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EVOLUTION OF A TWO-TYPE BELLMAN–HARRIS PROCESS GENERATED BY A LARGE NUMBER OF PARTICLES

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We investigate the evolution of a two-type critical Bellman–Harris branching process with the following properties: the tail of the life-length distribution of the first type particles is of order $o(t^{-2})$; the tail of the life-length distribution of the second type particles is regularly varying at infinity with index $-\beta$, $\beta \in (0,1]$; the process is generated at time $t = 0$ by a large number N of the second type particles and no particles of the first type. Letting $t = N^{\gamma}L(N)$, where $\gamma \in [0,\infty)$ and $L(N)$ is a function slowly varying at infinity, we show that the set of triples $(\beta,\gamma,L(N))$ may be divided into several regions within each of which the process at time t exhibits asymptotics (as $N,t \to \infty$) which is different from those in the other regions.

We investigate a two-type critical Bellman–Harris branching process $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$, $t \geq 0$ in which a particle of type $i \in \{1,2\}$ has the life–length distribution $G_i(t)$. At the end of her life she produces ξ_{i1} particles of the first type and ξ_{i2} particles of the second type in accordance with generating function $f_i(\mathbf{s}) = f_i(s_1,s_2) := \mathbf{E}s_1^{\xi_{i1}}s_2^{\xi_{i2}}$, $\mathbf{s} := (s_1,s_2) \in [0,1]^2$.

Define the two–dimensional vector-columns

$$\mathbf{G}(t) := (G_1(t), G_2(t))^{\dagger},\ \mathbf{f}(\mathbf{s}) = (f_1(\mathbf{s}), f_2(\mathbf{s}))^{\dagger}.$$

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For vectors $\mathbf{s} := (s_1, s_2) \in [0,1]^2$ and $\mathbf{z} := (z_1, z_2) \in \mathbb{Z}_+^2 := \{0,1,\dots\} \times \{0,1,\dots\}$ we write $\mathbf{s}^{\mathbf{z}} := s_1^{z_1} s_2^{z_2}$. Symbols $\mathbf{1}$ and $\mathbf{0}$ will be used to denote (depending on the context) either the vector-rows $(1,1)$ and $(0,0)$ or the vector-columns $(1,1)^\dagger$ and $(0,0)^\dagger$. Put

(1) $$m_{ij} := \mathbb{E}\xi_{ij},\ b^i_{jk} := \mathbb{E}\xi_{ij}\xi_{ik},\ i,j,k = 1,2,$$

$$\mathbf{M} := \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

We assume that $\mathbf{Z}(t)$ is indecomposable, aperiodic and critical. This means, in particular, that there exists a positive integer n_0 such that all the elements of the matrix $\mathbf{M}^{n_0}$ are positive, the Perron root of $\mathbf{M}$ is equal to 1 and there exist unique left and right eigenvectors $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$ such that

$$\mathbf{M}\mathbf{u}^\dagger = \mathbf{u}^\dagger,\ \mathbf{v}\mathbf{M} = \mathbf{v},\ \mathbf{v}\mathbf{u}^\dagger = 1,\ \mathbf{u} > \mathbf{0},\ \mathbf{v} > \mathbf{0},\ \mathbf{v}\mathbf{1} = 1.$$

In addition, we suppose that

$$B := 0.5 \sum_{i,j,k=1,2} v_i b^i_{jk} u_j u_k < \infty.$$

Along with the criticality we impose the following conditions on the tail behavior of the life-length distributions of particles:

(2) $$1 - G_1(t) = o(t^{-2}) \text{ and } 1 - G_2(t) = \ell(t) t^{-\beta},\ \beta \in (0,1],$$

where $\ell(t)$ is a function slowly varying at infinity. Here and hereafter all unspecified limiting relations are assumed to hold as $t \to \infty$ or $N, t \to \infty$.

We suppose that the two-type Bellman-Harris process is generated at time $t = 0$ by a large number N of the second type particles and no particles of the first type, i.e., $\mathbf{Z}(0) = (Z_1(0), Z_2(0)) = (0, N)$, and analyze the distribution of the population size $\mathbf{Z}(t) = (Z_1(t), Z_2(t))$, as $t \to \infty$. Note that a similar problem for a single-type critical Bellman-Harris branching process has been investigated in [3]. The critical multi-type Sevastyanov branching processes (see [1]) initiated by a large number of particles were (implicitly) considered in [2] under the assumption that the expected life-lengths of particles of all types are finite. In view of (2) our results do not follow from the results obtained in [2].

In this note we present a condensed and slightly generalized version of the results obtained in [5]. In particular, Theorem 7 below is more general than the corresponding statement in [5].

We show that the time axis $0 \le t < \infty$ splits into several regions whose ranges depend on β, γ and the ratio N/t within each of which the process at time t exhibits asymptotics (as $N, t \to \infty$) which is different from those in the other regions.

Let $\mu_i(t) := \int_0^t (1 - G_i(w))\mathrm{d}w$ and $\mu_i := \mu_i(\infty)$, $i = 1, 2$. We suppose that $\mu_2 = \infty$ and, consequently, $(1-\beta)\mu_2(t) \sim t^{1-\beta}\ell(t) = t(1 - G_2(t))$, if $\beta \in (0, 1)$, and $\mu_2(t) \sim \ell_1(t)$, if $\beta = 1$, where $\ell_1(t) := \int_0^t \ell(u)u^{-1}\mathrm{d}u \to \infty$ is a function slowly varying at infinity. Note that $\ell(t) = o(\ell_1(t))$. Put $R(t) := t\mu_2^{-1}(t)$,

$$F_i(t;\mathbf{s}) := \mathbf{E}_i\mathbf{s}^{\mathbf{Z}(t)},\ \mathbf{F}(t;\mathbf{s}) := (F_1(t;\mathbf{s}), F_2(t;\ \mathbf{s}))^\dagger,\ \mathbf{Q}(t;\mathbf{s}) := \mathbf{1} - \mathbf{F}(t,\mathbf{s}),$$
$$\mathbf{\Phi}(\mathbf{s}) = (\Phi_1(\mathbf{s}), \Phi_2(\mathbf{s}))^\dagger := \mathbf{M}\mathbf{s} - (\mathbf{1} - \mathbf{f}(\mathbf{1} - \mathbf{s})),$$
$$\mathbf{E}_{(0,N)}[\cdot] := \mathbf{E}[\cdot|\mathbf{Z}(0) = (0, N)],\ \mathbf{P}_{(0,N)}(\cdot) := \mathbf{P}(\cdot|\mathbf{Z}(0) = (0, N)),$$

where $\mathbf{E}_j[\cdot] := \mathbf{E}[\cdot|\mathbf{Z}(0) = (\delta_{1j}, \delta_{2j})]$, $\mathbf{P}_j(\cdot) := \mathbf{P}(\cdot|\mathbf{Z}(0) = (\delta_{1j}, \delta_{2j}))$, $j = 1, 2$, and δ_{ij} is the Kronecker symbol.

One of the main characteristics of any critical branching process is its survival probability. A specialization of Theorem 1 in [4] gives the asymptotic behavior of the survival probability of the two-type Bellman-Harris branching process which satisfies conditions (2): for any fixed $\mathbf{s} \in [0, 1]^2$

$$\mathbf{Q}(t;\mathbf{s}) = \mathbf{1} - \mathbf{F}\,(t;\mathbf{s}) \sim\ \mathbf{u}^\dagger\sqrt{\frac{v_2 u_2}{B}\,(1 - G_2(t))\,(1 - s_2)}. \tag{3}$$

In particular, $\mathbf{P}_i(Z_1(t) > 0) = o(Q_i(t))$, $i = 1, 2$. The last two asymptotic relations mean that if the two-type population survives up to a distant time t, then, with probability close to 1, the population at that time consists of the second type particles only.

Clearly,

$$\mathbf{E}_{(0,N)}\mathbf{s}^{\mathbf{Z}(t)} = F_2^N(t;\mathbf{s}) = e^{-N(1-F_2(t;\mathbf{s}))(1+o(1))} \tag{4}$$

provided that $\lim_{t\to\infty}(1 - F_2(t;\mathbf{s})) = 0$. Thus, to understand the asymptotic behavior of $\mathbf{Z}(t)$ under the present assumptions one has to investigate the behavior of $N(1 - F_2(t;\mathbf{s}))$, as $N, t \to \infty$, under a proper scaling of the components of $\mathbf{Z}(t)$. If N and t tend to infinity in such a way that $N\sqrt{1 - G_2(t)} \to 0$, then, in view of (3) the population becomes extinct. If, however, $N\sqrt{v_2 u_2\,(1 - G_2(t))\,/B} \to r \in (0, \infty)$, then

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)}\mathbf{s}^{\mathbf{Z}(t)} = e^{-ru_2\sqrt{1-s_2}}. \tag{5}$$

Thus, despite the indecomposability of the process there is only a finite number of the second type particles and no particles of the first type in the limit. This phenomenon has a natural intuitive explanation: at a distant time t the population only consists of the particles whose life-length distributions have heavy tails (see [3] where a similar effect is discussed for a single-type critical Bellman-Harris process).

Below we introduce basic assumptions of the paper borrowed from [7] and [6].

Hypothesis A. *The distribution functions $G_1(t)$ and $G_2(t)$ satisfy* (2). *In addition, if $\beta \in (0, 1/2]$ then there exist positive constants C and T_0 such that*

$$G_2(t+\Delta) - G_2(t) \leq C\Delta\ell(t)t^{-\beta-1}$$

for $t \geq T_0$ and any fixed $\Delta > 0$.

If the range of β is other than $(0, 1]$, then we write that Hypothesis $\mathbf{A}(a, b)$ or $\mathbf{A}(a, b]$ holds, meaning that we **only** consider the range $\beta \in (a, b)$ or $\beta \in (a, b]$ and require the validity of Hypothesis **A** for the indicated range.

Note that if Hypothesis **A** holds, then

$$N\mathbf{P}_2(Z_1(t) > 0) = N(1 - F_2(t; 0, 1)) \leq N\mathbf{E}_2\left[Z_1(t)\right] \leq CN/\mu_2(t) \tag{6}$$

which implies that there are no particles of the first type in the limit if $\mu_2(t) \gg N$. This means, in particular, that if, given $\mu_2(t) \gg N$, the limit

$$\Pi_2(\lambda) := \lim_{N,t\to\infty} N\big(1 - \mathbf{E}_2 e^{-\lambda Z_2(t)\psi(t)}\big) = \lim_{N,t\to\infty} N\big(1 - F_2\big(t; 0, e^{-\lambda\psi(t)}\big)\big)$$

exists for some function $\psi(t)$ and $\lambda > 0$, then, for any choice $s_1 = s_1(t) \in [0, 1]$

$$\lim_{N,t\to\infty} N\big(1 - \mathbf{E}_2 s_1^{Z_1(t)} e^{-\lambda Z_2(t)\psi(t)}\big) = \Pi_2(\lambda) \tag{7}$$

and vice versa.

An intuitive explanation of this effect is as follows. The branches of the genealogical trees generated by N initial particles consist of the rays which may be thought of as those generated by renewal processes with increments which (depending on the type of the corresponding particle) have the distribution function $G_1(t)$ or $G_2(t)$. As we know by (5), there are only a few surviving branches at a distant time t such that $\mu_2(t) \gg N$ and, as a result, not too many rays attain the time-level t. Since the life-length distribution of the first type particles has a light tail of order $o(t^{-2})$, particles of this type are present in the population at time t with probability which is negligible in comparison with the survival probability of the whole process up to this time.

It will be shown that there are several natural regions of $t = t(N)$ which correspond to essentially different limiting distributions of the vector $\mathbf{Z}(t)$, properly scaled, as $N, t \to \infty$. To describe the ranges of the regions in more detail we introduce three functions

$$y = \mathfrak{g}_1(N),\ y = \mathfrak{g}_2(N) \text{ and } y = \mathfrak{g}_3(N)$$

which are the inverse functions to

$$N(y) = (1-\beta)y^{\beta}\ell^{-1}(y),\ N(y) = y^{1-\beta}\ell(y)(1-\beta)^{-1}$$
$$\text{and } N(y) = [B^{-1}v_2u_2(1-G_2(y))]^{-1/2},$$

respectively, if $\beta \in (0,1)$ and to

$$N(y) = y\ell_1^{-1}(y), N(y) = \ell_1(y) \text{ and } N(y) = [B^{-1}v_2u_2(1-G_2(y))]^{-1/2}$$

respectively, if $\beta = 1$. By the properties of regularly varying functions we have

$$\mathfrak{g}_1(N) = N^{1/\beta}L_1(N),\ \mathfrak{g}_2(N) = N^{1/(1-\beta)}L_2(N),\ \mathfrak{g}_3(N) = N^{2/\beta}L_3(N)$$

for functions $L_i(\cdot)$, $i = 1, 2, 3$, slowly varying at infinity, where $\beta \in (0, 1]$, excluding the case $\beta = 1$ for $\mathfrak{g}_2$. Of course, $\mathfrak{g}_2(N) \gg N^k$ for all $k \in \mathbb{N}$, if $\beta = 1$. It can be checked that

$$\mathfrak{g}_2(N) \ll \mathfrak{g}_1(N) \ll \mathfrak{g}_3(N)$$

for $\beta \in [0, 1/2)$,

$$\mathfrak{g}_1(N) \ll \mathfrak{g}_2(N) \ll \mathfrak{g}_3(N)$$

for $\beta \in (1/2, 2/3)$, and

$$\mathfrak{g}_1(N) \ll \mathfrak{g}_3(N) \ll \mathfrak{g}_2(N)$$

for $\beta \in (2/3, 1]$, as $N \to \infty$.

We call the ranges of $t = t(N)$ satisfying $t = o(\mathfrak{g}_1(N))$, $t \sim \mathfrak{g}_1(Nr)$ and $t \gg \mathfrak{g}_1(N)$, as $N \to \infty$, the *early evolutionary stages* of the population, the *intermediate evolutionary stages* and the *final evolutionary stages*, respectively.

1. Early evolutionary stages

Denote by $\mathbf{D} = (D_{ij})_{i,j=1}^2$ a 2×2 matrix with positive entries $D_{ii} = \dfrac{1-m_{ii}}{1-m_{11}}$ and $D_{ij} = \dfrac{m_{ij}}{1-m_{11}}$ if $i \neq j$. Set $\Gamma_\beta = 1$ for $\beta = 1$ and $\Gamma_\beta := [\pi\beta(1-\beta)]^{-1}\sin\pi\beta$ for $\beta \in (0, 1)$.

Theorem 1. *Suppose that Hypothesis* $\boldsymbol{A}$ *holds and that* $t = o(\mathfrak{g}_1(N))$, $t = o(\mathfrak{g}_2(N))$. *If* $\beta = 1/2$, *assume additionally that*

$$\lim_{N,\,t\to\infty} \mu_2(t)N^{-1}\int_0^t \mathrm{d}w(1+\mu_2(w))^{-2} = 0. \tag{8}$$

Then, for any $\lambda_1,\ \lambda_2 \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda_1\frac{Z_1(t)\mu_2(t)}{N} - \lambda_2\frac{Z_2(t)}{N}} = e^{-\mu_1\beta\Gamma_\beta D_{21}\lambda_1 - D_{22}\lambda_2}.$$

Corollary 2. *Suppose that Hypothesis* $\boldsymbol{A}$ *holds and that* $t = o(\mathfrak{g}_1(N))$. *Then, for* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda\frac{Z_2(t)}{N}} = e^{-D_{22}\lambda}.$$

This corollary does not require $t = o(\mathfrak{g}_2(N))$, nor condition (8). Set

$$\mathbf{O}(s) := (O_1(s), O_2(s))^\dagger := \beta\Gamma_\beta \int_0^\infty \mathbf{D\Phi}(\mathbf{Q}(w; s, 1))\mathrm{d}w.$$

Theorem 3. *Suppose that Hypothesis* $\boldsymbol{A}(0, 0.5]$ *holds and that* $t = o(\mathfrak{g}_1(N))$ *and* $t \sim \mathfrak{g}_2(r^{-1}N)$, $r \in \mathbb{R}^+$. *If* $\beta = 1/2$, *assume additionally that*

$$\Upsilon := \int_0^\infty \mathrm{d}w(1+\mu_2(w))^{-2} < \infty. \tag{9}$$

Then, for any $s \in [0, 1]$ *and* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} s^{Z_1(t)} e^{-\lambda\frac{Z_2(t)}{N}} = e^{-r\beta\Gamma_\beta\mu_1 D_{21}\,(1-s) + rO_2(s) - D_{22}\lambda}.$$

Thus, in this case we asymptotically have a few individuals of the first type, while the number of individuals of the second type is still of order N. Moreover, $Z_1(t)$ and $Z_2(t)N^{-1}$ are asymptotically independent.

Remark. The assumptions of Theorem 3 hold if either $\beta = 1/2$ and $\lim_{t\to\infty} \ell(t) = \infty$ or $\beta < 1/2$.

Theorem 4. *Suppose that Hypothesis* $\boldsymbol{A}(0, 0.5]$ *holds and that* $t = o(\mathfrak{g}_1(N))$ *and* $t \gg \mathfrak{g}_2(N)$. *Then, for any* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} \left[e^{-\lambda\frac{Z_2(t)}{N}}; Z_1(t) = 0\right] = \lim_{N,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda\frac{Z_2(t)}{N}} = e^{-D_{22}\lambda}.$$

Remark. The assumptions of Theorem 4 hold for $\beta = 1/2$ only if $\ell(t) \to \infty$.

2. The intermediate evolutionary stages

There are three essentially different intermediate subranges which are characterized by one of the conditions $t \gg \mathfrak{g}_2(N)$, $t = o(\mathfrak{g}_2(N))$ or $t \sim \mathfrak{g}_2(Nr_2)$, $r_2 \in \mathbb{R}^+$, which are assumed to hold along with the defining property of the intermediate stages. We only analyze the first and the second subranges. The remaining case which implies $\beta = 1/2$ is not considered, because it requires much more delicate analysis.

Put

$$N_i(\mathbf{x}) := 0.5 \sum_{j,k=1,2} b^i_{jk} x_j x_k, \;\; \mathbf{N}(\mathbf{x}) := (N_1(\mathbf{x}), N_2(\,\mathbf{x}))^\dagger .$$

The system of equations

$$\boldsymbol{\Omega}(\lambda) = \mathbf{D}(0,\lambda)^\dagger - \Gamma_\beta \int_0^1 \frac{\mathbf{DN}\big(\,\boldsymbol{\Omega}\big(\lambda\,(1-w)^\beta\,\big)\big)}{(1-w)^{2\beta}} \mathrm{d}w^\beta, \quad \lambda > 0, \tag{10}$$

has a unique solution with non-negative components, and we denote this solution by $\boldsymbol{\Omega}(\lambda) := (\Omega_1(\lambda), \Omega_2(\lambda))^\dagger$.

Theorem 5. *Suppose that Hypothesis* $\boldsymbol{A}$ *holds and that* $t \sim \mathfrak{g}_1(Nr^{-1})$. *Then, for any* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,\,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda \frac{rZ_2(t)}{N}} = e^{-r\Omega_2(\lambda)}. \tag{11}$$

Furthermore, if Hypothesis $\boldsymbol{A}(0, 0.5]$ *holds together with* $t \sim \mathfrak{g}_1(Nr^{-1})$ *and* $t \gg \mathfrak{g}_2(N)$, *then, for any* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,\,t\to\infty} \mathbf{E}_{(0,N)} \left[e^{-\lambda \frac{rZ_2(t)}{N}}; Z_1(t) = 0 \right] = e^{-r\Omega_2(\lambda)}. \tag{12}$$

Observe that there are no first type particles in the limit under the asymptotic regime $t \gg \mathfrak{g}_2(N)$. Note also that the assumptions $t \sim \mathfrak{g}_1(Nr^{-1})$ and $t \gg \mathfrak{g}_2(N)$ entail $\beta = 1/2$ and $\lim_{t\to\infty} \ell(t) = \infty$, or $\beta < 1/2$.

Put

$$\mathbf{C}_\beta := \begin{pmatrix} D_{11}\beta\Gamma_\beta & D_{12} \\ D_{21}\beta\Gamma_\beta & D_{22} \end{pmatrix}.$$

The system of equations

$$\mathbf{H}(\theta,\lambda) = \mathbf{C}_\beta\big(1, \lambda\theta^{1-\beta}\big)^\dagger - \Gamma_\beta \theta^{2\beta-1} \int_0^1 \frac{\mathbf{DN}\,(\mathbf{H}(\theta(1-y),\lambda))}{(1-y)^{2-2\beta}} \mathrm{d}y^\beta, \tag{13}$$

for $\theta, \lambda > 0$ and $\beta \in (1/2, 1]$, has a unique solution with non-negative components, and we denote this solution by $\mathbf{H}(\theta,\lambda) = (H_1(\theta,\lambda), H_2(\theta,\lambda))^\dagger$.

Theorem 6. *Suppose that Hypothesis* $\boldsymbol{A}(0.5,1]$ *holds, and that* $t \sim \mathfrak{g}_1(Nr^{-1})$ *and* $t = o(\mathfrak{g}_2(N))$. *Then, for* $\lambda_1,\ \lambda_2 \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda_1 \frac{r\mu_2(t)Z_1(t)}{\mu_1 N} - \lambda_2 \frac{rZ_2(t)}{N}} = e^{-r\lambda_1 H_2\left(\lambda_1^{\frac{1}{2\beta-1}},\, \lambda_2 \lambda_1^{-\frac{\beta}{2\beta-1}}\right)}.$$

2.1. Final evolutionary stages

Recall that relations (3) and (4) imply that the population dies out whenever $N\sqrt{1-G_2(t)} \to 0$, whereas the limit distribution of the number of particles is given by formula (5) whenever $N\sqrt{1-G_2(t)} \to r \in (0,\infty)$. Therefore, if $t \gg \mathfrak{g}_1(N)$, we only investigate the case $N\sqrt{1-G_2(t)} \to \infty$. Observe that under the additional assumption $t \gg \mathfrak{g}_2(N)$ we have in Theorem 7 given below that $\beta = 2/3$ and $\lim_{t\to\infty} \ell(t) = \infty$, or $\beta < 2/3$.

For $\gamma \in [0,1)$, define $\psi(y) := y^{-\gamma}\ell(y)$ and assume that $\lim_{y\to\infty} \psi(y) = 0$. Denote by

$$y = \mathfrak{g}_{\gamma,1}(N) \text{ and } y = \mathfrak{g}_{\gamma,2}(N)$$

the inverse functions to

$$N = \sqrt{B[v_2 u_2 \psi(y)(1-G_2(y))]^{-1}} \text{ and } N = \mu_2(y)\psi^{-1}(y),$$

respectively. By the properties of regularly varying functions we have

$$\mathfrak{g}_{\gamma,1}(N) = N^{2/(\gamma+\beta)} L_{\gamma,1}(N) \text{ and } \mathfrak{g}_{\gamma,2}(N) = N^{1/(\gamma+1-\beta)} L_{\gamma,2}(N)$$

for some $L_{\gamma,1}(\cdot)$ and $L_{\gamma,2}(\cdot)$ slowly varying at infinity.

Theorem 7. *Suppose that Hypothesis* $\boldsymbol{A}(0,1)$ *holds, and that* $t \gg \mathfrak{g}_1(N)$, $t \gg \mathfrak{g}_{\gamma,2}(N)$ *and, for a fixed* $r \in (0,\infty)$ *set* $t \sim \mathfrak{g}_{\gamma,1}(Nr^{-1})$. *Then, for each* $\lambda \in \mathbb{R}^+$,

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} e^{-\lambda u_2 Z_2(t)\psi(t)} = e^{-ru_2\sqrt{\lambda}}. \tag{14}$$

Replacing $t \gg \mathfrak{g}_{\gamma,2}(N)$ *by a stronger condition* $t \gg \mathfrak{g}_2(N)$ *we also have*

$$\lim_{N,t\to\infty} \mathbf{E}_{(0,N)} \left[e^{-\lambda u_2 Z_2(t)\psi(t)}; Z_1(t) = 0\right] = e^{-ru_2\sqrt{\lambda}}. \tag{15}$$

Let us describe the conditions of Theorem 7 in terms of the restrictions imposed on β and γ. Set $t = N^{\nu}L^*(N)$ and $\nu \in (1/\beta, 2/\beta)$. According to Theorem 7 relations (14) and (15) hold if $\nu = \nu(\beta,\gamma) = 2/(\gamma+\beta)$ and $L^*(N)$ is a function slowly varying at infinity which is defined by the condition $t \sim \mathfrak{g}_{\gamma,1}(Nr^{-1})$.

For simplicity we omit the cases $\nu = 1/\beta, 2/\beta$, $\gamma = \beta$, $\gamma = 0$ and $\gamma = 2-3\beta$. Direct calculations show that ν increases when γ decreases and, moreover, $\gamma \in (0,\beta)$ for $\beta \le 2/3$. Besides, to demonstrate (14) we assume that $\gamma \in (3\beta-2,\beta)$ for $\beta \ge 2/3$. Finally, the assumption $t \gg \mathfrak{g}_2(N)$ needed to prove (15) is meaningful only for $t = o(\mathfrak{g}_3(N))$ or $\beta \le 2/3$ which gives $\gamma \in (0, 2-3\beta)$.

We think that relation (15) holds true without the condition $t \gg \mathfrak{g}_2(N)$, i.e., under the same assumptions that are formulated for (14). At the moment we only have a preliminary result: for $\beta \in (1/2, a)$, where $2/3 < a < 1$ is fixed, there exists slowly varying $L_i^*(t)$ such that $\mathbf{P}_i(Z_1(t) > 0) \sim t^{-\beta}L_i^*(t)$. We believe that a similar result holds for $a = 1$, too. If this were the case we would have

$$\begin{aligned} 0 &\le \mathbf{E}_{(0,N)} e^{-\lambda u_2 Z_2(t)\psi(t)} - \mathbf{E}_{(0,N)}\left[e^{-\lambda u_2 Z_2(t)\psi(t)}; Z_1(t) = 0\right] \\ &= \mathbf{E}_{(0,N)}\left[e^{-\lambda u_2 Z_2(t)\psi(t)}; Z_1(t) > 0\right] \le \mathbf{P}_{(0,N)}(Z_1(t) > 0) \\ &= 1 - \mathbf{P}_2^N(Z_1(t) = 0) = 1 - (1 - t^{-\beta}L_2^*(t)(1+o(1)))^N \sim t^{-\beta}L_2^*(t)N \end{aligned}$$

provided that $t^{-\beta}L_2^*(t)N \to 0$, as $t \to \infty$. On the other hand, under the conditions of Theorem 7 $t = N^{\nu}L^*(N)$ for $\nu \in (1/\beta, 2/\beta)$ which leads to

$$t^{-\beta}L_2^*(t)N = (N^{\nu}L^*(N))^{-\beta}L_2^*(N^{\nu}L^*(N))N \to 0,$$

as $t \to \infty$. Thus, it seems the assumption $t \gg \mathfrak{g}_2(N)$ given in the second part of Theorem 7 can be dispensed with.

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