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Dedicated to Prof. Dušan D. Repovš
on the occasion of his 65th birthday

A MULTIPLICITY THEOREM FOR PARAMETRIC SUPERLINEAR (p, q)-EQUATIONS

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Abstract. We consider a parametric nonlinear Robin problem driven by the sum of a p-Laplacian and of a q-Laplacian ((p, q)-equation). The reaction term is $(p-1)$-superlinear but need not satisfy the Ambrosetti–Rabinowitz condition. Using variational tools, together with truncation and comparison techniques and critical groups, we show that for all small values of the parameter, the problem has at least five nontrivial smooth solutions, all with sign information.

Keywords: superlinear reaction, constant sign and nodal solutions, extremal solutions, nonlinear regularity, nonlinear maximum principle, critical groups.

Mathematics Subject Classification: 35J20, 35J60.

1. INTRODUCTION

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric (p, q)-equation

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = \lambda f(z, u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 & \text{on } \partial\Omega,\ 1 < q < p < +\infty,\ \lambda > 0. \end{cases} \tag{P_λ}$$

For every $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}\left(|\nabla u|^{r-2}\nabla u\right) \quad \text{for all } u \in W^{1,r}(\Omega).$$

So, the differential operator in (P_λ) is the (p, q)-Laplacian plus a potential function $\xi \in L^\infty(\Omega)$, $\xi \geq 0$. The reaction term (right hand side of (P_λ)) is parametric with $\lambda > 0$ being the parameter. The function $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous) and we assume that $f(z,\cdot)$ is $(p-1)$-superlinear near $\pm\infty$, while it is $(q-1)$-superlinear near zero. These assumptions incorporate in our framework the case of competing concave and convex nonlinearities in the reaction (concave-convex problem). In the boundary condition, $\dfrac{\partial u}{\partial n_{pq}}$ denotes the conormal derivative of u corresponding to the (p,q)-Laplacian. This conormal derivative is interpreted using the nonlinear Green's identity (see Papageorgiou–Rădulescu–Repovš [21], Corollary 1.5.16). If $u \in C^1(\overline{\Omega})$, then

$$\frac{\partial u}{\partial n_{pq}} = \left[|\nabla u|^{p-2} + |\nabla u|^{q-2}\right] \frac{\partial u}{\partial n}$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(\cdot)$ is nonnegative.

Using variational tools from the critical point theory together with suitable truncation and comparison techniques and critical groups, we show that for all $\lambda > 0$ small problem (P_λ) has at least five nontrivial smooth solutions, all with sign information (two positive, two negative and the fifth nodal (sign changing)).

Parametric (p,q)-equations were studied primarily in the context of Dirichlet problems, using different conditions of the reaction. We mention the works of Benouhiba–Belyacine [4], Bhattacharya–Emamizadeh–Farjudian [5], Bobkov–Tanaka [6], Papageorgiou–Rădulescu [15], Papageorgiou–Rădulescu–Repovš [19, 20], Papageorgiou–Vetro–Vetro [22], Papageorgiou–Zhang [25, 26], Rădulescu [28], Tanaka [29].

Equations driven by the sum of a p-Laplacian and of a q-Laplacian ((p,q)-equations), arise naturally in many mathematical models of physical processes. In elasticity theory such equations describe composites consisting of two different materials with distinct hardening exponents. In their general form such anisotropic materials have energy functionals with unbalanced growth and were introduced and studied by Marcellini [13] and Zhikov [30, 31]. Recently important regularity results for minimizers of such functionals were obtained by Mingione and coworkers (see [1, 8, 9]). We encounter (p,q)-equations in other physical applications too. We mention the works of Bahrouni–Rădulescu–Repovš [2] (on transonic flows), Benci–D'Avenia–Fortunato–Pisani [3] (quantum physics), Cherfils–Il'yasov [7] (reaction-diffusion systems).

2. MATHEMATICAL BACKGROUND – HYPOTHESES

Let X be a Banach space. By X^* we denote the topological dual of X and by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|_X)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence."

The main spaces in the study of (P_λ) are the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq +\infty$.

By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ defined by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is ordered by the positive (order) cone

$$C_+ = \{u \in C^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure, we can define in the usual way the boundary Lebesgue spaces $L^s(\partial\Omega)$, $1 \leq s \leq +\infty$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u\big|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary values to all Sobolev functions. We know that the trace map $\gamma_0(\cdot)$ is compact into $L^s(\partial\Omega)$ for all $1 \leq s < \dfrac{(N-1)p}{N-p}$ if $1 < p < N$ and into $L^s(\partial\Omega)$ for all $1 \leq s < +\infty$ if $N \leq p$. Moreover, we have

$$\operatorname{im}\gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \quad \text{and} \quad \ker\gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$ are understood in the sense of traces.

As we already mentioned in the Introduction, we will also use critical groups in order to distinguish between solutions of (P_λ). So, let us recall the definition of critical groups. As before, X is a Banach space and $\varphi \in C^1(X,\mathbb{R})$. For $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} K_\varphi &= \{u \in X : \varphi'(u) = 0\} \quad \text{(the critical set of } \varphi\text{)}, \\ \varphi^c &= \{u \in X : \varphi(u) \leq c\}. \end{aligned}$$

For a topological pair (Y_1, Y_2) such that $Y_2 \subseteq Y_1 \subseteq X$, for every $k \in \mathbb{N}_0$ by $H_k(Y_1,Y_2)$ we denote the k^{th}-relative singular homology group for the pair (Y_1,Y_2) with integer coefficients. If $u \in K_\varphi$ is isolated, then the critical groups of φ at u are defined by

$$C_k(\varphi,u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with $c = \varphi(u)$ and U a neighborhood of u such that $\varphi^c \cap U \cap K_\varphi = \{u\}$. By convention $C_k(\varphi,u) = 0$ for all $k \in -\mathbb{N}$. The excision property of singular homology implies that this definition is independent of the choice of the isolating neighborhood U.

Next let us introduce the basic notation that we will use throughout this work. For every $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then given $u \in W^{1,p}(\Omega)$, we define $u^{\pm}(z) = u(z)^{\pm}$ for all $z \in \Omega$. We know that $u^{\pm} \in W^{1,p}(\Omega)$ and $u = u^+ - u^-$. If $v, u \in W^{1,p}(\Omega)$ and $v \leq u$, then we define

$$[v, u] = \{h \in W^{1,p}(\Omega) : v(z) \leq h(z) \leq u(z) \text{ for a.a. } z \in \Omega\}.$$

For $r \in (1, +\infty)$, by $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ we denote the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W^{1,r}(\Omega).$$

This map is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (hence maximal monotone too) and of type $(S)_+$, that is, "$u_n \xrightarrow{w} u$ in $W^{1,r}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0 \Rightarrow u_n \to u$ in $W^{1,r}(\Omega)$".

Recall that a set $S \subseteq W^{1,p}(\Omega)$ is downward (resp. upward) directed if for all $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1$, $u \leq u_2$ (resp. $u_1 \leq u$, $u_2 \leq u$).

Now let us introduce our hypotheses on the data of (P_λ).

$H(\xi)$: $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$.

$H(\beta)$: $\beta \in C^{0,\alpha}(\partial\Omega)$ with $0 < \alpha \leq 1$ and $\beta(z) \geq 0$ for all $z \in \partial\Omega$.

H_0: $\xi \not\equiv 0$ or $\beta \not\equiv 0$.

$H(f)$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$ with $a \in L^\infty(\Omega)$,
$$p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p \end{cases};$$

(ii) if $F(z, x) = \int_0^x f(z, s)ds$, then $\lim\limits_{x\to\pm\infty} \dfrac{F(z,x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $\mu \in \left((r-p)\max\left\{\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0 > 0$ such that
$$\beta_0 \leq \liminf_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\mu} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) $\lim\limits_{x\to 0} \dfrac{f(z,x)}{|x|^{q-2}x} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exist $\tau \in (1, q)$ and $0 < \widehat{\eta}_0 < \eta_0$ such that
$$0 \leq \liminf_{x\to 0} \frac{\tau F(z,x) - f(z,x)x}{|x|^p} \quad \text{uniformly for a.a. } z \in \Omega,$$
$$\widehat{\eta}_0 \leq \liminf_{x\to 0} \frac{f(z,x)}{|x|^{\tau-2}x} \leq \limsup_{x\to 0} \frac{f(z,x)}{|x|^{\tau-2}x} \leq \eta_0 \text{ uniformly for a.a. } z \in \Omega.$$

Remark 2.1. Hypothesis $H(f)$ (ii) implies that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega. \tag{2.1}$$

Therefore the reaction in problem (P_λ) is $(p-1)$-superlinear. However, we do not express this superlinearity of $f(z,\cdot)$ using the Ambrosetti–Rabinowitz condition (the AR-condition for short), which is very common in the literature. Recall that the AR-condition says that there exist $\vartheta > p$ and $M > 0$ such that

$$0 < \vartheta F(z,x) \le f(z,x)x \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M, \tag{2.2a}$$

$$0 < \operatorname*{ess\,inf}_{\Omega} F(\cdot, \pm M). \tag{2.2b}$$

Integrating (2.2a) and using (2.2b), we obtain the following weaker condition

$$c_0|x|^\vartheta \le F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \ge M, \text{ some } c_0 > 0. \tag{2.3}$$

Then (2.2a) and (2.3) show that under the AR-condition $f(z,\cdot)$ has at least $(\vartheta-1)$-polynomial growth. So, condition (2.1) holds. In the present work, instead of the AR-condition, we use a more general hypothesis (see $H(f)$ (ii)), which incorporates in our framework superlinear nonlinearities with slower growth near $\pm\infty$ (see the examples below). Note also that no sign condition is imposed on $f(z,\cdot)$.

Example 2.2. The following functions satisfy hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f_1(x) = \begin{cases} |x|^{\tau-2}x & \text{if } |x| \le 1, \\ |x|^{p-2}x \ln|x| + x & \text{if } 1 < |x|, \end{cases} \quad \text{with } 1 < \tau < q < p,$$

$$f_2(x) = \begin{cases} |x|^{r-2}x + k_- & \text{if } x < -1, \\ \dfrac{|x|^{\vartheta-2}x}{\ln(1+|x|)} & \text{if } |x| \le 1, \\ x^{r-1} + k_+ & \text{if } 1 < x, \end{cases} \quad \text{with } 1 < \vartheta < q+1,\ p < r < p^*,\ k_\pm = -1 \pm \frac{1}{\ln 2}.$$

From the above functions f_1 does not satisfy the AR-condition.

3. SOLUTIONS OF CONSTANT SIGN

Let $\gamma_p : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\gamma_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \quad \text{for all } u \in W^{1,p}(\Omega).$$

Hypotheses $H(\xi)$, $H(\beta)$, H_0, together with Lemma 4.11 of Mugnai–Papageorgiou [14] and Proposition 2.4 of Gasiński–Papageorgiou [10] imply that

$$\gamma_p(u) \ge c_1\|u\|^p \quad \text{for some } c_1 > 0, \text{ all } u \in W^{1,p}(\Omega). \tag{3.1}$$

For every $\lambda > 0$, we consider the C^1-functionals $\varphi_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda^\pm(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega F(z, \pm u^\pm)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Proposition 3.1. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda > 0$, then the functionals $\varphi_\lambda^\pm$ satisfy the C-condition.*

Proof. We do the proof for the functional φ_λ^+, the proof for φ_λ^- being similar.

We consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\varphi_\lambda^+(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \tag{3.2}$$

$$(1 + \|u_n\|)(\varphi_\lambda^+)'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{3.3}$$

From (3.3) we have

$$\begin{aligned}
&\left|\langle(\varphi_\lambda^+)'(u_n), h\rangle\right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+,\\
\Rightarrow\ &\Bigg|\langle A_p(u_n), h\rangle + \langle A_q(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n\, h\, dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n\, h\, d\sigma\\
&- \lambda \int_\Omega f(z, u_n^+)\, h\, dz\Bigg| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|}, \quad \text{for all } n \in \mathbb{N}.
\end{aligned} \tag{3.4}$$

In (3.4) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned}
&\gamma_p(u_n^-) + \|\nabla u_n^-\|_q^q \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N},\\
\Rightarrow\ &c_1\|u_n^-\|^p \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N} \text{ (see (3.1))},\\
\Rightarrow\ &u_n^- \to 0 \quad \text{in } W^{1,p}(\Omega) \text{ as } n \to +\infty.
\end{aligned} \tag{3.5}$$

From (3.2) and (3.5) it follows that

$$\gamma_p(u_n^+) + \frac{p}{q}\|\nabla u_n^+\|_q^q - \lambda \int_\Omega F(z, u_n^+)dz \leq M_2 \quad \text{for some } M_2 > 0, \text{ all } n \in \mathbb{N}. \tag{3.6}$$

In (3.4) we choose $h = u_n^+ \in W^{1,p}(\Omega)$ and obtain

$$-\gamma_p(u_n^+) - \|\nabla u_n^+\|_q^q + \lambda \int_\Omega f(z, u_n^+)u_n^+ dz \leq \varepsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{3.7}$$

We add (3.6) and (3.7) and recall that $q < p$. We obtain

$$\lambda \int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq M_3 \quad \text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}. \tag{3.8}$$

On account of hypotheses $H(f)$ (i),(ii) we have

$$\beta_1|x|^\mu - c_2 \leq f(z,x)x - pF(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } \beta_1 \in (0, \beta_0),\ c_2 > 0. \tag{3.9}$$

Using (3.9) in (3.8) we obtain

$$\|u_n^+\|_\mu^\mu \le M_4 \quad \text{for some } M_4>0, \text{ all } n\in\mathbb{N}. \tag{3.10}$$

First suppose that $N \neq p$. It is clear from hypothesis $H(f)$ (ii) that we may assume that $\mu \le r < p^*$. We choose $t\in[0,1)$ such that

$$\frac{1}{r}=\frac{1-t}{\mu}+\frac{t}{p^*}. \tag{3.11}$$

(recall that $p^*=+\infty$ if $N>p$). Using the interpolation inequality (see Papageorgiou–Winkert [24, p. 116]), we have

$$\begin{aligned} &\|u_n^+\|_r \le \|u_n^+\|_\mu^{1-t}\|u_n\|_{p^*}^t,\\ \Rightarrow\quad &\|u_n^+\|_r^r \le c_3\|u_n^+\|^{tr} \quad \text{for some } c_3>0, \text{ all } n\in\mathbb{N}, \end{aligned}$$

(here we have used (3.10) and the Sobolev embedding theorem).

From hypothesis $H(f)$ (i) we have

$$f(z,x)x \le c_4[1+|x|^r] \quad \text{for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } c_4>0. \tag{3.12}$$

In (3.4) we choose $h=u_n^+\in W^{1,p}(\Omega)$. Then for all $n\in\mathbb{N}$ we have

$$\begin{aligned} &\gamma_p(u_n^+)+\|\nabla u_n^+\|_q^q-\lambda\int_\Omega f(z,u_n^+)u_n^+dz \le \varepsilon_n,\\ \Rightarrow\quad &c_1\|u_n^+\|^p+\|\nabla u_n^+\|_q^q \le c_5[\lambda(1+\|u_n^+\|_r^r)+1] \qquad (3.13)\\ &\qquad \text{for some } c_5>0 \text{ (see (3.1) and (3.12))}\\ &\le c_6[\lambda(1+\|u_n^+\|^{tr})+1] \quad \text{for some } c_6>0. \end{aligned}$$

From (3.11) and our hypothesis on μ (see hypothesis $H(f)$ (ii)) we infer that $tr<p$. So, from (3.13) it follows that

$$\{u_n^+\}_{n\ge1}\subseteq W^{1,p}(\Omega) \text{ is bounded}. \tag{3.14}$$

Now suppose that $N=p$. In this case we know that $p^*=+\infty$. On the other hand by the Sobolev embedding theorem (see, for example, [21, Theorem 1.9.15]) we have that $W^{1,p}(\Omega)\hookrightarrow L^s(\Omega)$ for all $1\le s<+\infty$). So, for the previous argument to work, we need to replace p^* by $s>r$ big so that

$$tr=\frac{s(r-\mu)}{s-\mu}<p \quad \text{(recall } r-p<\mu, \text{ see } H(f)\text{ (ii))}.$$

Then with such a choice of $s>r$, the previous argument goes through and again we reach (3.12).

From (3.5) and (3.14) it follows that $\{u_n^+\}_{n\ge1}\subseteq W^{1,p}(\Omega)$ is bounded. So, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n\to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.15}$$

In (3.4) we choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.15). Then

$$\begin{aligned}
&\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] = 0,\\
\Rightarrow\quad &\limsup_{n\to+\infty} [\langle A_p(u_n), u_n - u\rangle + \langle A_q(u_n), u_n - u\rangle] \le 0 \ \text{ (recall that } A_q(\cdot) \text{ is monotone)},\\
\Rightarrow\quad &\limsup_{n\to+\infty} \langle A_p(u_n), u_n - u\rangle \le 0 \ \text{ (see (3.15))},\\
\Rightarrow\quad &u_n \to u \text{ in } W^{1,p}(\Omega) \ \text{ (since } A_p(\cdot) \text{ is of type } (S)_+).
\end{aligned}$$

This proves that φ_λ^+ satisfies the C-condition. In a similar fashion we show that φ_λ^- satisfies the C-condition. ☐

Let $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy (Euler) functional for problem (P_λ) defined by

$$\varphi_\lambda(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \lambda \int_\Omega F(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$. With small changes in the previous proof we can have the following result.

Proposition 3.2. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda > 0$, then φ_λ satisfies the C-condition.*

Now we are ready to produce constant sign solutions when $\lambda > 0$ is small.

Proposition 3.3. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda > 0$, then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least four nontrivial solutions of constant sign $u_0, \widehat{u} \in D_+$ and $v_0, \widehat{v} \in -D_+$.*

Proof. First we produce the positive solutions.

On account of hypotheses $H(f)$ (i),(iii) we have

$$F(z,x) \le c_7[|x|^\tau + |x|^r] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } c_7 > 0. \tag{3.16}$$

Then for all $u \in W^{1,p}(\Omega)$ we have

$$\begin{aligned}
\varphi_\lambda^+(u) &\ge \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \lambda c_8[\|u\|^\tau + \|u\|^r] \quad \text{for some } c_8 > 0 \text{ (see (3.16))},\\
&\ge \frac{c_1}{p}\|u\|^p - \lambda c_8[\|u\|^\tau + \|u\|^r]\\
&= \left[\frac{c_1}{p} - \lambda c_8[\|u\|^{\tau-p} + \|u\|^{r-p}\right]\|u\|^p.
\end{aligned} \tag{3.17}$$

Consider the function

$$\vartheta(t) = t^{\tau-p} + t^{r-p} \quad \text{for all } t > 0.$$

We have $\vartheta \in C^1(0,+\infty)$ and since $\tau < p < r$, we see that

$$\vartheta(t) \to +\infty \text{ as } t \to 0^+ \text{ and as } t \to +\infty.$$

Therefore we can find $t_0 \in (0,+\infty)$ such that

$$\begin{aligned} & \vartheta(t_0) = \inf_{t>0} \vartheta, \\ \Rightarrow \quad & \vartheta'(t_0) = 0, \\ \Rightarrow \quad & (p-\tau)t_0^{\tau-p-1} = (r-p)t_0^{r-p-1}, \\ \Rightarrow \quad & t_0 = \left[\frac{p-\tau}{r-p}\right]^{\frac{1}{r-\tau}}. \end{aligned}$$

We set $\lambda^* = \dfrac{c_1}{\lambda p c_8 \vartheta(t_0)} > 0$. Then for every $\lambda \in (0,\lambda^*)$, from (3.17) we have

$$\varphi_\lambda^+(u) \geq d_\lambda > 0 \quad \text{for all } \|u\| = t_0. \tag{3.18}$$

On the other hand using again hypotheses $H(f)$ (i),(ii), we see that given $\eta > 0$, we can find $c_\eta > 0$ such that

$$F(z,x) \geq \eta |x|^q - c_\eta |x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.19}$$

Let $\widehat{u} \in D_+$ and $t > 0$. We have

$$\begin{aligned} \varphi_\lambda^+(t\widehat{u}) &\leq \frac{t^p}{p}\gamma_p(\widehat{u}) + \frac{t^q}{q}\|\nabla \widehat{u}\|_q^q - \lambda t^q \eta \|\widehat{u}\|_q^q + \lambda c_\eta \|\widehat{u}\|_r^r \quad \text{(see (3.19))} \\ &\leq c_9[t^p + \lambda t^r] + \frac{t^q}{q}\left[\|\nabla \widehat{u}\|_q^q - \lambda q \eta \|\widehat{u}\|_q^q\right] \quad \text{for some } c_9 = c_9(\eta) > 0. \end{aligned}$$

Choosing $\eta > 0$ big, we see that

$$\varphi_\lambda^+(t\widehat{u}) \leq c_9[t^p + \lambda t^r] - c_{10}t^q \quad \text{for some } c_{10} > 0. \tag{3.20}$$

Since $q < p < r$, from (3.20) it follows that for $t \in (0,1)$ small we have

$$\varphi_\lambda^+(t\widehat{u}) < 0. \tag{3.21}$$

We consider the following minimization problem

$$\inf\left[\varphi_\lambda^+(u) : u \in \overline{B}_{t_0}\right] = m_\lambda^+. \tag{3.22}$$

The set $\overline{B}_{t_0} = \{u \in W^{1,p}(\Omega) : \|u\| \leq t_0\} \subseteq W^{1,p}(\Omega)$ is sequentially weakly compact (by the Eberlein–Šmulian theorem). Also the Sobolev embedding theorem and the compactness of the trace map imply that $\varphi_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$m_\lambda^+ = \varphi_\lambda^+(u_0) \quad \text{(see (3.22))}, \tag{3.23}$$

$$
\begin{aligned}
&\Rightarrow \quad \varphi_\lambda^+(u_0) < 0 = \varphi_\lambda^+(0) \quad \text{(see (3.21))},\\
&\Rightarrow \quad u_0 \neq 0.
\end{aligned}
$$

Moreover, from (3.18) we have that

$$
\begin{aligned}
& \quad 0 < \|u_0\| < t_0,\\
&\Rightarrow \quad (\varphi_\lambda^+)'(u_0) = 0 \quad \text{(see (3.22), (3.23))},\\
&\Rightarrow \quad \langle A_p(u_0), h\rangle + \langle A_q(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0\, h\, dz + \int_{\partial\Omega} \beta(z)|u_0|^{p-2}u_0\, h\, d\sigma\\
&\qquad = \lambda \int_\Omega f(z, u_0^+)\, h\, dz \quad \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}
\tag{3.24}
$$

In (3.24) we choose $h = -u_0^- \in W^{1,p}(\Omega)$. We obtain

$$
\begin{aligned}
& \gamma_p(u_0^-) + \|\nabla u_0^-\|_q^q = 0,\\
\Rightarrow \quad & c_1\|u_0^-\|^p \leq 0 \quad \text{(see (3.1))},\\
\Rightarrow \quad & u_0 \geq 0,\ u_0 \neq 0.
\end{aligned}
$$

Then from (3.24) we have

$$
\begin{cases}
-\Delta_p u_0(z) - \Delta_q u_0(z) + \xi(z)u_0(z)^{p-1} = \lambda f(z, u_0(z)) & \text{for a.a. } z \in \Omega,\\
\dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u_0|^{p-1} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{3.25}
$$

From (3.25) and Proposition 2.10 of Papageorgiou–Rădulescu [17], we have that $u_0 \in L^\infty(\Omega)$. Then from the nonlinear regularity theory of Lieberman [12] we have that $u_0 \in C_+ \setminus \{0\}$.

Let $\rho = \|u_0\|_\infty$. Hypotheses $H(f)$ (i),(iii) imply that we can find $\widehat{\xi}_\rho > 0$ such that

$$
\lambda f(z, x)x + \widehat{\xi}_\rho |x|^p \geq 0 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \rho. \tag{3.26}
$$

Then from (3.25) and (3.26) we have

$$
\begin{aligned}
& \Delta_p u_0(z) + \Delta_q u_0(z) \leq \left[\|\xi\|_\infty + \widehat{\xi}_\rho\right] u_0(z)^{p-1} \quad \text{for a.a. } z \in \Omega,\\
\Rightarrow \quad & u_0 \in D_+ \quad \text{(see Pucci–Serrin [27, pp. 111, 120])}.
\end{aligned}
$$

Also, if $\widetilde{u} \in D_+$, then on account of hypothesis $H(f)$ (ii) we have

$$
\varphi_\lambda^+(t\widetilde{u}) \to -\infty \quad \text{as } t \to +\infty. \tag{3.27}
$$

From Proposition 3.1 we know that

$$
\varphi_\lambda^+(\cdot) \text{ satisfies the } C\text{-condition.} \tag{3.28}
$$

Then (3.18), (3.27), (3.28) permit the use of the mountain pass theorem. So, we can find $\widehat{u} \in W^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\varphi_\lambda^+} \text{ and } \varphi_\lambda^+(u_0) < d_\lambda \leq \varphi_\lambda^+(\widehat{u}) \text{ (see (3.18)).} \tag{3.29}$$

From (3.29) we see that $\widehat{u} \neq u_0$ and

$$\begin{aligned} &\langle A_p(\widehat{u}), h\rangle + \langle A_q(\widehat{u}), h\rangle + \int_\Omega \xi(z)|\widehat{u}|^{p-2}\widehat{u}\, h\, dz + \int_{\partial\Omega} \beta(z)|\widehat{u}|^{p-2}\widehat{u}\, h\, d\sigma \\ &= \lambda \int_\Omega f(z, \widehat{u}^+)\, h\, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.30}$$

As before, choosing $h = -\widehat{u}^- \in W^{1,p}(\Omega)$ in (3.30), we infer that $\widehat{u} \geq 0$. Moreover, as before, the nonlinear regularity theory implies that $\widehat{u} \in C_+$. We will show that $\widehat{u} \neq 0$.

Since $\widehat{u}$ is a critical point of φ_λ^+ of mountain pass type, we have

$$C_1(\varphi_\lambda^+, \widehat{u}) \neq 0. \tag{3.31}$$

(see Papageorgiou–Rădulescu–Repovš [21, Theorem 6.5.8]).

For $u \in W^{1,p}(\Omega)$ with $\|u\| \leq 1$, we have

$$\begin{aligned} |\varphi_\lambda(u) - \varphi_\lambda^+(u)| &\leq \int_\Omega \lambda |F(z, u) - F(z, u^+)| dz \\ &\leq \lambda c_{11} \left[\|u\|^\tau + \|u\|^r\right] \quad \text{for some } c_{11} > 0 \text{ (see (3.16))} \\ &\leq 2\lambda c_{11} \|u\|^\tau \quad (\text{since } \|u\| \leq 1,\ \tau < r). \end{aligned} \tag{3.32}$$

Moreover, for all $h \in W^{1,p}(\Omega)$ we have

$$\begin{aligned} &|\langle \varphi_\lambda'(u) - (\varphi_\lambda^+)'(u), h\rangle| \\ &\leq \lambda \int_\Omega |f(z, u) - f(z, u^+)||h| dz \\ &= \lambda \int_\Omega |f(z, -u^-)||h| dz \\ &\leq \lambda c_{12} \left[\|u\|^{\tau-1} + \|u\|^{r-1}\right] \|h\| \quad \text{for some } c_{12} > 0 \\ &\leq 2\lambda c_{12} \|u\|^{\tau-1} \|h\| \quad (\text{since } \|u\| \leq 1,\ \tau < r). \end{aligned} \tag{3.33}$$

Then from (3.32), (3.33) and the C^1-continuity of critical groups (see Papageorgiou–Rădulescu–Repovš [21, Theorem 6.3.4]), we have

$$C_k(\varphi_\lambda, 0) = C_k(\varphi_\lambda^+, 0) \quad \text{for all } k \in \mathbb{N}_0. \tag{3.34}$$

Hypothesis $H(f)$ (iii) and Proposition 4.2 of Papageorgiou–Vetro–Vetro [23] imply that

$$\begin{aligned} & C_k(\varphi_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(\varphi_\lambda^+, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see (3.34))}. \end{aligned} \tag{3.35}$$

Comparing (3.31) and (3.35), we conclude that $\widehat{u} \neq 0$, that is, $\widehat{u} \in C_+ \setminus \{0\}$.

As we did for u_0, via the nonlinear regularity theory (see [12]) and the nonlinear maximum principle (see [27]), we have $\widehat{u} \in D_+$.

An inspection of the above proof reveals that it remains valid if we replace φ_λ^+ with φ_λ^-. So, for all $\lambda \in (0, \lambda^*)$ we can produce two nontrivial negative solutions $v_0, \widehat{v} \in -D_+$. ☐

Next we show that problem (P_λ) admits extremal constant sign solutions, that is, it has a smallest positive solution and a biggest negative solution ($\lambda \in (0, \lambda^*)$). These extremal solutions will be useful in producing nodal solutions (see Section 4).

Let $\lambda \in (0, \lambda^*)$ and $\widehat{\lambda}_1(q) \geq 0$ be the principal eigenvalue of the negative q-Laplacian with Robin boundary condition. On account of hypotheses $H(f)$ (i),(iii), given $\eta > \widehat{\lambda}_1(q)$, we can find $c_{13} = c_{13}(\eta) > 0$ such that

$$\lambda f(z,x)x \geq \eta |x|^q - c_{13}|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.36}$$

This unilateral growth estimate for the reaction of (P_λ) leads to the following auxiliary Robin problem

$$\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)|u(z)|^{p-2}u(z) = \eta |u(z)|^{q-2}u(z) - c_{13}|u(z)|^{r-2}u(z) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_{pq}} + \beta(z)|u|^{p-2}u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.37}$$

From Proposition 3.5 of Papageorgiou–Rădulescu [16], we have:

Proposition 3.4. *Problem* (3.37) *admits a unique positive solution* $\widetilde{u} \in D_+$ *and since the equation is odd,* $\widetilde{v} = -\widetilde{u} \in -D_+$ *is the unique negative solution of* (3.37).

We introduce the following sets:

$$\begin{aligned} S_\lambda^+ &= \text{ set of positive solutions of } (P_\lambda), \\ S_\lambda^- &= \text{ set of negative solutions of } (P_\lambda). \end{aligned}$$

From Proposition 3.3 and its proof, we know that

$$\emptyset \neq S_\lambda^+ \subseteq D_+ \text{ and } \emptyset \neq S_\lambda^- \subseteq -D_+ \text{ for all } \lambda \in (0, \lambda^*).$$

Moreover, from Papageorgiou–Rădulescu–Repovš [18] (proof of Proposition 7), we know that S_λ^+ is downward directed, while S_λ^- is upward directed.

Proposition 3.5. *If hypotheses* $H(\xi)$, $H(\beta)$, H_0, $H(f)$ *hold and* $\lambda \in (0, \lambda^*)$, *then* $\widetilde{u} \leq u$ *for all* $u \in S_\lambda^+$ *and* $v \leq \widetilde{v}$ *for all* $v \in S_\lambda^-$.

Proof. Let $u \in S_\lambda^+ \subseteq D_+$ and consider the Carathéodory function

$$g_+(z,x) = \begin{cases} \eta(x^+)^{q-1} - c_{13}(x^+)^{r-1} & \text{if } x \le u(z), \\ \eta u(z)^{q-1} - c_{13}u(z)^{r-1} & \text{if } u(z) < x. \end{cases} \tag{3.38}$$

We set $G_+(z,x) = \int_0^x g_+(z,s)ds$ and consider the C^1-functional $\psi_+ : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_+(u) = \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int\limits_\Omega G_+(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.11) and (3.38) it is clear that $\psi_+(\cdot)$ is coercive. Also, from the Sobolev embedding theorem and the compactness of the trace map, we have that $\psi_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass–Tonelli theorem we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$\psi_+(\widetilde{u}_0) = \inf\left[\psi_+(u) : u \in W^{1,p}(\Omega)\right]. \tag{3.39}$$

As in the proof of Proposition 3.3 (see (3.20)), we have

$$\begin{aligned} &\psi_+(\widetilde{u}_0) < 0 = \psi_+(0), \\ \Rightarrow\quad &\widetilde{u}_0 \ne 0. \end{aligned}$$

Using (3.38) we can easily check that

$$\begin{aligned} &K_{\psi_+} \subseteq [0,u] \cap C^1(\overline{\Omega}), \\ \Rightarrow\quad &\widetilde{u}_0 \in [0,u], \quad \widetilde{u}_0 \ne 0, \\ \Rightarrow\quad &\widetilde{u}_0 = \widetilde{u} \in D_+ \quad \text{(see (3.38) and Proposition 3.4)}, \\ \Rightarrow\quad &\widetilde{u} \le u \quad \text{for all } u \in S_\lambda^+ \subseteq D_+. \end{aligned}$$

Similarly we show that $v \le \widetilde{v}$ for all $v \in S_\lambda^- \subseteq -D_+$. □

Now we are ready to produce extremal constant sign solutions.

Proposition 3.6. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) has a smallest positive solution $u_\lambda^* \in D_+$ and a biggest negative solution $v_\lambda^* \in -D_+$.*

Proof. Recall that S_λ^+ is downward directed. So, invoking Hu–Papageorgiou [11, Lemma 3.10, p. 178], we can find a decreasing sequence $\{u_n\}_{n\ge1} \subseteq S_\lambda^+ \subseteq D_+$ such that

$$\inf_{n\ge1} u_n = \inf S_\lambda^+.$$

We have

$$\langle A_p(u_n),h\rangle + \langle A_q(u_n),h\rangle + \int\limits_\Omega \xi(z)|u_n|^{p-2}u_n\,h\,dz + \int\limits_{\partial\Omega} \beta(z)|u_n|^{p-2}u_n\,h\,d\sigma$$

$$= \lambda \int_\Omega f(z, u_n)\, h\, dz \quad \text{for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}, \tag{3.40}$$

$$\widetilde{u} \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N} \text{ (see Proposition 3.5)}. \tag{3.41}$$

In (3.40) we choose $h = u_n \in W^{1,p}(\Omega)$ and use (3.1), (3.41) and hypothesis $H(f)$ (i) to infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n \to u_\lambda^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{3.42}$$

In (3.40) we choose $h = u_n - u_\lambda^* \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (3.42). We obtain

$$\lim_{n\to+\infty} [\langle A_p(u_n), u_n - u_\lambda^*\rangle + \langle A_q(u_n), u_n - u_\lambda^*\rangle] = 0,$$

$$\Rightarrow \quad u_n \to u_\lambda^* \text{ in } W^{1,p}(\Omega) \quad \text{(as in the proof of Proposition 3.1)}. \tag{3.43}$$

Passing to the limit as $n \to +\infty$ in (3.40) and using (3.43), we obtain

$$\begin{aligned} &\langle A_p(u_\lambda^*), h\rangle + \langle A_q(u_\lambda^*), h\rangle + \int_\Omega \xi(z)(u_\lambda^*)^{p-1}\, h\, dz + \int_{\partial\Omega} \beta(z)(u_\lambda^*)^{p-1}\, h\, d\sigma \\ &= \lambda \int_\Omega f(z, u_\lambda^*)\, h\, dz \quad \text{for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{3.44}$$

Also from (3.41) we have

$$\widetilde{u} \leq u_\lambda^*. \tag{3.45}$$

From (3.44) and (3.45) we conclude that $u_\lambda^* \in S_\lambda^+ \subseteq D_+$ and $u_\lambda^* = \inf S_\lambda^+$.

Similarly for the negative solutions. Recall that $S_\lambda^- \subseteq -D_+$ is upward directed. Then reasoning as above we produce $v_\lambda^* \in S_\lambda^- \subseteq -D_+$ and $v_\lambda^* = \sup S_\lambda^-$. □

4. NODAL SOLUTIONS – MULTIPLICITY THEOREM

In this section we produce nodal solutions and then formulate the multiplicity theorem for problem (P_λ) establishing five nontrivial smooth solutions all with sign information when $\lambda \in (0, \lambda^*)$.

To generate a nodal (sign-changing) solution, we will use the extremal constant solutions $u_\lambda^* \in D_+$, $v_\lambda^* \in -D_+$ from Proposition 3.6. We form the set

$$[v_\lambda^*, u_\lambda^*] = \{h \in W^{1,p}(\Omega) : v_\lambda^*(z) \leq h(z) \leq u_\lambda^*(z) \text{ for a.a. } z \in \Omega\}.$$

Evidently, the extremality of u_λ^* and v_λ^* implies that any nontrivial solution of (P_λ) in $[v_\lambda^*, u_\lambda^*]$ distinct from u_λ^* and v_λ^*, must be nodal. So, our aim is to produce such a solution. To this end, using $u_\lambda^* \in D_+$ and $v_\lambda^* \in -D_+$ from Proposition 3.6, we introduce the following Charathéodory function

$$k_\lambda(z, x) = \begin{cases} \lambda f(z, v_\lambda^*(z)) & \text{if } x < v_\lambda^*(z), \\ \lambda f(z, x) & \text{if } v_\lambda^*(z) \leq x \leq u_\lambda^*(z), \\ \lambda f(z, u_\lambda^*(z)) & \text{if } u_\lambda^*(z) < x. \end{cases} \tag{4.1}$$

We also consider the positive and negative truncations of $k_\lambda(z,\cdot)$, namely the Carathéodory functions

$$k_\lambda^\pm(z,x) = k_\lambda(z,\pm x^\pm). \tag{4.2}$$

We set

$$K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds \quad \text{and} \quad K_\lambda^\pm(z,x) = \int_0^x k_\lambda^\pm(z,s)ds$$

and consider the C^1-functionals $e_\lambda, e_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} e_\lambda(u) &= \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\lambda(z,u)dz, \\ e_\lambda^\pm(u) &= \frac{1}{p}\gamma_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\lambda^\pm(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega). \end{aligned}$$

Using (4.1), (4.2) and extremality of $u_\lambda^* \in D_+$ and of $v_\lambda^* \in -D_+$, we can easily check that

$$K_{e_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}), \quad K_{e_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{e_\lambda^-} = \{0, v_\lambda^*\}. \tag{4.3}$$

Proposition 4.1. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold and $\lambda \in (0,\lambda^*)$, then problem (P_λ) admits a nodal solution $y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$.*

Proof. First we show that u_λ^* and v_λ^* are local minimizers of e_λ.

To this end, note that $e_\lambda^+(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\widehat{u}_\lambda^* \in W^{1,p}(\Omega)$ such that

$$e_\lambda^+(\widehat{u}_\lambda^*) = \inf\left[e_\lambda^+(u) : u \in W^{1,p}(\Omega)\right]. \tag{4.4}$$

As before (see the proof of Proposition 3.3 and in particular (3.20)), we have

$$\begin{aligned} & e_\lambda^+(\widehat{u}_\lambda^*) < 0 = e_\lambda^+(0), \\ \Rightarrow \quad & \widehat{u}_\lambda^* \neq 0, \\ \Rightarrow \quad & \widehat{u}_\lambda^* = u_\lambda^* \in D_+ \quad \text{(see (4.4) and (4.3))}. \end{aligned} \tag{4.5}$$

From (4.2) it is clear that $e_\lambda^+\Big|_{C_+} = e_\lambda\Big|_{C_+}$. So, from (4.5) it follows that

$$\begin{aligned} & u_\lambda^* \in D_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } e_\lambda, \\ \Rightarrow \quad & u_\lambda^* \in D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } e_\lambda, \\ & \text{(see Papageorgiou–Rădulescu [17, Proposition 2.12])}. \end{aligned} \tag{4.6}$$

Similarly, using this time e_λ^- we show that

$$v_\lambda^* \in -D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } e_\lambda. \tag{4.7}$$

We may assume that
$$e_\lambda(v_\lambda^*) \le e_\lambda(u_\lambda^*).$$

The reasoning is similar if the opposite inequality holds, using (4.7) instead of (4.6).

From (4.3) it is clear that we may assume that
$$K_{e_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \text{ is finite.} \tag{4.8}$$

Otherwise we already have an infinity of smooth nodal solutions. From (4.6), (4.8) and Theorem 5.7.6 of Papageorgiou–Rădulescu–Repovš [21], we know that we can find $\rho \in (0,1)$ small such that
$$e_\lambda(v_\lambda^*) \le e_\lambda(u_\lambda^*) < \inf\left[e_\lambda(u) : \|u - u_\lambda^*\| = \rho\right] = m_\lambda, \quad \|v_\lambda^* - u_\lambda^*\| > \rho. \tag{4.9}$$

Clearly $e_\lambda(\cdot)$ is coercive (see (3.1) and (4.1)). Hence we infer that
$$e_\lambda(\cdot) \text{ satisfies the } C\text{-condition,} \tag{4.10}$$

(see Papageorgiou–Rădulescu–Repovš [21, Proposition 5.1.15]). Then (4.9), (4.10) permit the use of the mountain pass theorem. So, we can find $y_0 \in W^{1,p}(\Omega)$ such that
$$y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}), \quad m_\lambda \le e_\lambda(y_0). \tag{4.11}$$

From (4.9) and (4.11) we see that
$$y_0 \notin \{u_\lambda^*, v_\lambda^*\}.$$

Also since y_0 is a critical point of e_λ of mountain pass type we have
$$C_1(e_\lambda, y_0) \neq 0 \tag{4.12}$$

(see Papageorgiou–Rădulescu–Repovš [21, Theorem 6.5.8]).

Since $u_\lambda^* \in D_+$, $v_\lambda^* \in -D_+$, we have $0 \in \operatorname{int}_{C^1(\overline{\Omega})}[v_\lambda^*, u_\lambda^*]$ (= the interior in the $C^1(\overline{\Omega})$-norm topology of $[v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega})$). Also from (4.1) it is clear that
$$e_\lambda\Big|_{[v_\lambda^*, u_\lambda^*]} = \varphi_\lambda\Big|_{[v_\lambda^*, u_\lambda^*]}.$$

It follows that
$$\begin{aligned}
& C_k(e_\lambda\big|_{C^1(\overline{\Omega})}, 0) = C_k(\varphi_\lambda\big|_{C^1(\overline{\Omega})}, 0) \quad \text{for all } k \in \mathbb{N}_0,\\
\Rightarrow \quad & C_k(e_\lambda, 0) = C_k(\varphi_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0,\\
& \text{(see Papageorgiou–Rădulescu–Repovš [21, Theorem 6.6.26]),}\\
\Rightarrow \quad & C_k(e_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see the proof of Proposition 3.3).}
\end{aligned} \tag{4.13}$$

Comparing (4.12) and (4.13), we conclude that $y_0 \neq 0$. On account of (4.11), we have that
$$y_0 \in [v_\lambda^*, u_\lambda^*] \cap C^1(\overline{\Omega}) \text{ is a nodal solution of } (P_\lambda),\ \lambda \in (0, \lambda^*).$$

□

Summarizing the situation for problem (P_λ), we can state the following multiplicity theorem.

Theorem 4.2. *If hypotheses $H(\xi)$, $H(\beta)$, H_0, $H(f)$ hold, then there exist $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least five nontrivial solutions $u_0, \widehat{u} \in D_+$, $v_0, \widehat{v} \in -D_+$, $y_0 \in C^1(\overline{\Omega})$ nodal; moreover, it has extremal constant sign solutions $u^*_\lambda \in D_+$ and $v^*_\lambda \in -D_+$ and we have $y_0 \in [v^*_\lambda, u^*_\lambda] \cap C^1(\overline{\Omega})$.*

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