$

$$u^l(x) = g^l(x) \ \text{ on } \ \partial\Omega \tag{2}$$

for $l = 1, \dots, N$.

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System (1) is strictly elliptic one, i.e. there are monotonously decreasing continuous function $\lambda(|u|) > 0$ and monotonously increasing one $\Lambda(|u|) > 0$, depending only on $|u| = \left(\left(u^1\right)^2 + \cdots + \left(u^N\right)^2\right)^{1/2}$, such that

$$\lambda(|u|)\left|\xi^l\right|^2 \leq \sum_{i,j=1}^{n} \frac{\partial a^{li}}{\partial p_j^l}(x, u^1, \ldots, u^N, p^l)\xi_i^l \xi_j^l \leq \Lambda(|u|)\left|\xi^l\right|^2 \tag{3}$$

holds for every u^l and $\xi^l = (\xi_1^l, \ldots, \xi_n^l) \in R^n$, $l = 1, 2, \ldots N$.

Coefficients $a^l(x,u,p)$, $F^l(x,u,p)$, $f^l(x)$ and $g^l(x)$ are supposed at least measurable functions in Ω with respect to x variable, and locally Liepschitz continuous with respect to u^l, u and p, i.e.

$$\begin{aligned} &\left|F^l(x,u,p) - F^l(x,v,q)\right| \leq C(K)\left(|u-v| + |p-q|\right), \\ &\left|a^l(x,u^l,p) - a^l(x,v^l,q)\right| \leq C(K)\left(\left|u^l - v^l\right| + |p-q|\right) \end{aligned} \tag{4}$$

holds for every $x \in \Omega$, $|u| + |v| + |p| + |q| \leq K$, $l = 1, \ldots, N$.

Furthermore we assume $a^l(x,u,p)$ and $F^l(x,u,p)$ to be differentiable on u^l and p^l, and

$$\frac{\partial a^{li}}{\partial p_j}, \frac{\partial a^{li}}{\partial u^k}, \frac{\partial F^l}{\partial p_l}, \frac{\partial F^l}{\partial u^k} \in L^1(\Omega).$$

The essence of the method of sub- and super-solution for general operator is the existence of lower (sub) and upper (super) solution, on one hand, and a kind of monotonicity of the operator, on the other hand. This way if the operator is increasing, one can construct monotonically increasing sequence of operators that are bounded from above, and the initial operator is the lower solution (in the opposite case the sequence is monotonically decreasing and is bounded from below). The feature that gives monotonicity of system (1) is the comparison principle. It is well studied for cooperative systems (see [1] or [3]), but the non-cooperative case is much more difficult to study. One result about the validity of the comparison principle is given in [2]. For sake of completeness in the following chapter "Comparison principle for quasi-linear elliptic systems" is recalled this result.

The construction of the monotonically decreasing sequence is given in details in [4]. For sake of completeness the main theorem and a sketch of the proof are recalled in section "Existence of clasical solutions" below.

Hereafter by $f^-(x) = \min(f(x), 0)$ and $f^+(x) = \max(f(x), 0)$ are denoted the non-negative and, respectively, the non-positive part of the function f. The same convention is valid for matrices as well. For instance, we denote by M^+ the non-negative part of M, i.e.$M^+ = \{m_{ij}^+(x)\}_{i,j=1}^N$.

2. Comparison principle for quasi-linear elliptic systems

Let $\underline{u}(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be classical sub-solution of (1), (2). Then

$$\int_\Omega \left(a^{li}(x, \underline{u}^l, D\underline{u}^l)\eta^l_{x_i} + F^l(x, \underline{u}^1, \dots, \underline{u}^N, D\underline{u}^l)\eta^l - f^l(x)\eta^l\right) dx \le 0$$

for $l = 1, \dots, N$ and for every non-negative vector-function $\eta \in \left(W^1_c(\Omega) \cap C(\overline{\Omega})\right)^N$ (i.e. $\eta = (\eta^1, \dots \eta^N)$, $\eta^l \ge 0$, $\eta^l \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega})$ and $\eta^l = 0$ on $\partial\Omega$).

Analogously, let $\overline{u}(x) \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)^N$ be a classical super-solution of (1), (2). Then

$$\int_\Omega \left(a^{li}(x, \overline{u}^l, D\overline{u}^l)\eta^l_{x_i} + F^l(x, \overline{u}^1, \dots, \overline{u}^N, D\overline{u}^l)\eta^l - f^l(x)\eta^l\right) dx \ge 0$$

for $l = 1, \dots, N$ and every non-negative vector-function $\eta \in \left(W^1_c(\Omega) \cap C(\overline{\Omega})\right)^N$.

Recall that the comparison principle holds for (1), (2), if $Q(\underline{u}) \le Q(\overline{u})$ in Ω and $\underline{u} \le \overline{u}$ on $\partial\Omega$ yields $\underline{u} \le \overline{u}$ in Ω.

Since $\underline{u}(x)$ and $\overline{u}(x)$ are sub- and super- solutions, then $\widetilde{w}(x) = \underline{u}(x) - \overline{u}(x)$ is weak sub-solution of the following problem

$$-\sum_{i,j=1}^n D_i\left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right) + \sum_{k=1}^N E^l_k \widetilde{w}^k + \sum_{i=1}^n H^l_i D_i \widetilde{w}^l = 0 \text{ in } \Omega$$

with non-positive boundary data on $\partial\Omega$, i.e.

$$\int_\Omega \left(\sum_{i,j=1}^n \left(B^{li}_j D_j \widetilde{w}^l + B^{li}_0 \widetilde{w}^l\right)\eta^l_{x_i} + \sum_{k=1}^N E^l_k \widetilde{w}^k \eta^l + \sum_{i=1}^n H^l_i D_i \widetilde{w}^l \eta^l\right) dx \le 0 \text{ in } \Omega$$

Here

$$B^{li}_j = \int_0^1 \frac{\partial a^{li}}{\partial p_j}(x, P^l) ds \;\; B^{li}_0 = \int_0^1 \frac{\partial a^{li}}{\partial u^l}(x, P^l) ds,$$

$$P^l = \left(v^l + s(u^l - v^l), Dv^l + sD(u^l - v^l)\right)$$

$$E^l_k = \int_0^1 \frac{\partial F^l}{\partial u^k}(x, S^l) ds, H^l_i = \int_0^1 \frac{\partial F^l}{\partial p_i}(x, S^l) ds,$$

$$S^l = \Big(v + s(u-v), Dv^l + sD(u^l - v^l)\Big)\,.$$

Therefore $\widetilde{w}_+(x) = \max\left(\widetilde{w}(x), 0\right)$ is weak sub-solution of

$$-\sum_{i,j=1}^{n} D_i\left(B_j^{li} D_j \widetilde{w}_+^l + B_0^{li}\widetilde{w}_+^l\right) + \sum_{k=1}^{N} E_k^l \widetilde{w}_+^k + \sum_{i=1}^{n} H_i^l D_i \widetilde{w}_+^l = 0 \text{ in } \Omega \tag{5}$$

with null boundary data on $\partial\Omega$.

Equation (5) is equivalent to

$$B_E \widetilde{w}_+ = (B+E)\widetilde{w}_+ = 0 \text{ in } \Omega, \tag{6}$$

where $B = \operatorname{diag}\left(B_1, B_2, \dots B_N\right)$, $B_l = -\sum_{i,j=1}^{n} D_i\left(B_j^{li} D_j\widetilde{w}_+^l + B_0^{li}\widetilde{w}_+^l\right) + \sum_{i=1}^{n} H_i^l D_i \widetilde{w}_+^l$ and $E = \{E_k^l\}_{l,k+1}^N$.

Then the following theorem (Theorem (8) in [2]) holds:

Theorem 1. *Let* (1), (2) *be quasi-linear system and corresponding system* B_{E^-} *in* (6) *is elliptic one. Then comparison principle holds for system* (1), (2) *if* B_{E^-} *is irreducible one and for every* $j = 1, \dots, n$ *hold*

$$(i)\quad \lambda + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x,p,q^l) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j)\right)^+ > 0 \quad \textit{for some} \quad x_0 \in \Omega,$$

$$(ii)\quad \lambda + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j) + \frac{\partial F^j}{\partial p^j}(x,p,q^j)\right)^+ \geq 0 \quad \textit{for every} \quad x \in \Omega,$$

where $p, q \in R^n$ *and* λ *is the first eigenvalue of operator* B_{E^-} *in* Ω*;*
or if B_{E^-} *is reducible one and for every* $j = 1, \dots, n$ *hold*

$$(i')\quad \lambda_j + \left(\sum_{k=1}^{N} \frac{\partial F^k}{\partial p^j}(x,p,q^j) + \sum_{i=1}^{N} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j)\right)^+ > 0 \quad \textit{for some} \quad x_0 \in \Omega,$$

$$(ii')\quad \lambda_j + \left(\sum_{i=1}^{n} D_i \frac{\partial a^{ji}}{\partial p^j}(x,p^j,q^j) + \frac{\partial F^j}{\partial p^j}(x,p,q^j)\right)^+ \geq 0 \quad \textit{for every } x \in \Omega,$$

$p, q \in R^n$ *and* λ_l *is the first eigenvalue of operator* B_l *in* Ω*.*

Note: *We remind the reader that* B_{E^-} *stands for the negative part of* B_E*. Irreducible matrix is one that can not be decomposed to matrices of lower rank, and respectively, the reducible matrix can be decomposed.*

3. Existence of classical solutions

In order to use the method of sub- and super- solutions we need some constraints on the growth of the coefficients. Assume that for every $l = 1, \dots N$

$$\left\{ \sum_{i=1}^{n} \left(\sum_{j=1}^{n} D_j B_j^{li} + \left(B_0^{li} + H_i^l \right) \right)^2, \left| \sum_{i=1}^{n} \left(D_i B_0^{li} \right) \right| \right\} \le b \tag{7}$$

holds for $x \in \overline{\Omega}$, where b is a positive constant,

$$\left[\sum_{i=1}^{n} \left(B_0^{li} + H_i^l \right) .p_i.u^l + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l + \sum_{k=1}^{n} E_k^l.u_k(x) \right] u^l \ge c_1 |u|^2 - c_2 \tag{8}$$

for every $x \in \Omega$, $l = 1, \dots N$ and arbitrary vectors u and p, where $c_1 = const > 0$ and $c_2 = const \ge 0$,

$$\left| \sum_{i=1}^{n} \left(B_0^{li} + H_i^l \right) .p_i.u^l + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l + \sum_{k=1}^{n} E_k^l.u_k(x) \right| \le \tag{9}$$

$$\le \varepsilon(C_M) + P(p, C_M)(1 + |p|^2),$$

where $P(p, C_M) \to 0$ for $|p| \to \infty$ and $\varepsilon(C_M)$ is sufficiently small and depends only on n, N, C_M, λ and Λ. λ and Λ are the constants from condition (3) and

$$C_M = \max \left\{ \max_{\partial\Omega} |u|, \frac{2 \max |f(x)|}{c_1 n}, \sqrt{\frac{2c_2}{c_1 n}} \right\}. \tag{10}$$

Then the following theorem holds

Theorem 2. *Suppose that system* (1), (2) *satisfies conditions* (3) *to* (9), *and* (*i*), (*ii*) *or* (*i*′), (*ii*′), *according to the structure of matrix* $E = (E_k^l)$. *Assume that* $v(x)$ *is a classical super-solution and* $w(x)$ *is a a classical sub-solution of* (1), (2). *Then there exists a classical* $C^2(\Omega) \bigcap C(\overline{\Omega})$ *solution* $u(x)$ *of the problem* (1), (2) *with null boundary data.*

Since the system (1) is a quasi-linear one, we assume in the following proof without loss of generality that $g(x) = 0$.

Sketch of the proof. Let us denote

$$\Phi_l^-(x, u^1, \dots, u^N) = \sum_{k=1}^{n} E_k^{l-} u^k + \sum_{i=1}^{n} \left(D_i B_0^{li} \right) u^l$$

and

$$\Phi_l^+(x, u^1, \ldots, u^N) = \sum_{k=1}^{n} E_k^{l+} u^k.$$

1. Consider the sequence of vector-functions $u_0, u_1, \ldots u_k, \ldots$, where $u_0 = v(x)$ and $u_k \in H_0^1(\Omega)$ defines u_{k+1} by induction as a solution of the problem

$$-\sum_{i,j=1}^{n} D_i \left(B_j^{li} D_j u_{k+1}^l + B_0^{li} u_{k+1}^l \right) + \sum_{i=1}^{n} H_i^l D_i u_{k+1}^l + \Phi_l^-(x, u_{k+1}^1, \ldots, u_{k+1}^N) + \sigma u_{k+1}^l =$$

$$= f^l(x) - \Phi_l^+(x, u_k^1, \ldots, u_k^N) + \sigma u_k^l \quad \text{in} \quad \Omega$$

with null boundary conditions

$$u_{k+1}^l(x) = 0 \quad \text{on} \quad \partial\Omega$$

for every $l = 1, \ldots, N$, $\sigma < 0$ is a constant.

2. $u_0^l \geq u_1^l \geq \cdots \geq u_{k+1}^l \geq \cdots$ by the comparison principle. $u_0^l \geq u_1^l$ since u_0^l is a super-solution of (1), (2).

3. The inequality $u_{k+1}(x) \geq w(x)$ holds for every k, since $w(x)$ is a sub-solution of the same system (1), (2).

4. The sequence of vector-functions $\{u_k\}$ is monotonously decreasing and bounded from below in Ω. Therefore there is a function u such that $u_k(x) \to u(x)$ point-wise in Ω. Furthermore, (13) yields $\{u_k\}$ is uniformly equicontinuous in $\overline{\Omega}$ and $\{u_k\} <$ const, since $u_k^l(x)$ is Holder continuous and therefore $|u_k^l(x) - u_k^l(x_0)| \leq c(|x - x_0|^\beta)$ for every $l = 1, \ldots, N$. By Arzela–Ascoli compactness criterion there is a sub-sequence $\{u_{k_j}\}$ that converges uniformly to $u \in C(\overline{\Omega})$. For convenience we denote $\{u_{k_j}\}$ by $\{u_k\}$.

Since $u \in C(\overline{\Omega})$ and all functions $\{u_{k_j}\}$ satisfy the null boundary conditions, then u satisfies the boundary conditions as well.

For more details about the smoothness of the limit function $u(x)$ see [4]. □

4. Model example

Our model example is due to R. de Azevedo Tribuzy and I. Guadalupe (see [7]). Consider the system in R^2

$$\left| \begin{array}{l} \left(K^2 - \chi^2\right)^{1/4} \Delta_2 \ln|\chi - K| = 2(2K - \chi) \\ \\ \left(K^2 - \chi^2\right)^{1/4} \Delta_2 \ln|\chi + K| = 2(2K + \chi) \end{array} \right. \tag{11}$$

where $\Delta_2 = \partial_x^2 + \partial_y^2$, $K^2 > \chi^2$, $K < 0$, $K = K(x, y)$ and $\chi = \chi(x, y)$, where $K(x, y)$ stands for the Gaussian curvature of the two-dimmensional non-super-conformal minimal surface M^2 in R^4, while $\chi(x, y)$ is the normal curvature of M^2.

Every couple of solutions (K, χ) define uniquely minimal non-super-conformal surface M^2 in R^4 with Gaussian curvature K and normal curvature χ (see [8]).

Let $K > \chi$. Then we denote

$$\left| \begin{array}{l} K - \chi = e^u \\ K + \chi = e^v \end{array} \right. \tag{12}$$

and transform (11) to

$$\left| \begin{array}{l} \Delta u = 3e^{(3u-v)/4} + e^{(3v-u)/4} \\ \Delta v = e^{(3u-v)/4} + 3e^{(3v-u)/4} \end{array} \right. \tag{13}$$

System (13) is quasi-linear, non-cooperative and elliptic one. First we investigate the validity of the comparison principle.

Assume that Ω is a local map from $M^2 \to M^2$. Since K is the Gaussian curvature and χ is the curvature of the normal connection on minimal non-super-conformal surface M^2 in R^4, by (12) and geometrical reasons we presume that u and v are bounded functions in Ω. In other words we suppose there is constant $c(\Omega)$ such that $e^{|u|} \le c(\Omega)$ and $e^{|v|} \le c(\Omega)$ and therefore condition (4) holds. In order to apply Theorem 1 we have to check weaker conditions (i'), (ii').

There are many results about the lower bound of the first eigenvalue of the Laplacian in Ω. The following one is given in [5] and [6]. Without loss of generality we may assume that Ω is a simply connected planar domain. Then $\lambda_1 \ge \dfrac{\alpha}{\rho^2}$ where α is a constant and ρ is the radius of the largest ball B_Ω inscribed in Ω, $\rho = \max\limits_{x\in\Omega} \min\limits_{y\in\partial\Omega} \{|x-y|\}$. Therefore, the smaller is ρ, the larger is the first eigenvalue λ_1. Since conditions (i'), (ii') for system (13) read

$$\lambda_1 + \left(\frac{\partial F^1}{\partial u} + \frac{\partial F^2}{\partial u}\right)^+ = \lambda_1 + \left(3e^{(3u-v)/4} - e^{(3v-u)/4}\right)^+ > 0,$$

$$\lambda_1 + \left(\frac{\partial F^1}{\partial v} + \frac{\partial F^2}{\partial v}\right)^+ = \lambda_1 + \left(-e^{(3u-v)/4} + 3e^{(3v-u)/4}\right)^+ > 0$$

for some $x_0 \in \Omega$, and

$$\lambda_1 + \left(\frac{\partial F^1}{\partial u}\right)^+ = \lambda_1 + \left(\frac{9}{4}e^{(3u-v)/4} - \frac{1}{4}e^{(3v-u)/4}\right)^+ \ge 0,$$

$$\lambda_1 + \left(\frac{\partial F^2}{\partial v}\right)^+ = \lambda_1 + \left(-\frac{1}{4}e^{(3u-v)/4} + \frac{9}{4}e^{(3v-u)/4}\right)^+ \geq 0$$

for every $x \in \Omega$, and by presumption $e^u \leq c(\Omega)$ and $e^v \leq c(\Omega)$, it is obvious that for ρ sufficiently small, the inequalities above hold, and therefore comparison principle is a property of system (13).

Next step is to check conditions of Theorem 2, namely (7), (8) and (9). For the linearization of the system (1) we use one sub- and one super-solution of the system. It is obvious that $\underline{u} = \underline{v} = 0$ is a sub-solution of (13), since $\Delta\underline{u} = \Delta\underline{v} = 0 < e^{(3\underline{u}-\underline{v})/4} + 3e^{(3\underline{v}-u)/4} = 3e^{(3\underline{u}-\underline{v})/4} + e^{(3\underline{v}-\underline{u})/4} = 4$, and $Q(0) \leq 0$.

On the other hand one super-solution of (13) is $\overline{u}$, $\overline{v}$, where

$$\left|\begin{array}{l} \Delta\overline{u} = 4c \\ \Delta\overline{v} = 4c \end{array}\right. \tag{14}$$

It is easy to check that $e^{(3u-v)/4} \leq e^{(3|u|+|v|)/4} \leq c^{3/4}.c^{1/4} = c$ and therefore $Q(\overline{u}, \overline{v}) \leq 4c$.

Remark. In some cases the explicit solution of the above system is well-known, for instance if Ω is a disk.

For system (13) the coefficients of (5) are as follows:

$$B_j^{li} = \int_0^1 \frac{\partial a^{li}}{\partial p_j}(x, P^l)ds = \delta_{i,j}, \;\; B_0^{li} = \int_0^1 \frac{\partial a^{li}}{\partial u^l}(x, P^l)ds = 0,$$

$$E_1^1 = \int_0^1 \frac{\partial F^1}{\partial u}(x, S^1)ds = \int_0^1 \left(\frac{9}{4}e^{(1-s)(3\overline{u}-\overline{v})/4} - \frac{1}{4}e^{(1-s)(3\overline{v}-\overline{u})/4}\right) ds,$$

$$E_2^1 = \int_0^1 \frac{\partial F^1}{\partial v}(x, S^1)ds = \int_0^1 \left(-\frac{3}{4}e^{(1-s)(3\overline{u}-\overline{v})/4} + \frac{3}{4}e^{(1-s)(3\overline{v}-\overline{u})/4}\right) ds,$$

$$E_1^2 = \int_0^1 \frac{\partial F^2}{\partial u}(x, S^2)ds = \int_0^1 \left(\frac{3}{4}e^{(1-s)(3\overline{u}-\overline{v})/4} - \frac{3}{4}e^{(1-s)(3\overline{v}-\overline{u})/4}\right) ds,$$

$$E_2^2 = \int_0^1 \frac{\partial F^2}{\partial v}(x, S^2)ds = \int_0^1 \left(-\frac{1}{4}e^{(1-s)(3\overline{u}-\overline{v})/4} + \frac{9}{4}e^{(1-s)(3\overline{v}-\overline{u})/4}\right) ds,$$

$$H_i^l = \int_0^1 \frac{\partial F^l}{\partial p_i}(x, S^l)ds = 0,$$

where $\delta_{i,j}$ is Kronecker delta (symbol). In this case $P^1 = ((1-s)\overline{u}, (1-s)D\overline{u})$, $S^1 = ((1-s)\overline{u}, (1-s)\overline{v}, (1-s)D\overline{u})$, $P^2 = ((1-s)\overline{v}, (1-s)D\overline{v})$ and $S^2 = \left((1-s)\overline{u}, (1-s)\overline{v}, (1-s)D\overline{v}^l\right)$.

Therefore the coefficients of the linearized system (13) are

$$B_j^{li} = \delta_{i,j}, B_0^{li} = H_i^l = 0,$$

$$\begin{aligned}
E_1^1 &= \frac{9}{3\overline{u}-\overline{v}}\left(e^{(3\overline{u}-\overline{v})/4}-1\right) - \frac{1}{3\overline{v}-\overline{u}}\left(e^{(3\overline{v}-\overline{u})/4}-1\right),\\
E_2^1 &= -\frac{3}{3\overline{u}-\overline{v}}\left(e^{(3\overline{u}-\overline{v})/4}-1\right) + \frac{3}{3\overline{v}-\overline{u}}\left(e^{(3\overline{v}-\overline{u})/4}-1\right),\\
E_1^2 &= \frac{3}{3\overline{u}-\overline{v}}\left(e^{(3\overline{u}-\overline{v})/4}-1\right) - \frac{3}{3\overline{v}-\overline{u}}\left(e^{(3\overline{v}-\overline{u})/4}-1\right),\\
E_2^2 &= -\frac{1}{3\overline{u}-\overline{v}}\left(e^{(3\overline{u}-\overline{v})/4}-1\right) + \frac{9}{3\overline{v}-\overline{u}}\left(e^{(3\overline{v}-\overline{u})/4}-1\right).
\end{aligned} \tag{15}$$

Note that if we consider equal boundary conditions for $\overline{u}$ and $\overline{v}$ in (14), the system reduces to two equations. Furthermore, since every couple super-solutions $\overline{u}$ and $\overline{v}$ is worthy for the linearization of system (13), we can choose a couple such that $3\overline{u} \neq \overline{v}$, $\overline{u} \neq 3\overline{v}$

The conditions of Theorem 2 are easy to check. By (15) we have $B_j^{li} = B_0^{li} = H_i^l = 0$ for $i \neq j$ and therefore (7) is a trivial inequality. As for (8), we can find possitive constants c_1 and c_2 such that

$$E_1^1.u^2 + E_1^2.uv + E_2^1.vu + E_2^2.v^2 = E_1^1.u^2 + E_2^2.v^2 \geq c_1(u^2+v^2) - c_2$$

since $E_1^2 = -E_2^1$, u, v are bounded in Ω by presumption, and Ω is small.

The last condition to check is (9). In this case it is

$$\left|E_1^1.u + E_2^1.v\right| \leq \frac{c^{3/4}}{2}\left(5|u| + 3|v|\right) \leq 4c^{3/4}\ln c,$$

$$\left|E_2^l.u + E_2^2.v\right| \leq \frac{c^{3/4}}{2}\left(3|u| + 5|v|\right) \leq 4c^{3/4}\ln c.$$

In summary, the smaller is the map Ω (by means of B_Ω), the smaller is $c(\Omega)$ and the larger is the first eigenvalue λ_1 of system (11). Therefore, if Ω is sufficiently small, conditions (i), (ii) (or (i'), (i'')) hold and by Theorem 1 comparison principle holds for system (13). Furthermore, conditions (7)–(9) hold as well. This way we construct (locally) a classical solution of system (11), and thereby the following theorem holds:

Theorem 3. *There is a classical $C^2(\Omega)\bigcap C(\overline{\Omega})$ solution (u, v) of system* (13) *in every sufficiently small local map $\Omega : M^2 \to M^2$.*

Since the solution (u, v) of system (13) is bounded in the local map Ω, it follows that there is corresponding solution (K, χ) of system (11), and (K, χ) define uniquely in the local map Ω a minimal non-super-conformal surface M^2 in R^4 with Gaussian curvature K and normal curvature χ.

Note that the construction above is applicable for any strongly elliptic operator, not Laplacian only. As it is well-known, the first eigenvalue of the Laplacian is given by min-max formula

$$\lambda_1 = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}.$$

Then by(3) we have

$$\lambda(|u|)\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2} \leq \frac{\int_\Omega \sum_{i,j=1}^n \frac{\partial a^{li}}{\partial p_j^l} u_{x_i}^l u_{x_j}^l}{\int_\Omega |u|^2} \leq \Lambda(|u|)\frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}$$

and the first eigenvalue of operator (1) is bounded by $\lambda.\lambda_1$ and $\Lambda.\lambda_1$.

Of course, there is another approach to system (11). Define $B_R = \{x^2 + y^2 < R^2\}$, $S_R = \partial B_R$ and consider Dirichlet boundary value problem for (11) with data $K|_{S_R} = B_1, \chi|_{S_R} = B_2, B_1 < B_2 < 0$. After the changes $K = -\frac{e^u + e^v}{2}, \chi = \frac{e^u - e^v}{2}$ and $4p = 3u - v$, $4q = 3v - u$ we reduce the solvability of (11) to the solvability of the following two scalar Dirichlet problems in B_R:

(16) $$\Delta_2 p + 2e^p = 0, p|_{S_R} = \frac{1}{4}\ln\frac{-(B_1 - B_2)^3}{|B_1 + B_2|}$$

and

(17). $$\Delta_2 q + 2e^q = 0, q|_{S_R} = \frac{1}{4}\ln\frac{|B_1 - B_2|^3}{|B_1 - B_2|}$$

Certainly, (16) and (17) are Liouville type PDE.

Solving if possible (16) and (17) we obtain that

$$K = -\frac{e^{\frac{p+q}{2}}(e^p + e^q)}{2}, \chi = \frac{e^{\frac{p+q}{2}}(e^p - e^q)}{2}.$$

Radial solutions of (11) with data $B_1 = \text{const}$, $B_2 = \text{const}$ are found in the paper [9].

It is interesting to point out that the general classical solution of the Liouville equation (16) can be written in the form

$$p = \log \frac{2|\phi'(z)|}{(1 + |\phi^2(z)|)}, \tag{18}$$

where $\phi'(z) \neq 0$, $z = x+iy$ and $\phi(z)$ is arbitrary analytic function in some domain $\Omega \subset \mathbb{C}^1$ (usually the considerations are local). More precisely, if a classical solution of (16) exists near some point (x_0, y_0) then one can find an analytic function $\phi(z)$ near $z_0 = x_0 + iy_0$ and such that $\phi'(z) \neq 0$ and (16) holds in a tiny neighbourhood of (x_0, y_0). Conversely, the function p given by (18) satisfies (16) in each domain Ω, where $\phi'(z) \neq 0$. In fact, $G(z)$ analytic near z_0, $G(z_0) \neq 0$ implies that $\Delta \log |G| = 0$.

Our main conclusion is that the system (11) possesses (at least locally) a general solution depending on two arbitrary analytic functions in $\mathbb{C}^1$.

Unfortunately, we are able to solve only the constant data Dirichlet problem for (11). One can easily see that (16), respectively (17) do not possess unique solution in general.

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