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# EQUIVALENCE RELATIONS AND OPERATORS ON ORDERED ALGEBRAIC STRUCTURES WITH DIFFERENCE

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# Preface

This work surveys the largest part of my research as Ph.D. student of the course "Informatica e Matematica del Calcolo" (Computer Science and Computational Mathematics) at the "Università degli Studi dell'Insubria" (Univesity of Insubria), done for most of the time at the "Dipartimento di Scienze Teoriche e Applicate" (Department of Theoretical and Applied Sciences) in Varese (Italy), with a visit of one month at the Department of Knowledge-Based Mathematical Systems of the Johannes Kepler University in Linz (Austria).

The work concerns mainly algebraic models of fuzzy and many-valued propositional logics, in particular Boolean Algebras, Heyting algebras, GBL-algebras and their dual structures, and partial algebras.

The central idea is the representation of complex structures through simpler structures and equivalence relations on them: in order to achieve this, a structure is often considered under two points of view, as total algebra and partial algebra. The equivalence relations which allow the representation are congruences of partial algebras.

The first chapter introduces D-posets, the partial algebraic structures used for this representation, which generalize Boolean algebras and MV-algebras.

The second chapter is a study of congruences on D-posets and the structure of the quotients, in particular for congruences induced by some kinds of idempotent operators, here called S-operators. The case of Boolean algebras and MV-algebras is studied more in detail.

The third chapter introduces GBL-algebras and their dual, and shows how the interplay of an S-operator with a closure operator gives rise to a dual GBL-algebra. Other results about the representation of finite GBL-algebras and GBL*algebras (GBL-algebras with monoidal sum), part of two papers that I wrote with my advisor and co-advisor [14, 15], are summarized and put in relation with the other results of this work.

# Acknowledgments

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# Contents

# Chapter 1

# D-posets

In this chapter we introduce D-posets, which are partial algebraic structures. Informally, a *partial algebra* is a set $A$ with some operations, each with its *arity*, such that every k-ary operation is defined for some, not necessarily all, ordered k-tuples of elements of $A$, that is, for a (possibly proper) subset of $A^k$. If, for a k-ary operation, the domain of definition is all $A^k$, then the operation is *total*. If all operations on $A$ are total, then $A$ is a *total algebra*, or simply an *algebra*.

For binary operations, the domain of definition is a binary relation on $A$ (a subset of $A \times A$).

D-posets have been introduced in [33] as models for quantum logic, and they have are essentially equivalent to effect algebras [18] and weak orthoalgebras [22], introduced for similar purposes.

As it will be shown in Section 1.1, D-posets are partial algebras with one binary operation, called *partial difference*, whose domain of definition is a relation of partial order.

One of the main concepts arising in quantum logic (and more in general, in quantum mechanics) is the notion of *measurement*. In the present work we will not be concerned with quantum logic, but we will keep the notion of measurement in our intended interpretation, in a twofold way.

On one hand, D-posets can represent universes with a collection of parts, ordered by inclusion, which can be object of measurement. The difference of two parts, one contained into the other, is what remains after removing the smaller part from the larger. A simple example is given by a collection of subsets of a set, closed by relative complement. More in general, Boolean algebras can be seen as D-posets, when the operation of difference is restricted to comparable elements. Another important example (not examined in depth in this work) is given by *orthomodular lattices*, for instance the lattice of linear subspaces of a Hilbert spaces.

On the other hand, D-posets can represent generalized systems of values for (finitely additive) measures. Indeed, the unit interval $[0,1]$ with difference is a D-poset, and a finitely additive probability can be seen as a morphism of D-posets to $[0,1]$

A useful distinction in D-posets is between *sharp* and *unsharp* elements: an element is sharp if it has a complement with respect to the order. A D-poset is *sharp* if all its elements are sharp. Sharp D-posets include Boolean algebras and orthomodular lattices, while $[0,1]$ is an example of unsharp D-poset. Informally, we can think of sharp D-poset as representing collections of parts (or substructures) of a universe, and unsharp D-posets as spaces of generalized measures.

A class which includes Boolean algebras and the unit interval is the class of *MV-algebras*, which will be treated most extensively in this work. An MV-algebra is a total algebra, but it is also determined by its partial structure of D-poset.

General references for partial algebras are [5] and [23, Chapter 2]. The latter book is mainly concerned with total algebras.

For lattice theory, we mainly refer to [24].

## 1.1 Definitions and first properties

For definitions and results on D-posets we refer the reader to [8].

**Definition 1.1.1.** *Let $(P, \leq)$ be a poset with a minimum $0_P$ and a maximum $1_P$. Let $\smallsetminus$ be a partial operation (partial difference) such that $b \smallsetminus a$ is defined for $a \leq b$. Then, $(P, \leq, \smallsetminus, 0_P, 1_P)$ is called a* D-poset *if the following conditions hold:*

*(D1)* $a \smallsetminus 0_P = a$;

*(D2) for all* $a, b, c \in P$ *with* $a \leq b \leq c$, $(c \smallsetminus b) \leq (c \smallsetminus a)$;

*(D3) for all* $a, b, c \in P$ *with* $a \leq b \leq c$, $(c \smallsetminus a) \smallsetminus (c \smallsetminus b) = b \smallsetminus a$.

Whenever there is no risk of confusion, we write just $0$ and $1$ for, respectively, the lower and the upper bound of a bounded poset.

A derived unary operation of D-posets is $\neg a = 1 \smallsetminus a$.

D-posets are partial algebras, since the operation of difference $\smallsetminus$ is defined only for pairs of elements which are comparable with respect to the order relation $\leq$. Therefore, $\leq$ stands for the domain of definition of the binary operation $\smallsetminus$.

We say that a D-poset $P$ is a *D-lattice* if it is a lattice with respect to the order relation $\leq$.

We say that $P$ is a *Boolean D-poset* if it is a D-lattice and, for all $a, b \in P$, it holds

(D4) $(a \vee b) \smallsetminus b = a \smallsetminus (a \wedge b)$.

A D-poset $P$ is an *orthomodular lattice* if it is a D-lattice and $a \vee (b \setminus a) = b$ for all $a, b \in P$ with $a \leq b$ (which is equivalent to the *orthomodular law*).

A D-poset $P$ is a *Boolean algebra* if it is both a Boolean D-poset and and an orthomodular lattice.

Every MV-algebra is a Boolean D-poset, by setting $b \setminus a = b \ominus a$ for $a \leq b$, where $\ominus$ is the usual, total, MV-operation of difference [8]. Conversely, every Boolean D-poset $P$ can be endowed in a unique way with the structure of MV-algebra, having the same order relation as $P$, and the MV-operation $\ominus$ such that $b \ominus a = b \setminus a$ for $a \leq b$ in $P$ [8]. For more details, see Section 1.3

**Remark 1.1.1.** *D-posets are often studied regarding their operation of partial sum, which can be derived from the partial difference. In this case they are most widely known as* effect algebras*, starting from [18]. In the context of effect algebras, D-lattices are called* lattice-ordered effect algebras *and Boolean D-posets* MV-effect algebras. *We prefer not to give details of this correspondence, in order not to create confusions betwen the used terminology. In the present work we will be more concerned with the operation of difference rather than sum, hence we use the terminology of D-posets rather than that of effect algebras. However, many of the concepts used in this work are mutuated from the theory of effect algebras.*

We conclude this section explaining in what sharp and unsharp elements of a D-poset differ.

Let $P$ be a D-poset and $a \in P$. We say that $a$ is *sharp* if the greatest lower bound of $a$ and $\neg a$ exists and it is $a \wedge \neg a = 0$. We say that $a$ is *unsharp* if it is not sharp.

We say that $P$ is sharp if *all* its element are sharp, and $P$ is unsharp if *some* of its elements are unsharp (notice that 0 and 1 are sharp in every D-poset).

Boolean algebras and orthomodular lattices are sharp D-lattices. A Boolean D-poset is sharp if and only if it is a Boolean algebra.

## 1.2 D-morphisms

A common requirement for a morphisms of a partial algebras $f : A \to B$ is that, if a k-ary operation of is defined for a k-tuple of $A$ and it has value $x$, the same operation is defined for the k-tuple of images, and for its value $y$ it holds $y = f(x)$. See [5, 23] for the general theory. For D-poset, we have the following definition [8].

**Definition 1.2.1.** *Let $P$ and $Q$ be D-posets. A function $f : P \to Q$ is a homomorphism of D-posets, (shortly,* D-morphism*), if*

*(M1)* $f(1_P) = 1_Q$*;*

*(M2) for all $a, b \in P$ with $a \leq b$, $f(a) \leq f(b)$;*

*(M3) for all $a, b \in P$ with $a \leq b$, $f(b \setminus a) = f(b) \setminus f(a)$.*

It follows immediately from Definition 1.2.1 that $f$ preserves $0$. Indeed, $f(1 \setminus 1) = f(1) \setminus f(1) = 0$.

We say that $f$ is *mono* if it is injective as a function, *epi* if it is surjective.

D-morphisms do not necessarily preserve existing joins (least upper bounds) or meets (greatest lower bounds).

Certain classes of D-morphisms are of special interest for the sequel.

**Definition 1.2.2.** *Let $P$ and $Q$ be D-posets and $f : P \to Q$ be a D-morphism. We say that $f$ is* regular *if, for all $a, b \in P$ with $f(a) \leq f(b)$, there are $c, d \in P$ such that $c \leq d$, $f(c) = f(a)$ and $f(d) = f(b)$.*

*Let $f$ be an epi D-morphism. We say that $f$ is* clopen *if, for all $a, b \in P$ with $f(a) \leq f(b)$, there is $c \in P$ such that $f(c) = f(a)$ and $c \leq b$, or, equivalently by duality, if, for all $a, b \in P$ with $f(a) \leq f(b)$, there is $d \in P$ such that $f(d) = f(b)$ and $a \leq d$.*

*We say that $f$ is* sectional *if there is a function $\sigma : P \to P$ such that following conditions hold, for all $a, b \in P$:*

*$\sigma(\sigma(a)) = \sigma(a)$;*

*$\sigma(a) = \sigma(b)$ if and only if $f(a) = f(b)$;*

*if $f(a) \leq f(b)$, then $\sigma(a) \leq \sigma(b)$.*

It follows from the definition that $f(\sigma(a)) = f(a)$ for all $a \in P$. The function $\sigma$ is monotone, but it is not, in general, a D-morphism. Clearly, if $f$ is a clopen or sectional epi D-morphism, then $f$ is regular.

We adopted the term "clopen" since a clopen epi D-morphism $f$ is both an open and a closed map, regarding the posets as finite (or Alexandrov) topological spaces.

We adopted the term "sectional" since a sectional epi D-morphism $f : P \to Q$, as a morphism of posets, has a section, that is a function $s : Q \to P$ such that $fs$ is the identity of $Q$. For such $s$, it holds $sf = \sigma$.

In Section 2.3 we will give an example of a clopen epi D-morphism which is not sectional. We do not have a proof whether every sectional epi D-morphism is clopen, though this fact seems plausible at least for finite D-posets.

An example of regular epi D-morphism which is nor clopen nor sectional is the following.

**Example 1.2.1.** *Let $P$ be the powerset of the three-element set $\{a, b, c\}$, with the usual order by set inclusion, and set difference as partial difference, and let $Q = \{0, 1/4, 1/2, 3/4, 1\}$ with the usual order and difference, which is a Boolean D-poset. Let $f : P \to Q$ such that:*

$f(\varnothing) = 0$;
$f(\{a\}) = f(\{b\}) = 1/4$;
$f(\{c\}) = f(\{a, b\}) = 1/2$;
$f(\{a, c\}) = f(\{b, c\}) = 3/4$;
$f(\{a, b, c\}) = 1$.
*It can be verified that $f$ is a regular epi D-morphism. However, $f$ is not clopen: $f(\{a\}) \leq f(\{c\})$, but there is not a subset of $\{c\}$ with the same image as $\{a\}$ through $f$. The D-morphism $f$ is not sectional either: indeed, $Q$ is a five-element chain, while the longest chains of $P$ have four elements.*

It is easier to find examples of non-regular D-morphisms. One is by taking as $P$ the powerset of the two-element set $\{a, b\}$, as $Q$ the Boolean D-poset $\{0, 1/3, 2/3, 1\}$ and as $f : P \to Q$ the function:
$f(\varnothing) = 0$;
$f(\{a\}) = 1/3$;
$f(\{b\}) = 2/3$;
$f(\{a, b\}) = 1$ .

**Remark 1.2.1.** *Our notions of regular, clopen and sectional D-morphisms are intermediate between the weaker notion of* full homomorphism *of partial algebras given in [5, 23] and the stronger notion of* strong homomorphism *[23] also called* closed homomorphism *[5] of partial algebras.*

We say that a D-morphism is an *embedding* if it is a regular mono D-morphism.

Let $P$ be a D-poset and $Q \subseteq P$ be a subset of $P$. We say that $Q$ is a sub D-poset of $P$ if

> $1_P \in Q$;
>
> for all $a, b \in Q$ with $a \leq b$, $(b \smallsetminus a) \in Q$.

It easily follows that $0_P \in Q$, $(Q \leq, \smallsetminus, 0_P, 1_P)$ is a D-poset and the inclusion of $Q$ in $P$ is an embedding. Conversely, the image of every embedding is a sub D-poset of the codomain.

A D-morphism $f : P \to Q$ which is an embedding and it is epi (in other words, a regular mono and epi D-morphism) is an *isomorphism*, and we say that $P$ and $Q$ are *isomorphic*.

## 1.3 MV-algebras

In this section we recall some basic facts about MV-algebra. We refer to the handbook [9], and to [8] for MV-algebras as Boolean D-posets.

An MV-algebra is a structure $(L, \oplus, \neg, 0)$ of type $(2, 1, 0)$ such that the following identities hold:

$a \oplus 0 = a$;

$\neg\neg a = a$;

$(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

$\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$;

$a \oplus \neg 0 = \neg 0$.

Commutativity of $\oplus$ follows from the axioms, hence $(L, \oplus, 0)$ is a commutative monoid. Derived operations are:

$a \to b = \neg a \oplus b$;

$a \odot b = \neg(\neg a \oplus \neg b)$;

$b \ominus a = \neg a \odot b$;

$a \vee b = a \oplus (b \ominus a)$;

$a \wedge b = \neg(\neg a \vee \neg b) = b \ominus (b \ominus a)$.

By setting $1 = \neg 0$, $(L, \odot, 1)$ is a commutative monoid, and $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.

If we set $b \setminus a = b \ominus a$ for $a \leq b$, then $(L, \leq, \setminus, 0, 1)$ is a Boolean D-poset [8]. On the other hand, if $(L, \leq, \setminus, 0, 1)$ is a Boolean D-poset (which is a lattice), by setting $b \ominus a = b \setminus (a \wedge b)$ for all $a, b \in M$, we have that $(L, \oplus, \neg, 0)$ is an MV-algebra, where $\neg a = 1 \ominus a$ and $a \oplus b = \neg(\neg b \ominus a)$. We will call $\ominus$ the *total difference* of $L$: it is a total operation which extend the partial difference $\setminus$.

In a Boolean D-poset $L$ an element $a$ is sharp if and only if $a \oplus a = a$. For this property, a sharp element in an MV-algebra is usually said an *idempotent*. We denote by $I(L)$ the subset of idempotents of $L$. For two idempotents $a, b \in I(L)$, it holds $a \oplus b = a \vee b$ and $a \odot b = a \wedge b$. It follows that $(I(L), \vee, \neg) = (I(L), \oplus, \neg)$ is a Boolean algebra. Further, if $L$, as a D-poset, is sharp (that is, we recall, all its elements are sharp), the MV-algebra $(L, \oplus, \neg, 0) = (L, \vee, \neg, 0)$ is a Boolean algebra.

We will write $b - a$ for the total difference in a Boolean algebra, that is $b - a = \neg a \wedge b$.

We say shortly *MV-morphism* for a homomorphisms of MV-algebras.

As we have seen, MV-algebras and Boolean D-posets are essentially the same thing, in the sense that the structure of MV-algebra can be recovered from the structure of Boolean D-poset and vice-versa. But the operation of difference is total in MV-algebras, while it is partial in Boolean D-posets. This means that part of the structure of MV-algebra is "implicit" in a Boolean D-poset, hence we may expect that MV-morphism between two algebras are also D-morphisms, but the converse does not hold. This statement is made precise in the following proposition.

**Proposition 1.3.1.** *Let $L$ and $M$ be two MV-algebras, and $f : L \to M$ be a function. Then, $f$ is an MV-morphisms if and only if $f$ is a D-morphism and it is a lattice-morphism.*

*Proof.* If $f$ is an MV-morphism, then it preserves $\vee$ and $\wedge$, since these are derived operations. Hence $f$ is a lattice-morphism. Furthermore, $f$ preserves the total difference $\ominus$, whence it preserves also the partial difference $\smallsetminus$, and we can conclude that $f$ is a D-morphism. Conversely, let $f$ be a D-morphism and lattice-morphism. Since $b \ominus a = b \smallsetminus (a \wedge b)$, $f$ preserves also the total difference $\ominus$, hence also the total sum $\oplus$, therefore it is an MV-morphism □

An example of D-morphism which is not an MV-morphism is given in Example 1.2.1.

Taking Proposition 1.3.1 into account, a convenient way to axiomatize MV-algebras in the setting of partial algebras is the following: an MV-algebra is a structure $(L, \vee, \wedge, \smallsetminus, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $(L, \leq, \smallsetminus, 0, 1)$ is a Boolean D-poset, that is it satisfies $(D1)$-$(D4)$. Here $\leq$ is the order relation in the lattice (defined as $a \leq b$ if and only if $a = a \wedge b$). It follows from the axioms that $L$, as a lattice, is distributive.

Summarizing, an MV-algebra is a Boolean D-poset where the lattice operations $\vee$ and $\wedge$ are added to the signature.

**Remark 1.3.1.** *We will often refer to a structure regardless to the operations considered in the signature. Therefore, we will say "MV-algebra" and "Boolean algebra" even when we are interested only in their structure of D-posets.*

The following proposition shows that, for MV-morphisms, the distinction we made about epi D-morphisms vanishes.

**Proposition 1.3.2.** *Let $L$ and $M$ be two MV-algebras, and $f : L \to M$ be an MV-morphism. Then $f$, as a D-morphism, is regular. If $f$ is epi, then $f$ is clopen and sectional.*

*Proof.* Let $f$ be epi. If $f(a) \leq f(b)$, then, since $f$ is a lattice morphism, it holds $f(a \wedge b) = f(a) \wedge f(b) = f(a)$, and, for $c = a \wedge b$, it holds that $f(c) = f(a)$ and $c \leq b$, which means $f$ is clopen.

The fact that $f$ is sectional comes from the existence of a section in the category of posets with monotone functions, for a surjective function $f$ between two lattices which is also a lattice morphism. But we will not give details of this implication.

Let $f : L \to M$ be any MV-morphism. Then, the image $f(L)$ is a subalgebra of $M$, and $f$ is an epi MV-morphism to $f(L)$, hence it is clopen, therefore regular. Then, $f$ is regular also as a D-morphism to $M$. □

It worths mentioning the simple fact that two MV-algebras are isomorphic as D-posets if and only if they are also isomorphic as MV-algebras.

The unit interval $[0,1]$, with the usual operation of difference between numbers ($b - a$ for $a \leq b$) as partial difference, is a Boolean D-poset. As an MV-algebra, $[0,1]$ is called the *standard MV-algebra*.

## 1.4 States and valuations

In this section we define valuations on bounded distributive lattices in terms of functions to D-posets, and we briefly show how D-morphism can be seen as a generalization of finitely additive probability measures on Boolean algebras, as well as of states of MV-algebras, as defined in [37].

We invite the reader to consult [4] for valuations to real numbers, and [40] for valuations to rings. We will use just the term "valuation" for what is often called "normalized monotone valuation", since we are only concerned with valutations of this kind. For similar reasons, in the sequel we say "distributive lattice" for "bounded distributive lattice", since all the lattices that we will consider are bounded.

Let $(L, \vee, \wedge, 0, 1)$ be a distributive lattice. A *valuation* on $L$ to the real unit interval $[0,1]$ is a function $f : L \to [0,1]$ which satisfies

- *normalization*: $f(0) = 0$ and $f(1) = 1$;
- *monotonicity*: for all $a, b \in L$ with $a \leq b$, $f(a) \leq f(b)$;
- *subtractivity*: for all $a, b \in L$, $f(a \vee b) - f(b) = f(a) - f(a \wedge b)$.

Usually, subtractivity is given in the equivalent additive form:

- *additivity*: for all $a, b \in L$, $f(a \vee b) + f(a \wedge b) = f(a) + f(b)$,

but we prefer the subtractive notation because we want to keep the range of operations within $[0,1]$, and the reason will soon be clear.

Notice that our notion of valuation corresponds to *isotone valuation* with values in $[0,1]$ of [4, Chapter X].

Since subtractivity involves only subtraction of smaller from greater numbers, it is clear how the definition of valuation can be generalized to D-posets.

**Definition 1.4.1.** *Let $L$ be a distributive lattice and $P$ be a D-poset. We call a function $f : L \to P$ a* valuation *to $P$ if $f$ satisfies*

*(N)* $f(0) = 0$ *and* $f(1) = 1$*;*

*(M) for all $a, b \in L$ with $a \leq b$, $f(a) \leq f(b)$;*

*(S)* *for all* $a, b \in L$, $f(a \vee b) \setminus f(b) = f(a) \setminus f(a \wedge b)$.

The following proposition is a simple reformulation of the definition of valuation for a Boolean algebra.

**Proposition 1.4.1.** *Let* $U$ *be a Boolean algebra,* $P$ *be a D-poset and* $f : U \to P$ *be a function satisfying* $(N)$ *and* $(M)$ *of Definition 1.4.1. Then,* $(S)$ *is equivalent to the following condition:*

*(S')* *for all* $a, b \in U$ *with* $a \leq b$, $f(b) \setminus f(a) = f(b \setminus a)$.

*Proof.* Let $(S)$ hold. If $a \leq b$, then $b = (b \setminus a) \vee a$ and $f(b) \setminus f(a) = f((b \setminus a) \vee a) \setminus f(a) = f(b \setminus a) \setminus f(0) = f(b \setminus a)$, since $(b \setminus a) \wedge a = 0$ and $f(0) = 0$.

Let now $(S')$ hold. Then, for all $a, b \in U$, $f(a \vee b) \setminus f(b) = f((a \vee b) \setminus b)$ and $f(a) \setminus f(a \wedge b) = f(a \setminus (a \wedge b))$, whence $(S)$ follows since, in a Boolean algebra, $(a \vee b) \setminus b = a \setminus (a \wedge b)$. □

A function which satisfies $(N)$, $(M)$ and $(S')$ has been previously called a D-morphism. Therefore, valuations on Boolean algebras are D-morphisms.

When $U$ is a Boolean algebra, $P = [0, 1]$ and $f : U \to P$ satisfies $(N)$, $(M)$ and $(S')$, $f$ is also called a *finitely additive probability measure*. Therefore, valuations of Boolean algebras in $[0, 1]$ are finitely additive probability measures.

States of MV-algebras were introduced in [37] as averaging values of Łukasiewicz truth-evaluations.

**Definition 1.4.2.** *A* state $s$ *of an MV-algebra* $L$ *is a map* $s : L \to [0, 1]$ *satisfying*

*(1)* $s(1) = 1$*;*

*(2)* $s(a \oplus b) = s(a) + s(b)$ *whenever* $a \odot b = 0$.

It is easy to see that a function $s : L \to [0, 1]$ is a state if and only if it is a D-morphism. Indeed, $a \oplus b$ can be any element $c$ with $a \leq c$, and for such $c$, if $a \odot b = 0$, it holds $c \ominus a = b$. Hence, condition $(2)$ can be reformulated as $(M1)$ in Definition 1.2.1. Further, $(2)$ implies monotonicity, that is $(M2)$, and $(1)$ is the same as $(M1)$.

On the other hand, if $s$ is a D-morphism, $a \odot b = 0$ implies $s(a) = s((a \oplus b) \ominus b) = s(a \oplus b) - s(b)$, that is $(2)$.

Condition $(2)$ can be replaced by

(2') $s(a \oplus b) - s(b) = s(a) - s(a \odot b)$.

Generalizing Definition 1.4.2, we call *state* of a D-poset $P$ any D-morphism $s : P \to [0,1]$.

As shown before, if $L$ is a Boolean algebra, states of $L$ coincide with valuations to $[0,1]$, and with finitely additive probability measures.

If $L$ is an MV-algebra, a D-morphism of $L$ to a D-poset $P$ is also a valuation on $L$, as a distributive lattice, to $P$. Indeed, $(N)$ and $(M)$ of Definition 1.4.1 are clearly satisfied, and

$$s(a \vee b) \setminus s(b) = s((a \vee b) \setminus b) = s(a \setminus (a \wedge b)) = s(a) \setminus s(a \wedge b).$$

On the other hand, a valuation on $L$ as a distributive lattice to a D-poset $P$ is not necessarily a D-morphsm from $L$ to $P$. For instance, consider the three-element MV-chain $L = \{0, 1/2, 1\}$ and $f : L \to [0,1]$ with
$f(0) = 0$;
$f(1/2) = 1/3$;
$f(1) = 1$.
Then, $f$ is a valuation on $L$, as a distributive lattice, to $[0,1]$, but it is not a state of $L$, since the image of $1/2$ through any state of $L$ has to be $1/2$.

# Chapter 2

# Equivalence relations on D-posets

In this chapter we are going to investigate several kinds of equivalence relations on D-posets.

For total algebras there is a canonical notion of congruence. Informally, a congruence on an algebra $A$ is an equivalence relation such that, for every k-ary operation $*$ of $A$, applying $*$ to a k-tuple $x \in A^k$ gives a result equivalent to the one obtained applying $*$ to a k-tuple $y \in A^k$ which is elementwise equivalent to $x$.

For partial algebras there are different concepts of congruence, which correspond to different ways of taking into account the partial domain of operations, that is a possibly proper subset of $A^k$. We will investigate some kinds of congruences on D-posets which allow a representation of D-posets as quotients, in particular for MV-algebras.

In our intended interpretation, where sharp D-poset can be thought of as collections of parts of a universe, which can be object of measurement, and unsharp D-posets as spaces of generalized measures, we would like represent unsharp D-posets in terms of sharp ones. The idea dates back to Euclid's Elements, where measure arises from an equivalence relation on some class of geometric entities satisfying five common notions, in particular the following three:

- things which equal the same thing also equal one another;
- if equals are subtracted from equals, then the remainders are equal;
- the whole is greater than the part.

The first notion is just the transitivity of the equivalence relation. The other two suggest us to interpret a congruence on a sharp D-poset as a generalization of Euclidean's relation of "equimeasurability".

Among the different classes of congruences on D-posets, we will investigate more in detail congruences induced by certain kinds of idempotent operators, that we call *S-operators*. An S-operator is an idempotent, monotone operator which represents the process of measure through the comparison with a prototype: think, for instance, of measuring lengths by comparing objects with the initial segments of a ruler.

We will study more in detail the properties of S-operators, and congruences induced by them, on Boolean algebras, which are the classical models to which measures apply. For a Boolean algebra $U$, we show that the quotient is an MV-algebra, and its algebraic structure can be recovered by only the order structure and difference of $U$: an operation between two classes is the minimum or the maximum class obtained by a corresponding Boolean operation, applied to elements in the two classes. For instance, starting with the operation of disjunction of a Boolean algebra, in the quotient we obtain two operation, the "weak disjunction" and the "strong disjunction", by respectively taking the minimum or the maximum. These correspond to two basic operations of MV-algebras.

## 2.1 D-congruences

For a general introduction to congruence relations in partial algebras see [5] and [23]

In the literature there are several definitions of congruence in the general setting of partial algebras. For the particular cases of effect algebras and D-postes, we invite the reader to consult [25] and [8] respectively.

Here we mutuate some definitions from other works and we adapt them to our context, with emphasis on the classes of equivalence relations that will be most useful in our model. All our definitions of congruences are based on morphisms: congruences are equivalence relations induced by certain classes of morphisms. We will not relate D-congruences with ideals in D-posets, as introduced in [8].

Let $P$ and $Q$ be D-posets and $f : P \to Q$ be a function. We write $\sim_f$ for the equivalence relation induced by $f$, that is $a \sim_f b$ if and only if $f(a) = f(b)$, for $a, b \in P$.

**Definition 2.1.1.** *A* congruence of D-poset *(shortly,* D-congruence*) $\sim$ on $P$ is an equivalence relation induced by some regular epi D-morphism, that is, $\sim$ is a D-congruence if there are a D-poset $Q$ and a regular epi D-morphism $f : P \to Q$ such that $\sim$ coincides with $\sim_f$. If $\sim$ is induced by a clopen epi D-morphism, we say that $\sim$ is a* clopen D-congruence. *If $\sim$ is induced by a sectional epi D-morphism, we say that $\sim$ is a* sectional D-congruence.

**Remark 2.1.1.** *If $P$ is an MV-algebra, then an MV-algebra congruence of $P$ is also a D-congruence, both clopen and sectional. Indeed, every congruence of an MV-algebra is induced by the projection $\pi$ to the quotient, which is an epi MV-morphism. By Proposition 1.3.1, $\pi$ is also an epi D-morphism and, by Proposition*

*1.3.2, $\pi$ is regular, clopen and sectional.*

*On the other hand, a D-congruence induced by a regular epi D-morphism which is not an MV-morphism, such as $f$ of Example 1.2.1, is not a congruence of MV-algebras.*

*For every Boolean algebra or MV-algebra $P$ we call* MV-congruences *the congruences of $P$ as a total algebra, to distinguish them from the aforementioned D-congruences, which form a larger class.*

It follows from Definition 2.1.1 that a congruence ~ satisfies the following property:

(A1) if $a \leq b$, $c \leq d$, $a \sim c$ and $b \sim d$, then $(b \setminus a) \sim (d \setminus c)$.

We call *weak D-congruence* an equivalence relation satisfying $(A1)$. Weak D-congruence are called simply "congruences" in [8].

Not every weak D-congruence is a D-congruence (Definition 2.1.1), as shown in the following example.

**Example 2.1.1.** *Let $P$ be the powerset of the three-element set $\{a, b, c\}$, with the usual order by set inclusion and set difference. Let ~ be the equivalence relation whose classes are:*
$\{\varnothing\}$;
$\{\{a\}, \{b, c\}\}$;
$\{\{b\}, \{a, c\}\}$;
$\{\{c\}, \{a, b\}\}$;
$\{\{a, b, c\}\}$.
*It can be verified that ~ is a weak D-congruence, but there is not a D-poset $Q$ and a D-morphism $f : P \to Q$ such that ~ is induced by $f$. Indeed, for such D-morphism it would hold*

$$f(\{a\}) \leq f(\{a, c\}) = f(\{b\}) \leq f(\{a, b\}) = f(\{c\}) \leq f(\{b, c\}) = f(\{a\}),$$

*hence $f(\{a\}) = f(\{b\}) = f(\{c\})$, but $\{a\}$, $\{b\}$ and $\{c\}$ belong to distinct equivalence classes.*

Let ~ be a D-congruence on a D-poset $P$, and let $a, b \in P$. We write $[a]$ for the equivalence class of $a$ in the quotient $P/\sim$, and $[a] \leq [b]$ if there are $c \sim a$ and $d \sim b$ such that $c \leq b$. We will soon see that the use of the symbol of partial order is justified.

If $[a] \leq [b]$, we set $[b] \oslash [a] = [d \setminus c]$, with $c$ and $d$ as before. By $(A1)$, the class $[d \setminus c]$ does not depend on the representatives chosen in $[a]$ and [b], hence $\oslash$ is well defined as a partial operation between equivalence classes.

The following proposition shows that the quotient of a D-poset $P$ with respect to a D-congruence ~ is a D-poset, with the order relation and the partial difference just defined.

**Proposition 2.1.1.** *Let $P$ be a D-poset and $\sim$ be a D-congruence on $P$. Then, $(P/\sim, \leq, \oslash, [0], [1])$ is a D-poset, and the projection $\pi : P \to P/\sim$, $x \mapsto [x]$ is a regular epi D-morphism. The congruence is clopen if and only if $\pi$ is clopen, and sectional if and only if $\pi$ is sectional.*

*Proof.* Since $\sim$ is a D-congruence, there is a D-poset $Q$ and a regular epi D-morphism $f : P \to Q$ such that $\sim$ coincides with $\sim_f$. By regularity of $f$, it holds $[a] \leq [b]$ if and only if $f(a) \leq f(b)$. Further,

(1) $x \in ([b] \oslash [a])$ if and only if $f(x) = f(b) \setminus f(a)$.

Let us first verify that $(P/\sim, \leq)$ is a poset, that is $\leq$ is reflexive, antisymmetric and transitive on $P/\sim$.

It is reflexive because $a \leq a$ implies $[a] \leq [a]$. It is antisymmetric because $[a] \leq [b]$ and $[b] \leq [a]$ imply $f(a) \leq f(b)$ and $f(b) \leq f(a)$, hence $f(a) = f(b)$ and $[a] = [b]$. It is transitive because $[a] \leq [b]$ and $[b] \leq [c]$ imply $f(a) \leq f(b)$ and $f(b) \leq f(c)$, hence $f(a) \leq f(c)$ and $[a] \leq [c]$. It is clear that $[0]$ and $[1]$ are, respectively, the lower and the upper bounds of $(P/\sim, \leq)$.

Let us now verify $(D1)$-$(D3)$ in the definition of D-poset for $(P/\sim, \leq, \oslash, [0], [1])$.

$(D1)$ is immediate: $[a] \oslash [0] = [a \setminus 0] = [a]$.

For $(D2)$ and $(D3)$, let $[a] \leq [b] \leq [c]$. Then, $f(a) \leq f(b) \leq f(c)$, hence $f(c) \setminus f(b) \leq f(c) \setminus f(a)$ and $(f(c) \setminus f(a)) \setminus (f(c) \setminus f(b)) = f(b) \setminus f(a)$. By (1), for $x \in [b] \oslash [a]$, $y \in [c] \oslash [b]$ and $z \in [c] \oslash [a]$ it holds $f(x) = f(b) \setminus f(a)$, $f(y) = f(c) \setminus f(b)$ and $f(z) = f(c) \setminus f(a)$. Therefore, $[y] \leq [z]$, that is $[c] \oslash [b] \leq [c] \oslash [a]$, and $[z] \oslash [y] = [x]$, that is $([c] \oslash [a]) \oslash ([c] \oslash [b]) = [b] \oslash [a]$.

The quotient $P/\sim$ is isomorphic to $Q$ through the isomorphism $\phi : P/\sim \to Q$ defined as $\phi([a]) = f(a)$ for all $a \in P$. Indeed $\phi$ is a bijective function, it is monotone since $[a] \leq [b]$ if and only if $f(a) \leq f(b)$ and, for $[a] \leq [b]$, $\phi([b] \oslash [a]) = \phi[b] \setminus \phi[a]$ since, by (1), $[b] \oslash [a] = [x]$ with $f(x) = f(b) \setminus f(a)$. Further, $\phi([0_P]) = f(0_P) = 0_Q$ and $\phi([1_P]) = f(1_P) = 1_Q$.

Therefore, the projection $\pi : P \to P/\sim$, which is the composition $\pi = \phi^{-1} f$, is a regular epi D-morphism and can be taken as $f$ in the definition of congruence. Then, it is immediate that $f$ is clopen or sectional if and only if $\pi$ is, respectively, clopen or sectional. $\square$

We would like to add some conditions to $(A1)$ in order to have an explicit definition of congruence, not relying on the existence of the D-morphism $f$. In the literature on effect algebra some conditions have been considered which characterize subclasses of congruences [7, 25, 38]. In the next section we will add some conditions to $(A1)$ which give what we will call *clopen* and *sectional* D-congruences.

## 2.2 Clopen D-congruences

In this section we introduce a class of congruence relation on D-posets which is considered the standard one in some works about effect algebras (see the references below). In our setting, it is convenient to use a new terminology, and start from the notion of *clopen D-equivalence*.

**Definition 2.2.1.** *Let $P$ be a D-poset. We call* clopen D-equivalence *an equivalence relation $\sim$ on $P$ such that:*

*(A1) if $a \leq b$, $c \leq d$, $a \sim c$ and $b \sim d$, then $(b \setminus a) \sim (d \setminus c)$;*

*(A2) if $a \leq b$ and $b \sim d$, then there is $c \in P$ such that $c \sim a$ and $c \leq d$.*

From $(A1)$ and $(A2)$, by duality, it follows

(A2') if $a \leq b$ and $a \sim c$, then there is $d$ such that $d \sim b$ and $c \leq d$.

By $(A1)$, a clopen D-equivalence is also a weak D-congruence. Condition $(A2')$ can be taken instead of $(A2)$ in the definition. As characterized in [11], an open partition of a poset $P$ is a partition such that the up-set of each class (also called *block*) is union of classes. Dually, a closed partition of $P$ is a partition such that the down-set of each class is union of classes. Condition $(A2)$ says that the partition of $P$ induced by $\sim$ is an open partition, and condition $(A2')$ says that the partition of $P$ induced by $\sim$ is a closed partition. Shortly, we can say that a weak D-congruence is a clopen D-equivalence if and only if the associated partition is clopen, that is both open and closed.

Clopen D-equivalences, as equivalence relations on effect algebras, have been often called simply "congruences". See, for instance, [7, 25, 38, 28]. If $P$ is a Boolean algebra, a clopen D-congruence on $P$ has been called *s-equivalence relation* in [41].

Let $\sim$ be a clopen D-equivalence on a D-poset $P$, and let $a, b \in P$. We write $[a]$, $[a] \leq [b]$ and $[b] \oslash [a]$ with the same meaning as for congruences. Again, by $(A1)$, $\oslash$ is well defined as a partial operation between equivalence classes. Further, by $(A2)$ and $(A2')$, we have $[a] \leq [b]$ if and only if there is $c \in P$ such that $c \sim a$ and $c \leq b$, if and only if there is $d \in P$ such that $d \sim b$ and $a \leq d$.

**Example 2.2.1.** *Let $P$ be the powerset of the four-element set $\{a, b, c, d\}$, with the usual order by set inclusion, and set difference as partial difference. Let $\sim$ be the equivalence relation whose classes are:*
$\{\varnothing\}$;
$\{\{a\}, \{b\}\}$;
$\{\{c\}, \{d\}\}$;
$\{\{a, b\}\}$;

*$\{\{c,d\}\}$;*
*$\{\{a,c\},\{b,d\}\}$;*
*$\{\{a,d\},\{b,c\}\}$;*
*$\{\{a,b,c\},\{a,b,d\}\}$;*
*$\{\{a,c,d\},\{b,c,d\}\}$;*
*$\{\{a,b,c,d\}\}$.*
*It can be verified that $\sim$ is a clopen D-equivalence.*

We are going to show that clopen D-equivalences and clopen D-congruences are the same thing. One implication is straightforward.

**Proposition 2.2.1.** *Let $P$ be a D-poset and $\sim$ be a clopen D-congruence on $P$. Then, $\sim$ is a clopen D-equivalence.*

*Proof.* We recall that $\sim$ is a clopen D-congruence when it is induced by a D-morphism $f$ from $P$ to some D-poset $Q$, and for all $a,b \in P$ with $f(a) \le f(b)$, there is $c \in P$ such that $f(c) = f(a)$ and $c \le b$. Then, $(A1)$ follows from the property of D-morphism of $f$, and $(A2)$ follows from the fact that $f$ is clopen. □

In order to show that clopen D-equivalences are clopen D-congruences, we need a few steps.

**Lemma 2.2.1.** *Let $\sim$ be a clopen D-equivalence on a D-poset $P$, and let $a,b \in P$. If $[a] \le [b]$, then $[b] \oslash [a] = [0]$ if and only if $a \sim b$.*

*Proof.* If $a \sim b$, then $[b] \oslash [a] = [b \setminus b] = [0]$. Let now $[a] \le [b]$ and $[b] \oslash [a] = [0]$. Then, there is $c$ such that $c \sim a$ and $c \le b$. It holds

$$a \sim c = b \setminus (b \setminus c) \sim b \setminus 0 = b.$$

□

**Proposition 2.2.2.** *Let $P$ be a D-poset and $\sim$ be a clopen D-equivalence on $P$. Then, $(P/\sim, \le, \oslash, [0], [1])$ is a D-poset, and the projection $\pi : P \to P/\sim$, $x \mapsto [x]$ is a clopen epi D-morphism.*

*Proof.* Let us first verify that $(P/\sim, \le)$ is a poset, that is $\le$ is reflexive, antisymmetric and transitive on $P/\sim$. Reflexivity is trivial, since $a \le a$ implies $[a] \le [a]$.

Antisimmetry: let $[a] \le [b]$ and $[b] \le [a]$. Then, by $(A2)$, there are $a' \sim a$ and $b' \sim b$ such that $a' \le b' \le a$. Since $[a] \oslash [a'] = [a \setminus a] = [0]$, it holds $a \setminus a' \sim 0$. Then, since $a \setminus b' \le a \setminus a'$ by $(D2)$, it holds $a \setminus b' \sim 0$, that is $[a] \oslash [b'] = [0]$. Then, by Lemma 2.2.1, $[b'] = [a]$, that is $[a] = [b]$.

Transitivity: let $[a] \le [b]$ and $[b] \le [c]$. Then, by $(A2)$, there are $a' \sim a$ and $b' \sim b$ such that $a' \le b' \le c$. Therefore, $[a] = [a'] \le [c]$.

It is clear that $[0]$ and $[1]$ are, respectively, the lower and the upper bounds of $(P/\!\sim, \leq)$.

Let us now verify $(D1)$-$(D3)$ in the definition of D-poset for $(P/\!\sim, \leq, \oslash, [0], [1])$.

$(D1)$ is immediate: $[a] \oslash [0] = [a \setminus 0] = [a]$.

For $(D2)$ and $(D3)$, let $[a] \leq [b] \leq [c]$. Then, by $(A2)$, there are $a' \sim a$ and $b' \sim b$ such that $a' \leq b' \leq c$. By $(D2)$, $c \setminus b' \leq c \setminus a'$, hence $[c \setminus b'] \leq [c \setminus a']$ and $[c] \oslash [b] \leq [c] \oslash [a]$.

By $(D3)$, $(c \setminus a') \setminus (c \setminus b') = b' \setminus a'$, hence $[c \setminus a'] \oslash [c \setminus b'] = [b' \setminus a']$ and $([c] \oslash [a]) \oslash ([c] \oslash [b]) = [b] \oslash [a]$.

The projection $\pi : P \to P/\!\sim$, defined as $\pi(x) = [x]$, is a D-morphism. Indeed, if $a \leq b$ then $[a] \leq [b]$, and $[b] \oslash [a] = [b \setminus a]$. The D-morphism $\pi$ is clearly epi, it is regular by definition of the order relation $\leq$ in $P/\!\sim$, and it is clopen by $(A2)$ in Definition 2.2.1. Summarizing, $\pi$ is a clopen epi D-morphism. ☐

**Corollary 2.2.1.** *An equivalence relation $\sim$ on a D-poset $P$ is a clopen D-equivalence if and only if it is a clopen D-congruence.*

*Proof.* One direction is given by Proposition 2.2.1. On the other hand, if $\sim$ is D-equivalence, let $Q = P/\!\sim$. Then, by Proposition 2.2.2, there is a clopen epi D-morphism $\pi : P \to Q$ which induces $\sim$. ☐

After Corollary 2.2.1, we will just write "clopen D-congruence" referring either to a clopen D-congruence or to a clopen D-equivalence.

Let us consider again Example 2.2.1. The relation $\sim$ is a clopen D-congruence, hence the quotient $Q = P/\!\sim$ is a D-poset, by Proposition 2.2.2.

**Remark 2.2.1.** *Example 2.2.1 also illustrates a crucial fact: the quotient of a D-lattice (or even a Boolean D-poset) modulo a clopen D-congruence is not necessarily a lattice. For instance, the classes $\{\{a\},\{b\}\}$ and $\{\{c\},\{d\}\}$ do not have a supremum. Starting from this observation, it seems desirable to determine some further conditions that, added to $(A1)$ and $(A2)$ of Definition 2.2.1, imply that $P/\!\sim$ be a lattice if $P$ is a lattice. Notice that this is the case when lattice operations belong to the signature. For instance, for every MV-congruence of an MV-algebra $P$, the quotient $P/\!\sim$ is an MV-algebra, then a lattice.*

In [41] the following condition is considered when $P$ is a Boolean algebra and $\sim$ is a clopen D-congruence on $P$ (there called *s-equivalence*):

(R) For all $a, b \in P$, the set $\{[c \vee d] \mid c \sim a, d \sim b\}$ contains a smallest element.

In the same [41] the quotient $P/\!\sim$ is called *residuable* if it fulfills $(R)$. Then it is proven a result equivalent to the following:

**Theorem 2.2.1.** *Let $P$ be a Boolean algebra and $\sim$ be an s-equivalence relation on $P$ such that $P/\sim$ is residuable (in our terms, a clopen D-congruence satisfying also $(R)$, or an equivalence relation satisfying $(A1)$, $(A2)$ and $(R)$). Then, $(U/\sim, \leq, \oslash, [0], [1])$ is a Boolean D-poset.*

In the same [41] sufficient conditions for residuability are given.

We would like to determine some conditions for a clopen D-congruence on a D-poset $P$ implying $(R)$ when $P$ is a Boolean algebra. The condition $(R)$ cannot be used in that form for every D-poset, since, if $P$ is not a lattice, the set in $(R)$ is not always defined.

We start with a preliminary definition.

**Definition 2.2.2.** *Let $P$ be a D-poset, $\sim$ be a clopen D-congruence on $P$ and $a, b \in P$. We say that $b$ is in general position with respect to $a$ (and to $\sim$) if:*

*(GP) for all $c \in P$ such that $[c] \leq [a]$ and $[c] \leq [b]$, there is $d \in P$ such that $d \sim c$, $d \leq a$ and $d \leq b$.*

Condition $(GP)$ is equivalent, by duality, to the following:

(GP') for all $c \in P$ such that $[a] \leq [c]$ and $[b] \leq [c]$, there is $d \in P$ such that $d \sim c$, $a \leq d$ and $b \leq d$.

**Definition 2.2.3.** *Let $P$ be a D-poset and $\sim$ be a clopen D-congruence on $P$. We say that $\sim$ is* residuable *if the following further condition holds:*

*(A3) for all $a, b \in P$, there is $d \in P$ such that $d \sim b$ and $d$ is in general position with respect to $a$.*

**Remark 2.2.2.** *We have chosen the term "residuable" in analogy with the residuability condition in [41]. Notice, however, that the property of residuability loses here its original meaning, related to residuated lattices.*

A trivial example of residuable clopen D-congruence is the equivalence relation whose classes are singletons. It can be shown that the clopen D-congruence of Example 2.2.1 is not residuable: we do not give details now, it will follow from the next statements.

**Lemma 2.2.2.** *Let $P$ be a D-poset and $\sim$ be a residuable clopen D-congruence on $P$. Then,*

*(1) If $P$ is a D-lattice then, for all $a, b \in P$, $[a]$ and $[b]$ have a least upper bound;*

*(2) If $P$ is a D-lattice then, for all $a, b \in P$, $[a]$ and $[b]$ have a greatest lower bound;*

*(3) If $P$ is a Boolean D-poset then, for all $a, b \in P$,*

$$([a] \vee [b]) \oslash [b] = [a] \oslash ([a] \wedge [b]).$$

*Proof.* $(1)$ Let $d \in P$ such that $d \sim b$ and $d$ is in general position with respect to $a$. Then, $[a \vee d] = [a] \vee [b]$. Indeed, if $[a] \leq [x]$ and $[b] \leq [x]$, then, by $(GP')$, there is $y \sim x$ such that $a \leq y$ and $d \leq y$. Therefore, $a \vee d \leq y$, hence $[a \vee d] \leq [x]$.

$(2)$ Dually to $(1)$, it holds $[a \wedge d] = [a] \wedge [b]$, where $d \sim b$ is in general position with respect to $a$.

$(3)$ Let $d \in P$ such that $d \sim b$ and $d$ is in general position with respect to $a$. Then, by the two previous points, $([a] \vee [b]) \oslash [b] = [a \vee d] \oslash [d] = [(a \vee d) \setminus d]$ and $[a] \oslash ([a] \wedge [b]) = [a] \oslash [a \wedge d] = [a \setminus (a \wedge d)]$. The equality in $(3)$ follows from $(D3)$ of the definition of Boolean D-poset. □

**Theorem 2.2.2.** *Let $P$ be a D-poset and $\sim$ be a residuable clopen D-congruence on $P$. Then,*

*(1) $(P/\!\sim, \leq \oslash, [0], [1])$ is a D-poset;*

*(2) if $P$ is a D-lattice, then $(P/\!\sim, \leq \oslash, [0], [1])$ is a D-lattice;*

*(3) if $P$ is a Boolean D-poset, then $(P/\!\sim, \leq \oslash, [0], [1])$ is a Boolean D-poset.*

*Proof.* $(1)$ It follows from Proposition 2.2.2, since $\sim$ is clopen.

$(2)$ It follows from $(1)$ and $(2)$ of Lemma 2.2.2.

$(3)$ The D-poset $P/\!\sim$ is a D-lattice by the previous point, and it is a Boolean D-poset by $(3)$ of Lemma 2.2.2. □

The following corollary is just a specialization of Theorem 2.2.2 to the case that $P$ is a sharp D-poset. We write it in order to emphasize how unsharp D-poset can arise from sharp D-poset via suitable equivalence relations.

**Corollary 2.2.2.** *Let $P$ be a sharp D-poset and $\sim$ be a residuable clopen D-congruence on $P$. Then,*

*(1) $(P/\!\sim, \leq \oslash, [0], [1])$ is a D-poset;*

*(2) if $P$ is an orthomodular lattice, then $(P/\!\sim, \leq \oslash, [0], [1])$ is a D-lattice;*

*(3) if $P$ is a Boolean algebra, then $(P/\!\sim, \leq \oslash, [0], [1])$ is a Boolean D-poset.*

Now it is clear why the clopen D-congruence of Example 2.2.1 is not residuable: the D-poset $P$ is a Boolean algebra, but $P/\!\sim$ is not a Boolean D-poset (it is not even a lattice).

**Remark 2.2.3.** *Corollary 2.2.2 suggests the inverse problem: given a D-poset $P$, can it be represented by a sharp D-poset Q modulo some kind of D-congruence? In other words, is $P$ homomorphic image of some D-morphism from a sharp D-poset $Q$? These questions can be answered in several ways, depending on the hypothesis done on congruences and morphisms. In [27, 28, 13] a solution is given when $P$ is an MV-algebra, with the use of the* free Boolean extension*, that we will define in Section 3.1.*

## 2.3 Sectional D-congruences and S-operators

In the previous section we studied clopen D-congruences, a class of congruences of D-posets, and we characterized the quotients. In this section we do a parallel analysis of another class of D-congruences, closely connected with the previous ones. These newly introduced congruences can be characterized as the equivalence relations induced by some internal operators, formally similar to the modal operator used in modal logic [6], or the internal state operators defined on MV-algebras [17].

**Definition 2.3.1.** *Let $P$ be a D-poset. We call* sectional operator *(shortly,* S-operator*) on $P$ a function $\sigma : P \to P$ such that, for all $a, b \in P$:*

*(S1)* $\sigma(\sigma(a)) = \sigma(a)$;

*(S2) if* $a \le b$, *then* $\sigma(a) \le \sigma(b)$;

*(S3) if* $a \le b$, *then* $\sigma(b \setminus a) = \sigma(\sigma(b) \setminus \sigma(a))$.

We say that an S-operator $\sigma : P \to P$ is *strict* if $\sigma(0) = 0$ and one of the following equivalent conditions hold:

$\sigma(a) = 0$ only for $a = 0$;

if $a < b$, then $\sigma(a) < \sigma(b)$.

If $\sigma$ is strict, it also holds $\sigma(a) = 1$ if and only if $a = 1$.

Notice that the identity $\iota : P \to P$ ($x \mapsto x$) is a strict S-operator.

**Definition 2.3.2.** *Let $P$ be a D-poset. We call* sectional D-equivalence *an equivalence relation $\sim$ on $P$ induced by some S-operator. In other words, $\sim$ is a sectional D-equivalence if there is $\sigma : P \to P$ such that $\sigma$ is an S-operator and $\sim$ coincides with $\sim_\sigma$.*

Let $\sim$ be a sectional D-equivalence. We say that $\sim$ is *strict* if it is induced by a strict S-operator. It is easy to see that $\sim$ is strict if and only if one of the following two conditions hold:

$[0] = \{0\}$;

if $a < b$, then $[a] \neq [b]$.

If $\sim$ is strict, it also holds $[1] = \{1\}$.

The sectional D-equivalence induced by the identity S-operator, which is strict, is the minimal equivalence relation (the one whose equivalence classes are singletons).

**Example 2.3.1.** *Let $P$ be the powerset of the three-element set $\{a, b, c\}$, with the usual order by set inclusion and set difference. Let $\sigma : P \to P$ be defined as follows:*
$\sigma(\emptyset) = \emptyset$*;*
$\sigma(\{a\}) = \sigma(\{b\}) = \sigma(\{c\}) = \{a\}$*;*
$\sigma(\{a, b\}) = \sigma(\{a, c\}) = \sigma(\{b, c\}) = \{a, b\}$*;*
$\sigma(\{a, b, c\}) = \{a, b, c\}$.
*It can be verified that $\sigma$ is a strict S-operator. It induces the strict sectional D-equivalence whose classes are:*
$\{\emptyset\}$*;*
$\{\{a\}, \{b\}, \{c\}\}$*;*
$\{\{a, b\}, \{a, c\}, \{b, c\}\}$*;*
$\{\{a, b, c\}\}$.
*Two subsets are equivalent by $\sim$ if they have the same cardinality.*

**Remark 2.3.1.** *In [17] a notion of* internal state *has been introduced for MV-algebras, in order to provide an algebraic theory of states of MV-algebras.*

*An MV-algebra with internal state (shortly, SMV-algebra) is a structure $(L, \sigma) = (L, \oplus, \neg, \sigma, 0)$, where $(L, \oplus, \neg, 0)$ is an MV-algebra and $\sigma : L \to L$ is a unary operator on $L$ satisfying, for all $a, b \in L$:*

*(IS1)* $\sigma(0) = 0$*;*

*(IS2)* $\sigma(\neg a) = \neg\sigma(a)$*;*

*(IS3)* $\sigma(a \oplus b) = \sigma(a) \oplus \sigma(b \ominus (a \odot b))$*;*

*(IS4)* $\sigma(\sigma(a) \oplus \sigma(b)) = \sigma(a) \oplus \sigma(b)$

*An SMV-algebra $(L, \sigma)$ is said to be faithful if it satisfies the quasi-equation: $\sigma(a) = 0$ implies $a = 0$.*

*It can be shown that, if $L$ is an MV-algebra and $\sigma : L \to L$ is an internal state of $L$, then $\sigma$ is an S-operator on $L$. If $(L, \sigma)$ is faithful, then the S-operator $\sigma$ is strict.*

*However, if $\sigma : L \to L$ is an S-operator, it is not necessarily an internal state. In particular $(IS2)$ does not always hold for an S-operator. For instance, in Example 2.3.1, $\sigma(\neg\{a\}) = \sigma(\{b, c\}) = \{a, b\}$, while $\neg\sigma(\{a\}) = \neg\{a\} = \{b, c\}$.*

*For an S-operator, it holds the weaker*

$$\sigma(\neg a) = \sigma(\neg\sigma(a)),$$

*which can be easily deduced by* $(S3)$ *since* $\neg a = 1 \setminus a$.

Now, following similar lines as for clopen D-congruences, we are going to show that sectional D-equivalences coincide with sectional D-congruences.

**Lemma 2.3.1.** *Let* $\sim$ *be a sectional D-equivalence on a D-poset* $P$. *Then,* $\sim$ *is a weak D-congruence.*

*Proof.* We verify $(A1)$. Let $a \le b$, $c \le d$, $a \sim c$ and $b \sim d$. Then, by $(S2)$,

$$\sigma(b \setminus a) = \sigma(\sigma(b) \setminus \sigma(a)) = \sigma(\sigma(d) \setminus \sigma(c)) = \sigma(d \setminus c),$$

hence $\sigma(b \setminus a) \sim (d \setminus c)$. □

We observe that, when $[a] \le [b]$, $[b] \oslash [a] = [\sigma(b) \setminus \sigma(a)]$.

**Proposition 2.3.1.** *Let* $P$ *be a D-poset and* $\sim$ *be a sectional D-congruence on* $P$. *Then,* $\sim$ *is a sectional D-equivalence.*

*Proof.* We recall that $\sim$ is a sectional D-congruence when it is induced by a D-morphism $f : P \to Q$ for some D-poset $Q$ and there is an idempotent function $\sigma : P \to P$ such that $\sigma(a) = \sigma(b)$ if and only if $f(a) = f(b)$, and $f(a) \le f(b)$ implies $\sigma(a) \le \sigma(b)$. It is clear from this definition that $\sim$ is induced by $\sigma$, that is $\sim$ coincides with $\sim_f$ and with $\sim_\sigma$. Then, property $(S1)$-$(S3)$ for $\sigma$ follow immediately from the properties of D-morphisms of $f$. □

Similarly as for clopen D-congruences, we have the following proposition.

**Proposition 2.3.2.** *Let* $P$ *be a D-poset and* $\sim$ *be a sectional D-equivalence on* $P$. *Then,* $(P/\!\sim, \le, \oslash, [0], [1])$ *is a D-poset, and the projection* $\pi : P \to P/\!\sim$, $x \mapsto [x]$ *is a sectional epi D-morphism.*

*Proof.* Let $\sigma : P \to P$ an S-operator associated with $\sim$. It holds $[a] \le [b]$ if and only if $\sigma(a) \le \sigma(b)$, hence the relation $\le$ on $P/\!\sim$ is isomorphic to the order relation $\le$ restricted to $\sigma(P)$, the image of $\sigma$. Therefore, $(P/\!\sim, \le)$ is a poset.

Clearly, $[0]$ and $[1]$ are, respectively, the lower and the upper bounds of $(P/\!\sim, \le)$.

Let us now verify $(D1)$-$(D3)$ in the definition of D-poset for $(P/\!\sim, \le, \oslash, [0], [1])$.

$(D1)$ is immediate: $[a] \oslash [0] = [a \setminus 0] = [a]$.

For $(D2)$ and $(D3)$, let $[a] \le [b] \le [c]$. Then, $\sigma(a) \le \sigma(b) \le \sigma(c)$, hence, by $(D2)$, $\sigma(c) \setminus \sigma(b) \le \sigma(c) \setminus \sigma(a)$, therefore $[c] \oslash [b] = [\sigma(c) \setminus \sigma(b)] \le [\sigma(c) \setminus \sigma(a)] = [c] \oslash [a]$.

From the two equalities

$$([c] \oslash [a]) \oslash ([c] \oslash [b]) = [(\sigma(c) \setminus \sigma(a)) \setminus (\sigma(c) \setminus \sigma(b))],$$

and

$$[b] \oslash [a] = [\sigma(b) \setminus \sigma(a)]$$

it follows, by $(D3)$, that

$$([c] \oslash [a]) \oslash ([c] \oslash [b]) = [b] \oslash [a].$$

The projection $\pi : P \to P/\sim$, defined as $\pi(x) = [x]$, is a D-morphism. Indeed, if $a \leq b$ then $[a] \leq [b]$, and $[b] \oslash [a] = [b \setminus a]$. It is obvious that $\pi$ is epi and it is sectional by the same function $\sigma$. □

Summarizing Proposition 2.3.1 and Proposition 2.3.2, we have the following corollary.

**Corollary 2.3.1.** *An equivalence relation $\sim$ on a D-poset $P$ is a sectional D-equivalence if and only if it is a sectional D-congruence.*

After Corollary 2.3.1, we will just write "sectional D-congruence" referring either to a sectional D-congruence or to a sectional D-equivalence.

The following results show that sectional D-congruences are similar to residuable clopen D-congruences (considered in Section 2.2) from the point of view of the structure of the quotients.

**Lemma 2.3.2.** *Let $P$ be a D-poset and $\sim$ be a sectional D-congruence on $P$. Then,*

*(1) If $\sigma(a)$ and $\sigma(b)$ have a least upper bound, then $[a]$ and $[b]$ have a least upper bound;*

*(2) If $\sigma(a)$ and $\sigma(b)$ have a greatest lower bound, then $[a]$ and $[b]$ have a greatest lower bound;*

*(3) If both $\sigma(a) \vee \sigma(b)$ and $\sigma(a) \wedge \sigma(b)$ exist, and $(\sigma(a) \vee \sigma(b)) \setminus \sigma(b) = \sigma(a) \setminus (\sigma(a) \wedge \sigma(b))$, then $([a] \vee [b]) \oslash [b] = [a] \oslash ([a] \wedge [b])$.*

*Proof.* $(1)$ We show that $[\sigma(a) \vee \sigma(b)] = [a] \vee [b]$. If $[a] \leq [c]$ and $[b] \leq [c]$, then $\sigma(a) \leq \sigma(c)$ and $\sigma(b) \leq \sigma(c)$, hence $\sigma(a) \vee \sigma(b) \leq \sigma(c)$, which implies $[\sigma(a) \vee \sigma(b)] \leq [c]$.

$(2)$ It is dual to $(1)$.

$(3)$ It holds, by $(1)$,

$$([a] \vee [b]) \oslash [b] = [\sigma(a) \vee \sigma(b)] \oslash [b] = [(\sigma(a) \vee \sigma(b)) \setminus \sigma(b)].$$

On the other hand, by $(2)$,

$$[a] \oslash ([a] \wedge [b]) = [a] \oslash [\sigma(a) \wedge \sigma(b)] = [\sigma(a) \setminus (\sigma(a) \wedge \sigma(b))].$$

Then, by hypothesis, it follows $([a] \vee [b]) \oslash [b] = [a] \oslash ([a] \wedge [b])$. □

**Theorem 2.3.1.** *Let $P$ be a D-poset and $\sim$ be a sectional D-congruence on $P$. Then,*

*(1) $(P/\sim, \leq \oslash, [0], [1])$ is a D-poset;*

*(2) if $P$ is a D-lattice, then $(P/\sim, \leq \oslash, [0], [1])$ is a D-lattice;*

*(3) if $P$ is a Boolean D-poset, then $(P/\sim, \leq \oslash, [0], [1])$ is a Boolean D-poset.*

*Proof.* $(1)$ It is part of Proposition 2.3.2.

$(2)$ It follows from $(1)$ and $(2)$ of Lemma 2.3.2.

$(3)]$ The D-poset $P/\sim$ is a D-lattice for the previous point, and it is a Boolean D-poset for $(3)$ of Lemma 2.3.2. □

**Remark 2.3.2.** *Theorem 2.3.1 implies that a clopen D-congruence is not always a sectional D-congruence. One case is given in Example 2.2.1. There, the quotient $P/\sim$ is not a lattice, therefore $\sim$ cannot be sectional. For the same reasons, the projection $\pi : P \to P/\sim$ to the quotient is an example of clopen D-morphism which is not sectional.*

As for residuable clopen D-congruences, we can specialize Theorem 2.3.1 to sharp D-posets and conclude that, if $P$ is a sharp D-poset,

- $(P/\sim, \leq \oslash, [0], [1])$ is a D-poset;
- if $P$ is an orthomodular lattice, then $(P/\sim, \leq \oslash, [0], [1])$ is a D-lattice;
- if $P$ is a Boolean algebra, then $(P/\sim, \leq \oslash, [0], [1])$ is a Boolean D-poset.

We now focus our attention on the images of S-operators.

We write $\sigma(P)$ for the image of an S-operator $\sigma : P \to P$. For all $\sigma(a) \leq \sigma(b)$, we write $\sigma(b) \oslash \sigma(a) = \sigma(\sigma(b) \setminus \sigma(a))$. Notice that, for $a \leq b$, it holds $[b] \oslash [a] = [\sigma(b) \oslash \sigma(a)]$, and, in general, $\sigma(b) \oslash \sigma(a) \neq \sigma(b) \setminus \sigma(a)$. For instance, in 2.3.1, $\sigma(\{a,b\}) \oslash \sigma(\{a\}) = \{a\}$, while $\sigma(\{a,b\}) \setminus \sigma(\{a\}) = \{b\}$.

With this notation, the following proposition is an immediate consequence of Theorem 2.3.1 and Proposition 2.3.2.

**Theorem 2.3.2.** *Let $P$ be a D-poset and $\sigma : P \to P$ be an S-operator. Then,*

- *$(\sigma(P), \leq, \oslash, \sigma(0), \sigma(1))$ is a D-poset;*

- *if $P$ is a D-lattice, then $(\sigma(P), \leq \oslash, \sigma(0), \sigma(1))$ is a D-lattice;*
- *if $P$ is a Boolean D-poset, then $(\sigma(P), \leq \oslash, \sigma(0), \sigma(1))$ is a Boolean D-poset;*
- *the function $m : P \to \sigma(P)$, $x \mapsto \sigma(x)$ is a sectional epi D-morphism.*

Notice that, when $P$ is a D-lattice, the least upper bound of $\sigma(a)$ and $\sigma(b)$ in $\sigma(P)$ is $\sigma(\sigma(a) \vee \sigma(b))$ and not, in general, $\sigma(a) \vee \sigma(b)$. It is $\sigma(a) \vee \sigma(b)$ when $\sigma$ is strict.

We recall that an unary operator $\sigma$ on a D-poset $P$ is a strict S-operator if it satisfies:

(S1') $\sigma(0) = 0$;

(S2') if $a < b$, then $\sigma(a) < \sigma(b)$;

(S3') if $a \leq b$, then $\sigma(b \setminus a) = \sigma(\sigma(b) \setminus \sigma(a))$.

In this formulation we do not need to require the idempotence of $\sigma$, which follows from $(S1') - (S3')$. Indeed, $\sigma(\sigma(a)) = \sigma(\sigma(a) \setminus \sigma(0)) = \sigma(a \setminus 0) = \sigma(a)$.

**Proposition 2.3.3.** *Let $P$ be a D-lattice and $\sigma : P \to P$ be a strict S-operator. Then $(\sigma(P), \leq \oslash, 0, 1)$ is a D-lattice, and $(\sigma(P), \vee, \wedge, 0, 1)$ is a sublattice of $(P, \vee, \wedge, 0, 1)$.*

*Proof.* First we observe that, if $\sigma(a) \leq a$, then $\sigma(a) = a$. Indeed, suppose $\sigma(a) < a$. From $(S2')$ and idempotence of $\sigma$ it would follow that $\sigma(a) = \sigma\sigma(a) < \sigma(a)$, which cannot hold. Then we have $\sigma(1) = 1$, since $\sigma(1) \leq 1$ as $1$ is the upper bound of $P$. Summarizing, the bounds of $\sigma(P)$ are $0$ and $1$.

It holds $\sigma(a) = \sigma\sigma(a) \leq \sigma(\sigma(a) \vee \sigma(b))$, and similarly $\sigma(b) \leq \sigma(\sigma(a) \vee \sigma(b))$. Hence, $\sigma(a) \vee \sigma(b) \leq \sigma(\sigma(a) \vee \sigma(b))$. From the previous observation, the equality follows, $\sigma(a) \vee \sigma(b) = \sigma(\sigma(a) \vee \sigma(b))$. We have already proved that $\sigma(\sigma(a) \vee \sigma(b))$ is the least upper bound in $(\sigma(P), \leq)$, therefore $\sigma(a) \vee \sigma(b)$ is the least upper bound in $(\sigma(P), \leq)$. Similarly, $\sigma(a) \wedge \sigma(b)$ is the greatest lower bound. We can conclude that $(\sigma(P), \vee, \wedge, 0, 1)$ is a sublattice of $(P, \vee, \wedge, 0, 1)$. $\square$

For clarity we specify Theorem 2.3.2 to sharp D-posets and strict S-operators.

**Corollary 2.3.2.** *Let $P$ be a sharp D-poset and $\sigma : P \to P$ be a strict S-operator. Then,*

- *$(\sigma(P), \leq, \oslash, 0, 1)$ is a D-poset;*
- *if $P$ is an orthomodular lattice, then $(\sigma(P), \leq \oslash, 0, 1)$ is a D-lattice;*
- *if $P$ is a Boolean algebra, then $(\sigma(P), \leq \oslash, 0, 1)$ is a Boolean D-poset;*

- *the function* $m : P \to \sigma(P)$, $x \mapsto \sigma(x)$ *is a sectional epi D-morphism.*

Let us consider again Example 2.3.1. The D-poset $P$ is a Boolean algebra, and $\sigma$ is a strict S-operator. Therefore, $\sigma(P)$ is a Boolean D-poset. Indeed, it is isomorphic to the MV-chain $\{0, 1/3, 2/3, 1\}$.

## 2.4 Boolean algebras with S-operators

In this chapter we have studied different kinds of equivalence relations on D-posets.

In the sequel, we will be concerned only with S-operators and sectional D-congruences on Boolean algebras. As we have seen previously, both the quotient of a Boolean algebra $U$ modulo a sectional D-congruence and the image of an S-operator on $U$ have a structure of MV-algebra.

The operations of this MV-algebra have a simple interpretation in terms of minima and maxima, as in the following proposition.

**Proposition 2.4.1.** *Let* $(U, \vee, \wedge, \smallsetminus, 0, 1)$ *be a Boolean algebra and let* $\sim$ *be a sectional D-congruence on* $U$*. Then, for all* $[a]$ *and* $[b]$ *in the MV-algebra* $U/\sim$*, we have*

*(1)* $[b] \ominus [a] = \min\{[y - x] \mid x \sim a, y \sim b\}$*;*

*(2)* $[a] \vee [b] = \min\{[x \vee y] \mid x \sim a, y \sim b\}$*;*

*(3)* $[a] \wedge [b] = \max\{[x \wedge y] \mid x \sim a, y \sim b\}$*;*

*(4)* $[a] \oplus [b] = \max\{[x \vee y] \mid x \sim a, y \sim b\}$*;*

*(5)* $[a] \odot [b] = \min\{[x \wedge y] \mid x \sim a, y \sim b\}$*;*

*(6)* $[a] \to [b] = \max\{[x \to y] \mid x \sim a, y \sim b\}$.

*Proof.* (1) It holds $[b] \ominus [a] = [b] \oslash ([a] \wedge [b]) = [\sigma(b)] \oslash [\sigma(a) \wedge \sigma(b)]$. On the other hand, $[b - a] = [b \smallsetminus (a \wedge b)] = [\sigma(b)] \oslash [\sigma(a \wedge b)]$. For the monotonicity of $\sigma$, it holds $\sigma(a \wedge b) \leq \sigma(a) \wedge \sigma(b)$, hence $[\sigma(a \wedge b)] \leq [\sigma(a) \wedge \sigma(b)]$. Then, from the antitonicity of $\oslash$, it follows $[b] \ominus [a] \leq [b - a]$. It also holds

$$[b] \ominus [a] = [\sigma(b)] \oslash [\sigma(a) \wedge \sigma(b)],$$

and this is equal to

$$[\sigma(b) \smallsetminus (\sigma(a) \wedge \sigma(b))] = [\sigma(b) - \sigma(a)],$$

hence $[b] \ominus [a] = [y - x]$ for some $x, y \in L$.. We can conclude that $[b] \ominus [a] = \min\{[y - x] \mid x \sim a, y \sim b\}$.

$(2)$ By the property of least upper bound of $[a] \vee [b]$, it holds $[a] \vee [b] \leq [x \vee y]$ for all $x \sim a$ and $y \sim b$. Furthermore, $[a] \vee [b] = [\sigma(a) \vee \sigma(b)]$, hence $[a] \vee [b]$ is the minimum of the set at second term.

$(3)$, $(4)$, $(5)$ and $(6)$ follow from $(1)$ and $(2)$, since these three operation can be derived from $\ominus$ and $\vee$, for instance, the following steps:
$\neg x = 1 \ominus x$;
$x \wedge y = \neg(\neg x \vee \neg y)$;
$x \odot y = y \ominus (\neg x)$;
$x \oplus y = \neg(\neg x \odot \neg y)$;
$x \to y = \neg x \oplus y$. □

**Remark 2.4.1.** *Proposition 2.4.1 provides an interpretation of operations of an MV-algebra as minima or maxima of sets of classes taken in a Boolean algebra. In this context, $\oplus$ can be viewed as a* strong disjunction *and $\vee$ as a* weak disjunction*; $\wedge$ can be viewed as a* weak conjunction *and $\odot$ as a* strong conjunction.

*Notice that, when the D-congruence in 2.4.1 is a congruence of Boolean algebra (an MV-congruence), the quotient is again a Boolean algebra, and the minima and maxima in 2.4.1 are trivial, since there is only one class in every set to minimize or maximize. In this case, strong disjunction and weak disjunction coincide, as well as weak conjunction and strong conjunction. The possibility to represent an MV-algebra starting from the "simpler" structure of Boolean algebra is given by the fact that the equivalence relation is weaker, in general, than an MV-congruence.*

We formulate Proposition 2.4.1 in terms of strict S-operators.

**Proposition 2.4.2.** *Let $(U, \vee, \wedge, \setminus, 0, 1)$ be a Boolean algebra. Let $\sigma : U \to U$ be a strict S-operator. Then, for all $\sigma(a)$ and $\sigma(b)$ in the MV-algebra $\sigma(U)$, we have*

$$\sigma(b) \ominus \sigma(a) = \min\{\sigma(y - x) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\};$$

$$\sigma(a) \vee \sigma(b) = \min\{\sigma(x \vee y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\};$$

$$\sigma(a) \wedge \sigma(b) = \max\{\sigma(x \wedge y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\};$$

$$\sigma(a) \oplus \sigma(b) = \max\{\sigma(x \vee y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\};$$

$$\sigma(a) \odot \sigma(b) = \min\{\sigma(x \wedge y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\};$$

$$\sigma(a) \to \sigma(b) = \max\{\sigma(x \to y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\}.$$

The result for general S-operators is analogous, but in the sequel we will be concerned only with strict S-operators.

# Chapter 3

# GBL-algebras and their dual

Generalized BL-algebras, shortly GBL-algebras, are the divisible residuated lattices. They have been introduced in [2] as a generalization of BL-algebras (short term for *Basic Logic algebras* [26, 10]). GBL-algebras can provide the semantics for a generalization of Basic Logic where the axiom of prelinearity does not hold. The variety of GBL-algebras contains both the variety of BL-algebras (and so MV-algebras) and that of Heyting algebras, and can be seen as a *fuzzy extension* of the latter. Informally, GBL-algebras generalize Heyting algebras in a similar way as MV-algebras generalize Boolean algebras.

In [14] finite GBL-algebras are represented in terms of Heyting algebras with an equivalence relation, there called *indiscernibility relation*. Regarding this relation, two assumptions are made:

- it is an equivalence relation given by a subgroup $G$ of the automorphism group of the Heyting algebra: two elements are *indiscernible* if there is an automorphism in $G$ mapping one to the other;
- if two chains are *elementwise indiscernible*, then there is a unique automorphism in $G$ bringing one chain to the other.

The pair made of a Heyting algebra and an indiscernibility relation is called *GBL-pair*. This name is justified by the correspondence between finite GBL-pairs and finite GBL-algebras. Indeed, every finite GBL-algebra can be represented as a GBL-pair $(H, G)$. In this representation, the elements of a GBL-algebra are equivalence classes with respect to the relation induced by $G$. The operations of the so obtained GBL-algebra have a particularly intuitive meaning in this model: they are infima or suprema of the sets of classes which result from operations of $H$ between elements chosen in the two initial classes. This fact is analogous to what we have shown in Section 2.4 for MV-algebras.

We report the main results of [14] and we show that an indiscernibility relation is a clopen D-congruence when $H$ is a Boolean algebra.

We then follow another approach to represent GBL-algebras. In Section 2.3 we have shown that an S-operator on a Boolean algebra gives rise to an MV-algebra. An inverse of this result is proven in [27]: every MV-algebra $L$ can be represented in this way, by considering the *free Boolean extension* of $L$. In [36] it was proven that closed element in a closure algebra (a Boolean algebra with a closure operator) form a Brouwerian algebra (the dual structure of a Heyting algebra). Conversely, every Brouwerian algebra $L$ is the algebra of closed elements of a closure algebra, where the Boolean algebra is the free Boolean extension of $L$. These two results, about MV-algebras and Brouwerian algebras, suggest us to consider the interplay between an S-operator and a closure operator on a Boolean algebra, with some hypothesis of commutation between the operators. We prove that the image of the composition of two such operators has a structure dual to a GBL-algebra.

GBL*algebras were introduced in [15] as the structures obtained by adding an operation of monoidal sum on GBL-algebras. The motivation was mainly to have a good definition of state of a GBL-algebra, which generalizes states of MV-algebras as defined in [37], and finitely additive probability measures on Gödel algebras as defined in [1]. Under some hypothesis, the operation of monoidal sum can be defined in a unique way on finite GBL-algebras. It coincides with the usual sum in MV-algebras, and with the join in Heyting algebras. Therefore, GBL*algebras, as GBL-algebras, are a common generalization of MV-algebras and Heyting algebras. We report some results of [15] about GBL*algebras, in particular the unicity of the monoidal sum on finite GBL-algebra (hence, a bijection between finite GBL-algebras and GBL*algebras), the definition of states of a GBL*algebra, and their characterization in the finite case: for every finite GBL*algebra $X$ the states of $X$ are determined by their restriction to the subalgebra of idempotents of $X$, which is a valuation of distributive lattice.

Finally, we prove that, for an S-operator and a closure operator on a Boolean algebra, the image of the composition of the two operators has a structure dual to a GBL*algebra, strengthening the previously mentioned result.

## 3.1 Closure and interior algebras

This section is a survey of some results concerning interior and closure operator, that will be used in the sequel for the representation of GBL-algebras and dual GBL-algebras.

Closure algebras have been introduced in [35] as an apparatus for an algebraic treatment of point-set topology. They have been further developed in [36] by the same authors, and by several others (see, for instance, [3]), often considering the dual concept of *interior algebra*.

**Definition 3.1.1.** *A* Brouwerian algebra *is a structure*

$$(L, \vee, \wedge, \dot{-}, 1, 0)$$

*of type* $(2,2,2,0,0)$ *such that* $(L,\vee,\wedge,1,0)$ *is a bounded lattice, and, for all* $a,b,c \in L$, $c \le a \vee b$ *if and only if* $c \dot{-} a \le b$.

It follows from the definition that

$$b \dot{-} a = \min\{x \in L \mid b \le a \vee x\}.$$

Notice that many authors (see, for instance, [20]) call "Brouwerian algebras" other structures, different from the one just defined here, though related. Our definition of Brouwerian algebra follows [36], with the only difference that we include both bounds in the signature.

**Definition 3.1.2.** *Let U be a Boolean algebra. A* closure operator *on U is a function* $\mu : U \to U$ *such that the following identities hold:*

$\mu(0) = 0$;

$a \le \mu(a)$;

$\mu\mu(a) = \mu(a)$;

$\mu(a \vee b) = \mu(a) \vee \mu(b)$.

An element $x \in U$ is said *closed* if $\mu(x) = x$. The pair $(U,\mu)$ is called *closure algebra*. It is easy to prove that, for a closure operator $\mu$,

if $a \le b$, then $\mu(a) \le \mu(b)$;

$\mu(1) = 1$;

$\mu(a) \wedge \mu(b)$ is closed.

Therefore, $(\mu(U),\vee,\wedge,0,1)$ is a sublattice of $U$.

The following result is given in [36].

**Theorem 3.1.1.** *Let* $(U,\vee,\wedge,-,1,0)$ *be a Boolean algebra. Set, for all* $a,b \in U$, $\mu(b) \dot{-} \mu(a) = \mu(b-a)$. *Then,* $(\mu(U),\vee,\wedge,\dot{-},0,1)$ *is a Brouwerian algebra.*

Theorem 3.1.1 has a converse: every Brouwerian algebra is the algebra of closed elements in some closure algebra. To establish this result, in [36], it is used the concept of *free Boolean extension* of a distributive lattice, that we briefly explain here.

Let $U$ be a Boolean algebra and $L \subseteq U$ be a sublattice of $U$ (sharing the bounds with $U$). Since $U$, as a lattice, is distributive, $L$ is distributive as well.

We say that $L$ *generates* $U$ if every element $x \in U$ can be obtained with a finite number of lattice operations $\vee$, $\wedge$ and difference $-$ applied to a finite number of elements of $L$.

Let now $L$ be a distributive lattice (we remind, we will always assume a lattice to be bounded). Up to isomorphism, there is a unique Boolean algebra $U$ such that $L$ is a sublattice of $U$, and $L$ generates $U$ [24].

This Boolean algebra is called the free Boolean algebra generated by $L$, more shortly the *free Boolean extension* of $L$, and we will denote it by $U(L)$. The denomination "free" is justified by the fact that $U(L)$ satisfies the universal property of the free object with respect to the embedding of $L$ as a lattice. Let us denote by $e : L \to U(L)$ the embedding of $L$ in $U(L)$. Then, $U(L)$ satisfies:

- for every Boolean algebra $V$ and every bounded lattice morphism $f : L \to V$, there is a unique morphism of Boolean algebras $g : U \to V$ such that $f = ge$.

For details, see [24].

A more direct characterization of $U(L)$ is given in [34], one of the first works where the free Boolean extension is obtained, while in [40] the free Boolean extension is considered in the context of valuations to rings (the free Boolean extension can be obtained from a particular case of valuation ring, when the ring has two elements)

The following is a fundamental property of free Boolean extensions. For every element $x \in U(L)$, there are $x_1, x_2, \ldots, x_{2n}$ with $x_1 < x_2 < \ldots < x_{2n}$ such that

$$x = (x_{2n} - x_{2n-1}) \vee \ldots \vee (x_4 - x_3) \vee (x_2 - x_1).$$

In other words, every element of $U(L)$ can be represented by an even-cardinality chain of elements of $L$ [24]. We say that $(x_1, x_2, \ldots, x_{2n})$ is a *chain-representation* of $x$.

Now we can state the result, given in [36], which completes Theorem 3.1.1.

**Theorem 3.1.2.** *Let $(L, \vee, \wedge, \dot{-}, 0, 1)$ be a Brouwerian algebra and $U(L)$ be the free Boolean extension of its lattice reduct. Let $x \in U(L)$ and $(x_1, x_2, \ldots, x_{2n})$ be a chain-representation of $x$, with $x_1 < x_2 < \ldots < x_{2n}$. Then, the value*

$$(x_{2n} \dot{-} x_{2n-1}) \vee \ldots \vee (x_4 \dot{-} x_3) \vee (x_2 \dot{-} x_1).$$

*does not depend on the choice of the chain in the representation of $x$. Further, set $\mu : U(L) \to U(L)$ as*

$$\mu(x) = (x_{2n} \dot{-} x_{2n-1}) \vee \ldots \vee (x_4 \dot{-} x_3) \vee (x_2 \dot{-} x_1)$$

*for every $x \in U(L)$ and any chain-representation of $x$. Then, $\mu$ is a closure operator, $\mu(U(L)) = L$ and $\mu(b) \dot{-} \mu(a) = \mu(b - a)$.*

The concepts dual to Brouwerian algebra and closure operator are, respectively, *Heyting algebra* and *interior operator*.

**Definition 3.1.3.** *A Heyting algebra is a structure*

$$(L, \wedge, \vee, \to, 1, 0)$$

*of type* $(2, 2, 2, 0, 0)$ *such that* $(L, \vee, \wedge, 1, 0)$ *is a bounded lattice, and, for all* $a, b, c \in L$, $a \wedge b \le c$ *if and only if* $b \le a \to c$.

It follows from the definition that

$$a \to b = \max\{x \in L \mid a \wedge x \le b\}.$$

**Definition 3.1.4.** *Let* $U$ *be a Boolean algebra. An* interior operator *on* $U$ *is a function* $\lambda : U \to U$ *such that the following identities hold:*

$\lambda(1) = 1$;

$\lambda(a) \le a$;

$\lambda\lambda(a) = \lambda(a)$;

$\lambda(a \wedge b) = \lambda(a) \wedge \lambda(b)$.

An element $x \in U$ is said *open* if $\lambda(x) = x$. The pair $(U, \lambda)$ is called *interior algebra*.

All the result concerning Heyting algebras and interior operators can be obtained from the analogous ones concerning, respectively, Brouwerian algebras and closure operators, by reversing the order relation, exchanging $\vee$ with $\wedge$ and replacing $\dot{-}$ with $\to$, and vice-versa.

If $\mu : U \to U$ is a closure operator, the operator $\lambda : U \to U$ defined as $\lambda(x) = \neg\mu(\neg x)$ is an interior operator, and the analogous fact holds by exchanging the role of $\mu$ and $\lambda$.

For an algebraic study of interior algebras, see, for instance, [3].

## 3.2 GBL-pair representation of finite GBL-algebras

In this section we introduce GBL-algebras, we recall some basic facts about commutative bounded GBL-algebras and the representation of finite GBL-algebras as GBL-pairs. We do not prove most of the statements in this section. For the algebraic aspects, we refer the reader to [32, 21, 19] for a general treatment of the subject, [29, 30] (especially for the finite case), [31] for results about representation of GBL-algebras, and the handbook [20]. For the representation of finite GBL-algebras as GBL-pairs, we refer to [14].

**Definition 3.2.1.** *A* commutative bounded GBL-algebra *is a structure*

$$(X, \vee, \odot, \rightarrow, 1, 0)$$

*of type* $(2, 2, 2, 0, 0)$ *such that* $(X, \odot, 1)$ *is a commutative monoid, the following identities hold:*

$a \rightarrow a = 1$*;*

$a \odot (a \rightarrow b) = b \odot (b \rightarrow a)$*;*

$(a \odot b) \rightarrow c = b \rightarrow (a \rightarrow c)$*,*

*and* $(X, \wedge, \vee, 1, 0)$ *is a lattice, where* $a \wedge b = a \odot (a \rightarrow b)$*.*

It can be proven that, for every GBL-algebra $X$, the lattice $(X, \wedge, \vee, 1, 0)$ is distributive.

In every GBL-algebra $X$, for all $a, b \in X$,

$$a \rightarrow b = \max\{x \in X \mid a \odot x \leq b\}.$$

This property shows that $\rightarrow$ is the *residuum* of $\odot$.

GBL-algebras have a weaker axiomatization, which does not require commutativity and bound. However, as shown in [29], all finite GBL-algebras are commutative (and they are obviously bounded). In the sequel, we are mainly concerned with finite GBL-algebras, and by "GBL-algebra" we will always mean "commutative bounded GBL-algebra".

**Remark 3.2.1.** *We choose to axiomatize GBL-algebras as in Definition 3.2.1 in order to rely on the definition of* hoop. *A hoop [20] is a structure* $(X \odot, \rightarrow, 1)$ *of type* $(2, 2, 0)$ *where* $(X, \odot, 1)$ *is a commutative monoid and the following identities hold:*

$a \rightarrow a = 1$*;*

$a \odot (a \rightarrow b) = b \odot (b \rightarrow a)$*;*

$(a \odot b) \rightarrow c = b \rightarrow (a \rightarrow c)$*.*

*Every hoop* $X$ *has a meet-semilattice reduct, and it has a lattice reduct if and only if* $X$ *is the join-free reduct of a GBL-algebra [20, 19].*

In the sequel it will be sometimes convenient to extend the signature and define a GBL-algebra as a structure

$$(X, \wedge, \vee, \odot, \rightarrow, 1, 0)$$

of type $(2, 2, 2, 2, 0, 0)$ such that $(X, \odot, 0)$ is a commutative monoid, $(X, \vee, \wedge, 0, 1)$ is a lattice and the following identities hold:

$a \to a = 1$;

$a \odot (a \to b) = a \wedge b$;

$(a \odot b) \to c = b \to (a \to c)$.

We now briefly introduce idempotent elements in GBL-algebras.

**Definition 3.2.2.** *An element $a$ of a GBL-algebra $X$ is* idempotent *if $a \odot a = a$.*

We denote by $I(X)$ the poset of idempotents of $X$, endowed with the order relation induced by $X$.

**Proposition 3.2.1.** *Let $X$ be a finite GBL-algebra. Then:*

*$I(X)$ is closed with respect to $\odot, \vee, \to, 1, 0$;*

*$a \odot b = a \wedge b$ for all $a \in I(X)$, $b \in X$;*

*$(I(X), \wedge, \vee, \to, 1, 0)$ is a* Heyting algebra.

**Remark 3.2.2.** *If all elements of $X$ are idempotents, then $X$ coincides with $I(X)$, and it is a Heyting algebra. On the other hand, every Heyting algebra $H$ is a commutative bounded GBL-algebra by setting $a \odot b = a \wedge b$ for all $a, b \in H$. Thus, Heyting algebras are special cases of commutative bounded GBL-algebras, and finite Heyting algebras are special cases of finite GBL-algebras.*

*If in $X$ it holds the further identity (*prelinearity*) $(a \to b) \vee (b \to a) = 1$, then $X$ is a* Basic Logic algebra *(shortly, BL-algebra). As we have mentioned, GBL-algebras were originally defined as generalizations of BL-algebras where prelinearity does not hold. If $X$ is both a BL-algebra and a Heyting algebra, it is called* Gödel algebra *[10, 26].*

*The commutative integral GBL-algebras which satisfy the identity $(a \to b) \to b = (b \to a) \to a$ are term-equivalent to MV-algebras.*

We now introduce a class of structures obtained from Heyting algebras and equivalence relations on them.

Let $H$ be a Heyting algebra and $G$ be a subgroup of the automorphism group of $H$.

For $a, b \in H$ we write $a \sim b$ if there is $f \in G$ such that $f(a) = b$. It is easy to verify that $\sim$ is an equivalence relation on $H$.

We write $[a]$ for the equivalence class of $a$, and $H/G$, for the the quotient modulo the equivalence relation $\sim$.

In a similar way as we did for congruences on D-posets, we endow $H/G$ with a relation, writing $[a] \leq [b]$ if there are $x \in [a]$ and $y \in [b]$ with $x \leq y$ in $H$. It can be easily verified that $(H/G, \leq)$ is a partially ordered set.

**Definition 3.2.3.** *Let $H$ be a Heyting algebra and $G$ be a group of automorphisms of $H$.*

*We say that $G$ is* chain-transitive *if, for every pair of sequences $a_1 \leq a_2 \leq \ldots \leq a_k$ and $b_1 \leq b_2 \leq \ldots \leq b_k$ with $a_i \sim b_i$ for all $i$, $1 \leq i \leq k$, there is $f \in G$ such that $f(a_i) = b_i$ for all $i$, $1 \leq i \leq k$.*

In other words, $G$ is chain-transitive when every two chains of elements equivalent in pairs are equivalent with respect to one single automorphism.

In [14] the following structures have been introduced.

**Definition 3.2.4.** *A* GBL-pair *is a pair $(H, G)$, where $H$ is a Heyting algebra and $G$ is a chain-transitive subgroup of the automorphism group of $H$.*

**Remark 3.2.3.** *The definition of GBL-pair recalls the one of MV-pair in [28] and [13]. In those works, however, the definition applies to Boolean algebras, and an analogue of the condition of chain-transitivity is given for chains of two elements.*

Regarding $H$ as a model of intuitionistic propositional logic, the relation $\sim$ can be thought of as an *indiscernibility relation* between propositions. This relation, indeed, can be interpreted as follows: a subject is not able to discern among certain classes of propositions, due to vagueness or ambiguity of the language. Note that the indiscernibility in our definition is a particular case of the *indiscernibility relation on a poset* of [12]. The fact that the equivalence relation is given by a group of automorphisms can be interpreted as follows: when two propositions are equivalent, they are also equivalent with respect to their relation with the rest of the model. In other words, if the subject is not able to discern between two propositions, the role of these propositions in the logic model cannot be discerned either. Chain transitivity can be interpreted as follows: if two chains of deductions in $H$ are step-by-step equivalent by the action of $G$, then they are equivalent as a whole. That is, the subject is not able to discern between two *lines of reasoning* made of indiscernible steps.

**Remark 3.2.4.** *Chain-transitivity prevents two strictly comparable elements $a < b$ in $H$ to be equivalent. Indeed, suppose that $a \sim b$, besides the trivial $a \sim a$. Then, since $a \leq a$, $a \leq b$, by chain-transitivity there would be an automorphism $f \in G$ such that $f(a) = a$ and $f(a) = b$, which cannot hold.*

In [14] it was proven that, for a GBL-pair $(H, G)$ with $H$ finite, the quotient $H/G$ can be endowed with the structure of GBL-algebra (whence the term "GBL-pair" comes from), and the operations between classes can be expressed as minima and maxima of sets of classes.

**Proposition 3.2.2.** *Let $(H, G)$ be a GBL-pair, where $H$ is a finite Heyting algebra. The following sets exist in $H/G$:*

$$[a] \wedge [b] = \max\{[x \wedge y] \mid x \sim a, y \sim b\};$$

$$[a] \vee [b] = \min\{[x \vee y] \mid x \sim a, y \sim b\};$$

$$[a] \odot [b] = \min\{[x \wedge y] \mid x \sim a, y \sim b\};$$

$$[a] \to [b] = \max\{[x \to y] \mid x \sim a, y \sim b\}.$$

*The structure* $(H/G, \wedge, \vee, \odot, \to, [1], [0])$ *is a GBL-algebra.*

**Remark 3.2.5.** *Proposition 3.2.2 provides an interpretation of logical operations as minima or maxima of sets of classes. In this context,* $\odot$ *can be viewed as a* strong conjunction *while* $\wedge$ *is a* weak conjunction. *As* $\to$ *is the adjoint of* $\odot$*, it can be viewed as a* strong implication*, while* $\vee$ *is a* weak disjunction. *To complete the set of operations, we can define two other operations, a* strong disjunction $\oplus$ *and a* weak implication $\Rightarrow$*:*

$$[a] \oplus [b] = \max\{[x \vee y] \mid x \sim a, y \sim b\};$$

$$[a] \Rightarrow [b] = \min\{[x \to y] \mid x \sim a, y \sim b\}.$$

*The operation* $\oplus$ *will be considered again in Section 3.4. We do not investigate further* $\Rightarrow$*: just note that it is not, as one may expect, the residuum of* $\wedge$*.*

The final result of [14] is a representation of finite GBL-algebras as GBL-pairs.

**Theorem 3.2.1.** *Let* $X$ *be a finite GBL-algebra. Then, there is a GBL-pair* $(H, G)$ *such that* $(X, \vee, \odot, \to, 1, 0)$ *is isomorphic to* $(H/G, \vee, \odot, \to, [1], [0])$.

When $H$ is a Boolean algebra, Proposition 3.2.2 recalls Proposition 2.4.1, which concerned sectional D-congruences. Indeed, it is shown in [14] that, when $H$ is a finite Boolean algebra, $H/G$ is an MV-algebra. Here we show that the equivalence relation induced by $G$ on a Boolean algebra $H$ in a GBL-pair $(H, G)$ is a clopen D-congruence.

**Proposition 3.2.3.** *Let* $(H, G)$ *be a GBL-pair, where* $H$ *is a Boolean algebra. Then, the equivalence relation* $\sim$ *induced by* $G$ *is a clopen D-congruence.*

*Proof.* Let $a, b, c, d \in H$. If $a \leq b$, $c \leq d$, $a \sim c$ and $b \sim d$, then, by chain-transitivity, there is $f \in G$ such that $f(a) = c$ and $f(b) = d$. Since $g$ is an automorphism of $H$, it preserves the difference, therefore $f(b \smallsetminus a) = f(b) \smallsetminus f(a) = d \smallsetminus c$, hence $(b \smallsetminus a) \sim (d \smallsetminus c)$ and $(A1)$ of Definition 2.2.1 holds.

If $a \leq b$ and $b \sim d$, then there is $f \in G$ such that $f(b) = d$. If we set $c = f(a)$, then $c \sim a$ and $c \leq d$, therefore $(A2)$ of Definition 2.2.1 holds. □

## 3.3 Dual GBL-algebras

In this section we introduce a class of algebraic structures, *dual GBL-algebras*, which are the structures obtained by GBL-algebras by reversing the order relation, and changing the binary operations $\odot$ with $\oplus$ and $\to$ with $\oslash$. For definitions and results about dual GBL-algebras we refer to the analogous ones about GBL-algebras in Section 3.2 and its references. Here we choose to work with the dual of GBL-algebras, because our results involve S-operators, that we introduced with respect to an operation of difference analogous to the operation $\ominus$ of dual GBL-algebras.

**Definition 3.3.1.** *A* dual GBL-algebra *is a structure*

$$(X, \wedge, \oplus, \ominus, 0, 1)$$

*of type* $(2, 2, 2, 0, 0)$ *such that* $(X, \oplus, 0)$ *is a commutative monoid, the following identities hold:*

$a \ominus a = 0$*;*

$(b \ominus a) \oplus a = (a \ominus b) \oplus b$*;*

$c \ominus (a \oplus b) = (c \ominus a) \ominus b$*,*

*and* $(X, \vee, \wedge, 0, 1)$ *is a lattice, where* $a \vee b = (b \ominus a) \oplus a$*.*

It can be proven that, for every dual GBL-algebra $X$, the lattice $(X, \vee, \wedge, 0, 1)$ is distributive, and, for all $a, b \in X$,

$$b \ominus a = \min\{x \in X \mid b \leq a \oplus x\}.$$

We say that $a \in X$ is idempotent if $a \oplus a = a$. The bounds $0$ and $1$ are idempotent in every dual GBL-algebra. If all $a \in X$ are idempotent, then $a \oplus b = a \vee b$ for all $a, b \in X$, and $(X, \vee, \wedge, 0, 1)$ is a Brouwerian algebra.

Every MV-algebra is a dual GBL-algebra, as well as a GBL-algebra, by De Morgan's duality.

For these and other results about dual GBL-algebras, see the analogous ones about GBL-algebras, in Section 3.2 and the relative references.

**Remark 3.3.1.** *It is more convenient for the sequel to extend the signature and define a GBL-algebra as a structure*

$$(X, \vee, \wedge, \oplus, \ominus, 0, 1)$$

*of type* $(2, 2, 2, 2, 0, 0)$ *such that* $(X, \oplus, 0)$ *is a commutative monoid,* $(X, \vee, \wedge, 0, 1)$ *is a lattice and the following identities hold:*

*(G1)* $a \ominus a = 0$;

*(G2)* $(b \ominus a) \oplus a = a \vee b$;

*(G3)* $c \ominus (a \oplus b) = (c \ominus a) \ominus b$.

In this section we show how the combination of an S-operator and a closure operator can give rise to a dual GBL-algebra.

We will always assume S-operators to be strict. The results could be generalized to S-operators not necessarily strict, but the proofs would be longer and the notation more cumbersome.

**Definition 3.3.2.** *Let $X$ be a set and $f, g : X \to X$ be two functions. We say that $f$ and $g$* commute *if $gf = fg$. We say that the $f$ and $g$* strongly commute *if they commute and, for all $a, b \in X$ such that $gf(a) = fg(b)$, there is $c \in X$ such that $f(c) = f(a)$ and $g(c) = g(b)$.*

Let now $U$ be a Boolean algebra, $\sigma : U \to U$ be a strict S-operator and $\mu : U \to U$ be a closure operator.

We will consider only the equivalence relation induced by $\sigma$, and not that induced by $\mu$. Thus, we say that $a$ and $b$ are equivalent, and we write $a \sim b$, whenever $\sigma(a) = \sigma(b)$.

The following lemma shows that, if a strict S-operator and a closure operator commute, then all the elements equivalent to a closed element are closed.

**Lemma 3.3.1.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be a strict S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ commute. For all $a, b \in U$, if $a \sim b$ and $b$ is closed, then $a$ is closed.*

*Proof.* Let us suppose that $a$ is not closed. This means that $a < \mu(a)$, hence, since $\sigma$ is strict, $\sigma(a) < \sigma\mu(a)$ and $\sigma(a) < \mu\sigma(a)$ by commutation. Since $a \sim b$ and $b$ is closed, it holds $\sigma(a) = \sigma\mu(b)$ and, by the previous inequality and commutation,

$$\mu\sigma(b) = \sigma\mu(b) < \mu\sigma\mu(b) = \mu\sigma(b),$$

which cannot hold. We conclude that $a$ is closed. □

The following lemma shows that, under the hypothesis of strong commutation, $\mu$ distributes over the operation $\oplus$ of the MV-algebra $\sigma(U)$. This property generalizes the distributivity of a closure operator over joins.

**Lemma 3.3.2.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be a strict S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ commute. Then, for all $a, b \in U$,*

$$\mu(\sigma(a) \oplus \sigma(b)) = \mu\sigma(a) \oplus \mu\sigma(b),$$

*where $\oplus$ is taken in the MV-algebra $\sigma(U)$.*

*Proof.* By Proposition 2.4.2 we have

$$\sigma(a) \oplus \sigma(b) = \max\{\sigma(x \vee y) \mid \sigma(x) = \sigma(a), \sigma(y) = \sigma(b)\},$$

hence $\sigma(a) \oplus \sigma(b) = \sigma(a' \vee b')$ for some $a' \sim a$ and $b' \sim b$. Then,

$$\mu(\sigma(a) \oplus \sigma(b)) = \mu\sigma(a' \vee b') = \sigma\mu(a' \vee b') = \sigma(\mu(a') \vee \mu(b')),$$

where the last equality follows from the closure property. Therefore, by monotonicity of $\mu$,

$$\mu(\sigma(a) \oplus \sigma(b)) = \max\{\sigma(\mu(a') \vee \mu(b')) \mid a' \sim a, b' \sim b\}.$$

On the other hand,

$$\sigma\mu(a) \oplus \sigma\mu(b) = \max\{\sigma(x \vee y) \mid x \sim \mu(a), y \sim \mu(b)\}.$$

The equality in (2) follows from the fact that, for all $a \in U$, the sets $\{x \mid x \in U, x \sim \mu(a)\}$ and $\{\mu(a') \mid a' \in U, a' \sim a\}$ are equal. Indeed, clearly the second set is included in the first. On the other hand, from Lemma 3.3.1 it follows that, if $x \sim \mu(a)$, then $x = \mu(y)$ for some $y \in U$. Then, $\sigma\mu(y) = \mu\sigma(a)$ and, by strong commutation, there is $a' \in U$ such that $\mu(a') = \mu(y)$ and $\sigma(a') = \sigma(a)$. Therefore, $x = \mu(a')$ for an $a' \sim a$, hence every $x$ in the first set is also an element of the second. □

Let $U$ be a Boolean algebra, $\sigma : U \to U$ be a strict S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$, and $\mu$ strongly commute. Let $\eta = \mu\sigma = \sigma\mu$. We set, for all $a, b \in U$,

$\eta(a) \boxplus \eta(b) = \sigma\mu(a) \oplus \sigma\mu(b)$, with $\oplus$ in the MV-algebra $\sigma(U)$;

$\eta(b) \boxminus \eta(a) = \mu(\sigma(b) \ominus \sigma(a))$, with $\ominus$ in the MV-algebra $\sigma(U)$.

Clearly, since $\sigma$ and $\mu$ are both idempotent, also $\eta$ is idempotent, that is $\eta\eta(a) = \eta(a)$ for all $a \in U$.

The following lemma shows that $\eta(U)$ is closed by lattice operations, by $\boxplus$ and $\boxminus$.

**Lemma 3.3.3.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be an S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ strongly commute. Let $\eta = \mu\sigma = \sigma\mu$. Then, for all $a, b \in U$,*

*(1) There is $c \in U$ such that $\eta(a) \vee \eta(b) = \eta(c)$;*

*(2) There is $c \in U$ such that $\eta(a) \wedge \eta(b) = \eta(c)$;*

*(3) There is $c \in U$ such that $\eta(a) \boxplus \eta(b) = \eta(c)$;*

*(4) There is $c \in U$ such that $\eta(b) \boxminus \eta(a) = \eta(c)$.*

*Proof.* (1) Since $\mu$ distributes over joins, $\eta(a) \vee \eta(b) = \mu(\sigma(a) \vee \sigma(b))$, and $\sigma(a) \vee \sigma(b) = \sigma(c)$ for some $c \in U$, because $\sigma(U)$ is closed by join.

(2) Since both $\sigma(U)$ and $\mu(U)$ are closed by $\wedge$, $\eta(a) \wedge \eta(b) = \sigma(x) = \mu(y)$ for some $x, y \in U$. This means that $\mu(y) \sim x$, and by Lemma 3.3.1 $x$ is closed, that is $x = \mu(x)$, and we conclude $\eta(a) \wedge \eta(b) = \sigma\mu(x) = \eta(x)$.

(3) By commutation, distributivity of $\mu$, and since $\sigma(U)$ is closed by $\oplus$,

$$\eta(a) \boxplus \eta(b) = \mu\sigma(a) \oplus \mu\sigma(b) = \mu\sigma(\sigma(a) \oplus \sigma(b)),$$

hence $\eta(a) \boxplus \eta(b)$ is of the form $\eta(c)$ for some $c \in U$.

(4) Since $\sigma(U)$ is closed by $\ominus$,

$$\eta(b) \boxminus \eta(a) = \mu(\sigma(b) \ominus \sigma(a)) = \mu\sigma(\sigma(b) \ominus \sigma(a)),$$

hence $\eta(b) \boxminus \eta(a)$ is of the form $\eta(c)$ for some $c \in U$. □

**Theorem 3.3.1.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be an S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ strongly commute. Let $\eta = \mu\sigma = \sigma\mu$. Then,*

$$(\eta(U), \vee, \wedge, \boxplus, \boxminus, 0, 1)$$

*is a dual GBL-algebra.*

*Proof.* By Lemma 3.3.3, $\eta(U)$ is closed by $\boxplus$, therefore $(\eta(U), \boxplus, 0)$ is a monoid. Commutativity of this monoid comes from the commutativity of $\oplus$ in the MV-algebra $\sigma(U)$, and $0$ is the neutral element of the monoid and it is $0 = \eta(0)$.

By Lemma 3.3.3, $\eta(U)$ is closed by $\vee$ and $\wedge$, and it contains $0 = \eta(0)$ and $1 = \eta(1)$. Therefore, $(\eta(U), \vee, \wedge, 0, 1)$ is a sublattice of $U$.

By Lemma 3.3.3, $\eta(U)$ is also closed by $\boxminus$.

We now verify $(G1)$, $(G2)$ and $(G3)$ of Remark 3.3.1.

$(G1)$ It holds $\eta(a) \boxminus \eta(a) = \mu(\sigma(a) \ominus \sigma(a)) = \mu(0) = 0$.

$(G2)$ It holds

$$(\eta(b) \boxminus \eta(a)) \boxplus \eta(a) = \mu(\sigma(b) \ominus \sigma(a)) \boxplus \mu\sigma(a),$$

and this is equal, by Lemma 3.3.2, to $\mu((\sigma(b) \ominus \sigma(a)) \boxplus \sigma(a))$. Then, by definition of MV-algebra, by Lemma 3.3.2 and the closure property, we have

$$(\eta(b) \boxminus \eta(a)) \boxplus \eta(a) = \mu(\sigma(a) \vee \sigma(b)) = \eta(a) \vee \eta(b).$$

$(G3)$ By Lemma 3.3.2 and by definition of $\boxplus$ and $\boxminus$,

$$\eta(c) \boxminus (\eta(a) \boxplus \eta(b)) = \mu(\sigma(c) \ominus (\sigma(a) \oplus \sigma(b))).$$

By definition of MV-algebra, this is equal to $\mu((\sigma(c) \ominus \sigma(b)) \ominus \sigma(a))$, hence

$$\eta(c) \boxminus (\eta(a) \boxplus \eta(b)) = \mu(\sigma(c) \ominus \sigma(a)) \boxminus \eta(b) = (\eta(c) \boxminus \eta(a)) \boxminus \eta(b).$$

□

If $\mu$ in Theorem 3.3.1 is the identity, then $\eta = \sigma$, and the theorem reduces to Corollary 2.3.2, hence $\eta(U)$ is an MV-algebra. On the other hand, if $\sigma$ is the identity, then $\eta = \mu$, and the theorem reduces to Theorem 3.1.1, hence $\eta(U)$ is a Brouwerian algebra.

We now show a simple, non-degenerate example.

**Example 3.3.1.** *Let $U$ be the powerset of the three-element set $\{a, b, c\}$, with the usual order by set inclusion and set difference, which is a Boolean algebra. Let $\sigma : U \to U$ be defined as follows:*
$\sigma(\varnothing) = \varnothing$;
$\sigma(\{a\}) = \{a\}$;
$\sigma(\{b\}) = \sigma(\{c\}) = \{b\}$;
$\sigma(\{a, b\}) = \sigma(\{a, c\}) = \{a, b\}$;
$\sigma(\{b, c\}) = \{b, c\}$;
$\sigma(\{a, b, c\}) = \{a, b, c\}$.
*Let $\mu : U \to U$ be defined as follows:*
$\mu(\varnothing) = \varnothing$;
$\mu(\{a\}) = \{a\}$;
$\mu(\{b\}) = \mu(\{a, b\}) = \{a, b\}$;
$\mu(\{c\}) = \mu(\{a, c\}) = \{a, c\}$;
$\mu(\{b, c\}) = \mu(\{a, b, c\}) = \{a, b, c\}$.
*The example is small enough to allow a direct verification of the hypothesis of Theorem 3.3.1. The resulting dual GBL-algebra is $(X, \vee, \wedge, \oplus, \ominus, 0_X, 1_X)$ on the chain*

$$X = \{\varnothing, \{a\}, \{a, b\}, \{a, b, c\}\},$$

*where $\varnothing = 0_X$, , $\{a, b, c\} = 1_X$, $\{a\}$ is an idempotent element and $\{a, b\} \oplus \{a, b\} = 1_X$.*

Theorem 3.3.1 can be reformulated, mutatis mutandis, in terms of GBL-algebras instead of dual GBL-algebras, and interior operators instead of closure operators.

## 3.4 GBL*algebras and dual GBL*algebras

In [15] a new class of algebras has been introduced, defining on GBL-algebras a further operation, which generalizes the monoidal sum of MV-algebras.

**Definition 3.4.1.** *We call* GBL*algebra *a structure*

$$(X, \wedge, \vee, \odot, \oplus, 1, 0)$$

*of type* $(2, 2, 2, 2, 0, 0)$ *such that* $(X, \wedge, \vee, \odot, \rightarrow, 1, 0)$ *is a GBL-algebra,* $(X, \oplus, 0)$ *is a commutative monoid and* $\oplus$ *satisfies the following identities:*

$$a \rightarrow (a \oplus b) = 1;$$

$$(a \oplus b) \rightarrow b = a \rightarrow (a \odot b).$$

The class of GBL*algebras has as its subclasses both MV-algebras and Heyting algebras. Every MV-algebra is a GBL*algebra with $\oplus$ as the usual monoidal sum. As shown in [15], $a \oplus b = a \vee b$ for all $a, b \in I(X)$. Thus, every Heyting algebra is a GBL*algebra with $a \oplus b = a \vee b$ (in particular this equality holds for every Gödel algebra).

In a GBL*algebra, in general,

$$a \oplus b \neq \neg(\neg a \odot \neg b),$$

and the equality holds only if the GBL*algebra is an MV-algebra. This makes $\oplus$ different from the operation $\oplus'$ defined as

$$a \oplus' b = \neg(\neg a \odot \neg b),$$

introduced in [39] for BL-algebras. The operations $\oplus$ and $\oplus'$ coincide only in MV-algebras.

The following result was proven in [15].

**Proposition 3.4.1.** *Let* $(X, \wedge, \vee, \odot, \rightarrow, 1, 0)$ *be a finite GBL-algebra. Then, there is a unique operation* $\oplus$ *such that* $(X, \wedge, \vee, \odot, \oplus, \rightarrow, 1, 0)$ *is a GBL*algebra.*

Since, on the other hand, every GBL*algebra satisfies the identities of a GBL-algebra, Proposition 3.4.1 implies that there is a bijection between finite GBL-algebras and finite GBL*algebras.

In a finite GBL*algebra, the operation $\vee$ can be expressed in terms of the other GBL-algebra operations and $\oplus$, as stated in the following proposition, proven in [15].

**Proposition 3.4.2.** *Let* $a, b \in X$ *be two elements of a finite GBL-algebra. Then,*

$$a \vee b = ((a \rightarrow b) \rightarrow b) \wedge (a \oplus b).$$

By commutativity of $\vee$, it is also $a \vee b = ((b \rightarrow a) \rightarrow a) \wedge (a \oplus b)$.

As mentioned before, every hoop has a meet-semilattice reduct, and it has a lattice reduct if and only if it is the join-free reduct of a GBL-algebra. Every finite meet-semilattice is complete, hence it is the join-free reduct of a lattice (in a unique way). Then, every finite hoop is the join-free reduct of a unique GBL-algebra, and if we add the axioms relative to the operation $\oplus$, the join operation can be removed from the language, by Proposition 3.4.2. Therefore, we have the following proposition.

**Proposition 3.4.3.** *Let* $(X, \odot, \oplus, \rightarrow, 1, 0)$ *be a structure such that* $(X, \odot, 1)$ *and* $(X, \oplus, 0)$ *are commutative monoids and the following identities hold:*

$$a \rightarrow (a \oplus b) = 1;$$

$$a \odot (a \rightarrow b) = b \odot (b \rightarrow a);$$

$$(a \odot b) \rightarrow c = b \rightarrow (a \rightarrow c);$$

$$(a \oplus b) \rightarrow b = a \rightarrow (a \odot b).$$

*Let* $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge (a \oplus b)$ *and* $a \wedge b = a \odot (a \rightarrow b)$. *Then,*

$$(X, \wedge, \vee, \odot, \oplus, \rightarrow, 1, 0)$$

*is a GBL*algebra.*

Notice, however, that Proposition 3.4.2 was proven in [15] relying on a representation theorem in [30], valid only for finite GBL-algebras. This means that, for a generic GBL*algebra, we cannot (as far as we know) omit $\vee$ from the signature.

One of the motivations for introducing GBL*algebras in [15] was to have a notion of state of GBL-algebras. In [15] a *state* of GBL*algebra is defined as follows.

**Definition 3.4.2.** *Let* $X$ *be a GBL*algebra. A* state *of* $X$ *is a function* $s : X \rightarrow [0, 1]$ *which satisfies*

- normalization*:* $s(0) = 0$ *and* $s(1) = 1$*;*
- monotonicity*: for all* $a, b \in L$ *with* $a \leq b$, $s(a) \leq s(b)$*;*
- additivity*: for all* $a, b \in L$, $s(a \oplus b) + s(a \odot b) = s(a) + s(b)$.

If $X$ is an MV-algebras, the definitions of state of MV-algebra [37, 16] and state of GBL-algebra coincide. If $X$ is a Heyting algebra, states of $X$ are valuations to $[0, 1]$ on $L$ as a distributive lattice. In particular, if $X$ is a Gödel algebra, states of $X$ as a GBL*algebra are the same as states of $X$ as a Gödel algebra, as they are defined in [1].

If $X$ is the GBL-algebra reduct of a GBL*algebra, then we say that a function $s : X \rightarrow [0, 1]$ is a state of $X$ if it is a state of $X$ as a GBL*algebra.

In [15] the following result is proven.

**Theorem 3.4.1.** *Let $X$ be a finite GBL-algebra and $s : X \to [0,1]$ be a state of $X$. Then, the restriction of $s$ to $I(L)$ is a valuation (on the distributive lattice reduct of $I(X)$).*

*Conversely, every valuation $f : I(L) \to [0,1]$ can be extended to a state $s : L \to [0,1]$ in a unique way.*

Similarly as we introduced dual GBL-algebras as the algebraic structures order-dual to GBL-algebras, we now introduce dual GBL*algebras. In the definition of GBL*algebra, we exchange the order relation, the pair of operations $\wedge, \vee$ and $\odot, \oplus$, and we replace $\to$ with $\ominus$. We get to the following definition.

**Definition 3.4.3.** *We call* dual GBL*algebra *a structure*

$$(X, \vee, \wedge, \oplus, \odot, 0, 1)$$

*of type $(2,2,2,2,0,0)$ such that $(X, \oplus, 0)$ and $(X, \odot, 1)$ are commutative monoids, $(X, \vee, \wedge, 0, 1)$ is a lattice and the following identities hold:*

*(P1)* $(a \odot b) \ominus a = 0$;

*(P2)* $(b \ominus a) \oplus a = a \vee b$;

*(P3)* $c \ominus (a \oplus b) = (c \ominus a) \ominus b$;

*(P4)* $(a \oplus b) \ominus b = a \ominus (a \odot b)$

Let now $U$ be a Boolean algebra, $\sigma : U \to U$ be a strict S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ strongly commute, as defined in Section 3.3. Let $\eta = \mu\sigma = \sigma\mu$. As in Section 3.3 we set, for all $a, b \in U$,

$\eta(a) \boxplus \eta(b) = \sigma\mu(a) \oplus \sigma\mu(b)$, with $\oplus$ in the MV-algebra $\sigma(U)$;

$\eta(b) \boxminus \eta(a) = \mu(\sigma(b) \ominus \sigma(a))$, with $\ominus$ in the MV-algebra $\sigma(U)$.

Furthermore, we set

$\eta(a) \boxdot \eta(b) = \sigma\mu(a) \odot \sigma\mu(b)$, with $\odot$ in the MV-algebra $\sigma(U)$.

We first show that $\eta(U)$ is closed by the newly defined operation $\boxdot$.

**Lemma 3.4.1.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be an S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ strongly commute. Let $\eta = \mu\sigma = \sigma\mu$. Then, for all $a, b \in U$, there is $c \in U$ such that $\eta(a) \boxdot \eta(b) = \eta(c)$.*

*Proof.* By Proposition 2.4.2 we have

$$\sigma\mu(a) \odot \sigma\mu(b) = \min\{\sigma(x \wedge y) \mid x \sim \mu(a), y \sim \mu(b)\}.$$

By Lemma 3.3.1, we have then

$$\sigma\mu(a) \odot \sigma\mu(b) = \min\{\sigma(\mu(a') \wedge \mu(b')) \mid \mu(a') \sim \mu(a), \mu(b') \sim \mu(b)\}.$$

Since $\mu(U)$ is closed by $\wedge$, the second term of equality is $\sigma\mu(c)$ for some $c \in U$, therefore $\eta(a) \boxdot \eta(b) = \eta(c)$. □

We can strengthen the statement of Theorem 3.3.1 as follows.

**Theorem 3.4.2.** *Let $U$ be a Boolean algebra, $\sigma : U \to U$ be an S-operator and $\mu : U \to U$ be a closure operator such that $\sigma$ and $\mu$ strongly commute, as defined in Section 3.3. Let $\eta = \mu\sigma = \sigma\mu$. Then,*

$$(\eta(U), \vee, \wedge, \boxplus, \boxdot, \boxminus, 0, 1)$$

*is a dual GBL*algebra.*

*Proof.* By Theorem 3.3.1 we know that $\eta(U)$ is a dual GBL-algebra, hence the structure $(\eta(U), \vee, \wedge, 0, 1)$ is a lattice, $(\eta(U), \boxplus, 0)$ is a commutative monoid and $(P2)$ and $(P3)$ of Definition 3.4.3 hold, since they are the same as, respectively, $(G2)$ and $(G3)$ of Definition 3.3.1.

The identity $(P1)$ for $\boxdot$ is stonger than $(G1)$, but it easily follows from the monotonicity of $\odot$ in $\sigma(U)$.

Further, by Lemma 3.4.1, $(\eta(U), \boxdot, 1)$ is a submonoid of $(\sigma(U), \boxdot, 1)$, hence it is a commutative monoid.

It remains to prove $(P4)$. By Lemma 3.3.2,

$$(\eta(a) \boxplus \eta(b)) \boxminus \eta(b) = \mu((\sigma(a) \oplus \sigma(b)) \ominus \sigma(b)).$$

This is equal to $\mu(\sigma(a) \ominus (\sigma(a) \odot \sigma(b)))$, since in every MV-algebra the identity $(x \oplus y) \ominus y = x \ominus (x \odot y)$ holds. By idempotence of $\mu$, for every $x \in U$ it holds $\eta(x) = \sigma\mu(x) = \sigma\mu\mu(x) = \eta\mu(x)$. Therefore

$$(\eta(a) \boxplus \eta(b)) \boxminus \eta(b) = (\eta\mu(a) \boxplus \eta\mu(b)) \boxminus \eta\mu(b),$$

and from our previous computation we obtain

$$(\eta(a) \boxplus \eta(b)) \boxminus \eta(b) = \mu(\eta(a) \ominus (\eta(a) \boxdot \eta(b))),$$

which is finally equal to $\eta(a) \boxminus (\eta(a) \boxdot \eta(b))$. To prove this last identity, observe that $\eta(a) \boxdot \eta(b)$ is of the form $\eta(c)$ for some $c \in U$ (since $\eta(U)$ is closed by $\boxdot$) and that

$$\mu(\sigma\mu(a) \ominus \sigma\mu(c)) = \eta\mu(a) \boxminus \eta\mu(c) = \eta(a) \boxminus \eta(c).$$

□

Similarly as for Theorem 3.3.1, Theorem 3.4.2 can be reformulated, mutatis mutandis, in terms of GBL*algebras instead of dual GBL*algebras, and interior operators instead of closure operators.

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