Universitá degli Studi dell'Insubria
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Flow-Induced Torsion

Ph.D. Thesis

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Introduction

The purpose of this thesis is to summarize and extend the connections between the geometry and the stability of dynamical systems generated by flows; the main results are presented through very general assumptions, without many restrictions on the classes of systems involed. Aiming to give clear connections between the possible *theoretical* and *numerical* approaches, the adopted formalism is kept as *simple* as possible throughout the chapters, showing directly or referring to explicitly computable recipes and approximations for almost all the involved quantities.

The main motivation for such an *hybrid* approach lies in the generally extreme complexity of the systems involved and can be justified by a two-fold consideration: the limitations of the present formal theory are unknown, following a naturally slow but obviously reliable evolution; on the other hand, since the widespread of efficient numerical techniques, the gap between numerical and theoretical results is increasingly thinner, requiring more and more care in the analysis and interpretation of the data.

Operatively, the present work is then *intentionally* loose on both the formal and computational sides of rigor, targeting the connections between what can be calculated by hands and what can be computed by machines. Along the reading, such deliberate lack of rigor can be possibly felt by both communities of mathematicians and mathematical physicists but, exactly for this reason, it should be regarded as a fair attempt to *spread* knowledge and suggestions. Our implicit hope is that, through a wider and more general view of the subject, each scientific community can get significant advantages in the own development of both sides of such a useful and beautiful theory.

Chapter 1

Preliminaries

1.1 The Phase Space

Throughout the chapters several conventions hold; the first of these is the geometry of *phase-space*, the space in which the dynamics takes place: we consider N-dimensional *smooth riemannian manifolds* $\mathcal{M}$, over which infinitesimal (tangent) distances can be measured through the *inner product* of tangent vectors induced by the *metric tensor*. Such 2-tensor, which we represent by its components $g_{\mu\nu}$, is symmetric and induces the duality between the upper-indexed components of *contravariant* vectors x^μ and the lower-indexed ones of *covariant* vectors, $x_\mu := g_{\mu\nu}x^\nu$. In doing this, we make use of the Einstein notation (sums over repeated indexes) and, if not otherwise specified, the Andrews convention:

- *greek* dummy indices label *coordinate* frames
- *latin* dummy indices label *local* frames.

The use of unbold upper/lowercase letters is reserved for scalar quantites 'a', bold lowercases for vectors '$\mathbf{a}$' and bold uppercases for matrices '$\mathbf{A}$'. In Euclidean phase-spaces $\mathcal{M} \subseteq \mathbb{R}^N$, the metric is represented by the $N \times N$ identity matrix $\mathbf{I}$ and co/contravariant vectors are equivalent.

1.1.1 An example

As an illustrative case, consider the length of a tangent vector $d\mathbf{x} \in T_\mathbf{x}\mathcal{M}$ attached to the phase-space point $\mathbf{x} \in \mathcal{M}$:

$$\|d\mathbf{x}\|^2 := dx_\mu dx^\mu = g_{\mu\nu}dx^\mu dx^\nu \quad . \tag{1.1}$$

When the ambient manifold is flat (i.e. $g_{\mu\nu} = 1$ iff $\mu = \nu$), the index *lowering* of a contravariant vector simply corresponds to transposing it from column to row; to implicitly mean this fact, in the following chapters the simpler notation will be used:

$$a_\mu b^\mu \equiv \mathbf{a} \cdot \mathbf{b} \tag{1.2}$$

for any $\mathbf{a}$, $\mathbf{b} \in T\mathcal{M}$, understood as either the index contraction by the metric or transposition.

1.2 A Flow

A more interesting example comes from the notion of *flows* associated to *vector-fields*: given a vector-field $\mathbf{u}(\mathbf{x})$, locally defined as a vector-valued function inside some compact subset of phase-space $\mathcal{U} \subset \mathcal{M}$, the associated flow is a one-parameter *family of curves* $\mathbf{g}^s(\mathbf{x})$ that are *tangent* to the vector-field in each point $\mathbf{x} \in \mathcal{U}$ for any value s in the open interval $\mathcal{V} :=]-\varepsilon, \varepsilon[$. This statement is summarized by locally defining the flow $\mathbf{g}^s : \mathcal{U} \times \mathcal{V} \to \mathcal{M}$ as the solution to an ordinary differential equation (ODE), that is, the Cauchy problem:

$$\tfrac{d}{ds}\mathbf{g}^s \;=\; \mathbf{u} \circ \mathbf{g}^s \quad , \quad \mathbf{g}^0 \;=\; \mathrm{id} \tag{1.3}$$

The constraint at $s = 0$ implies that, for each $\mathbf{x} \in \mathcal{U}$, the flow traces a path starting exactly at $\mathbf{x} \equiv \mathbf{g}^{(0)}(\mathbf{x})$; such ODE can be formally solved by associating to the vector-field $\mathbf{u}$ a suitable, purely *imaginary* differential operator $\hat{\mathbf{u}}$ acting on functions $f \in \mathcal{C}^1(\mathbb{R}^N)$:

$$\hat{\mathbf{u}} \;:=\; -i\, u^\mu \partial_\mu \quad \Rightarrow \quad \hat{\mathbf{u}}[f](\mathbf{x}) \;=\; i\, u^\mu(\mathbf{x})\partial_\mu f|_{\mathbf{x}} \tag{1.4}$$

with the imaginary unit multiplying what is usually meant for *vector-field* in the context of differential geometry; then one can write the formal solution of (1.3) as the *exponential* of the differential operator $\hat{\mathbf{u}}$, by making use of the usual Taylor expansion and $\hat{\mathbf{u}}^0 := \hat{\mathrm{id}}$:

$$\hat{\mathbf{g}}^s \;=\; \mathrm{e}^{is\hat{\mathbf{u}}} \;:=\; \sum_{q=0}^{\infty} \tfrac{1}{q!}\,(i\,s)^q\,\hat{\mathbf{u}}^q \quad ; \tag{1.5}$$

Once such exponential operator is applied to any point $\mathbf{x} \in \mathcal{U}$, chosen as *initial condition*, the functional dependence of the flow components is recovered as a power series in s:

$$(\mathbf{g}^s(\mathbf{x}))^\mu \;\equiv\; \mathrm{e}^{is\hat{\mathbf{u}}}\,[x^\mu] \;=\; x^\mu \;+\; s\,u^\mu(\mathbf{x}) \;+\; \tfrac{1}{2}\,s^2\,u^\nu(\mathbf{x})\partial_\nu u^\mu|_{\mathbf{x}} \;+\; \ldots \tag{1.6}$$

By the mutual independence of coordinates x^μ encoded by the relation $\partial_\nu x^\mu = \delta^\mu_\nu$, equation (1.6) is the infinte expansion obtained by all the higher order derivatives of equation (1.3) w.r.t. parameter s. Indeed, by considering a function $f \in \mathcal{C}^\infty(\mathcal{U})$ instead of x^μ, expansion (1.5) yields additional terms: these already appear by applying the square operator:

$$-\hat{\mathbf{u}}^2[f] \;=\; u^\nu\partial_\nu\,(u^\mu\partial_\mu f) \;=\; u^\nu(\partial_\nu u^\mu)\partial_\mu f \;+\; u^\nu u^\mu\,(\partial_\nu\partial_\mu f) \tag{1.7}$$

through the *second* derivatives of f; such term is identically zero once $f \equiv x^\sigma$ is chosen for some fixed σ, so that expansion (1.6) holds. By extending this argument, the application of

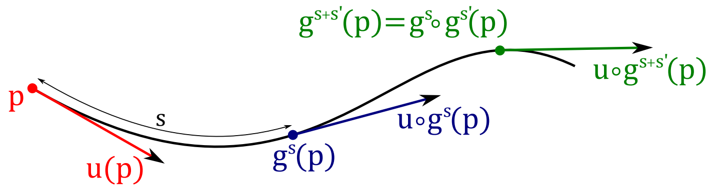

Figure 1.1: Schematic picture illustrating the action of a flow $\mathbf{g}^s$ generated by the vector-field $\mathbf{u}$ by starting at point $\mathbf{p}$; two intermediate points, at arc lengths s and $s + s'$, are shown toghether with the local direction of the normalized vector-field.

the full exponential operator $\hat{\mathbf{g}}^s$ leads to *compose* any $\mathcal{C}^\infty$ function f with the represented flow $\mathbf{g}^s$, in the sense of series expansions (1.5) and (1.6):

$$\hat{\mathbf{g}}^s[f] \;=\; f \circ \mathbf{g}^s \quad ; \tag{1.8}$$

in turn, this enables to check again the correctness of the formal solution (1.5):

$$\tfrac{d}{ds}\left(\hat{\mathbf{g}}^s[x^\mu]\right) \;=\; i\,\hat{\mathbf{u}}\,\hat{\mathbf{g}}^s[x^\mu] \;=\; i\,\hat{\mathbf{g}}^s\,\hat{\mathbf{u}}[x^\mu] \;=\; \hat{\mathbf{g}}^s[u^\mu] \;=\; u^\mu \circ \mathbf{g}^s \quad . \tag{1.9}$$

Here we made use of the fact that the operators $\hat{\mathbf{u}}$ and $\hat{\mathbf{g}}^s$ *commute*; we will come back later on this point, after giving the proper notions of *linearized* flow.

1.2.1 Two remarks

When the vector-field $\mathbf{u}$ is chosen to be everywhere normalized, $\|\mathbf{u}\| = 1\,,\, \forall\, \mathbf{x} \in \mathcal{U}$, the parameter s corresponds exactly to the *arc length* of the *integral curves* of $\mathbf{u}$ (equivalently called the *orbits* of the associated flow), as can be seen by the relation between curves at small parameter ds and their infinitesimal length dl:

$$dl \;:=\; \|d\mathbf{x}\| \;=\; \|\mathbf{u}\|\, ds \quad , \quad \|\mathbf{u}\| \;=\; 1 \;\Leftrightarrow\; ds \;=\; dl \quad . \tag{1.10}$$

In addition, the apparently redundant presence of the imaginary unit in the definition of operator $\hat{\mathbf{u}}$ is justified by the fact that it becomes a *self-adjoint operator* whenever two specific conditions are fullfilled; these can be derived by the notion of L^2 inner product between functions f, h in the (functional) Hilbert space $\mathcal{H}(\mathbb{C}, \mathcal{U})$ associated to subset $\mathcal{U}$:

$$\langle f|h\rangle \;:=\; \int_{\mathcal{U}} d\mu\,(f^* h) \quad ; \tag{1.11}$$

with * the complex conjugation and $d\mu$ a volume measure that is compatible with $\mathcal{M}$; if $\hat{\mathbf{u}}^\dagger$ is the adjoint operator for $\hat{\mathbf{u}}$, then $\langle \hat{\mathbf{u}}^\dagger f|h\rangle = \langle f|\hat{\mathbf{u}}h\rangle$, which can be expressed as:

$$\langle f|\hat{\mathbf{u}}h\rangle \;=\; \langle \hat{\mathbf{u}}f|h\rangle - i\left(\int_{\partial\mathcal{U}} d\mu|_{\partial\mathcal{U}}\,(f^* h)\, u^\mu n_\mu \;-\; \int_{\mathcal{U}} d\mu\,(f^* h)\,\mathrm{div}(\mathbf{u})\right) \tag{1.12}$$

Here $\partial\mathcal{U}$ is the boundary set of $\mathcal{U}$, $\mathbf{n}$ is the unit vector perpendicular to $\partial\mathcal{U}$ in each of its points, while the *divergence* of $\mathbf{u}$ is defined as $\mathrm{div}(\mathbf{u}) = \partial_\mu u^\mu$; from (1.12) we thus deduce that, in order to get $\mathbf{u}^\dagger = \mathbf{u}$ for all f, h over $\mathcal{U}$, two conditions must be sathisfied:

- vector-field $\mathbf{u}$ must be *divergence-free*, $\mathrm{div}(\mathbf{u}) = 0\ \forall\, \mathbf{x} \in \mathcal{U}$
- domain $\mathcal{U}$ must be *invariant* under the action of the associated flow, $\mathrm{e}^{is\hat{\mathbf{u}}}(\mathcal{U}) \equiv \mathcal{U}$.

The second condition implies that $u^\mu n_\mu = 0\ \forall\, \mathbf{x} \in \mathcal{U}$, so that $\mathbf{u}^\dagger = \mathbf{u}$ is self-adjoint for any pair f, h defined over the subset $\mathcal{U}$. This property of operators and subsets, respectively seen as *generators* and *domains* of a flow, is very important in many contexts of mathematical physics: if the total amount of some quantity that is *advected* by the flow is required to be globally preserved inside its domain, the flow generators must be self-adjoint over such domain. Whenever this condition is not fulfilled, balance between losses and gains can still be obtained, but the amount of quantity inside the domain won't be preserved by the flow. Moreover, having a self-adjoint operator over some domain implies a fundamental property for the associated flow as an operator itself: if $\hat{\mathbf{u}}$ is self-adjoint over $\mathcal{U}$, its exponential $\hat{\mathbf{g}}^s = \mathrm{e}^{is\hat{\mathbf{u}}}$ is automatically a *unitary* operator:

$$(\mathrm{e}^{is\hat{\mathbf{u}}})^{-1} \;=\; \mathrm{e}^{-is\hat{\mathbf{u}}} \;=\; (\mathrm{e}^{is\hat{\mathbf{u}}})^{\dagger} \quad . \tag{1.13}$$

This in turn assures that the action of the flow is a one-parameter *group action*, with the composite actions of the flow at different arc-lengths simply sum up for any pair $s,\, s'$:

$$\mathrm{e}^{is\hat{\mathbf{u}}}\,\mathrm{e}^{is'\hat{\mathbf{u}}} \;=\; \mathrm{e}^{i(s+s')\hat{\mathbf{u}}} \quad \forall\, s,\, s' \quad \sim \quad \mathbf{g}^{s+s'} \;=\; \mathbf{g}^{s} \circ \mathbf{g}^{s'} \tag{1.14}$$

and the flow can be extended to arbitrary arc-length values, including *negative* ones. This can also be true for non-unitary flows, but requires to be proven case by case. In the generic case, property (1.14) is fulfilled for positive arc-lengths only and the flow acts as a *semi-group*.

1.3 Local Parameters

Suppose now to have exactly N vector-fields $\mathbf{u}_j$, all normalized and *linearly independent* for any $\mathbf{x} \in \mathcal{U}$ (but not necessarily diverence-free); this means that we can choose such N-tuple of vector-fields as a basis to express any tangent vector in $T_\mathbf{x}\mathcal{U}$:

$$dx^\mu \; = \; u^\mu_j ds^j \quad \Leftrightarrow \quad d\mathbf{x} \; = \; \mathbf{u}_j(\mathbf{x}) ds^j \quad . \tag{1.15}$$

In a strictly local sense, one can be tempted to interpret the arc lengths s^j (associated to the flow generated by each operator $\hat{\mathbf{u}}_j$) as a new set of *local coordinates* for $\mathcal{U}$: organizing the latter inside a column vector $\mathbf{s}$ and the (column) vector-fields $\mathbf{u}_j$ inside a row vector, i.e a matrix $\mathbf{U} := [\mathbf{u}_1 .. \mathbf{u}_n]$ or $(\mathbf{U})^\mu_j := u^\mu_j$, we can rewrite (1.15) as:

$$d\mathbf{x} \; = \; \mathbf{U} d\mathbf{s} \quad . \tag{1.16}$$

Moreover, by exploiting the point-wise dependence of the local basis $\mathbf{U}$ induced by the flows, we can also consider the derivations w.r.t. the arc length parameters, $\frac{\partial}{\partial s^j} \; := \; \partial_j$:

$$\partial_j \; = \; u^\mu_j \partial_\mu \quad \Leftrightarrow \quad \nabla_\mathbf{s} \; = \; \mathbf{U} \nabla_\mathbf{x} \quad , \tag{1.17}$$

organizing the derivations in column vectors $\nabla_{\mathbf{x}/\mathbf{s}}$ in a fashion analogous to (1.16). The main consequence of this relation is that the local basis, induced by the generators $\mathbf{u}_j$, in turn induces a new metric tensor for the local parameters s^j; by inserting (1.15) in (1.1):

$$g_{\mu\nu} dx^\mu dx^\nu \; = \; \underbrace{(g_{\mu\nu} u^\mu_j u^\nu_k)}_{\eta_{jk}} ds^j ds^k \; = \; \eta_{jk} ds^j ds^k \quad . \tag{1.18}$$

The new metric tensor has thus the role to connect the local co- and contravariant vectors, $ds_j = \eta_{jk} ds^k$, so that infinitesimal lengths can be also measured by using local parameters.

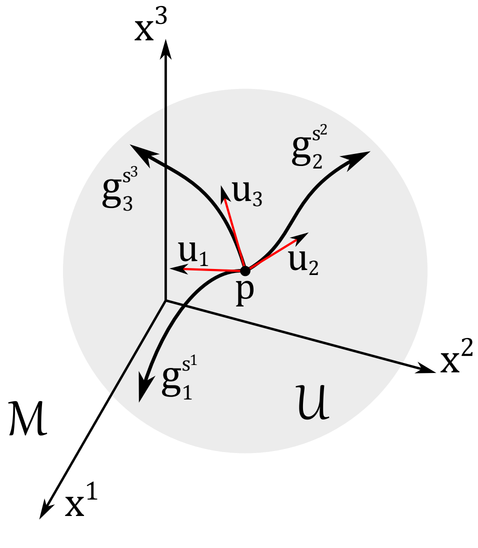

Figure 1.2: System of curvilinear parameters in a neighborhood $\mathcal{U}$ of point $\mathbf{p} \in \mathcal{M}$, induced by the flows $\mathbf{g}^{s^j}_j$ generated by vector-fields $\mathbf{u}_j$, a local basis for tangent space.

1.3.1 Dual Basis

By consequence of equation (1.18), through the definition of the *inverse* metric tensor η^{jk} such that $\eta^{jm}\eta_{mk} = \delta^j_k$, it is possible to define the *dual basis*:

$$w^j_\mu \;:=\; g_{\mu\nu}\eta^{jk}\,u^\nu_k \quad\Rightarrow\quad \begin{cases} w^j_\mu u^\mu_k \;=\; \delta^j_k \\ w^j_\mu u^\nu_j \;=\; \delta^\nu_\mu \end{cases} \quad . \tag{1.19}$$

Notice that covariant vectors are *linear functionals* that map usual (contravariant, or column) vectors to scalars, as is apparent from the position of the greek index in the r.h.s. of the last definition; this is natural in a differential framework but, for our discussions to follow, it would be more a source of confusion than an advange. For this reason we will always consider their contravariant counterparts, i.e. without lowered greek index: $\left(\mathbf{w}^j\right)^\mu = w^{j\mu} \;:=\; \eta^{jk}\,u^\mu_k$ and refer to these as *dual vectors*, with possible abuse of nomenclature but assuring their identification by keeping upper latin indexes.
The dual vectors $\mathbf{w}^j$ are thus orthogonal to any $\mathbf{u}_k$ with $k \neq j$, so that the first equation in (1.19) can be rewritten as $\mathbf{w}^j \cdot \mathbf{u}_k = \delta^j_k$; we immediately notice that, unlike vectors $\mathbf{u}_j$, the dual vectors *are not* normalized, as can be deduced from the usual relation:

$$\mathbf{w}^j \cdot \mathbf{u}_j \;=\; \|\mathbf{w}^j\|\,\|\mathbf{u}_j\|\cos(\alpha_j) \quad\Rightarrow\quad \|\mathbf{w}^j\| \;=\; \tfrac{1}{\cos(\alpha_j)} \quad . \tag{1.20}$$

In the case of flat phase-space, the above-mentioned relation between linear functionals w^j_μ (known also as 1-forms) and corresponding contravectors $w^{j\mu}$ simply coincides with transposition; thus, if the $\mathbf{u}_k$ are seen as columns of a matrix $\mathbf{U}$, the $\mathbf{w}^j$ correspond to the column vectors of the matrix $\mathbf{U}^{-T}$; this would be useful in deriving relations that are explicitly compatible with the numerical computation of such objects.
Finally, the meaning of the dual basis as a set of 1-forms can be understood in constructing the relations dual to (1.15) and (1.17), i.e. their versions with flipped indexes:

$$dx_\mu \;=\; w^j_\mu ds_j \quad , \quad \partial^j \;=\; w^j_\mu \partial^\mu \quad . \tag{1.21}$$

It should be noticed that the entire frame-work constructed so far involves *first-order* differential relations only; it turns out that this is not enough to allow the promotion of parameters s^j to *true* local coordinates. Indeed, such an operation requires to go up to *second-order* relations, at least; we can thus pass now into the details of *when* this can be effectively done, through the concept of commutativity of vector-fields.

1.4 Jacobian Matrices

The *Jacobian matrix* associated to any $\mathcal{C}^1$ vector-field $\mathbf{u}$ is defined as the square matrix $\mathbf{J_u}$ of the partial derivatives of each of its entries:

$$(\mathbf{J_u})^\mu_\nu \ := \ \partial_\nu u^\mu \quad . \tag{1.22}$$

by the convention adopted here, the row/column index refers respectively to the vector-field/gradient components. Two important properties characterize these matrices: the *trace* of the Jacobian matrix of a vector-field corresponds to its divergence:

$$\mathrm{Tr}(\mathbf{J_u}) \ := \ \partial_\mu u^\mu \ = \ \mathrm{div}(\mathbf{u}) \quad , \tag{1.23}$$

implying that the Jacobian matrix of a divergence-free vector-field is *trace-less*; furthermore, in presence of a *normalized* vector-field, one has also the following relation:

$$\|\mathbf{u}\| \ = \ 1 \quad \Rightarrow \quad \mathbf{J}^T_\mathbf{u}\mathbf{u} \ = \ \mathbf{0} \quad , \tag{1.24}$$

where $\mathbf{0}$ is the vector with zero components; in such cases, the Jacobian matrix thus *projects* any vector onto some (locally) linear subspace that is *orthogonal* to the vector-field itself. By the second derivative of the flow, this has another interpretation:

$$\tfrac{d}{ds}\left(\mathbf{g}^s\right) \ \equiv \ \mathbf{u} \circ \mathbf{g}^s \quad \Rightarrow \quad \tfrac{d^2}{ds^2}\left(\mathbf{g}^s\right) \ = \ \left(\mathbf{J_u u}\right) \circ \mathbf{g}^s \quad , \tag{1.25}$$

while $\mathbf{u}$ is the vector tangent to the flow orbits *by definition*, vector $\mathbf{J_u u}$ corresponds to the orbits' *normal vector*, whose magnitude $\kappa := \|\mathbf{J_u u}\|$ is the *curvature* of the orbit and whose direction is perpendicular to the tangent vector $\mathbf{u}$, as (1.24) implies.

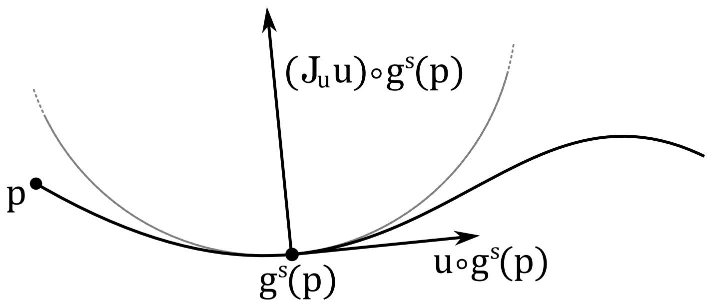

Figure 1.3: Normal vector to a curve, $\mathbf{J_u u}$, expressed through the Jacobian matrix of the generator $\mathbf{u}$, as in equation (1.25), to which is orthogonal. Its modulus equals κ, the curvature of the curve in $\mathbf{g}^s(\mathbf{p})$; partially, in gray, the corresponding *osculating* circle.

1.4.1 Cocycle Relation

An important Jacobian matrix that will be considered in the following is the one associated to the flow itself, for which we reserve a separate symbol:

$$\mathbf{\Gamma}^s \;:=\; \mathbf{J}_{\mathbf{g}^s} \quad \Rightarrow \quad (\mathbf{\Gamma}^s)^{\nu}_{\mu} \;=\; \partial_\mu (\mathbf{g}^s)^\nu \quad . \tag{1.26}$$

For these objects, by the group-property of the flow and the composition-rule for derivations, it is easy to prove the fundamental *cocycle* relation:

$$\mathbf{\Gamma}^{s+s'} \;=\; \mathbf{\Gamma}^{s'} \circ \mathbf{g}^s \, \mathbf{\Gamma}^s \quad , \quad \mathbf{\Gamma}^0 \;\equiv\; \mathbf{1} \quad . \tag{1.27}$$

This allows to express any cocycle $\mathbf{\Gamma}^{s+\sigma}$ as the product of other cocycles, corresponding to two (or more) *intermediate* steps. In particular, this highlights the structural difference between matrices $\mathbf{J_u}$ and $\mathbf{\Gamma}^s$: while the former is evaluated at a point, the latter depends on both the point and the arc-length value. This implies that $\mathbf{\Gamma}^s$ *is not* a *local* function of phase space, but actually depends on entire arcs of orbit (of length s, in this case) and, in general, can be ill- or not-defined in the limits $s \to \pm\infty$. Indeed, such matrix encodes all the information about the action of the flow upon tangent spaces, being essentially its *push-forward* and covering a central role in all the main results illustrated in this thesis.

1.4.2 Evolution of Generators

The basic relation between the Jacobian matrix of a flow and its generator (even if not normalized) is the following *evolution* equation:

$$\mathbf{\Gamma}^s \mathbf{u} \;=\; \mathbf{u} \circ \mathbf{g}^s \quad ; \tag{1.28}$$

that is, applying the flow Jacobian matrix at s to its generator *evaluates* this at $\mathbf{g}^s$; notice that we are not just saying that $\mathbf{\Gamma}^s$ maps tangent vectors to the tangent space in $\mathbf{g}^s$ (that would be simply the definition of push-forward), but also that the resulting vector is the *same function* evaluated at such point. This property, abusively called *covariance* and formally defined as *left-invariance*, may hold also for vectors different from $\mathbf{u}$; in the *numerical* branch of Dynamical Systems these are all called *covariant Lyapunov vectors* (CLV, these are actually left-invariant contravectors, the improper shorthand stands for their particular evolution law along the flow). In general, their existence has to be proven from case to case, as discussed in deep in the next chapters.
Notice that relation (1.28) could be derived also from the earlier comment on equation (1.9) about the commutativity between the operators $\hat{\mathbf{g}}^s$ and $\hat{\mathbf{u}}$; the fact that these commutes follows from $\hat{\mathbf{g}}^s$ being a differentiable function of $\hat{\mathbf{u}}$ so that, by definitions only, one has:

$$\begin{aligned} \hat{\mathbf{u}}\, \hat{\mathbf{g}}^s[x^\mu] \;&=\; \hat{\mathbf{u}}[(\mathbf{g}^s)^\mu] \;=\; -i\, u^\nu \partial_\nu (\mathbf{g}^s)^\mu \;\equiv\; -i(\mathbf{\Gamma}^s \mathbf{u})^\mu \;= \\ =\; \hat{\mathbf{g}}^s\, \hat{\mathbf{u}}[x^\mu] \;&=\; -i\, \hat{\mathbf{g}}^s\, [u^\mu] \;\equiv\; -i\, (\mathbf{u} \circ \mathbf{g}^s)^\mu \quad , \end{aligned} \tag{1.29}$$

the component-wise statement of equation (1.28), additionally multiplied by $-i$; the latter equation is obtained equivalently by partial differentiation of the flow ODE.

Evolution of Dual Vectors

It is easy to see that a property similar to evolution (1.28) holds for vectors that have constant *projection* along the orbit, and thus along $\mathbf{u}$:

$$\mathbf{w}\cdot\mathbf{u} \;=\; c \;=\; (\mathbf{w}\cdot\mathbf{u})\circ\mathbf{g}^s \quad , \quad \forall\, s \quad ,$$

$$\Rightarrow \qquad (\boldsymbol{\Gamma}^s)^{-T}\,\mathbf{w} \;=\; \mathbf{w}\circ\mathbf{g}^s \quad , \tag{1.30}$$

this is analogous to (1.28), but involving the *inverse-transpose* of the flow Jacobian matrix; when $\mathbf{u}$ is seen as an element of a local tangent basis $\{\mathbf{u}_j\}$, as in (1.16), such type of evolution would be the correct one for *each* of the corresponding dual basis vectors $\{\mathbf{w}^j\}$ (see (1.20)) along *any* of the flows induced by elements of the local tangent basis.

1.4.3 Cocycle Evaluation

The last property that we need to illustrate about Jacobian matrices relates the cocycle of a flow to the Jacobian matrix of its generator. By deriving definition (1.26) w.r.t. s:

$$\tfrac{d}{ds}\boldsymbol{\Gamma}^s \;=\; (\mathbf{J}_\mathbf{u}\circ\mathbf{g}^s)\;\boldsymbol{\Gamma}^s \quad , \quad \boldsymbol{\Gamma}^0 \;=\; \mathbf{I} \quad , \tag{1.31}$$

we obtain a Cauchy problem for $\boldsymbol{\Gamma}^s$ that involves the Jacobian matrix $\mathbf{J}_\mathbf{u}$ by multiplication from the left; notice that it is completely analogous to the original ODE for the flow in (1.3), and that the initial conditions for the latter, $\mathbf{g}^0 = \mathrm{id}$, induces the initial conditions for the present problem, $\boldsymbol{\Gamma}^0 = \mathbf{I}$, as also stated by cocycle relation (1.27). Indeed, the ODE (1.31) can be formally integrated by writing its infinitesimal evolution:

$$\boldsymbol{\Gamma}^{ds}\circ\mathbf{g}^s \;\simeq\; (\mathbf{I} \;+\; ds\,\mathbf{J}_\mathbf{u}\circ\mathbf{g}^s) \quad , \tag{1.32}$$

and by making use of cocycle relation (1.27) to deduce that a sequential application (from the left) of *infinitesimal* evolutions yields any possible *finite* one, corresponding to the standard definition of *time-ordered exponential*; this is denoted by T-exp, and reads:

$$\boldsymbol{\Gamma}^s \;=\; \lim_{ds\to 0}\prod_{k=0}^{s/ds-1}\left(\mathbf{I} + ds\,\mathbf{J}_\mathbf{u}\circ\mathbf{g}^{k\,ds}\right) \;:=\; \text{T-}\exp\left(\int_0^s \mathbf{J}_\mathbf{u}\circ\mathbf{g}^{s'}\,ds'\right) \quad . \tag{1.33}$$

Although formal, this operation is fundamental and, in general, enables to deduce useful informations on the cocycle once the structure of the Jacobian matrix $\mathbf{J}_\mathbf{u}$ is known.

Commuting Cocycles

In the particular case that the cocycles of a flow *commute* at different parameters, the ordered exponential becomes a regular matrix exponential. By defining the *commutator* $[\![\cdot,\cdot]\!]$ for any pair of matrices $\mathbf{A},\mathbf{B}$:

$$[\![\,\mathbf{A},\mathbf{B}\,]\!] \;:=\; \mathbf{AB}-\mathbf{BA} \quad , \tag{1.34}$$

and by introducing the *average along the flow* of any function W at s as the integral:

$$\langle W\rangle^s \;:=\; \tfrac{1}{s}\int_0^s W\circ\mathbf{g}^{s'}\,ds' \tag{1.35}$$

we can re-write relation (1.33) under the commuting hypothesis:

$$[\![\,\boldsymbol{\Gamma}^s,\boldsymbol{\Gamma}^{s'}\,]\!] \;=\; 0\;,\;\forall\; s\neq s' \quad\Rightarrow\quad \boldsymbol{\Gamma}^s \;=\; \exp\left(s\,\langle\mathbf{J_u}\rangle^s\right) \quad . \tag{1.36}$$

This remains a non-local function, depending again on the whole arc of orbit from $s'=0$ to $s'=s$; notice that in all the above calculations we omitted the obvious dependence on the chosen initial point in phase-space from which the orbit should start.
Finally, a further simplification can be found when the Jacobian matrix $\mathbf{J_u}$ of the generator is even *constant* along the flow: in such case the average coincides with its argument and thus the solution to (1.31) at any s can be written as a regular exponential:

$$\boldsymbol{\Gamma}^s \;=\; \mathrm{e}^{s\,\mathbf{J_u}} \quad , \quad \forall\; s \quad . \tag{1.37}$$

In the following it will be discussed when and how this last particular case can produce important preliminary information about the phase-space structures induced by a flow.

1.5 Lie Brackets

What has been defined in the previous section as 'local parameters', for each point of a certain domain, can be seen as a set of so-called *curvilinear coordinates*, that is, a coordinate system with arbitrary curvilinear axes; this kind of objects, extensively studied in the wide context of differential geometry, have particularly interesting and non-trivial features that need some introduction. Starting from the natural coordinates x^μ, technically called *holonomic* (from the greek, 'one rule for all') because they share the same properties everywhere, we recall the commuting property of their derivations:

$$\partial_\mu \partial_\nu \;=\; \partial_\nu \partial_\mu \quad ; \tag{1.38}$$

such relation, although of obviously differential nature, implies that we can arbitrarily order any sequence of 'moves' in space that we wish to apply: if we first translate a point along axis μ and then translate it again along axis ν we get to the same point that we would reach following the reverted sequence, provided that also the amounts of translation are exchanged. Using the notation introduced in the previous section:

$$\mathrm{e}^{s\partial_\mu}\mathrm{e}^{s'\partial_\nu} \;=\; \mathrm{e}^{s'\partial_\nu}\mathrm{e}^{s\partial_\mu} \;\equiv\; \mathrm{e}^{s\partial_\mu + s'\partial_\nu} \quad \forall\, s,\, s' \quad . \tag{1.39}$$

Being the trivial consequence of (1.38), this can be generalized through the definition of the *Lie brackets* $[\cdot,\cdot]$ for vector-fields intended as first order differential operators:

$$[\,\hat{\mathbf{u}}\,,\hat{\mathbf{v}}\,] \;:=\; \hat{\mathbf{u}}\hat{\mathbf{v}} - \hat{\mathbf{v}}\hat{\mathbf{u}} \quad . \tag{1.40}$$

This is skew-symmetric, $[\hat{\mathbf{u}},\hat{\mathbf{v}}] = -[\hat{\mathbf{v}},\hat{\mathbf{u}}]$, and written in component-wise notation reads:

$$[\,\hat{\mathbf{u}}\,,\hat{\mathbf{v}}\,] \;=\; -\,(u^\mu\partial_\mu v^\nu \;-\; v^\mu\partial_\mu u^\nu)\,\partial_\nu \;\equiv\; -(\mathbf{J_v u} - \mathbf{J_u v})^\nu\partial_\nu \quad , \tag{1.41}$$

showing explicitly, also by definition (1.22), that the Lie brackets of vector-fields is again a vector-field; notice that such vector directly depends on the phase-space dependence of the

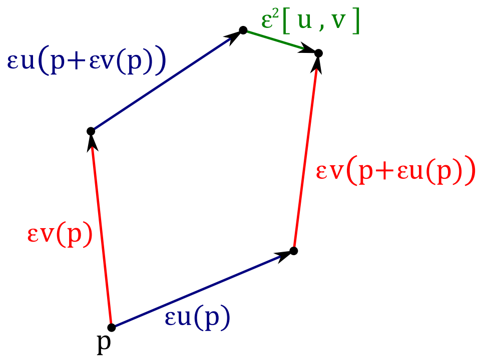

Figure 1.4: Geometrical meaning of the Lie brackets for vector-fields, understood as first order differential operators; with the approximations $\mathrm{e}^{i\varepsilon\hat{\mathbf{u}}}\mathbf{p} \simeq \mathbf{p} + \varepsilon\mathbf{u}(\mathbf{p})$ and the analog for $\mathrm{e}^{i\varepsilon\hat{\mathbf{v}}}\mathbf{p}$, the deviation for a pair of vector-fields from being *holonomic* is quantified by their Lie brackets $[\mathbf{v},\mathbf{u}]$, i.e. the II order term in the BCH formula (1.42).

two vectors under brackets, i.e. how 'much' curvilinear are the two associated axes (see figure 1.4), but in addition takes into account the possibility that such distortions exactly cancel out; the latter case would correspond to figure (1.4) with the red/blue vectors forming a closed quadrilateral (i.e. $[\hat{\mathbf{u}}, \hat{\mathbf{v}}] = 0$) but still *not* a parallelogram.

1.5.1 Baker-Campbell-Hausdorff

The last point can be stated through the general formula for the product of two exponential operators, the reknown *Baker-Campbell-Hausdorff* (BCH) formula, which consists in a series expansion in powers of the arc length, with coefficients made up by the two operators and all their possible Lie brackets; written up to 2-nd order, it reads:

$$\exp(i\varepsilon\hat{\mathbf{v}})\exp(i\varepsilon\hat{\mathbf{u}}) \;=\; \exp\left(i\varepsilon\left(\hat{\mathbf{v}}+\hat{\mathbf{u}}\right) - \tfrac{1}{2}\varepsilon^2[\,\hat{\mathbf{u}}\,,\hat{\mathbf{v}}\,] + \mathcal{O}\left(\varepsilon^3\right)\right) \quad , \tag{1.42}$$

from which we deduce the schematic picture represented in figure (1.4) for very small values of parameter ε; notice that the minus sign in front of the Lie brackets in the last exponential, as well as in the component-wise expression in (1.41), is completely due to our definition for the differential operators (1.4). Taking out all the imaginary units such minus signs disappear, but nevertheless, in possible presence of self-adjoint operators, our convention is consistent with the fact that their Lie brackets are *anti-self-adjoint*:

$$\hat{\mathbf{u}}^\dagger = \hat{\mathbf{u}}\ ,\ \hat{\mathbf{v}}^\dagger = \hat{\mathbf{v}} \quad \Rightarrow \quad [\,\hat{\mathbf{u}}\,,\hat{\mathbf{v}}\,]^\dagger = [\,\hat{\mathbf{v}}^\dagger\,,\hat{\mathbf{u}}^\dagger\,] = -[\,\hat{\mathbf{u}}\,,\hat{\mathbf{v}}\,] \quad ; \tag{1.43}$$

said this, all the calculations can be converted to different conventions at any stage. To make such distinction more precise, we adopt a parallel notation "without hats" for which the Lie brackets are defined for vector-fields meant as phase-space functions only:

$$[\,\mathbf{u}\,,\mathbf{v}\,]^\nu \;=\; (\mathbf{J_v u} - \mathbf{J_u v})^\nu \quad ; \tag{1.44}$$

by components, it is again explicit that the brackets are a vector-field on their own.

1.5.2 Jacobi Identity

The fundamental property that, along with the bi-linearity and the skew-symmetry (1.43), characterizes the Lie brackets completely, is the *Jacobi identity*; this is defined by considering three distinct vector-fields $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$:

$$[\,\mathbf{u}\,,[\,\mathbf{v}\,,\mathbf{w}\,]\,] \;+\; [\,\mathbf{v}\,,[\,\mathbf{w}\,,\mathbf{u}\,]\,] \;+\; [\,\mathbf{w}\,,[\,\mathbf{u}\,,\mathbf{v}\,]\,] \;=\; \mathbf{0} \quad ; \tag{1.45}$$

the proof of this relation is left to the reader, being of pure pedagogical interest; in the operatorial notation (with "hats") it holds equivalently, and turns out to be essential in the following sections.

1.5.3 Obstruction to local coordinates: Torsion

As can be expected, the fact that two (or more) vector-fields have non-zero Lie brackets (for which we say that, intuitively, they do not *commute*) poses a serious obstruction to the identification of the associated arc-lengths as *proper* local-coordinates. In general, the main request upon *coordinates* is the ability to parametrize any *differentiable* function by a Taylor series in their powers; such ability is lost starting from the *second* derivative, as can be checked by:

$$-\partial_j \partial_k f \;=\; (\hat{\mathbf{u}}_j \hat{\mathbf{u}}_k)\,[f] \;=\; (\hat{\mathbf{u}}_k \hat{\mathbf{u}}_j \;+\; [\hat{\mathbf{u}}_j, \hat{\mathbf{u}}_k])\,[f] \;\neq\; -\partial_k \partial_j f \tag{1.46}$$

implying that the cross-terms in the power series for f in the $\{s^j\}$ are not unique, also for higher orders, making the expansion impossible. By $\mathbf{J}_k := \mathbf{J}_{\mathbf{u}_k}$, the responsible term:

$$[\hat{\mathbf{u}}_j, \hat{\mathbf{u}}_k][f] \;=\; (\mathbf{J}_k \mathbf{u}_j - \mathbf{J}_j \mathbf{u}_k) \cdot \boldsymbol{\nabla} f \quad , \tag{1.47}$$

is called the *torsion* induced by the vector-fields, being the net contribution to the torsion tensor *solely* due to the non-linearity of the fields, in opposition to the 'background' term due to the metric tensor, which *may* be present only in curved phase-spaces. Notice that this does not avoid the construction of a metric tensor by generators $\mathbf{u}_j$ (being it a first-order object), but prevents the promotion of the arc-lengths to local coordinates. Briefly, commuting generators can induce a parametrization of higher-dimensional sub-sets of phase-space: such extension is not possible in the case of non-zero Lie brackets.

Chapter 2

The Flow

2.1 Velocity

Let us focus on the study of just one particular flow generated by the vector-field $\mathbf{v}$, standing as an abbreviation for *velocity* field, with the assumptions that it is time-independent (or, *autonomous*) and at least two-times differentiable, i.e. each of its components is a function in $\mathcal{C}^2(\mathcal{M},\mathbb{R})$. The corresponding flow should be globally defined in the whole phase space $\mathcal{M}$, so the latter is typically selected among a set of possible spaces that are *compatible* with the choice of the velocity-field; in general such field is not normalized to 1 nor divergence free, so that its choice is rather free.
In writing the Cauchy problem (1.3) that operatively defines the associated flow:

$$\tfrac{d}{dt}\mathbf{f}^t \;=\; \mathbf{v}\circ\mathbf{f}^t \quad , \quad \mathbf{f}^0 \;=\; \mathrm{id} \quad , \tag{2.1}$$

we now tie to the ODE variable t the promoted role of *time*, taking the action of the flow $\mathbf{f}^t$ as the natural *evolution* in time of the whole phase-space; the latter is thus interpreted as the full set of parameters $\mathbf{x}$ necessary to specify the state of a system whose *infinitesimal*-time evolution is supposed to be described by velocity field $\mathbf{v}(\mathbf{x})$:

$$\mathbf{x}_t := \mathbf{f}^t(\mathbf{x}) \quad , \quad \mathbf{x}_{t+dt} \;\simeq\; \mathbf{x}_t \;+\; dt\,\mathbf{v}(\mathbf{x}_t) \quad . \tag{2.2}$$

2.1.1 Invariant Measure

Such evolution is additionally required to preserve some function μ specifically constructed to measure the volume (or probability) of any sub-set of phase-space; this is due to the need to quantify the properties of sub-sets in a way that is compatible with the underlying dynamical evolution. The formal statement defining such measure as a function of any sub-set $\mathcal{A}\subset\mathcal{M}$ reads:

$$\mu(\mathcal{A}) \;\equiv\; \int_{\mathcal{A}} d\mu(\mathbf{x}) \;:=\; \mu\left(\mathbf{f}^{-t}(\mathcal{A})\right) \quad , \quad \forall\, t \quad , \tag{2.3}$$

and makes explicit use of the flow at negative *times*, i.e. its inverse, implying that for non-invertible flows the measure $\mu(\mathbf{f}^{-t}(\mathcal{A}))$ refers to the union of the *preimages* of the set $\mathcal{A}$ w.r.t.

$\mathbf{f}^t$; in such way the measure may be defined for any possible dynamical case. Due to its own definition, the function μ is called the *invariant measure* for the flow $\mathbf{f}^{(t)}$ and, together with the flow and the phase-space $\mathcal{M}$, it completes the triple $(\mathbf{f}^t, \mathcal{M}, \mu)$ which constitutes the rigorous definition of any *dynamical system*.
One of the most important issues in the theory of dynamical systems regards the proof of *existence* of an invariant measure for a prescribed time evolution; while for continuous (in phase space) flows on compact manifold the existence is assured by the *Krylov-Bogolyubov* theorem, there is no standard recipe to achieve such result for a generic evolution law, and in many cases it remains an open problem. Since almost all of the results that we are going to illustrate follow directly (or makes sense) by the existence of an invariant measure, this can be incorporated in the basic hypotheses of the present work: any flow considered here is assumed to admit an invariant measure.
As will be shown also in the following chapters, a fundamental class of dynamical systems that *automatically* admits an invariant measure is the one generated by divergence-free velocity-fields: in such case the invariant measure is simply the usual N-dimensional euclidean volume $d^N x$; for obviuos reasons, such type of flows is called *volume-preserving*.

2.1.2 Observables

As the entire phase-space evolves under the action of the flow $\mathbf{f}^t$, any non-costant function will inherit some kind of time dependence from it; for any $\mathcal{C}^1$ time-independent function $W : \mathcal{M} \to \mathbb{R}$ and vector-field $\mathbf{w} \in \mathfrak{X}(\mathcal{M})$ we define their *total* time derivative:

$$\dot{W} \quad := \quad \tfrac{d}{dt}\left(W \circ \mathbf{f}^t \right) \quad = \quad (\mathbf{v} \cdot \nabla W) \circ \mathbf{f}^t \tag{2.4}$$

$$\dot{\mathbf{w}} \quad := \quad \tfrac{d}{dt}\left(\mathbf{w} \circ \mathbf{f}^t \right) \quad = \quad (\mathbf{J_w v}) \circ \mathbf{f}^t \tag{2.5}$$

implicitly meaning the phase-space dependence of the function's derivatives and of the velocity and flow as well; the "dot" above is then reserved for quantities whose time dependence follows from being advected by the flow. This type of derivation, called *material derivative*, is naturally extended to explicitly time-dependent functions of phase space by simply adding the partial derivative w.r.t. time:

$$\tfrac{d}{dt}\left(W_t \circ \mathbf{f}^t \right) \; = \; \left(\dot{W}_t \; + \; \partial_t W_t \right) \circ \mathbf{f}^t \quad . \tag{2.6}$$

The distinction between the two types of time-derivatives is quite fundamental: any equation involving the material derivative of time-independent functions implies for them a PDE in the coordinates and thus, in general, a fairly more complex problem than an ODE in the single time variable.

2.2 Periodic Orbits and Invariant Manifolds

Before approaching the main topic of the present work, that is the structure induced by a flow upon each tangent space, it is important to introduce what can be considered as the dynamical skeleton of phase-space: *periodic orbits*. In general, the action of a flow can change significantly from point to point, with the typical (but not uniform) result that *small differences* in the choice of the initial conditions turn into huge changes in the nature of the generated orbits. In the set of all possible initial conditions may exist a very small, usually countable sub-set of them generating *closed orbits*, whose dynamics brings the initial point $\mathbf{x}_P$ exactly back to itself after a specific time P, naturally called the *period* of the orbit. These then represent *fixed points* of the flow $\mathbf{f}^P$ evaluated at the corresponding period:

$$\mathbf{f}^P(\mathbf{x}_P) \;=\; \mathbf{x}_P \quad , \tag{2.7}$$

and, by consequence, they are privileged with respect to arbitrary non-periodic points in the sense that a wide range of results applies to them, independently of the period. Among these results, probably the most important is the existence of particular sub-manifolds in phase-space that are *invariant* under the action of the flow, exactly as the periodic orbits themselves. For a sub-set, *flow-invariance* means that any of its points is transported by the flow into another of its points. Then, to be also proper manifolds, a certain level of regularity is required: a manifold should resemble an Euclidean space in *each* of its neighborhoods; this rules out pathologies such as discontinuities and self-intersections, even if restricted to isolated points. The classification of invariant manifolds of periodic orbits is then induced by their *dynamical stability*: given a periodic orbit, the associated *stable/unstable* invariant manifolds are defined as the sets of points that, under the action of the flow, converge to such periodic orbit as time goes respectively to $+/-\infty$. This can be easily visualized for period 0 points $\mathbf{x}_0$, i.e. true fixed-points: in such case the associated stable/unstable manifolds are constituted by all the points that eventually collapse on $\mathbf{x}_0$ as time goes respectively to $+/-\infty$. In the generic case of non-zero periods, points from the invariant manifolds follow only *asymptotically* the dynamics of the associated periodic orbit in the respective limits. From a global point of view of phase-spce, periodic orbits and the associated invariant manifolds put natural constraints upon the geometry of a system:

- the *topology* of the closed orbits imposes a skeleton upon phase-space; by continuity of the velocity field, nearby orbits must resemble such structure at short times;
- invariant manifolds associated to periodic orbits may induce *barriers*, meaning that the phase-space can be *divided* by them into full-measure sub-sets whose inter-communication by orbits is ruled by the properties of the invariant manifolds.

In many cases, these considerations allow for an analysis of the dynamics of the whole phase-space through a set-theoretic approach, giving way to extremely powerful results such as the coding of orbits by symbolic dynamics and the explicit evaluation of the transport properties of the system [Wiggins et al.].

On the other hand, in his *Ergodic Theory of Dynamical Systems*, Y. Pesin pushed the definitions beyond periodic orbits, by extending the concept of invariant manifold to generic points through the notion of *local invariant manifolds*. In analogy with the periodic case, these are defined again as sub-sets that asymptotically converge into smaller and smaller regions neighboring the (never-closing) reference orbits. By consequence, these do not induce separations of phase-space but, instead, are governed by the underlying structure of proper invariant manifolds. For this reason, their main applications do not concern the schemes followed by the dynamics, but the relative behaviour of *families* of orbits, or, of *perturbations* around reference orbits.

This finally connects with our main approach, exposed through the following chapters: the *natural generalization* to arbitrary points of some of the standard techniques initially conceived in the study of periodic and fixed points. In particular, such approach points to understand the properties of the *linearization* of local invariant manifolds, with the local *generators* of such objects as main targets.

Chapter 3

Flow Linear Stability

By *linear stability* it is meant the sensitivity of the flow upon *infinitesimal* modifications on its state at a prescribed time, at which the system is left free to evolve.
The typical situation is represented by having already knowledge on the behaviour of the system for just one or a few (possibly periodic) points and then asking which is its reaction once small initial perturbations are applied.
More specifically, one might ask how to setup such infinitesimal perturbations in order to selectively obtain different types of behaviour, being these either very similar or completely different from the already known dynamics.
Working in an arbitrary number of dimensions, implying that the system needs N parameters (coordinates) to be fully described at any time, means that the range of possible perturbations resides on an (infinitesimal radius) hyper-sphere centered at the point to perturb; by consequence, there is no mean by which all the possible perturbations can be exhausted, e.g. by trial and error, even for a single point.
What emerges from the analysis of linear stability is that any $\mathcal{C}^1$ measure-preserving flow induces a natural structure that is *complementary* to the dynamical skeleton induced by periodic orbits and invariant manifolds: for μ-almost all points in phase-space there exist a unique tangent basis, called the *Oseledets' splitting*, whose elements are the exact directions along which perturbations must be implemented to select a specific dynamical behaviour. The range of choice, in turn, is also completely induced by the system.
The expression 'μ-almost' simply means that the subset $\mathcal{X}$ whose point do not admit such tangent basis has zero measure, $\mu(\mathcal{X}) = 0$, and thus can be considered *non-observable.* By contrast, the presence of such type of points can in general affect greatly the *finite-time* dynamics of the system; nevertheless, the smoothness of the flow allows to gather information also through the analysis of the splitting in the neighborhoods of points in which the splitting fails to exist.
As mentioned before, the machinery involved in the study of such fundamental objects corresponds to a generalization of the standard methods of analysis for fixed and periodic points; for this reason, we first begin by addressing the periodic orbits stability problem.

3.1 Stability Matrix

The Jacobian matrices of the velocity field and flow represent the central objects in the dynamics induced by the flow on tangent spaces; for this reason we reserve special symbols for both of them. By the notation introduced in (1.22), let us define respectively:

$$\mathbf{M} \;:=\; \mathbf{J}_{\mathbf{v}} \quad , \quad \mathbf{F}^t \;:=\; \mathbf{J}_{\mathbf{f}^t} \quad , \tag{3.1}$$

calling $\mathbf{M}$ the *stability matrix* of the flow; as shown in section 1.4, matrix $\mathbf{F}^t$ corresponds to the ordered exponential of the stability matrix along the flow until time t:

$$\mathbf{F}_t(\mathbf{x}) \;=\; \text{T-exp}\left(\int_0^t \mathbf{M}\circ\mathbf{f}^\tau\, d\tau\right) \;\equiv\; \lim_{\tau\to 0}\prod_{k=0}^{t/\tau-1}\left(\mathbf{1}+\tau\mathbf{M}\circ\mathbf{f}^{k\tau}\right) \quad , \tag{3.2}$$

which is the generic formal solution to the Jacobian matrix equation introduced in (1.31):

$$\tfrac{d}{dt}\mathbf{F}^t \;=\; \left(\mathbf{M}\circ\mathbf{f}^t\right)\mathbf{F}^t \quad , \quad \mathbf{F}^0 \;=\; \mathbf{I} \quad . \tag{3.3}$$

The exponential structure of the flow Jacobian matrix fulfills the fundamental *cocycle relation*, as in (1.27), connecting tangent spaces at points along the same orbit (figure 3.1):

$$\mathbf{F}^{t+\tau} \;=\; \mathbf{F}^\tau\circ\mathbf{f}^t\;\mathbf{F}^t \quad ; \tag{3.4}$$

in the following section it will be shown how this property is crucial to prove the existence of the Oseledets' tangent basis; for this reason we simply call $\mathbf{F}^t$ *the cocycle* of the flow.

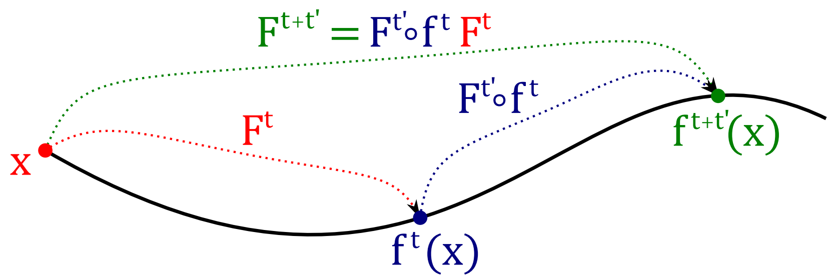

Figure 3.1: Schematic behaviour of the Jacobian matrix $\mathbf{F}^t$ of the flow for different points along the same orbit; the cocycle property, equation (3.4), assures that the matrix for the total path is the ordered product of the matrices of the two sub-paths.

3.2 Periodic Orbits Stability

The knowledge of the cocycle is important primarily because it determines how any possible perturbation of the flow evolves, as can be seen by perturbing a generic point:

$$\mathbf{x}_t := \mathbf{f}^t(\mathbf{x}_0) \quad \Rightarrow \quad \delta\mathbf{x}_t \;=\; \mathbf{F}^t(\mathbf{x}_0)\,\delta\mathbf{x}_0 \quad . \tag{3.5}$$

with $\delta\mathbf{x}_0$ the arbitrary perturbation to be applied to the initial condition $\mathbf{x}_0$. When this is chosen to be a periodic (or fixed) point $\mathbf{x}_P$ of period P, the cocycle condition (3.4) induces to study the matrix $\mathbf{F}^P(\mathbf{x}_P)$ alone, by considering times nP that are *integer multiples* of the period:

$$\mathbf{f}^P(\mathbf{x}_P) \;=\; \mathbf{x}_P \quad \Rightarrow \quad \delta\mathbf{x}_{nP} \;=\; \left(\mathbf{F}^P(\mathbf{x}_P)\right)^n \delta\mathbf{x}_0 \quad , \quad \forall\, n \in \mathbb{Z} \quad , \tag{3.6}$$

in this way the n-th power of the cocycle over a single period P gives the cocycle after n periods; notice that the integer n may be negative for invertible flows only.
Historically, the Jacobian matrix of the flow *evaluated* on periodic points is called the *Floquet operator* (although its introduction appeared in the context of periodically-driven linear ODE) and the solutions of equation (3.6) for perturbations are produced by its diagonalization; by setting up the characteristic equation for $\mathbf{F}^P_* \equiv \mathbf{F}^P(\mathbf{x}_P)$ and by defining the matrices $\mathbf{E}_* = [\mathbf{e}_1..\mathbf{e}_n]$ and $\mathbf{\Sigma}_* = \mathbf{Diag}[\Sigma_1..\Sigma_n]$:

$$\mathbf{F}^P_*\,\mathbf{E}_* \;=\; \mathbf{E}_*\,\mathbf{\Sigma}_* \quad \sim \quad \mathbf{F}^P_*\,\mathbf{e}_k \;=\; \mathbf{e}_k\,\Sigma_k \tag{3.7}$$

we identify each $\mathbf{e}_k$ as an *eigenvector* of cocycle $\mathbf{F}^P_*$ associated to the *eigenvalue* Σ_k, with no sum over k implied: this is the eigen-decomposition of matrix $\mathbf{F}^P_*$, which may be inserted directly into equation (3.6); indeed, when the columns of $\mathbf{E}_*$ do form a basis for the tangent space in $\mathbf{x}_P$, these can decompose any initial perturbation by linear combination, making its evolution automatic:

$$\delta\mathbf{x}_0 \;=\; \sum_{k=1}^{n} \mathbf{e}_k\, c_0^k \quad \Rightarrow \quad \delta\mathbf{x}_{nP} \;=\; \sum_{k=1}^{n} \mathbf{e}_k\, \underbrace{c_0^k\,(\Sigma_k)^n}_{c^k_{nP}} \quad . \tag{3.8}$$

By organizing each projection c^k as a component of the vector $\mathbf{c}_0 := \mathbf{E}_*^{-1}\delta\mathbf{x}_0$, the evolution of such vector is said to be *diagonal*, obviously because matrix $\mathbf{\Sigma}_*$ is diagonal:

$$\mathbf{c}_{nP} \;=\; (\mathbf{\Sigma}_*)^n\mathbf{c}_0 \quad . \tag{3.9}$$

and the evolution of each component is *decoupled* from the others. As the number n of turns around the periodic orbit grows, each c_0^k is multiplied by a factor that is *exponential* in n; to quantify its stability it is thus natural to consider its logarithm, appropriately normalized by n :

$$\tfrac{1}{n}\log\left|c^k_{nP}\right| \;=\; \log|\Sigma_k| \;+\; \tfrac{1}{n}\log\left|c_0^k\right| \quad . \tag{3.10}$$

It is then apparent that, while the second term on the r.h.s. tends to zero as $n \to \infty$, the first is constant and therefore characterizes the *asymptotic rate*:

$$\Rightarrow \quad \lim_{n\to\infty} \tfrac{1}{n}\log\left|c^k_{nP}\right| \;=\; \log|\Sigma_k| \tag{3.11}$$

by which the corresponding component of the perturbation evolves, *independently* of the initial value c_0^k , provided that this is non-zero, $c_0^k \neq 0$. Notice that $\log|\Sigma_k|$, called the *Floquet exponent*, can have both *real* and *imaginary* part. Thus, while the latter is the *frequency* with which the component rotates in the complex plane, the former quantifies its *exponential rate* of growth/decay, depending on its sign or, equivalently, on whether the Floquet eigenvalue has *modulus* greater of smaller than 1.
The exponential rates of perturbation growth/decay are universally called the *Lyapunov exponents* (LE), Λ_k , and so, for perturbations of periodic points, they are the real parts of the Floquet exponents:

$$\Lambda_k \equiv \mathrm{Re}\left(\log|\Sigma_k|\right) \quad . \tag{3.12}$$

If we consider again the full perturbation $\delta\mathbf{x}_0$ with the hypothesis that its projection on each eigenvector is non-zero, $c_0^k \neq 0 \;\forall\; k$, its norm in phase-space will also evolve under an exponential law in n. If, moreover, the Floquet eigenvalues are labeled in decreasing order w.r.t. their modulus, $|\Sigma_j| \geqslant |\Sigma_k|$ *iff* $j < k$, we can deduce that the perturbation norm grows asymptotically as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{n}\log\|\delta\mathbf{x}_n\| \;=\; \mathrm{Re}\left(\log|\Sigma_1|\right) \;\equiv\; \Lambda_1 \quad , \tag{3.13}$$

while for $n \to -\infty$ (and for invertible flows only) the opposite limit holds:

$$\lim_{n\to-\infty} \frac{1}{n}\log\|\delta\mathbf{x}_n\| \;=\; \mathrm{Re}\left(\log|\Sigma_N|\right) \;\equiv\; \Lambda_N \quad . \tag{3.14}$$

The two exponents, $\Lambda_1 := \max_k(\Lambda_k)$ and $\Lambda_N := \min_k(\Lambda_k)$ are respectively called the *largest* and the *smallest* Lyapunov exponent and, in practice, govern the asymptotic behaviour of any *randomly chosen* perturbation in the corresponding time limits.
We emphasize that the choice of initial perturbation should be random because, in such case, it is almost sure (by probabilistic arguments) that each projection on the eigenvectors is non-zero; this naturally suggests an inverse approach:
by choosing the perturbations to be made up by combinations of only certain selected eigenvectors, it would be possible to *control* their groth rate.
Although useful in a variety of applications, such approach will always depend on the range of growth rates that the system ‘allows’: any initial perturbation can asymptotically evolve according to only one of the Lyapunov exponents Λ_k, obviously identified as the greatest among those contained in the initial perturbation. The range of choice, composed by all the Lyapunov exponents ordered by decreasing magnitude, is therefore discrete and is called the *Lyapunov spectrum* of the reference periodic orbit.

3.2.1 Determinants & Divergences

As a final remark for this chapter, notice that we made use of the inverse $\mathbf{E}_*^{-1}$ of the matrix whose columns are the cocycle eigenvectors; stated in an equivalent way, it means that *any* perturbation can be decomposed on those vectors, i.e. they do form a basis for tangent space. For this to be true it is necessary and sufficient that the cocycle matrix $\mathbf{F}_*^P$ is itself invertible, or, more specifically, that its determinant is non-zero. This can be always verified, also for non-periodic orbits, by a useful property of the determinant of a matrix exponentials combined with relations (1.23) and (1.35):

$$\det\left(\mathbf{F}^t\right) \;=\; \exp\left(t\left\langle \mathrm{Tr}\left(\mathbf{M}\right)\right\rangle^t\right) \;=\; \exp\left(t\left\langle \mathrm{div}(\mathbf{v})\right\rangle^t\right) \quad . \tag{3.15}$$

It then follows that the condition $\det\left(\mathbf{F}^t\right) \neq 0$ is fulfilled for any velocity field as long as its divergence, averaged along the flow, does not drop to $-\infty$; this is always true for any $\mathcal{C}^1$ velocity and any associated periodic orbit, but in general it should be proven case by case. As a byproduct, we deduce that for any divergence-free (also called *solenoidal*) velocity, the corresponding cocycle determinant is identically 1 at any time. Since the phase space volume transforms by multiplication of the Jacobian of the transoformation (also in generic curved manifolds), this proves what has been introduced ahead: divergence-free velocity fields always generate volume-preserving flows.

Chapter 4

The Oseledets' Splitting

We now introduce some brief notion about the existence and uniqueness of the Lyapunov spectrum and the Oseledets' splitting. In the literature, many rigorous results are present on both topics and their formal details are well established [Oseldets', Ruelle, Barreira-Pesin]. A clear and technical statement of the Oseledets' theorem (to which we refer in this work) can be found in Scholarpedia, curated by Dr. V. Oseledets himself; his original paper is also available online, in russian. For these reasons, this exposition is focused on the main mechanisms involved: the scope here is to give an overview of the subject pushing its limits of validity; by choosing the weakest possible assumptions instead of a specific class of problems, we seek for explanations also for results that, at present, are supported only by numerical arguments (in particular, the convergence of the splitting algorithm also when no tangent exponential scaling is present).
Non-trivial systems are in general *non-uniform* across phase space, with different trajectories encountering very different sequences of essentially local properties. To reflect such feature, we set up a sketch of the actual proofs whose structure works for both the existence and the numerical convergence of the local tangent bases, allowing to make it eventually rigorous for specific cases. By the numerical observation of single orbits, such bases appear to be very complicated matrix-functions *evaluated* along the trajectories under study; the class of smoothness of such functions is assumed to be the same of the generator of the flow.

4.1 Kingman's Theorem & Ky Fan Norms

To first prove the existence of a Lyapunov spectrum for μ-almost any initial condition of the flow, it is standard to make use of the *Kingman's sub-additive ergodic theorem*; for a simple proof and references therein see [Steele].
Stated in terms of discrete dynamical systems, it affirms that, given a measure-preserving transformation T over the probability space (Ω, σ, μ), for any *sub-additive* sequence of integrable functions $\{g_n, 1 \leqslant n \leqslant \infty\}$, i.e. a sequence $g_n : \Omega \to \mathbb{R}$ such that:

$$g_{n+m} \leqslant g_m \circ T^n + g_n \tag{4.1}$$

the limit exists:

$$\exists \lim_{n\to\infty} \tfrac{1}{n} g_n \;=\; g \;\geqslant -\infty \tag{4.2}$$

and the limit function g is T-invariant, i.e. $g \circ T = g$ for μ-almost any $\mathbf{x} \in \Omega$.

For our purposes, it is sufficient to consider μ the invariant measure of the flow, the phase-space $\mathcal{M}$ as the *space of events* Ω, σ as the *Borel algebra* of sub-sets of $\mathcal{M}$ induced by μ and, finally, the flow $\mathbf{f}^\tau$ as the transformation T , for some fixed time τ. The sub-additive sequence $\{g_n\}$ we choose is induced by the so-called *Ky Fan k-norms* $\|\mathbf{A}\|^{(k)}$ of an $N \times N$ matrix $\mathbf{A}$, defined as the the sum of its k largest *singular values*:

$$\|\mathbf{A}\|^{(k)} \;:=\; \sum_{q=1}^{k} \sigma_q \quad , \quad \{\sigma_q,\}_{q=1..N} \;:=\; \mathrm{eig}\left(\sqrt{\mathbf{A}^T\mathbf{A}}\right) \tag{4.3}$$

by definition, the singular values of $\mathbf{A}$ are the eigenvalues of the associated matrix $\sqrt{\mathbf{A}^T\mathbf{A}}$, labeled by decreasing magnitude, $\sigma_j \geqslant \sigma_k > 0$ *iff* $j < k$; the matrix under analysis here is, by our choice, the cocycle $\mathbf{F}^{n\tau}$ associated to the flow at time $n\tau$ with $n \in \mathbb{Z}^+$ and $\tau \in \mathbb{R}$. For each Ky Fan norm, we induce a sequence of integrable functions:

$$g_n^{(k)} \;:=\; \ln \|\mathbf{F}^{n\tau}\|^{(k)} \quad , \quad g_n^{(k)} \;\geqslant\; g_n^{(k-1)} \quad \forall\, n,\, k = 1..N \quad . \tag{4.4}$$

Sub-additivity of $g_n^{(k)}$ is then naturally induced by the sub-additivity of the Ky Fan norms combined with the cocycle property; indeed, since we have:

$$\|\mathbf{AB}\|^{(k)} \;\leqslant\; \|\mathbf{A}\|^{(k)}\,\|\mathbf{B}\|^{(k)} \quad \forall\, k = 1..N \quad , \tag{4.5}$$

and, by the cocycle relation (1.27) or (3.4), also:

$$\mathbf{F}^{(n+m)\tau} \;=\; \mathbf{F}^{m\tau} \circ \mathbf{f}^{n\tau}\; \mathbf{F}^{n\tau} \quad , \tag{4.6}$$

it follows that the desired property (4.1) holds for each $k = 1..N$:

$$g_{n+m}^{(k)} \;\leqslant\; g_m^{(k)} \circ T^n \;+\; g_n^{(k)} \tag{4.7}$$

highlighting the only limitation of this procedure: to have $T^n \equiv \mathbf{f}^{n\tau}\,\forall\, n$ it is necessary that the flow is *autonomous*. Finally, the integrability of functions $g_n^{(k)}$ follows from the non-negativity and smoothness of the singular values, in turn induced by the invertibility and smoothness of the cocycle at any finite time. We thus conclude that the limits exist:

$$\exists \lim_{n\to\infty} \tfrac{1}{n} g_n^{(k)} \;=\; g^{(k)} \;\geqslant -\infty \quad , \quad k = 1..N \quad , \tag{4.8}$$

and are all $\mathbf{f}^\tau$-invariant for any finite choice of $\tau \neq 0$.

4.1.1 Lyapunov Spectrum

The existence and invariance of such limits can be now transferred to the singular values; the motivation for which this can be done only at this stage is that the singular values *do not* fulfill the sub-additive property (4.1) (for interesting discussions about, see the Terry Tao web-page, while for rigorous proofs see [Knutson&Tao]). Nevertheless, these can be expressed at any finite time $n\tau$ by inversion of the Ky Fan norm definition; indeed, their logarithms read:

$$\ln \left| \sigma_n^{(k)} \right| \;=\; \ln \left| e^{g_n^{(k)}} - e^{g_n^{(k-1)}} \right| \quad ; \tag{4.9}$$

The existence of the limit $n \to \infty$ for this sequence then follows from the smoothness of the logarithmic and exponential functions, and it is finite ($\neq -\infty$) provided that $g^{(k)}$ is strictly greater than $g^{(k-1)} < g^{(k)}$; at any finite time, such condition trivially implies to have non-zero singular values, i.e. matrix $\mathbf{F}^{n\tau}$ (and so the flow $\mathbf{f}^{n\tau}$) should be invertible.

The first part of the Oseledets' Theorem then states that the *Lyapunov spectrum* $\mathbf{\Lambda} := \{\Lambda_k\}_{k=1..N}$ is defined exactly by such limits:

$$\exists\ \Lambda_k \;:=\; \lim_{n\to\infty} \ln \left| \sigma_n^{(k)} \right| \tag{4.10}$$

whose existence is thus proved by the existence of the limits for the logarithms of the Ky-Fan norms. Historically, indeed, the first methods to approximate the Lyapunov spectrum were based on the explicit calculation of the singular values of matrix $\mathbf{F}^{n\tau}$ and (the logarithm of) their limits for large times. Instead of that, we can now pass to show how to also prove the existence of specific bases (for the tangent space of each point in phase-space) that are in complete correspondence with the Lyapunov spectrum; in turn, this also yields a much more convenient method to approximate the spectrum itself.

4.2 Existence & Convergence of the Oseledts' Orthonormal Bases

By considering a single (possibly non-periodic) orbit with initial condition $\mathbf{x}_0$, we *associate* to its tangent space an orthonormal basis whose elements are represented by the columns of an orthogonal matrix $\widehat{\mathbf{Q}}_0 \equiv (\widehat{\mathbf{Q}}_0)^{-T} \in O(N)$; here and from now on, the superscript $^{-T}$ stands for *inverse-transpose*. For now, no matrix function is evaluated at $\mathbf{x}_0$, only an arbitrary orthonormal basis is chosen; we then associate to the orbit $\{\mathbf{x}_t\}$ a sequence of bases $\{\mathbf{Q}_t\}$ obtained by application of the QR matrix decomposition:

$$\mathbf{F}^t(\mathbf{x}_0)\ \widehat{\mathbf{Q}}_0 \;\equiv\; \widehat{\mathbf{Q}}_t\ \widehat{\mathbf{\Gamma}}^t \;:=\; \mathrm{QR}\left[\mathbf{F}^t(\mathbf{x}_0)\ \widehat{\mathbf{Q}}_0\right] \quad , \tag{4.11}$$

with $\{\widehat{\mathbf{\Gamma}}^t\}$ the corresponding sequence of *upper-triangular* matrices; since the QR decomposition is unique up to basis permutations, once the *initial* $\widehat{\mathbf{Q}}_0$ is chosen both sequences are completely induced by the cocycle. The first consequence of such construction is that, $\forall\ \widehat{\mathbf{Q}}_0\,,\ t,$ matrices $\mathbf{F}^t$ and $\widehat{\mathbf{\Gamma}}^t$ share the same set of *singular values*; these are said to be

isospectral. Moreover, any sequence $\mathbf{\Gamma}^t$ obtained by (4.11) fulfills the cocycle property (3.4), introducing the concept of *transformed cocycle*; we come back later on this.
Notice that the sub/superscript 't' here is reserved respectively for functions evaluated at $\mathbf{f}^t$ / cocycles evolved until time t; it is thus never intended as an exponential operation.

4.2.1 Non-marginal Hypothesis

At this stage, we add the further hypothesis that the time-dependence of cocycle $\mathbf{F}^t$ is *expanding/contracting* over some linear sub-spaces of tangent space in $\mathbf{x}$. This is a fundamental hypothesis, necessary at any level of rigor for this proof: the sub-spaces are *unknown* and nothing else is said about the behaviour of the cocycle, except that it stretches *and/or* shrinks some specific tangent sub-space. Notice that this relies on the existence of Lyapunov exponents, but is much more general than the typical request to have all or some of them strictly non-zero. By such considerations and by the general fact that every triangular matrix carries its eigenvalues on its diagonal, we can decompose each $\widehat{\mathbf{\Gamma}}^t$ into a *bounded* upper-triangular part $\widehat{\mathbf{B}}^t$, with all diagonal elements set to 1, and a *growing* diagonal part $\widehat{\mathbf{X}}^t$, such that:

$$\widehat{\mathbf{\Gamma}}^t \;=\; \widehat{\mathbf{X}}^t\,\widehat{\mathbf{B}}^t \quad . \tag{4.12}$$

Notice that we do not know anything about the growth properties of matrix $\widehat{\mathbf{X}}^t$, except that it contains the eigenvalues of $\widehat{\mathbf{\Gamma}}^t$ and thus, in general, is a complex diagonal matrix with some entries, by the hypothesis on $\mathbf{F}^t$, having modulus different from 1; these can be possibly grouped in degenerate families, but should grow *or* shrink by some unknown (possibly exponential) time-law. This is the weakest possible set of assumptions that can induce some formal result; the motivations behind such choice of approach are that:

- it is valid for any possible flow or map satisfying the non-marginal hypothesis 4.2.1;
- it corresponds to a computable numerical algorithm and thus, as a byproduct, it explains also the mechanism and the range of validity of the latter.

Suppose now to redo the same procedure by setting a different orthogonal matrix $\check{\mathbf{Q}}_0$ and generate two new sequences of orthogonal $\check{\mathbf{Q}}_t$ and triangular matrices $\check{\mathbf{\Gamma}}^t$ as in (4.11):

$$\mathbf{F}^t(\mathbf{x}_0)\;\check{\mathbf{Q}}_0 \;\equiv\; \check{\mathbf{Q}}_t\;\check{\mathbf{\Gamma}}^t \;:=\; \mathrm{QR}\left[\mathbf{F}^t(\mathbf{x}_0)\;\check{\mathbf{Q}}_0\right] \quad . \tag{4.13}$$

We can quantify the deviation between the two orthogonal sequences by the product:

$$\mathbf{D}_0 \;:=\; \check{\mathbf{Q}}_0{}^T\widehat{\mathbf{Q}}_0 \;\equiv\; \mathbf{D}_0{}^{-T} \quad , \tag{4.14}$$

that is again an orthogonal matrix; whenever such matrix equals $\mathbf{I}$, the $N \times N$ identity, we deduce that the two orthogonal matrices *coincide*.
By making use of the evolution equations (4.11), (4.13), the original cocycle can be removed:

$$\mathbf{F}^t(\mathbf{x}_0) \;=\; \check{\mathbf{Q}}_t\;\check{\mathbf{\Gamma}}^t\;\check{\mathbf{Q}}_0{}^T \;=\; \widehat{\mathbf{Q}}_t\;\widehat{\mathbf{\Gamma}}^t\;\widehat{\mathbf{Q}}_0{}^T \quad , \tag{4.15}$$

and an evolution for $\mathbf{D}_t := \check{\mathbf{Q}}_t{}^T\widehat{\mathbf{Q}}_t$ can be deduced, taking into account its orthogonality:

$$\mathbf{D}_t \;=\; \check{\mathbf{\Gamma}}^t\;\mathbf{D}_0\;(\widehat{\mathbf{\Gamma}}^t)^{-1} \;\equiv\; (\check{\mathbf{\Gamma}}^t)^{-T}\;\mathbf{D}_0\;(\widehat{\mathbf{\Gamma}}^t)^T \;=\; (\mathbf{D}_t)^{-T} \quad . \tag{4.16}$$

The next steps critically depend on the last equation: the decomposition into bounded and growing parts for both the triangular sequences leads to define the generic matrix:

$$\breve{\mathbf{B}}^t := \check{\mathbf{B}}^t \; \mathbf{D}_0 \; (\widehat{\mathbf{B}}^t)^{-1} \quad , \tag{4.17}$$

which, by definition, is *bounded* through any sub-multiplicative norm:

$$\|\breve{\mathbf{B}}^t\| \;\leqslant\; \|\check{\mathbf{B}}^t\| \; \|\mathbf{D}_0\| \; \|(\widehat{\mathbf{B}}^t)^{-1}\| \;\leqslant\; 1 \quad , \tag{4.18}$$

implying that also each of its entries and those of its inverse are all bounded:

$$|(\breve{\mathbf{B}}^t)^{\mu}_{\nu}| \leqslant 1 \quad , \quad |((\breve{\mathbf{B}}^t)^{-1})^{\mu}_{\nu}| \leqslant 1 \quad , \quad \forall \; \mu \; , \; \nu \quad ; \tag{4.19}$$

Evolution (4.16) can thus be re-written through the bounded/growing decompositions:

$$\mathbf{D}_t \;=\; \check{\mathbf{X}}^t \; \breve{\mathbf{B}}^t \; (\widehat{\mathbf{X}}^t)^{-1} \;\equiv\; (\check{\mathbf{X}}^t)^{-1} \; (\breve{\mathbf{B}}^t)^{-T} \; \widehat{\mathbf{X}}^t \;=\; (\mathbf{D}_t)^{-T} \quad . \tag{4.20}$$

Since $\widehat{\mathbf{X}}^t$, $\check{\mathbf{X}}^t$ are both diagonal, it is convenient to re-write again (4.20) component-wise:

$$(\mathbf{D}_t)^{\mu}_{\nu} \;=\; (\check{\mathbf{X}}^t)^{\mu}/(\widehat{\mathbf{X}}^t)^{\nu} \; (\breve{\mathbf{B}}^t)^{\mu}_{\nu} \;\equiv\; (\widehat{\mathbf{X}}^t)^{\nu}/(\check{\mathbf{X}}^t)^{\mu} \; ((\breve{\mathbf{B}}^t)^{-T})^{\mu}_{\nu} \quad , \tag{4.21}$$

so that, by making use of bounds (4.19), the fundamental inequalities are finally implied:

$$|(\mathbf{D}_t)^{\mu}_{\nu}| \;\leqslant\; \begin{cases} |(\check{\mathbf{X}}^t)^{\mu}/(\widehat{\mathbf{X}}^t)^{\nu}| \\ \\ |(\widehat{\mathbf{X}}^t)^{\nu}/(\check{\mathbf{X}}^t)^{\mu}| \end{cases} \tag{4.22}$$

We can now see clearly what happens to $\mathbf{D}_t$ when both the QR evolutions (4.11) and (4.13), with respective initial conditions $\widehat{\mathbf{Q}}_0$ and $\check{\mathbf{Q}}_0$, are evolved along the flow :

- if for some pair $\mu \neq \nu$ the corresponding $|(\check{\mathbf{X}}^t)^{\mu}|$, $|(\widehat{\mathbf{X}}^t)^{\nu}|$ have *distinct* time-laws
- if *at least* one of these time-laws either grows to ∞ or shrinks to 0 as $t \to \infty$

then *one* of the two inequalities in (4.22) can be used to imply that $|(\mathbf{D}_t)^{\mu}_{\nu}| \to 0$.
Notice that if *all* the factors $|(\mathbf{X}^t)^{\nu}|$ fulfill both conditions, then $|(\mathbf{D}_t)^{\mu}_{\nu}| \to 0$ for all $\mu \neq \nu$, and the orthogonality of matrix $\mathbf{D}_t$ implies that it converges to the identity:

$$\lim_{t\to\infty} \mathbf{D}_t \to \mathbf{I} \quad \Rightarrow \quad t \to \infty \quad , \quad \check{\mathbf{Q}}_t \;\to\; \widehat{\mathbf{Q}}_t \quad , \tag{4.23}$$

and the two sequences of local orthogonal bases thus converge to the same one for arbitrary initial conditions $\widehat{\mathbf{Q}}_0 \neq \check{\mathbf{Q}}_0$; this implies that any sequence $\widehat{\mathbf{Q}}_t$ calculated by the QR evolution (4.11), starting with some $\widehat{\mathbf{Q}}_0$ at point $\mathbf{x}$, converges to a sequence that is *uniquely* associated to the orbit of that point. Such procedure then induces the construction of a matrix-valued function $\mathbf{Q}_+$ *evaluated* at each orbit's point $\mathbf{f}^t(\mathbf{x})$:

$$\widehat{\mathbf{Q}}_0 \neq \mathbf{Q}_+(\mathbf{x}) \quad , \quad t \to \infty \quad , \quad \widehat{\mathbf{Q}}_t \simeq \mathbf{Q}_+ \circ \mathbf{f}^t(\mathbf{x}) \quad , \tag{4.24}$$

to which *any* initial matrix $\widehat{\mathbf{Q}}_0$ converges. This is essential for the realistic case in which we do not actually know the value $\mathbf{Q}_+(\mathbf{x})$ of such matrix function at the initial point $\mathbf{x}$.

4.3 Oseledets' Splitting by Matrices

The matrix-valued function $\mathbf{Q}_+(\mathbf{x})$, to which we refer as a *phase-space dependent* tangent basis, is called the *forward Oseledets' orthonormal basis*, in virtue of the fact that it is induced by the action of the linearized flow for $t \to +\infty$. In the same fashion, one can define the *backward* analogous, by substitution of $\mathbf{F}^t$ with its inverse $\mathbf{F}^{-t} = (\mathbf{F}^t)^{-1} \circ \mathbf{f}^t$, to which another orthonormal basis $\mathbf{Q}_-(\mathbf{x})$, then called the *backward Oseledets' orthonormal basis.*
In the statement of his theorem, Oseledets' makes use of the notion of forward /backward *filtrations* of tangent spaces, defined as the sequences of *nested* sub-sets of tangent directions which experience instabilities *weaker* than a prescribed exponent among the spectrum. In addition to the existence of the Lyapunov spectrum $\mathbf{\Lambda}$ and of the two phase-space dependent (left-invariant) bases $\mathbf{Q}_\pm$, the most subtle part of the theorem then states a precise relation between all of them:
the forward/backward filtrations have exactly the forward/backward Oseledets' orthonornal bases as respective *normal bases* and, as a consequence, the tangent subspace that evolves exactly under the action of a specific exponent is the *intersection* between the associated forward/backward subspaces of the two filtrations.
To get a clear algebraic (and then also numerical) view of such statement, we re-write it by matrices only; to be a normal basis for the forward filtration means that, by taking *upper-triangular* matrices $\mathbf{R}$, the forward filtration is generated by the matrix product:

$$\mathbf{U}_+ \;=\; \mathbf{Q}_+ \, \mathbf{R} \;=\; [\mathbf{u}_{1+}..\mathbf{u}_{N+}] \quad \Rightarrow \quad \lim_{t\to+\infty} \tfrac{1}{t} \ln \|\mathbf{F}^t\, \mathbf{u}_{k+}\| \;\leqslant\; \Lambda_k \quad , \tag{4.25}$$

whose each k-th column has Lyapunov exponents *equal/less* than the k-th exponent Λ_k; the same holds for the backward filtration, by considering *lower-triangular* matrices $\mathbf{L}$:

$$\mathbf{U}_- \;=\; \mathbf{Q}_- \, \mathbf{L} \;=\; [\mathbf{u}_{1-}..\mathbf{u}_{N-}] \quad \Rightarrow \quad \lim_{t\to-\infty} \tfrac{1}{t} \ln \|\mathbf{F}^t\, \mathbf{u}_{k-}\| \;\leqslant\; \Lambda_k \quad . \tag{4.26}$$

Here the definition of $\mathbf{Q}_\pm$ to be the respective *normal basis* for the forward/backward filtrations is translated into the upper/lower shape of the *arbitrary* coefficients matrices $\mathbf{R}/\mathbf{L}$, by saying that the products (4.25), (4.26) *span* the corresponding filtration.
Then, the *definition* of the subspaces associated *exactly* to a given exponent Λ_k can be deduced quite intuitively: the intersection between the filtrations comes as a constraint upon the coefficients to *generate* exactly $\mathbf{U}_+ \equiv \mathbf{U}_-$, i.e. the same matrix $\mathbf{U}$ of vectors evolving under each Λ_k in *both* temporal limits:

$$\mathbf{U} \;:=\; \mathbf{Q}_+\mathbf{R}_+ \;=\; \mathbf{Q}_-\mathbf{L}_- \;=\; [\mathbf{u}_1..\mathbf{u}_N] \tag{4.27}$$

$$\Rightarrow \qquad \lim_{t\to\pm\infty} \tfrac{1}{t} \ln \|\mathbf{F}^t\, \mathbf{u}_k\| \;\equiv\; \Lambda_k \quad . \tag{4.28}$$

This is precisely the algebraic definition of the set of *covariant Lyapunov vectors* (CLV) composing the *Oseledets' splitting.* Operatively, it means that, to calculate the splitting once both the orthonormal bases are known, one also needs to identify the correct $\mathbf{R}_+$, $\mathbf{L}_-$ among *all* the possible upper/lower matrices of coefficients. Equivalently and more functionally, one can regard to relation (4.28) as a definition for the orthonormal/triangular pairs of matrices:

$$\mathbf{Q}_+\, \mathbf{R}_+ \;:=\; QR[\mathbf{U}] \;\equiv \mathbf{Q}_-\, \mathbf{L}_- \;:=\; QL[\mathbf{U}] \quad . \tag{4.29}$$

throught the process of QR and QL orthogonalization acting upon the CLV matrix $\mathbf{U}$; this is obviously a formal statement, since there is no general possibility to know in advance the splitting $\mathbf{U}$ for non-trivial systems, and also, in such improbable case, there would be no need then to calculate the orthogonal/triangular decompositions.
That is to say, the fundamental relations (4.29) should be considered as pinpointing the *exact* structure of the splitting as it comes from the Oseledets' theorem, and can be formally exploited at any stage to relate the Lyapunov vectors to the Oseledets' bases.

4.4 Coordinates Change

Before introducing our general interpretation of the Oseledets' splitting and its underlying algebraic mechanism, it is useful to illustrate how a generic *coordinates* transformation affects the cocycle of a flow. If some new coordinates $\widetilde{\mathbf{x}}$ are defined:

$$\widetilde{\mathbf{x}} = \mathbf{h}(\mathbf{x}) \quad \Rightarrow \quad \widetilde{\mathbf{x}}_t = \mathbf{h}(\mathbf{x}_t) \quad , \tag{4.30}$$

and the transformation $\mathbf{h}$ is *invertible*, it is straightforward to deduce the new flow $\widetilde{\mathbf{f}}^t$:

$$\widetilde{\mathbf{x}}_t = \widetilde{\mathbf{f}}^t(\widetilde{\mathbf{x}}_0) \quad \Rightarrow \quad \widetilde{\mathbf{f}}^t = \mathbf{h} \circ \mathbf{f}^t \circ \mathbf{h}^{-1} \quad . \tag{4.31}$$

By deriving this with respect to time t and by defining the transformation Jacobian matrix $\mathbf{H} := \mathbf{J}_\mathbf{h}$, also the expression for the new velocity field follows:

$$\tfrac{d}{dt}\widetilde{\mathbf{f}}^t = \widetilde{\mathbf{v}} \circ \widetilde{\mathbf{f}}^t \quad \Rightarrow \quad \widetilde{\mathbf{v}} = (\mathbf{H}\mathbf{v}) \circ \mathbf{h}^{-1} \quad ; \tag{4.32}$$

while the partial derivatives of (4.31) w.r.t. $\widetilde{\mathbf{x}}_0$ produce the new cocycle:

$$\widetilde{\mathbf{F}}^t := \frac{\partial \widetilde{\mathbf{x}}_t}{\partial \widetilde{\mathbf{x}}_0} = \left(\mathbf{H} \circ \mathbf{f}^t \, \mathbf{F}^t \, \mathbf{H}^{-1} \right) \circ \mathbf{h}^{-1} \quad . \tag{4.33}$$

We are mainly interested in the *structure* of such relation: in general, as can be seen, cocycles transform by being multiplied from the *right* and from the *left* by matrices (similarly to any linear transformation of linear operators) that *in addition* should correspond to the tangent space of respectively the *initial* and *final* point of the orbit. Notice that the same rule holds even if no actual coordinates transformation exists behind it, e.g. even if one prescribes a transformation law *for tangent spaces only* that *changes* from point to point, i.e. it is phase-space dependent. This kind of concept is exactly what is needed to *generalize* to non-periodic orbits the approach described in the previous section: in the periodic case, the Floquet eigen-basis is exactly mapped onto itself after a period, being just anisotropically scaled by the Floquet eigen-values. It would be useful to have a similar contruction also for points whose period diverges (i.e. does not exists), at least for the same reasons for which periodic orbit stability, as shown, can help in controlling perturbations. More specifically, the analogy would be achieved by findind a matrix valued function $\mathbf{H}$ (or even better, but even harder, a coordinates change $\mathbf{h}$ whose Jacobian matrix is $\mathbf{H}$) such that the new cocycle $\widetilde{\mathbf{F}}^t$ is *diagonal*: we call such procedure the '*cocycle diagonalization*'. To make the implications of such an achievement clear, we write the orbit equation (2.2) in the new coordinates: by simple substitutions, we see that also equation (1.28) holds exactly:

$$\widetilde{\mathbf{v}}^t \circ \widetilde{\mathbf{f}}^t \;=\; \widetilde{\mathbf{F}}^t \widetilde{\mathbf{v}} \quad . \tag{4.34}$$

Since the cocycle contains the partial derivatives of the coordinates at time t w.r.t. to coordinates at time zero, and since such definition holds also in the new coordinates (see (4.33)), to have a diagonal cocycle means to have completely *decoupled* the equations of motion, i.e. each coordinate at time t only depends on itself at time zero.
This is why, in general, we consider the transformations $\mathbf{h}$ or $\mathbf{H}$, in terms of *explicit* functions of phase-space, as practically *impossible* to find for non-trivial velocity fields.

4.5 Transformed Cocycles

Now we can illustrate a formal procedure that can be associated in general to the search for covariant Lyapunov vectors; let us start by considering a *point-wise* transformation of each tangent-space, similar to (4.33) but with no coordinate change behind it.
For generic perturbations $\delta\mathbf{x} \in T_{\mathbf{x}}\mathcal{M}$, in analogy with the Floquet decomposition in (3.8), let us define:

$$\delta\widetilde{\mathbf{x}} \;:=\; \mathbf{E}^{-1}(\mathbf{x})\delta\mathbf{x} \quad ; \tag{4.35}$$

with $\mathbf{E}(\mathbf{x})$ a phase-space dependend tangent basis; notice that the resulting $\delta\widetilde{\mathbf{x}} \in T_{\mathbf{x}}\mathcal{M}$ remains attached to the point $\mathbf{x}$ while matrix $\mathbf{E}(\mathbf{x})$, the equivalent of the Floquet eigen-basis in (3.7), would correspond to the inverse of the Jacobian matrix $\mathbf{H}$ in (4.32). The transformation (4.33) for the cocycle can be recovered, but we choose its inverse form:

$$\mathbf{F}^t \;=\; \mathbf{E} \circ \mathbf{f}^t \; \widetilde{\mathbf{F}}^t \; \mathbf{E}^{-1} \quad . \tag{4.36}$$

with the unknown central term $\widetilde{\mathbf{F}}^t$, the transformed cocycle, not necessarily diagonal yet. This is where our approach points to: the search for a phase-space dependent basis $\mathbf{E}$ which is *induced* by the request to transform the cocycle into some *prescribed shape*.

4.6 Cocycle Generators

In general, the idea to modify each tangent basis to get a new cocycle involves *non-local* considerations only, meaning to work with the flow $\mathbf{f}^t$ at arbitrary long times. In opposition to this, exactly as stability matrix $\mathbf{M}$ generates the original cocycle by the time-ordered exponential (3.2), one can consider the generator of the transformed cocycle, thus corresponding to some *transformed* stability matrix $\widetilde{\mathbf{M}}$:

$$\tfrac{d}{dt}\mathbf{F}^t \;=\; \mathbf{M}\circ \mathbf{f}^t \; \mathbf{F}^t \quad \Rightarrow \quad \tfrac{d}{dt}\widetilde{\mathbf{F}}^t \;=\; \widetilde{\mathbf{M}}\circ \mathbf{f}^t \; \widetilde{\mathbf{F}}^t \quad . \tag{4.37}$$

Combination of the generic transformation law (4.36) with the last two equations yields:

$$\mathbf{M} \circ \mathbf{f}^t \; \mathbf{F}^t \; \mathbf{E} \;=\; \dot{\mathbf{E}} \circ \mathbf{f}^t \; \widetilde{\mathbf{F}}^t \;+\; \mathbf{E} \circ \mathbf{f}^t \; \widetilde{\mathbf{M}} \circ \mathbf{f}^t \; \widetilde{\mathbf{F}}^t \quad . \tag{4.38}$$

and by evaluating this at $t=0$, recalling that for *any* cocycle we have $\widetilde{\mathbf{F}}^0 \equiv \mathbf{1}$, yields:

$$\mathbf{M}\,\mathbf{E} \;=\; \dot{\mathbf{E}} \;+\; \mathbf{E}\,\widetilde{\mathbf{M}} \quad . \tag{4.39}$$

This can be considered as the *fundamental* equation for any type of *symmetry* of the system, but also for more general generators, as will be shown them in the following.
To begin, let us consider proper symmetries of the tangent dynamics, that is, transformations $\mathbf{E}$ for which the new stability matrix $\widetilde{\mathbf{M}} \equiv \mathbf{M}$ remains the same:

$$\dot{\mathbf{E}} \;=\; \mathbf{M}\,\mathbf{E} \;-\; \mathbf{E}\,\mathbf{M} \;\equiv\; [\![\mathbf{M},\mathbf{E}]\!] \quad ; \tag{4.40}$$

recalling from (2.6) that $\dot{\mathbf{E}} \equiv v^\mu \partial_\mu \mathbf{E}$, last equation turns out to be a PDE for the matrix-valued function $\mathbf{E}(\mathbf{x})$ in phase-space. Moreover, if we look for *uniform* symmetries of the system (independent from the point) equation (4.40) turns into the intuitive relation:

$$[\![\mathbf{M},\mathbf{E}]\!] \;=\; \mathbf{0} \quad ; \tag{4.41}$$

which tells that the uniform symmetries are all the linear transformations that *commute* with stability matrix of the system. Both the results (4.40),(4.41) can be traced back to Sophus Lie [Stephani].

4.7 Triangular Cocycles

To make clear our point on cocycle transformations, consider again the QR evolution (4.11) to which the orthogonal Oseledets' bases obey; by leaving the original cocycle alone on the left hand side of that equation, we get:

$$\mathbf{F}^t \;=\; \mathbf{Q}_\pm \circ \mathbf{f}^t \; \mathbf{\Gamma}^t_\pm \; \mathbf{Q}^T_\pm \quad . \tag{4.42}$$

with $\mathbf{Q}_\pm$ respectively the forward/backward Oseledets' orthonormal bases, $\mathbf{\Gamma}^t_+$ the upper- and $\mathbf{\Gamma}^t_-$ the lower-triangular transformed cocycles; equation (4.42) should be recognized in the form (4.36). We can thus interpret the Oseledets' bases as those (point-dependent) transformations of the tangent spaces able to change the original cocycle $\mathbf{F}^t$ respectively into upper/lower-triangular shape $\widetilde{\mathbf{F}}^t \equiv \mathbf{\Gamma}^t_\pm$. The transformed stability matrices $\widetilde{\mathbf{M}}_\pm$ are then also upper/lower-triangular, both sharing the property:

$$\begin{aligned} \mathbf{\Gamma}^t_\pm \;&\equiv\; \text{T-exp}\left(\int_0^t \widetilde{\mathbf{M}}_\pm \circ \mathbf{f}^\tau \mathrm{d}\tau\right) \\ \Rightarrow \qquad \mathbf{Diag}\left(\mathbf{\Gamma}^t_\pm\right) \;&=\; \exp\left(t\left\langle \mathbf{Diag}\left(\widetilde{\mathbf{M}}_\pm\right)\right\rangle^t\right) \quad . \end{aligned} \tag{4.43}$$

for which the diagonal of any triangular cocycle is the *regular* exponential of the average along the flow of the diagonal of the corresponding triangular stability matrix. This is due to the fact that, in the multiplication of triangular matrices, the diagonal terms (i.e. the eigenvalues) are decoupled from all the off-diagonal ones.
Notice that the growing diagonal part $\mathbf{X}^t$, involved in (4.12) the sketch of a proof for the existence of the Oseledets' basis, here is precisely the diagonal matrix $\mathbf{Diag}\left(\mathbf{\Gamma}^t_\pm\right)$.

4.8 Diagonal Cocycle

Exactly the same approach can be then applied in the search for a cocycle in *diagonal* shape, making complete the Floquet analogy. Given the definition of covariant Lyapunov vectors in (4.28) and the fact that the bases $\mathbf{Q}_\pm$ transform the cocycle into upper/lower triangular form, the full meaning of the Oseledets' theorem may be now exploited. Indeed, by considering that the cocycle transforms in *some* way by the CLV matrix $\mathbf{U}$:

$$\mathbf{F}_t\mathbf{U} \;=\; \mathbf{U}\circ\mathbf{f}^t\,\widetilde{\mathbf{F}}^t \quad , \tag{4.44}$$

with no assumptions on the new cocycle $\widetilde{\mathbf{F}}^t$. By applying to this transformation the relations (4.28) and (4.42) for the forward Oseledets' basis:

$$\begin{aligned} \mathbf{F}_t\,(\mathbf{Q}_+\,\mathbf{R}_+) \;&=\; \left(\mathbf{Q}_+\circ\mathbf{f}^t\,\mathbf{\Gamma}^t_+\right)\mathbf{R}_+ \;=\; \left(\mathbf{Q}_+\circ\mathbf{f}^t\,\mathbf{R}_+\circ\mathbf{f}^t\right)\widetilde{\mathbf{F}}^t \\ \Rightarrow \qquad \mathbf{\Gamma}^t_+\,\mathbf{R}_+ \;&=\; \mathbf{R}_+\circ\mathbf{f}^t\,\widetilde{\mathbf{F}}^t \end{aligned} \tag{4.45}$$

as well as for the backward one:

$$\begin{aligned} \mathbf{F}_t\,(\mathbf{Q}_-\,\mathbf{L}_-) \;&=\; \left(\mathbf{Q}_-\circ\mathbf{f}^t\,\mathbf{\Gamma}^t_-\right)\mathbf{L}_- \;=\; \left(\mathbf{Q}_-\circ\mathbf{f}^t\,\mathbf{L}_-\circ\mathbf{f}^t\right)\widetilde{\mathbf{F}}^t \\ \Rightarrow \qquad \mathbf{\Gamma}^t_-\,\mathbf{L}_- \;&=\; \mathbf{L}_-\circ\mathbf{f}^t\,\widetilde{\mathbf{F}}^t \end{aligned} \tag{4.46}$$

we get two expressions for the *same* transformed cocycle $\widetilde{\mathbf{F}}^t$, that thus should *coincide*:

$$\widetilde{\mathbf{F}}^t \;=\; (\mathbf{R}_+\circ\mathbf{f}^t)^{-1}\,\mathbf{\Gamma}^t_+\,\mathbf{R}_+ \;\equiv\; (\mathbf{L}_-\circ\mathbf{f}^t)^{-1}\,\mathbf{\Gamma}^t_-\,\mathbf{L}_- \tag{4.47}$$

Finally, by recalling that $\mathbf{R}_+$ and $\mathbf{\Gamma}^t_+$ are upper-triangular while $\mathbf{L}_+$ and $\mathbf{\Gamma}^t_l$ are lower-triangular, we arrive to the fundamental conclusion that, in order to fulfill equations (4.47), the transformed cocycle $\widetilde{\mathbf{F}}^t$ induced by the CLV basis $\mathbf{U}$ should be *diagonal*.
If we take into account that, by the Oseledets' theorem again, the infinite-time limits of such diagonal cocycle should correspond to the Lyapunov spectrum, we can write it as:

$$\widetilde{\mathbf{F}}^t(\mathbf{x}) \;\equiv\; \mathrm{e}^{t\mathbf{\Lambda}^t(\mathbf{x})} \quad\Rightarrow\quad \lim_{t\to\pm\infty}\mathbf{\Lambda}^t(\mathbf{x}) \;=\; \mathbf{\Lambda}\{\mathbf{x}\} \tag{4.48}$$

with the diagonal matrix $\mathbf{\Lambda}^t$ containing the *finite-time Lyapunov exponents* (FTLE), representing the not-yet-converged exponential factors of expansion/contraction upon tangent sub-spaces. As implied by Kingmans' theorem, these become flow-invariant in the infinite-time limit, as denoted by $\{\mathbf{x}\}$ meaning "orbit-wise" dependence.
Relations (4.46) and (4.47) then bring the important additional information:

$$\mathbf{\Gamma}^t_+\,\mathbf{R}_+ \;=\; \mathbf{R}_+\circ\mathbf{f}^t\,\mathrm{e}^{t\mathbf{\Lambda}^t} \quad , \tag{4.49}$$

$$\mathbf{\Gamma}^t_-\,\mathbf{L}_- \;=\; \mathbf{L}_-\circ\mathbf{f}^t\,\mathrm{e}^{t\mathbf{\Lambda}^t} \quad , \tag{4.50}$$

that is to say, the columns of the upper/lower-triangular coefficients matrices $\mathbf{R}_+/\mathbf{L}_-$ are *left-invariant* under the action of the corresponding triangular cocycle $\mathbf{\Gamma}^t_+/\mathbf{\Gamma}^t_-$, exactly as the original CLV matrix $\mathbf{U}$ is left-invariant under $\mathbf{F}^t$. As explained in the following, this property is at the core of the algorithm to approximate the CLV matrix.
We remark that the finite-time behaviour of $\mathrm{e}^{t\mathbf{\Lambda}^t(\mathbf{x})}$ may *not* be exponential at all: such cases would correspond to the *generic* time-laws $\mathbf{X}^t$ mentioned in (4.22) with which the Oseledets' bases may also converge; nevertheless, the infinite-time limits of the exponents $\mathbf{\Lambda}^t \equiv \frac{1}{t}\ln\|\mathbf{X}^t\|$ converge to zero for any $\mathbf{X}^t$ having *sub-exponential* time-law.

4.9 Diagonal Cocycle Generators

It should be noted that, being essentially the logarithm of a cocycle, the FTLE matrix $\mathbf{\Lambda}^t$ should obey to an *additive* version of the cocycle relation (which is multiplicative):

$$(t+t')\,\mathbf{\Lambda}^{t+t'} \;=\; t'\,\mathbf{\Lambda}^{t'}\circ\mathbf{f}^t \;+\; t\,\mathbf{\Lambda}^t \quad . \tag{4.51}$$

This relation allows for a two-fold consideration: in the case we are able to calculate FTLE only for very short times Δt in *each* of the points $\mathbf{f}^{q\Delta t}$ of some numerically integrated orbit, by (4.51) we then also have the possibility to calculate FTLE for arbitrary longer times $n\Delta t$, for any $n \in \mathbb{N}$, by the discrete time average:

$$\mathbf{\Lambda}^{n\Delta t} \;=\; \tfrac{1}{n}\sum_{q=0}^{n-1}\mathbf{\Lambda}^{\Delta t}\circ\mathbf{f}^{q\Delta t} \quad . \tag{4.52}$$

In turn, last relation can be made continuous in time, by keeping the product $n\Delta t \equiv t$ finite while performing the limits $\Delta t \to 0$ and $n \to \infty$:

$$\mathbf{\Lambda}^t \;=\; \tfrac{1}{t}\int_0^t \boldsymbol{\lambda}\circ\mathbf{f}^{t'}\,\mathrm{d}t' \quad , \tag{4.53}$$

and by defining the diagonal matrix $\boldsymbol{\lambda}(\mathbf{x})$ of so-called local Lyapunov exponents (LLE):

$$\boldsymbol{\lambda} \;:=\; \tfrac{d}{dt}(t\,\mathbf{\Lambda}^t)_{t=0} \quad . \tag{4.54}$$

This is a completely local *function* of phase-space, whose regularity properties require careful and specific treatment, to be determined from case to case.
Notice that the additive cocycle relation (4.51) brings two additional considerations; first, and naturally, the Lyapunov exponent can be defined thrsough the local one as:

$$\mathbf{\Lambda}\{\mathbf{x}\} \;=\; \lim_{t\to\infty}\tfrac{1}{t}\int_0^t \boldsymbol{\lambda}\circ\mathbf{f}^{t'}(\mathbf{x})\,\mathrm{d}t' \quad ; \tag{4.55}$$

second, we get a useful relation between the forward- and backward- time exponents:

$$\mathbf{\Lambda}^{-t} \;=\; \mathbf{\Lambda}^t\circ\mathbf{f}^{-t} \quad . \tag{4.56}$$

This is obtained by simply setting $t' \equiv -t$ into relation (4.51); in turn, this tells us that, in the case of invertible flows, the forward and backward limits of the finite-time Lyapunov exponents do coincide, as their flow-invariant property should already suggests.

4.9.1 Pesin Formula

A remarkable result from the Ergodic Theory concerns the production of *entropy* by the action of a dynamical system: from information theory, entropy quantifies the minimum cost, in terms of digits, with which information can be compressed, codified through a certain alphabet; in turn, this can be connected to the degree of *unpredictability* of a certain random variable. The fundamental result, first introduced by G. Margulis and then formalized by Y. Pesin, is that the entropy associated to the invariant measure of a dynamical system corresponds to the *sum* of its distinct *positive* Lyapunov exponents. This is the central result which typically brings to say that a chaotic system *produces information*, in the sense that its action dilates the amount of digits needed to specify its state, independently of the chosen alphabet.

4.10 Im-Proper Symmetry Generators

Once placed in the context of cocycle generators, the matrix $\boldsymbol{\lambda}$ of local Lyapunov exponents has a profound meaning: it corresponds to the diagonal transformed stability matrix $\boldsymbol{\lambda} \equiv \widetilde{\mathbf{M}}$ appearing in relation (4.39) when a diagonal transformed cocycle is considered; such equation may be also obtained by finally writing the CLV evolution:

$$\mathbf{F}^t \, \mathbf{U} \;=\; \mathbf{U} \circ \mathbf{f}^t \; e^{t\boldsymbol{\Lambda}^t} \tag{4.57}$$

and by deriving it with respect to $t = 0$, getting the equation:

$$\mathbf{M} \, \mathbf{U} \;=\; \dot{\mathbf{U}} \;+\; \mathbf{U} \, \boldsymbol{\lambda} \quad . \tag{4.58}$$

This is again a PDE in the phase-space coordinates (recall that $\dot{\mathbf{U}} \equiv v^\mu \partial_\mu \mathbf{U}$) with the CLV matrix $\mathbf{U}$ as unknown matrix function; it can be considered as the *local*, or point-wise, definition of the CLV, in opposition to its *integrated* counterpart (4.57) which works at arbitrary times t. While the latter is the convenient relation to approximate the CLV numerically and turns into the Floquet eigen-problem (3.7) for periodic points $\mathbf{f}^t(\mathbf{x}) \equiv \mathbf{x}$, the local equation (4.58) coincides with the stability matrix eigen-problem on fixed-points ($\dot{\mathbf{U}} = \mathbf{0}$), but it also highlights the true nature of each column of $\mathbf{U}$:

$$\mathbf{M} \, \mathbf{u}_k \;=\; \dot{\mathbf{u}}_k \;+\; \mathbf{u}_k \, \lambda_k \quad , \quad \text{no sum on } k \quad . \tag{4.59}$$

To all respects, these are *vector-fields*, each defined by equation (4.59) as the *generator* of some transformation; the latter corresponds to a *proper symmetry* of the system when some local exponent $\lambda_k = 0$ is exactly null, or *improper* symmetry in any other case. We will come back soon on this point, connecting it with the local invariant manifolds.

4.11 Algorithms' History

In the course of our sketch for the proof of convergence of the Oseledets' bases, we made use of the elegant QR evolution equation (4.11); the true origin of such relation dates back to David Ruelle ('79), who comes across them in his foundational 'Ergodic Theory of Differentiable Dynamical Systems', briefly commenting about by saying that they 'do not transform simply' under the action of the flow [Ruelle '79, p.34, remark 1.8]. That was to say that each column of the Oseledets' orthonormal bases is not left-invariant under the action of the cocycle, in opposition to the splitting (lately renamed covariant Lyapunov vectors), being all coupled by what we call *QR evolution*. The latter has been made public one year later by Giancarlo Benettin [Benettin *et al.* '80], in the two-fold publication on the Lyapunov characteristic exponents; such method had an initial hard way in becoming popular, basically because of the huge computational cost needed, quite expensive for those times. In the

beginning, indeed, such method was typically restricted to only a few of the first orthogonal vectors by means of the so-called *thin QR decomposition* which, for systems with N degrees of freedom, yields an arbitrary number $m < N$ of the first orthonormalized vectors in $\mathbb{R}^N$ plus an $m \times m$ upper triangular matrix. For very complicated and large systems it was already a great advance, because it allowed to approximate also the first m Lyapunov vectors.
It is remarkable that an analogous approach (called by *singular vectors*) has been used until very recently as the only seriously accepted way to implement multiple initial conditions for the weather forecast simulations, by perturbing the detected data (coming from survey stations) exactly along those first m orthonormal vectors. The main reason behind such scheme was the belief that each of those vectors would evolve with its own corresponding exponent, and thus its own *magnitude* of instability, allowing to discriminate between the most probable upcoming scenarios; one of the problems of this approach is that, as obscurely pointed out by Ruelle and as we already saw in the previous section, such vectors are *mixes* (linear combinations) of the first actual CLV, and thus the largest instability of the system always shows up eventually in their evolutions. On the other hand, the computation of CLV for such large systems presents serious computational costs also today, because of the structural constraints to which the algorithm is subject. The causes of this fact are described in the following paragraph.

4.12 Ginelli's Algorithm

The rationale behind the numerical procedure for the complete Oseledets' splitting, published in 2007 by Francesco Ginelli and his colleagues [Ginelli *et al.* '07], is based on a two-fold consideration: the Benettin's QR evolution yields a set of orthonormal vectors that already provides a discrimination upon the possible intrinsic in-stabilities of a system; in turn, such discrimination is operated with the aid of a sequence of upper-triangular matrices, which we identify as a transformed cocycle by itself, directly connected to the original one by the Benettin's basis itself. While the last observation leads quite naturally to the idea that a tangent evolution through a triangular cocycle would automatically *preserve* the upper-triangular shape of the evolved objects, the central role is played here by the former point, by which the k-th Oseledets' orthonormal vector is a linear combination of the first k CLV:

$$\mathbf{u}_k \;=\; \sum_{j=1}^{k} \mathbf{q}_j \, \mathbf{R}_{jk} \quad \Leftrightarrow \quad \mathbf{q}_k \;=\; \sum_{j=1}^{k} \mathbf{u}_j \, (\mathbf{R}^{-1})_{jk}; \tag{4.60}$$

While the forward evolution of $\mathbf{q}_k$ (the k-th Oseledets' forward vector) is dominated by the *strongest* instability in the mix, i.e. the first exponent, its backward evolution lets prevail the *weakest* instability, corresponding in this case to the k-th exponent:

$$\mathbf{F}^t \, \mathbf{q}_k \;=\; \sum_{j=1}^{k} \mathbf{u}_j \circ \mathbf{f}^t \, \mathrm{e}^{t\Lambda_j^t} \, (\mathbf{R}^{-1})_{jk} \quad \Rightarrow \quad \lim_{t\to\infty} \tfrac{1}{t} \ln \|\mathbf{F}^t \, \mathbf{q}_k\| \;=\; \Lambda_1$$

$$\mathbf{F}^{-t} \, \mathbf{q}_k \;=\; \sum_{j=1}^{k} \mathbf{u}_j \circ \mathbf{f}^{-t} \, \mathrm{e}^{-t\Lambda_j^{-t}} \, (\mathbf{R}^{-1})_{jk} \quad \Rightarrow \quad \lim_{t\to-\infty} \tfrac{1}{t} \ln \|\mathbf{F}^t \, \mathbf{q}_k\| \;=\; \Lambda_k \quad ;$$

this fact follows from the decreasing ordering $\Lambda_j > \Lambda_k$ *iff* $j < k$ and the boundedness of the coefficients $\mathbf{R}_{jk}$ (which, we recall, are functions of phase-space, not of time).
This can be formalized by combining the Oseledets' theorem results (4.29) with both the evolutions for the Oseledets' (4.42) and the Lyapunov (4.57) vectors, dropping the $_+$:

$$\begin{aligned} \mathbf{F}^t\,\mathbf{U} \quad &\equiv\; \mathbf{F}^t\,(\mathbf{Q}\,\mathbf{R}) \;=\; \big(\mathbf{Q}\circ\mathbf{f}^t\big)\,\mathbf{\Gamma}^t\,\mathbf{R} \;= \\ &=\; \big(\mathbf{U}\circ\mathbf{f}^t\big)\,\mathrm{e}^{t\mathbf{\Lambda}^t} \;\equiv\; \big(\mathbf{Q}\circ\mathbf{f}^t\big)\,\big(\mathbf{R}\circ\mathbf{f}^t\big)\,\mathrm{e}^{t\mathbf{\Lambda}^t} \quad ; \end{aligned} \tag{4.61}$$

meaning to consider the forward Oseledets' basis $\mathbf{Q}_+$, so that both coefficients $\mathbf{R}$ and cocycle $\mathbf{\Gamma}^t$ are upper triangular; the equivalences $\equiv$ denote the use of decomposition $\mathbf{U}\equiv\mathbf{QR}$. The factor $\mathbf{Q}\circ\mathbf{f}^t$ can then be canceled from (4.61), leaving the evolution:

$$\mathbf{\Gamma}^t\,\mathbf{R} \;=\; \big(\mathbf{R}\circ\mathbf{f}^t\big)\,\mathrm{e}^{t\mathbf{\Lambda}^t} \quad ; \tag{4.62}$$

which is completely analogous to the original evolution (4.57) for the CLV, with the correspondence $\mathbf{F}^t \mapsto \mathbf{\Gamma}^t$, $\mathbf{U}\mapsto\mathbf{R}$; in other words, the columns of $\mathbf{R}$ are left-invariant under the transformed cocycle $\mathbf{\Gamma}^t$. We can then turn last evolution into backward form:

$$\mathbf{R}\circ\mathbf{f}^{-t} \;=\; \big(\mathbf{\Gamma}^t\circ\mathbf{f}^{-t}\big)^{-1}\,\mathbf{R}\,\mathrm{e}^{t\mathbf{\Lambda}^t\circ\mathbf{f}^{-t}} \quad ; \tag{4.63}$$

notice that, rather than substituting $t\mapsto -t$, we change the point $\mathbf{x}$ into point $\mathbf{f}^{-t}(\mathbf{x})$; in such a way, the cocycle $\mathbf{\Gamma}^t$ is exactly the one from the forward QR evolution, but evaluated running the orbit in reverse order. Since the coefficients $\mathbf{R}$ are unknown at any starting point, we need to show that the procedure (4.63) converges; to do that, consider two different initial conditions $\widehat{\mathbf{R}}$ and $\check{\mathbf{R}}$, both evolving under (4.63), together with the matrix product $\mathbf{P} := (\widehat{\mathbf{R}})^{-1}\check{\mathbf{R}}$. For the latter we then obtain the evolution:

$$\begin{aligned} \mathbf{P}\circ\mathbf{f}^{-t} \;&=\; \mathrm{e}^{-t\mathbf{\Lambda}^t\circ\mathbf{f}^{-t}}\,\mathbf{P}\,\mathrm{e}^{t\mathbf{\Lambda}^t\circ\mathbf{f}^{-t}} \quad , \\ \Rightarrow\quad (\;\mathbf{P})_{jk}\circ\mathbf{f}^{-t} \;&=\; (\mathbf{P})_{jk}\;\mathrm{e}^{-t(\Lambda_j-\Lambda_k)\circ\mathbf{f}^{-t}} \quad ; \end{aligned} \tag{4.64}$$

whose component-wise expression shows that the non-diagonal, non-zeros entries of $\mathbf{P}$ (which is also upper-triangular) converge to zero for $t\to\infty$ whenever we have $\Lambda_j\neq\Lambda_k$:

$$\lim_{t\to\infty}(\;\mathbf{P})_{jk}\circ\mathbf{f}^{-t} \;=\; 0 \quad , \quad \forall\, j<k \quad | \quad \Lambda_j\neq\Lambda_k \quad . \tag{4.65}$$

This is because the difference $(\Lambda_j-\Lambda_k)>0$ is positive when $j<k$ (due to the ordering), i.e. precisely when $(\mathbf{P})_{jk}\neq 0$; instead, when $j>k$ one has $(\Lambda_j-\Lambda_k)<0$ and the exponential weights diverge, but the entries $(\mathbf{P})_{jk} := 0$ are zero by definition. To finally rule out the diagonal terms (where no exponential acts) recall that the CLV matrix $\mathbf{U}$ is required to be *normalized* and, due to $\mathbf{Q}$ orthonormal, so is matrix $\mathbf{R}$, meaning that:

$$\mathbf{Diag}(\mathbf{U}^T\mathbf{U}) \;\equiv\; \mathbf{Diag}(\mathbf{R}^T\mathbf{R}) \;:=\mathbf{1} \quad ; \tag{4.66}$$

in turn, since we just show that $\mathbf{P}$ converges to some diagonal matrix $\mathbf{P}_*$, one has that:

$$\begin{aligned} \mathbf{Diag}(\check{\mathbf{R}}^T\check{\mathbf{R}})_{-t} \quad &=\; \mathbf{Diag}(\mathbf{P}_*\,(\widehat{\mathbf{R}}^T\widehat{\mathbf{R}})_{-t}\,\mathbf{P}_*) \;= \\ &=\; \mathbf{P}_*\,\mathbf{Diag}(\widehat{\mathbf{R}}^T\widehat{\mathbf{R}})_{-t}\,\mathbf{P}_* \;=\; \mathbf{P}_*^2 \;:=\; \mathbf{1} \quad . \end{aligned} \tag{4.67}$$

Having $(\check{\mathbf{R}}\circ\mathbf{f}^{-t}) \simeq (\widehat{\mathbf{R}}\circ\mathbf{f}^{-t})\mathbf{P}_*$ for sufficently long t, it follows that evolution (4.63) lets converge any initial condition $\mathbf{R}$ to the actual coefficients of the CLV basis upon the Oseledets' basis. In turn, this procedure highlights the following points to be essential:

- the upper-triangular shape for both $\mathbf{R}$ and $\mathbf{\Gamma}^t$;
- the backward re-iteration along the chosen reference orbit;
- the *non-degeneracy* condition upon FTLE, for which $\Lambda_j \neq \Lambda_k$.

The last point implies that there can be no convergence, and thus no distinction, upon vectors with *exactly* the same exponent; this is coherent with the Oseledets' theorem: to each degenerate exponent corresponds a specific *higher dimensional* linear subspace. By consequence, the possible structure of the latter cannot be determined by arguments involving Lyapunov exponents. Notice that, in analogy with our sketch of a proof of the Oeledets' theorem, the exponential factors may be replaced by more general expanding/contracting functions of time: also in such case, the requirement to have *distinct* time laws $(\mathbf{X}^t)^\mu$ between subspaces is essential for their separate convergence.

Chapter 5

A Differential Setting

We now attempt to interpret the cocycle diagonalization by a different approach; to start, we consider the concept of *local invariant manifold* as described by Pesin [Pesin&Barreira], involving particular phase-space sub-sets $\mathcal{W}^{\pm}_{\mathbf{x}}$ associated to a point $\mathbf{x}$. These are defined as the sets which *converge* to the orbit of $\mathbf{x}$ in one of the two limits:

$$\mathcal{W}^{\pm}_{\mathbf{x}} \;:=\; \{\mathbf{y} \in \Omega \mid \lim_{t\to\mp\infty} \|\mathbf{f}^t(\mathbf{x}) - \mathbf{f}^t(\mathbf{y})\| = 0\} \quad ; \tag{5.1}$$

intuitively, the sets $\mathcal{W}^{\pm}_{\mathbf{x}}$ can be identified as the *span* of the stable/unstable CLV from the splitting in $\mathbf{x}$; this is suggested also by checking their *left-invariance* under the flow:

$$\mathbf{f}^t(\mathcal{W}^{\pm}_{\mathbf{x}}) \;=\; \mathcal{W}^{\pm}_{\mathbf{f}^t(\mathbf{x})} \quad , \tag{5.2}$$

As already mentioned for vector-fields, any left-invariant object may be visualized as *transported* along the flow while still keeping its properties; to make clear the difference between left-invariance and *proper invariance*, put $\mathcal{I}_{\mathbf{x}}$ to be an invariant sub-set; then:

$$\mathbf{f}^t(\mathcal{I}_{\mathbf{x}}) \;\subseteq\; \mathcal{I}_{\mathbf{x}} \tag{5.3}$$

that is, any point of an invariant sub-set is mapped into that *same* sub-set, over which the dynamics is thus confined; in the left-invariant case, instead, a different set is defined for *each* point of an orbit, and the entire sequence of sets should obey relation (5.2). At any time t, the local mainfolds of point $\mathbf{x}$ may be arbitrarily *deformed* by the flow, but such deformation is equivalenty encoded by the corresponding local manifolds for point $\mathbf{f}^t$. Notice that, to promote any sub-set to the grade of (sub-)*manifold*, this must be also required to have sufficient regularity, to allow for the notion of *differentiability*.

Then, the idea that the covariant Lyapunov vectors may span the local (left-)invariant manifolds is made more precise by linearizing them: take points $\mathbf{y}^{\pm} := (\mathbf{x} + d\mathbf{y}^{\pm}) \in \mathcal{W}^{\pm}_{\mathbf{x}}$ in one of the local manifolds of $\mathbf{x}$ and very close to it, then make use of definition (5.1):

$$\lim_{t\to\mp\infty} \|\mathbf{f}^t(\mathbf{x}) - \mathbf{f}^t(\mathbf{x} + d\mathbf{y}^{\pm})\| \;\simeq\; \lim_{t\to\mp\infty} \|\mathbf{F}^t(\mathbf{x})d\mathbf{y}^{\pm}\| \;:=\; 0 \quad ; \tag{5.4}$$

it follows that, for differentiable flows or maps, the vectors $d\mathbf{y}^{\pm}$ respectively tangent to $\mathcal{W}^{\pm}_{\mathbf{x}}$ *should* be some linear combinations of the stable/unstable CLV. This raises the question: can

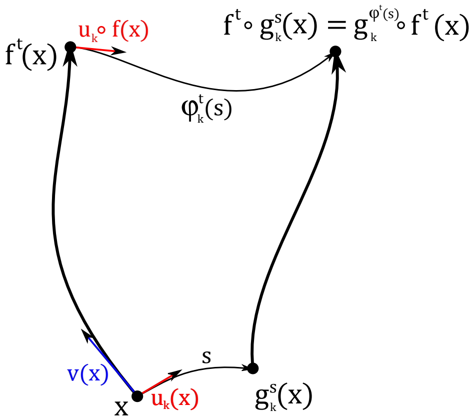

Figure 5.1: Schematic picture of the conjugation relation (5.6); the integral curve of the CLV $\mathbf{u}_k$ at $\mathbf{x}$, with arc-length s, is mapped into the integral curve of the same CLV at $\mathbf{f}^t(\mathbf{x})$, with mapped arc-length $\varphi^t(s)$.

we employ the Lyapunov vectors to induce a *parametrization* of the respective manifolds? From the introduction we can guess that any proper answer should eventually deal with the *commuting* properties of the CLV. More precisely, we can already say that *any* of the $d\mathbf{y}^{\pm}$ fullfilling (5.4) *is* a linear combination of the CLV; what may then be obstructed is the *extension* of the arc-lengths associated to each CLV flow to work as *proper* coordinates, that is, beyond first order relations.

5.1 Left-Invariant Curves

By starting from the property of left-invariance and by the fact that it is shared by each local invariant manifold *and* also each of the CLV, we wish to deduce the structure of the latter from a somewhat more detailed relation. To this end, we can first consider each separate flow $\mathbf{g}_k^s$ generated by the CLV, naturally taking them as normalized:

$$\tfrac{d}{ds}\mathbf{g}_k^s \;=\; \mathbf{u}_k \circ \mathbf{g}_k^s \quad , \quad \mathbf{g}_k^0 = \mathrm{id} \tag{5.5}$$

the orbits of these flows may be formally defined only for arc-length values s restricted to some interval but, since the regularity properties of the CLV descend from those of the velocity field, this in not an issue to be addressed at this stage. In particular, what can trouble more is that, by moving along a CLV orbit (thus changing initial condition), one may encounter different types of stability, with likely discontinuous behaviour. We argue that this is not strictly connected with the integrability properties of a single CLV flow, since the orbit-wise stability arises only after the asymptotics values of time-averages are achieved, while the Lyapunov vector-fields can be defined by the local equation (4.59).
Having this clear in mind, we thus conjecture that each of the CLV paths $\mathbf{g}_k^s$ is *conjugated* to the velocity flow $\mathbf{f}^t$ by a relation that generalizes the notion of local invariant manifold:

$$\mathbf{f}^t \;\circ\; \mathbf{g}_k^s \;=\; \mathbf{g}^{\varphi_k^t(s)} \;\circ\; \mathbf{f}^t \quad , \tag{5.6}$$

where $\varphi_k^t : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ encodes the action of the flow upon the arc-length of the k-th CLV flow at time t; the presence of such conjugating one-dimensional flows is natural, since the parameters s_k represent the CLV flows arc-lengths both before and after the action of $\mathbf{f}^t$, as illustrated in figure 5.1. In addition, the conjugating flows are also phase-space dependent: $\varphi_k^t(s) \equiv \varphi_k^t(s, \mathbf{x})$, since both the flows $\mathbf{f}^t$ and $\mathbf{g}_k^s$ act non-uniformly. This can be seen more precisely by deriving the conjugation (5.6) with respect to $s \equiv 0$:

$$\mathbf{F}^t\,\mathbf{u}_k \;=\; \mathbf{u}_k \circ \mathbf{f}^t\;\left(\tfrac{d}{ds}\varphi_k^t|_{s=0}\right) \quad\Rightarrow\quad \tfrac{d}{ds}\varphi_k^t|_{s=0} \;\equiv\; \mathrm{e}^{t\Lambda_k^t} \quad , \tag{5.7}$$

and by making use of relation (4.57) for the k-th CLV, or, column of matrix $\mathbf{U}$; we can thus deduce the formal solution to the above ODE in s for the function $\varphi_k^t(s)$:

$$\varphi_k^t(s) \;=\; \int_0^s \mathrm{e}^{t\left(\Lambda_k^t \circ \mathbf{g}_k^{s'}\right)}\,\mathrm{d}s' \quad , \tag{5.8}$$

with Λ_k^t the k-th finite-time Lyapunov exponent; this expression then shows how the phase-space dependence of $\varphi_k^t(s)$ comes from the k-th FTLE dependence. Expression (5.8) is physically reasonable, taking into account the contributions from all the exponential factors encountered by the CLV orbit $\mathbf{g}_k^{s'}$ for $0 \leqslant s' \leqslant s$; notice that such contributions correspond to an entire family of distinct orbits of the original flow $\mathbf{f}^t$.
In the nice but restricted case in which Λ_k^t is *constant* along its CLV flow, (5.8) simplifies:

$$\varphi_k^t(s) \;=\; s\,\mathrm{e}^{t\Lambda_k^t} \quad , \tag{5.9}$$

into a bare scaling of the arc-length s by the CLV exponential factor, as may be expected; trivially, this also tells that, when a CLV corresponds to a uniformly *null* exponent, the original flow commutes with that CLV flow, which is thus a symmetry of the system.
By defining the average along the CLV flow $\langle\;\rangle_{\mathbf{g}_k}^s$ of the *local* Lyapunov exponent:

$$\tfrac{d}{dt}\varphi_k^t(s)|_{t=0} \;\equiv\; s\,\langle\lambda_k\rangle_{\mathbf{g}_k}^s \;:=\; \int_0^s \lambda_k \circ \mathbf{g}_k^{s'}\,\mathrm{d}s' \quad , \tag{5.10}$$

we can similarly derive the conjugation (5.6) with respect to $t \equiv 0$ and get the relation:

$$\mathbf{v} \circ \mathbf{g}_k^s \;=\; \mathbf{\Gamma}_k^s\left(\,\mathbf{v} \;+\; s\,\mathbf{u}_k\,\langle\lambda_k\rangle_{\mathbf{g}_k}^s\,\right) \tag{5.11}$$

where we make also use of $\mathbf{\Gamma}_k^s$, the cocycle associated to the CLV flow, as defined in (1.26), and the relative property $\mathbf{u}_k \circ \mathbf{g}_k^s \equiv \mathbf{\Gamma}_k^s\,\mathbf{u}_k$. Equation (5.11) then states that the original velocity field $\mathbf{v}$ *is not* left-invariant under the action of the CLV flow $\mathbf{g}_k^s$ *unless* the averaged local exponent $\langle\lambda_k\rangle_{\mathbf{g}_k}^s$ vanishes for the prescribed arc of CLV orbit; moreover, the deviation is always parallel to the generator $\mathbf{u}_k$.

5.2 Consistency

To confirm that the conjugation equation (5.6) is consistent with the previously obtained results it is sufficient to derive it both with respect to time t and arc-length s; since it is in completely *parametric* form, the two operations should commute, that is, one can choose to derive relation (5.7) w.r.t. time t or relation (5.11) w.r.t. the arc-length s. By setting $s \equiv t \equiv 0$ in both cases, the results are the same:

$$\mathbf{M}\,\mathbf{u}_k \;=\; \dot{\mathbf{u}}_k \;+\; \mathbf{u}_k\,\lambda_k \quad , \tag{5.12}$$

and coincide with equation (4.59), thus proving that the exposed arguments lead to the correct local relation; on the other hand, the validity of conjugation (5.6) crucially depends on the actual range for the arc-lenght over which the CLV flows $\mathbf{g}_k^s$ can be integrated. In general, each of the CLV-fields is expected to separately share the same integrability conditions of the original flow $\mathbf{f}^t$ but, to our knowledge, this has never been proven. Once such conditions are known, one can then obtain the conjugation by direct integration of the CLV-defining equation (5.12), even ignoring the definitions of local invariant manifolds. The existence of the latter comes from Ergodic Theory [Pesin, Ruelle], where it is proven for a wide class of non-uniform systems; at the same time, no information is given about the intrinsic *parametrization* of such higher-dimensional sub-sets. The possible *separate* integrability of each CLV flow may suggest the use of the corresponding arc-lengths as *coordinates* for the local manifolds but, again, we remark that this is not enough: in addition, the set of CLV seen as generators should *commute*.

5.3 Local Exponent Properties

At first glance, equation (5.12) may rise concerns about the definition of the scaling factors λ_k, i.e. the local Lyapunov exponents, which in general are phase-space dependent function; indeed, some remark should be made:

- from the stability point of view, when the factor λ_k is changed by the addition of any *regular* time-derivative $\dot{W}$ the Lyapunov exponent Λ does not change:

  $$\lambda \mapsto \widetilde{\lambda} = (\lambda + \dot{W}) \quad \Rightarrow \quad \Lambda^t \mapsto \widetilde{\Lambda}^t = \Lambda^t + \tfrac{1}{t}(W \circ \mathbf{f}^t - W) \quad ,$$

  since the difference between the FTLE Λ^t and $\widetilde{\Lambda}^t$ vanishes in the infinite-time limit;

- any CLV-field is *normalized* since any bounded factor r adds to λ_k a time-derivative:

  $$\mathbf{M}(r\mathbf{u}_k) = (\dot{r\mathbf{u}_k}) + (r\mathbf{u}_k)\lambda_k \;\Rightarrow\; \mathbf{M}\mathbf{u}_k = \dot{\mathbf{u}}_k + \mathbf{u}_k\left(\lambda_k + (\dot{\ln|r|})\right) \quad ;$$

  solutions to (5.12) are then represented by *linear sub-spaces* of tangent space *independently* from $r \neq 0$, in full analogy with the solutions of any eigen-problem;

- due to normalization of $\mathbf{u}_k$, the factor λ_k can be obtained exactly as an eigen-value:

  $$\|\mathbf{u}_k\| = 1 \quad \Rightarrow \quad \mathbf{u}_k \cdot \dot{\mathbf{u}}_k = 0 \quad \Rightarrow \quad \lambda_k \;\equiv\; \mathbf{u}_k \cdot \mathbf{M}\mathbf{u}_k \quad .$$

  making clear that the degrees of freedom for the solutions are, again, the same of an eigen-problem; we stress here that the the term $\dot{\mathbf{u}}$ is the only difference between equation (5.12) and an eigen-problem, changing the latter into an algebraic PDE.

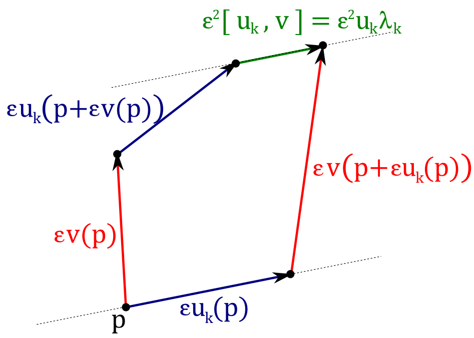

Figure 5.2: Pictorial representation of the commutation relation (5.14) between the velocity field $\mathbf{v}$ and one of the CLV-fields $\mathbf{u}_k$: such vector-field is intrinsically defined by having Lie brackets with the flow generator *parallel* to itself, with a scaling factor corresponding to λ_k, the local Lyapunov exponent associated to $\mathbf{u}_k$.

By consequence, the last point allows to *eliminate* the local Lyapunov exponent $\lambda_k \equiv \mathbf{u}_k \cdot \mathbf{M}\mathbf{u}_k$ from equation (5.12), at the cost of turning it into a *projective* PDE:

$$\dot{\mathbf{u}}_k \;:=\; v^\mu \partial_\mu \mathbf{u}_k \;=\; \left(\mathbf{1} - \mathbf{u}_k \mathbf{u}_k^T\right) \mathbf{M}\mathbf{u}_k \quad . \tag{5.13}$$

Notice that, in this form, it is no more possible to write a single equation for the whole matrix $\mathbf{U}$ of CLV but, instead, a distinct equation holds for each separate column; we are not aware of explicit methods to solve this type of equation *intended* as a PDE, i.e. such to produce phase-space dependent vector-valued solutions. To our knowledge, the only available solution to (5.12) is numerical, obtained by the standard CLV algorithm [Benettin *et al.*, Ginelli *et al.*]. This also implies that it is unknown how to obtain proper solutions defined over some full-measure sub-set of phase-space: the CLV algorithm yields excellent approximations of the unknown vector-fields *evaluated* along some orbit but, even in the case of chaotic trajectories that are *ergodic* (also called *space-filling*) in some sub-set, these would always have zero-measure.

5.4 Ladder-like Equation

The local equation (5.12), whose solutions define each CLV-field across phase-space, can be now re-stated by the language introduced in the first chapter; indeed, it should be noted that the term $\dot{\mathbf{u}}_k \equiv v^\mu \partial_\mu \mathbf{u}_k$, responsable for the PDE nature of the problem, can be written as $v^\mu \partial_\mu \mathbf{u}_k \equiv \mathbf{J}_k \mathbf{v}$, with $\mathbf{J}_k$ the k-th CLV jacobian matrix. By combining this with the Lie bracket definition in (1.41) "without hats", equation (5.12) corresponds to:

$$[\, \mathbf{u}_k \, , \, \mathbf{v} \,] \;=\; \mathbf{u}_k \, \lambda_k \quad , \tag{5.14}$$

a commutation relation between the velocity field and the CLV-field; in particular, such equation has the characteristic structure that defines $\mathbf{u}_k$ as a *ladder* (differential) operator with respect to the generator of the flow. This reminds the early relation we found in (4.41) for the symmetries of the stability matrix: here, any CLV-field associated to a null

local exponent $\lambda_k = 0$ *commutes* with the velocity filed, and thus it generates a CLV-flow corresponding to a *proper symmetry* of the main flow. Again, the origin of equation (5.14) can be ascribed to S. Lie in his early works on symmetries of PDE but, to our knowledge, no connection with invariant manifolds has ever been pointed out.
A more suggestive and precise view of the CLV local equation (5.14) comes from its operatorial version "with hats"; by considering again the associated differential operators $\hat{\mathbf{v}} := -i\, v^\mu \partial_\mu$ and $\hat{\mathbf{u}} := -i\, u^\mu \partial_\mu$, their commutation relation corresponds to:

$$[\, \hat{\mathbf{v}} \,,\, \hat{\mathbf{u}}_k \,] \;=\; i\, \lambda_k\, \hat{\mathbf{u}}_k \quad ; \tag{5.15}$$

notice the exchange in the bracket order, due to the sign flip, and the position of the function λ_k on the left side of the operator $\hat{\mathbf{u}}$, since this acts from the left; in this form, the analogy with the ladder operators of quantum mechanics is made stronger: if one can consider an eigen-function ϕ of $\hat{\mathbf{v}}$, defining it by $\hat{\mathbf{v}}[\phi] = E_\phi\, \phi$ with $E_\phi \in \mathbb{C}$ a complex eigenvalue, then can also obtain that the function $\phi_k := \hat{\mathbf{u}}_k[\phi]$ is still an eigen-function of $\hat{\mathbf{v}}$ corresponding to $\hat{\mathbf{v}}[\phi_k] = (E_\phi + i\, \lambda_k)\, \phi_k$ provided that the local exponent is a *constant*; in any other case the new factor $(E_\phi + i\, \lambda_k)$ becomes itself a function of phase-space. We are not aware of whether this has ever been related to the theory of Lyapunov functions.

5.4.1 The Velocity field is a CLV

A trivial solution for equation (5.14) is the velocity field itself, implying that it is also a CLV: this easily follows from $[\mathbf{v}, \mathbf{v}] = \mathbf{0}$ and, usually, is proven by the integral relation (1.28) with the cocycle: $\mathbf{F}^t \mathbf{v} = \mathbf{v} \circ \mathbf{f}^t$, as well known. In addition, it can be checked that *any* vector-field $r\mathbf{v}$ *proportional* to the velocity by some phase-space function r is also a CLV, satisfying: $[r\mathbf{v}, \mathbf{v}] = \mathbf{v}\dot{r}$, still with zero infinite-time Lyapunov exponent. In particular, one may choose $r = \frac{1}{\|\mathbf{v}\|}$ so to get a properly normalized CLV parallel to $\mathbf{v}$; the vanishing of the infinite-time exponent is then assured by the boundedness of the velocity. This confirms that the FTLE may always have a $\sim \frac{1}{t}$ vanishing bias, since the λ_k are unique up to time-derivatives.

5.5 Lie Point-Symmetries

The only expression related to (5.14) we were able to find regards, indeed, proper symmetries of differential systems [Stephani]: take the ODE problem $\frac{d}{dx}y = \omega(x, y)$ and define the operator $\widehat{\mathbf{a}} := \partial_x + \omega \partial_y$ which induces the PDE problem $\widehat{\mathbf{a}}[f] = 0$ for the first integrals f of the ODE; then the generators of *Lie point-symmetries* for the PDE (and thus for the ODE) are all the vector-fields $\widehat{\mathbf{w}} := \xi \partial_x + \eta \partial_y$ that satisfy:

$$[\widehat{\mathbf{w}}, \widehat{\mathbf{a}}] \;=\; \chi\, \widehat{\mathbf{a}} \quad , \tag{5.16}$$

with ξ, η, χ all functions of x, y; in this setting, the generator of symmetry $\widehat{\mathbf{w}}$ maps solutions f into new first integrals $\widetilde{f} := \widehat{\mathbf{w}}[f]$. Notice that, in our notation 'without hats', the operator $\widehat{\mathbf{a}}$ corresponds to the vector $\mathbf{a} = (1, \omega)^T$; an identical formulation holds for higher order ODE, with associated PDE and vector-fields in higher dimensions. However, in the present context it should be noted that:

- the commutation relation (5.16) is opposite to (5.14), since the proportionality of the Lie brackets falls on the main generator $\widehat{\mathbf{a}}$, not over the symmetry generator $\widehat{\mathbf{w}}$;
- equation (5.14) corresponds to (5.16) *iff* $\lambda_k = 0$ and the vector-field $\mathbf{u}_k$ is added by some multiple of the velocity field $\mathbf{v}$; this re-confirms that CLV with $\lambda_k = 0$ are generators of proper symmetries but, on the other hand, a shift of the CLV along $\mathbf{v}$ *does not* yield a CLV, and thus the analogy with (5.16) works only for $\chi = 0$.

From the stability point of view, the shift along $\mathbf{v}$ with $\chi \neq 0$ would correspond to picking a specific linear combination of CLV that is, for $\mathbf{u}_k$, a vector from the k-th sub-set of the Osledets' backward filtration.

5.6 What is the Geometry of Local Manifolds?

By the consideration made and the relations obtained so far, we arrive to the following overall picture: to any differentiable vector-field $\mathbf{v}$, considered as *the* global velocity field defining a dynamical system, may be associated a family of vector-fields $\{\mathbf{u}_k\}$ each defined as a solution to equation (5.14); depending on the value of each local Lyapunov exponent λ_k, these can be divided into two main classes: the generators of proper symmetries, which have $\lambda_k = 0$ and the generator of improper (or *non-*) symmetries, characterized by $\lambda_k \neq 0$ (again, modulo time-derivatives). In the language of stability, the former class of generators are grouped into the so-called *marginal* or *neutral* sub-set $\mathcal{N}$, while the former are divided into the *unstable* $\mathcal{U}$ and *stable* sub-set $\mathcal{S}$. These three types of sub-sets are said to *decompose* the tangent space $T_{\mathbf{x}}\mathcal{M}$ into a direct sum $T_{\mathbf{x}}\mathcal{M} = \mathcal{U}_{\mathbf{x}} \oplus \mathcal{N}_{\mathbf{x}} \oplus \mathcal{S}_{\mathbf{x}}$ corresponding to tangent directions along which perturbations respectively grow, remain the same and shrink:

$$\mathbf{U} \;=\; [\mathbf{U}_+ \,, \mathbf{U}_0 \,, \mathbf{U}_-] \;\sim\; \mathcal{U} \oplus \mathcal{N} \oplus \mathcal{S} \quad , \quad \mathbf{v} \in \mathcal{N}$$

In other words, one can associate $\mathcal{U}/\mathcal{S}$ to the positive/negative part of the Lyapunov spectrum, while $\mathcal{N}$ is obviously associated to the null exponents. The ordering induced on the CLV labeling follows from the association with the exponents in *decreasing* order; we remark here that such ordering operation is not possible on the sole basis of the values of the *local* exponents, since these can *even* change sign across phase-space: the ordering is thus induced only once the time-averages of these, i.e. the FTLE, are evaluated at *sufficiently* long times. To overcome this fact in the following formal manipulations, we can simply set the ordering of the columns of $\mathbf{U}$ and assume that it already coincides with the correct order induced by the asymptotic exponents. Apart from this issue about the *orientation* of the tangent basis, a far more deep question may be raised: which is the *geometry* of the invariant manifolds? What can be said is only that, looked in the neighborhoods of each of their points, these resemble an Euclidean space, in agreement with the picture of the CLV as a local tangent basis. In general, without specifications upon the system, little more can be said about such geometry and one can argue that, even for the weakly non-linear cases, the *global* geometry and topology of the manifolds can be highly non-trivial. This is what we wish to probe here, by the assumption that the CLV, once properly interpreted as *time-independent* and completely *flow-induced* vector-fields, constitute the natural generators of the underlying geometry.

5.7 Lie Brackets of CLV-fields

As we introduced in section 1, what carries all the information on the nature of any *local parametrization* induced by a family of vector-fields $\{\mathbf{u}_k\}$ is their ability to *commute*; in turn, we saw that this is completely encoded by their mutual Lie brackets.
Since we assume that the family $\{\mathbf{u}_k\}$, taken as the columns of the CLV matrix $\mathbf{U}$, does form a tangent basis in almost each point of phase space and, also, each Lie brackets is a vector-field on its own, it is possible to expand them over such basis almost everywhere:

$$[\,\mathbf{u}_j, \mathbf{u}_k\,] \;\equiv\; \mathbf{J}_k\mathbf{u}_j - \mathbf{J}_j\mathbf{u}_k \;=\; K^q_{jk}\mathbf{u}_q \quad . \tag{5.17}$$

The coefficients of these linear combinations are called *structure functions*, in analogy with the term 'structure constants' used in the analysis of Lie algebras: in this setting, these are not constants but *functions* of phase-space expressed by the scalar products:

$$K^q_{jk} \;=\; \mathbf{w}^q \cdot (\mathbf{J}_k\mathbf{u}_j - \mathbf{J}_j\mathbf{u}_k) \quad . \tag{5.18}$$

with $\mathbf{w}^q$ the q-th element of the CLV dual basis, i.e. a column of $\mathbf{U}^{-T}$. Notice that, for the sake of generality, we can restrict the existence of the CLV basis to only some particular sub-set of phase-space, but we always assume that this has full measure and is invariant under the flow. For non-trivial systems, anyway, there can exist zero-measure sub-sets where the CLV matrix is singular, in which the tangent basis $\mathbf{U}$ is not defined.
To know a priori the phase-space dependence of all the structure functions K^q_{jk} would bring to characterize *completely* the encoded geometry (notice that, by the anti-symmetry over j, k, these are a total of $N^2(N-1)/2$ functions); in practice, working with specific examples, this is an extremely hard result to achieve, even for quite simple systems. It is thus impossible here, working in a generic setting, to give an explicit description of these objects; nevertheless, what we can do is to analyze the *action* of the associated flow upon them: inded, this can still bring interesting results.

5.8 Local Lie Brackets of CLV

Given the relation (5.14) that *defines* the CLV-fields once the velocity, and thus the flow, is chosen, we start by looking at the relation between the vector-field $[\mathbf{u}_j, \mathbf{u}_k]$ and the velocity field; we then need the Jacobi identity:

$$\begin{aligned}
[\,[\,\mathbf{u}_j, \mathbf{u}_k\,]\,, \mathbf{v}\,] &= -[\,[\,\mathbf{u}_k, \mathbf{v}\,]\,, \mathbf{u}_j\,] - [\,[\,\mathbf{v}, \mathbf{u}_j\,]\,, \mathbf{u}_k\,] \\
&= [\,\mathbf{u}_j, [\,\mathbf{u}_k, \mathbf{v}\,]\,] + [\,[\,\mathbf{u}_j, \mathbf{v}\,]\,, \mathbf{u}_k\,] \\
&= [\,\mathbf{u}_j, \mathbf{u}_k\,\lambda_k\,] + [\,\mathbf{u}_j\,\lambda_j, \mathbf{u}_k\,] \\
&= [\,\mathbf{u}_j, \mathbf{u}_k\,](\lambda_j + \lambda_k) \;+\; \mathbf{u}_k\,\partial_j\lambda_k \;-\; \mathbf{u}_j\,\partial_k\lambda_j \quad .
\end{aligned} \tag{5.19}$$

Here we make use of the Lie brackets anti-symmetry, the CLV defining relation (5.14) and also define the shorthand $\partial_j\lambda_k \equiv \mathbf{u}_j \cdot \boldsymbol{\nabla}\lambda_k$ for the *directional derivative* along the j-th CLV-field of the k-th local Lyapunov exponent; rewriting altogether:

$$[\,[\,\mathbf{u}_j, \mathbf{u}_k\,]\,, \mathbf{v}\,] \;=\; [\,\mathbf{u}_j, \mathbf{u}_k\,](\lambda_j + \lambda_k) \;+\; \mathbf{u}_k\,\partial_j\lambda_k \;-\; \mathbf{u}_j\,\partial_k\lambda_j \quad . \tag{5.20}$$

we can deduce a first clear result: whenever *both* the local Lyapunov exponents λ_j, λ_k are constant respectively along the k-th/j-th CLV direction, the Lie bracket vector-field $[\,\mathbf{u}_j, \mathbf{u}_k\,]$ is *again* a CLV with its *own* local Lyapunov exponent the *sum* of the two exponents. Notice that it is not needed that the two local exponents are *constants*, but only that they fulfill a precise *reciprocal* symmetry by being *one* invariant under the action of the CLV-field of the *other*; moreover, such conditions should be fulfilled *separately*, since the two CLV involved are assumed to be linearly independent and thus the difference $\mathbf{u}_k\,\partial_j\lambda_k - \mathbf{u}_j\,\partial_k\lambda_j$ vanishes *iff* each of its coefficients is zero.

5.9 Operator Evolution

The relation obtained in (5.20) suggests the search for its *integrated* version, in analogy with the connection between equations (4.58) and (4.57) for the CLV-fields; to make things more clear and, mainly, less cumbersome, we highlight first a nice relation for the differential operators induced by the CLV, i.e. their versions "with hats": $\widehat{\mathbf{u}}_k := -i\, u_k^\mu \partial_\mu \equiv -i\,\mathbf{u}_k \cdot \boldsymbol{\nabla}$, which are just the directional derivatives operators along the CLV-fields. By the convenient evolution equation for $\boldsymbol{\nabla}$, the gradient operator:

$$\boldsymbol{\nabla} \;:=\; \tfrac{\partial}{\partial \mathbf{x}} \;=\; \tfrac{\partial \mathbf{f}^t}{\partial \mathbf{x}}\,\tfrac{\partial}{\partial \mathbf{f}^t} \;\equiv\; \left(\mathbf{F}^t\right)^T \left(\boldsymbol{\nabla}\circ \mathbf{f}^t\right)$$

$$\Rightarrow \qquad \boldsymbol{\nabla}\circ \mathbf{f}^t \;=\; \left(\mathbf{F}^t\right)^{-T} \boldsymbol{\nabla} \quad , \tag{5.21}$$

which is a standard result of differential geometry (regarding any type of *dual* vector, see also equation (1.30)), we can then make use of (4.57) and write down the consequent evolution equation for the CLV-fields directional derivatives:

$$\widehat{\mathbf{u}}_k \circ \mathbf{f}^t \quad := \; \left(-i\,\mathbf{u}_k \circ \mathbf{f}^t\right) \cdot \left(\boldsymbol{\nabla}\circ \mathbf{f}^t\right) \;=$$

$$=\; \mathrm{e}^{-t\Lambda_k^t}\,(-i\,\mathbf{u}_k)\cdot \left(\mathbf{F}^t\right)^T \left(\mathbf{F}^t\right)^{-T} \boldsymbol{\nabla} \;=\; \mathrm{e}^{-t\Lambda_k^t}\,(-i\,\mathbf{u}_k\cdot\boldsymbol{\nabla})$$

$$\Rightarrow \qquad \widehat{\mathbf{u}}_k \circ \mathbf{f}^t \;=\; \mathrm{e}^{-t\Lambda_k^t}\,\widehat{\mathbf{u}}_k \quad . \tag{5.22}$$

This clean outcome may then lighten some manipulations one needs but, nevertheless, we think that it may also had obscured the details of the operations performed until now; that is why we introduced it only at this stage. At any moment, it would be easy to go back to the previous notation "without hats".

5.10 Tangent Evolution of CLV Lie Brackets

Having at hand the convenient evolution (5.22) for the CLV directional derivatives, one can induce the evolution of their Lie brackets also; remember that, for any point in phase-space, the resulting operator:

$$[\,\widehat{\mathbf{u}}_j\,,\widehat{\mathbf{u}}_k\,] \;=\; \widehat{\mathbf{u}}_j\,\widehat{\mathbf{u}}_k \;-\; \widehat{\mathbf{u}}_k\,\widehat{\mathbf{u}}_j \;\equiv\; -[\,\mathbf{u}_j\,,\mathbf{u}_k\,]\cdot\boldsymbol{\nabla} \;\equiv\; -[\,\mathbf{u}_j\,,\mathbf{u}_k\,]^\mu\partial_\mu \tag{5.23}$$

is just a *commutator*, that is again a differential operator; then, to evaluate it along the flow, it is sufficient to insert the evolved factors and let them act on each other:

$$[\,\hat{\mathbf{u}}_j\,,\hat{\mathbf{u}}_k\,]\circ\mathbf{f}^t \;=\; [\,\hat{\mathbf{u}}_j\circ\mathbf{f}^t\,,\hat{\mathbf{u}}_k\circ\mathbf{f}^t\,] \;=\; [\,\mathrm{e}^{-t\Lambda_j^t}\,\hat{\mathbf{u}}_j\,,\mathrm{e}^{-t\Lambda_k^t}\,\hat{\mathbf{u}}_k\,] \;=$$

$$= \qquad \mathrm{e}^{-t(\Lambda_j^t+\Lambda_k^t)}\Big([\,\hat{\mathbf{u}}_j\,,\hat{\mathbf{u}}_k\,] \;-\; i\,t\Big(\partial_j\Lambda_k^t\,\hat{\mathbf{u}}_k \;-\; \partial_k\Lambda_j^t\,\hat{\mathbf{u}}_j\Big)\Big) \tag{5.24}$$

obtaining an equation that is consistent with its local counterpart (5.20); with the same notation, the CLV derivatives ∂_k now act upon the finite-time Lyapunov exponents Λ_j^t. Going back to the vector (or, "without hat") notation for (5.24), it is convenient to move the exponential factor to the left and make explicit all the $\boldsymbol{\nabla}$ operators: on the two hand sides of the evolution, these are evaluated at different locations (the orbit end-points).
Consequently, inserting the identity $\left(\mathbf{F}^t\right)^T\left(\mathbf{F}^t\right)^{-T}$ inside the product with $\boldsymbol{\nabla}$ on the right:

$$\mathrm{e}^{t(\Lambda_j^t+\Lambda_k^t)}\,[\,\hat{\mathbf{u}}_j\,,\hat{\mathbf{u}}_k\,]\circ\mathbf{f}^t \;\equiv\; -\,\mathrm{e}^{t(\Lambda_j^t+\Lambda_k^t)}\left([\,\mathbf{u}_j\,,\mathbf{u}_k\,]\circ\mathbf{f}^t\right)\cdot\left(\boldsymbol{\nabla}\circ\mathbf{f}^t\right) \;=$$

$$= \qquad -\Big(\,\mathbf{F}^t\Big([\,\mathbf{u}_j\,,\mathbf{u}_k\,] \;-\; i^2\,t\Big(\mathbf{u}_k\,\partial_j\Lambda_k^t \;-\; \mathbf{u}_j\,\partial_k\Lambda_j^t\Big)\Big)\,\Big)\cdot\left(\boldsymbol{\nabla}\circ\mathbf{f}^t\right) \tag{5.25}$$

allows us to eliminate the factor $\cdot\left(\boldsymbol{\nabla}\circ\mathbf{f}^t\right)$ on both sides, getting to the vector evolution:

$$[\,\mathbf{u}_j\,,\mathbf{u}_k\,]\circ\mathbf{f}^t\;\mathrm{e}^{t(\Lambda_j^t+\Lambda_k^t)} \;=\; \mathbf{F}^t\left([\,\mathbf{u}_j\,,\mathbf{u}_k\,] \;+\; t\left(\mathbf{u}_k\,\partial_j\Lambda_k^t \;-\; \mathbf{u}_j\,\partial_k\Lambda_j^t\right)\right) \quad , \tag{5.26}$$

that is now completely analogous to the CLV evolution equation $\mathbf{u}_k\circ\mathbf{f}^t\;\mathrm{e}^{t\Lambda_k^t} \;=\; \mathbf{F}^t\,\mathbf{u}_k$, though with an additive term that depends on the CLV derivatives of the finite-time Lyapunov exponents. Such term does not change between notations and, again, it agrees with the local relation (5.20) involving local exponents. In addition to that, it encodes a more clear and interesting fact: even starting an orbit from a point in which two of the CLV-fields commute (and so do their flows, inducing *proper coordinates* for a left-invariant surface), the action of the flow *breaks* such property as time evolves, to an extent that is proportional to the *non-uniformity* of one associated FTLE with respect to the CLV-field of the other. The *commutativity* of all the potential local coordinates thus depends on the directional derivatives of each FLTE along each CLV-field: pictorially, this is a subtle interplay between the CLV directions and the *level sets* geometry of the FTLE, which are hyper-surfaces orthogonal to the FTLE gradients.

5.11 Flow-induced Torsion

Since the Lie brackets quantify the *torsion* associated to a local parametrization given by flows generated by non-commuting vector-fields, and since formula (5.26) tells that the non-uniform deformations induced by a flow bring the brackets to be non-zero eventually, we refer to the additive term:

$$\mathbf{u}_k\,\partial_j\Lambda_k^t - \mathbf{u}_j\,\partial_k\Lambda_j^t \quad , \tag{5.27}$$

as a *flow-induced torsion*; it should be noticed that, in spite of the second-order nature of the Lie brackets (which can be seen as a *mutual* curvature of vector-fields), their evolution *does not* involve the second derivatives of the flow; compared to the information they bring, this is quite a virtue, especially from the computational point of view. The key ingredients of such evolution are then represented by the coefficients:

$$\partial_k \Lambda_j^t \;\equiv\; \mathbf{u}_k \cdot \boldsymbol{\nabla} \Lambda_j^t \quad , \tag{5.28}$$

which, in turn, directly depend upon the *magnitudes* $\|\boldsymbol{\nabla}\Lambda_j^t\|$ as well as on the *angle* between the CLV and the exponents' gradients; by assuming to choose an orbit where some CLV pair commutes, one of the following conditions should be satisfied to get again commutation after some time t:

1. both the associated finite-time Lyapunov exponents are *constant* along the CLV of the other, i.e. each exponent gradient is *orthogonal* to the other's Lyapunov vector;

2. the gradients of the associated finite-time Lyapunov exponents are both zero.

Condition (ii) is stronger than (i) because it requires to have both exponents *constant* along *any* direction in phase-space. Though this may happen in particular isolated points, in the general, non-uniform cases the exponents' gradients are expected to be non-zero, leaving condition (i) as the only relevant: a pair $\mathbf{u}_j, \mathbf{u}_k$ commutes *again* after a time t if the FTLE gradient $\boldsymbol{\nabla}\Lambda_j^t/\boldsymbol{\nabla}\Lambda_k^t$ is respectively orthogonal to the CLV $\mathbf{u}_k/\mathbf{u}_j$.
We remark that the CLV Lie brackets are *local* vector-valued functions of phase-space, exactly as the set of Lyapunov vectors; this is a clarification to the possible feeling that such objects might have an *explicit* time dependence, induced for example by the presence of the factor t. This is not the case and, although we avoid the explicit calculations here, it can be checked that, by combining the cocycle relations for $\mathbf{F}^t$ and $\boldsymbol{\Lambda}^t$ and the evolutions for CLV and gradients, the Lie brackets are completely left-invariant *with respect to* evolution (5.26); In turn this also means that, in the event one can prove that the CLV fields actually commute everywhere inside some flow-invariant sub-set, this also implies that the induced torsion (5.27) vanishes identically in that sub-set, and one of the two conditions (i) and/or (ii) holds there.

5.12 The Coefficients

The numerical approach to formula (5.26) may follow two possible ways: one can choose to employ computed CLV only and, by suitable phase-space discretizations, then approximate the CLV Jacobian matrices by finite-differences; this allows to compute the Lie brackets explicitly and locally, e.g. only inside prescribed regions. Alternatively, one can make use of an explicit formula for the coefficients (5.28), which can be obtained by direct differentiation of definition (4.53):

$$t\,\partial_k \Lambda_j^t \;\equiv\; \int_0^t \mathbf{u}_k \cdot \boldsymbol{\nabla}(\lambda_j \circ \mathbf{f}^\tau)\,\mathrm{d}\tau \;=\; \int_0^t \mathrm{e}^{\tau \Lambda_k^\tau}\,(\partial_k \lambda_j) \circ \mathbf{f}^\tau\,\mathrm{d}\tau \quad , \tag{5.29}$$

here the derivation can enter in the integral by assuming λ to be $\mathcal{C}^1$-regular and by considering finite times only. At this stage, one may thus restrict the analysis to the local Lyapunov exponent derivatives $\partial_k \lambda_j$, obtained again by discretization and finite-differences: this would detect the regions in which the integrand in (5.29) vanishes, to be then compared with the location of selected trajectories. In addition, notice first that evolution (5.26) can be expanded, by the action of the cocycle on the r.h.s., to give:

$$\begin{aligned} [\,\mathbf{u}_j\,,\mathbf{u}_k\,] \circ \mathbf{f}^t \qquad\qquad &= \mathbf{F}^t\,[\,\mathbf{u}_j\,,\mathbf{u}_k\,]\,\mathrm{e}^{-t(\Lambda_j^t+\Lambda_k^t)} \;+ \\ &+ (\mathbf{u}_k \circ \mathbf{f}^t)\left(t\,\partial_j \Lambda_k^t\right)\mathrm{e}^{-t\Lambda_j^t} \;-\; (\mathbf{u}_j \circ \mathbf{f}^t)\left(t\,\partial_k \Lambda_j^t\right)\mathrm{e}^{-t\Lambda_k^t} \end{aligned} \tag{5.30}$$

which is just a trivial restating, analogous to $\mathbf{u}_k \circ \mathbf{f}^t = \mathbf{F}^t \mathbf{u}_k \mathrm{e}^{-t\Lambda_k^t}$ for a single CLV; in such setting, by making use of relation (5.29), the relevant coefficients become:

$$\begin{aligned} \left(t\,\partial_k \Lambda_j^t\right)\mathrm{e}^{-t\Lambda_k^t} &= \int_0^t \mathrm{e}^{-(t\Lambda_k^t - \tau\Lambda_k^\tau)}\,(\partial_k \lambda_j) \circ \mathbf{f}^\tau\,\mathrm{d}\tau \;= \\ &= \int_0^t \left(\mathrm{e}^{-(t-\tau)\Lambda_k^{t-\tau}}\,\partial_k \lambda_j\right) \circ \mathbf{f}^\tau\,\mathrm{d}\tau \quad , \end{aligned} \tag{5.31}$$

with the second equation due to the additive cocycle property (4.51) of the FTLE: $(t - \tau)(\Lambda_k^{t-\tau} \circ \mathbf{f}^\tau) = t\Lambda_k^t - \tau\Lambda_k^\tau$; contrary to (5.29), which is a weighted average, the last expression for the coefficients is a pure average and makes explicit the fact that, for unstable FTLE $\Lambda_k^t > 0$, the most relevant contributions come from points in the *final* sector of the orbit, where one has $\tau \simeq t$. That is to say, when the directional derivative is along unstable directions and the associated FTLE $\Lambda_k^{t-\tau} \circ \mathbf{f}^\tau > 0$ is already sufficiently strong along the flow at small $(t - \tau) \simeq 0$, one can consider the zeroth-order estimate:

$$\left(t\,\partial_k \Lambda_j^t\right)\mathrm{e}^{-t\Lambda_k^t} \;\simeq\; (\partial_k \lambda_j) \circ \mathbf{f}^t \quad . \tag{5.32}$$

Notice that this is a very rough approximation which mainly justifies why one may prefer to analyze the local coefficients $\partial_k \lambda_j$ first; on the other hand, its derivation highlights new aspects upon the effects induced by non-uniform finite-time Lyapunov exponents: whenever the FTLE deviates from its asympotic value, in particular when it goes near-zero, the exponential factor brings new relevant terms in the integrand, which can be localized at arbitrary times along the orbit. This typically happens in systems with *large deviations* properties, where the convergence of time-averages, such as the Lyapunov exponent, may require extremely long times to reach a steady asymptotic value.

Chapter 6

Hamiltonian Systems

In this chapter we try to give a clear understanding of the constraints imposed by *Hamiltonian*, and thus *symplectic*, systems on the numerical calculation of the associated Oseledets' splitting. In brief, the symplectic structure needs to be transferred upon the whole CLV machinery *before* implementing it: this is because the standard algorithm leads to compute *redundant* quantities which, in turn, may also induce significant losses in the symplectic structure. By simple cosiderations upon degrees of freedom, we obtain a version of Benettin's and Ginelli's algorithms based on the algebraic Mœbius transformation for discretized time, whose equations of motion correspond to the algebraic Riccati equation. This yields the correct dimensional reduction for symplectic systems and is not just a convenient alternative to the usual method but, instead, it is the only robust way to preserve the symplectic structure of the splitting. The standard algorithm, as it stands, generates Oseledets' and CLV matrices distorted by unnecessary permutations, which are completely due to redundant degrees of freedom.

6.1 Hamiltonian Systems

The main point characterizing Hamiltonian systems is the structure of phase-space: indeed, the velocity field, and thus the dynamics, involves both the manifold $\mathcal{M}$ (the physical *configurations*) *and* its tangent spaces $T_{\mathbf{q}}\mathcal{M}, \forall\, \mathbf{q} \in \mathcal{M}$ (the physical *velocities*); that is, any *Hamiltonian flow* induces an evolution rule upon the *tangent bundle* $T\mathcal{M}$ associated to the chosen manifold. This is formalized through the set of *canonical* coordinates $(\mathbf{q}, \mathbf{p}) \in T\mathcal{M}$ with $\mathbf{q} \in \mathcal{M}$ called the *positions* and $\mathbf{p} \in T_{\mathbf{q}}\mathcal{M}$ their *conjugate momenta*; the dynamics then arises from the fundamental principle of *least action*, prescribing that the only paths $\boldsymbol{\gamma}(t) = (\mathbf{q}(t), \mathbf{p}(t))$ acceptable as *orbits* of the system are those *minimizing* the scalar functional S, called the *action*:

$$S[\boldsymbol{\gamma}] \;:=\; \int_{t1}^{t2} \big(\; \mathbf{p}(\tau) \cdot \tfrac{d}{dt}\mathbf{q}(\tau) \;-\; H(\mathbf{q}(\tau), \mathbf{p}(\tau)) \;\big)\, \mathrm{d}\tau \quad ; \tag{6.1}$$

with the real, scalar function H, called the Hamilton function or *Hamiltonian*, encoding all the properties of the system. By perturbing the path $\boldsymbol{\gamma}$ and requiring that the induced perturbation of the action *vanishes* identically, through the Euler-Lagrange formalism one arrives to the conditions for such path to be an orbit, called the *Hamilton equations*:

$$\forall\ \delta\boldsymbol{\gamma}\ ,\ \delta S[\boldsymbol{\gamma},\delta\boldsymbol{\gamma}]\ =\ 0 \quad \Rightarrow \quad \left\{ \begin{array}{rcl} \frac{d}{dt}\mathbf{q} & = & \boldsymbol{\nabla}_{\mathbf{p}}H \\ \frac{d}{dt}\mathbf{p} & = & -\boldsymbol{\nabla}_{\mathbf{q}}H \end{array} \right. \tag{6.2}$$

Thus, by considering $T\mathcal{M}$ as the actual phase-space and $\mathbf{x} := (\mathbf{q},\mathbf{p})$ as one of its points, the Hamilton equations can be reformulated through $\frac{d}{dt}\mathbf{x}_t = \mathbf{v}(\mathbf{x}_t)$, with the velocity field defined by $\mathbf{v} := \mathbf{Y}\boldsymbol{\nabla}_{\mathbf{x}}H$ and $\mathbf{x}_t \equiv \mathbf{f}^t(\mathbf{x})$ the Hamiltonian flow, while $\mathbf{Y}$ is the matrix representation of the canonical skew-symmetric form (see equation (6.5)); the latter turns the tangent bundle into a proper manifold: this is called a *symplectic* manifold. By extension, any possible symplectic object, e.g. a basis or a coordinates change, takes its definition from the property to *preserve* the symplectic structure of the phase-space.

6.2 Symplectic Matrices

A *symplectic* matrix $\mathbf{M} \in \mathrm{Symp}(2N) \subset \mathbb{R}^{2N\times 2N}$ has to be represented by *square* blocks:

$$\mathbf{M}\ =\ \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \quad , \quad \mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D} \in \mathbb{R}^{N\times N} \quad ; \tag{6.3}$$

These four blocks are *constrained* by the geometric request upon $\mathbf{M}$ to *preserve* the canonical 2-form $\omega(\mathbf{v}_1,\mathbf{v}_2)$ acting on any pair of tangent vectors $\mathbf{v}_1,\mathbf{v}_2 \in \mathbb{R}^{2N}$:

$$\omega(\mathbf{v}_1,\mathbf{v}_2)\ :=\ \mathbf{v}_1^T\mathbf{Y}\mathbf{v}_2\ =\ -\omega(\mathbf{v}_2,\mathbf{v}_1)\ \in\ \mathbb{R} \quad ; \tag{6.4}$$

that is to say, the *skew* product induced by the skew-symmetric matrix represention $\mathbf{Y}$:

$$\mathbf{Y}\ :=\ \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \quad , \quad \mathbf{0},\mathbf{1} \in \mathbb{R}^{N\times N} \quad , \tag{6.5}$$

is *invariant* under the application of matrix $\mathbf{M}$:

$$\omega(\mathbf{M}\mathbf{v}_1,\mathbf{M}\mathbf{v}_2)\ =\ \omega(\mathbf{v}_1,\mathbf{v}_2) \quad \Leftrightarrow \quad \mathbf{M} \in \mathit{Symp}(2N) \quad . \tag{6.6}$$

independently on whether such matrix is interpreted as a *basis* or as an *operator*.
By consequence, the fundamental relation defining the set of symplectic matrices arises:

$$\Rightarrow \quad \mathbf{M}^T\mathbf{Y}\mathbf{M}\ =\ \mathbf{Y} \quad . \tag{6.7}$$

and, by multiplying the last equation by $-\mathbf{Y} = \mathbf{Y}^T$ from left and right separately, having $\mathbf{Y}^T\mathbf{Y} = \mathbf{1}$, we get to the important property for which *every* symplectic matrix has an *inverse*, defined through its *transpose* by the relation:

$$\Rightarrow \quad \mathbf{M}^{-1}\ =\ \mathbf{Y}\mathbf{M}^T\mathbf{Y}^T\ =\ \mathbf{Y}^T\mathbf{M}^T\mathbf{Y} \quad . \tag{6.8}$$

In addition, every symplectic matrix has *determinant* equal to 1, as can be deduced from last equation; this allows then to check that also $\mathbf{M}^T$ sathisfies relation (6.7):

$$\Rightarrow \quad \mathbf{M}\mathbf{Y}\mathbf{M}^T\ =\ \mathbf{Y} \quad ; \tag{6.9}$$

implying that, once $\mathbf{M}$ is symplectic, also $\mathbf{M}^T$ is; since the product of two symplectic matrices is also symplectic, as can be easily checked, property (6.8) implies that the family of symplectic matrices is a *group*. The structure of the latter can be approached by expanding block-wise equation (6.7) or (6.9), respectively into one of the two *equivalent* sets of constraints upon the four square blocks:

$$\begin{cases} \mathbf{A}^T\mathbf{D} - \mathbf{C}^T\mathbf{B} = \mathbf{1} \\ \mathbf{A}^T\mathbf{C} = \mathbf{C}^T\mathbf{A} \\ \mathbf{D}^T\mathbf{B} = \mathbf{B}^T\mathbf{D} \end{cases} \Leftrightarrow \begin{cases} \mathbf{D}\mathbf{A}^T - \mathbf{B}\mathbf{C}^T = \mathbf{1} \\ \mathbf{A}\mathbf{B}^T = \mathbf{B}\mathbf{A}^T \\ \mathbf{D}\mathbf{C}^T = \mathbf{C}\mathbf{D}^T \end{cases} . \tag{6.10}$$

By the analysis of the degrees of freedom of one of the two linear systems above, it can be shown that the *dimension* of the group of symplectic matrices is $2N^2+N$; indeed, considering without loss of generality the left set of equations in (6.10), the first represents N^2 constraints, while both the second and the third each represent $N(N-1)/2$ constraints, for a total of $(2N)^2 - (N^2 + 2 \times N(N-1)/2) = 2N^2 + N$ degrees of freedom.

6.3 Degrees of Freedom

In comparison with a full matrix, uniquely identified by *four* blocks (dim $= 4N^2$), a symplectic matrix needs only *two* blocks (dim $= 2N^2$) plus an N-dimensional vector (or a diagonal $N \times N$ matrix) to be completely described. This is a well known but fundamental fact, since it implies that any *explicit* product of symplectic matrices automatically involves *redundant* operations. It is important here to understand that, in turn, such redundancy can affect any numerical approximation of symplectic matrices, every time the structure of the degrees of freedom is not taken into account.

6.3.1 Operative Example

This is a point that can be addressed through an example: suppose one has to approximate some $N \times N$ matrix function along a flow and it is known in advance that, for the class of systems under study, such matrix has actually only $m < N^2$ degrees of freedom; once supposed to explicitly know also the functional relations encoding such freedom reduction, the blind evolution of the full matrix function would be not only a waste of resources *by sizes* (from m to N^2 entries to be evolved) but also *by times* since, in the full-matrix case, the numerical errors span a space with larger dimension while (supposedly) converging to zero.

In the symplectic case, this brings simply to say that the full CLV algorithm is redundant and does not preserves nor induces the correct number of degrees of freedom (d.o.f.).

6.4 Symplectic Scalings

A first drastic consequence of considering symplectic systems is well illustrated through the structure induced by equations (6.10) on a diagonal symplectic matrix $\mathbf{\Sigma}$:

$$\mathbf{\Sigma} := \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}^{-1} \end{pmatrix} \quad , \quad \boldsymbol{\sigma} \in \mathrm{Diag}(N) \quad . \tag{6.11}$$

We remark that this has to be the form of any of the scaling matrices involved in the computation of symplectic CLV and related bases/coefficients. One typical application of diagonal matrices is to scale, or normalize, the columns of another matrix by multiplying it from the right. The immediate consequence for a symplectic matrix is that, once a first half of its columns is normalized, the magnitude of the second half is completely determined and cannot be chosen *iff* also the associated scaling matrix has to be symplectic. This is completely natural, given that the number of d.o.f. of a symplectic diagonal matrix (N) is just the half of the number of scalings needed to normalize all the columns of a $2N \times 2N$ matrix ($2N$).

6.5 Symplectic Rotations

To understand the structure of symplectic CLV we first consider a *symplectic Oseledets' orthonormal basis*, that, for now, is just the set of columns of some symplectic rotation matrix, defined by $\mathbf{Q} = \mathbf{Q}^{-T}$; this is then imposed to obey symplectic condition (6.8):

$$\mathbf{Q} \;=\; \mathbf{Y}^T\mathbf{Q}\mathbf{Y} \;=\; \mathbf{Y}\mathbf{Q}\mathbf{Y}^T \quad . \tag{6.12}$$

Since also matrix $\mathbf{Y}$ itself is a symplectic rotation and since these form a sub-group, the definition above can be restated through the simple fact that $\mathbf{Q}$ can be any rotation in $SO(2N)$ that *commutes* with rotation $\mathbf{Y}$, that is:

$$\mathbf{Y}\mathbf{Q} = \mathbf{Q}\mathbf{Y} \quad \Leftrightarrow \quad \mathbf{Q} \in (Sp(2N) \cap SO(2N)) \quad . \tag{6.13}$$

Writing explicitly definition (6.12) we obtain the reknown shape of a symplectic rotation:

$$\mathbf{Q} \;:=\; \begin{pmatrix} \mathbf{A} & -\mathbf{B} \\ \mathbf{B} & \mathbf{A} \end{pmatrix} \quad , \quad \mathbf{A}, \mathbf{B} \in \mathbb{R}^{N \times N} \quad , \tag{6.14}$$

which depends on the two square blocks $\mathbf{A}, \mathbf{B}$; due to orthonormality of $\mathbf{Q}$ or, equivalently, because of the first relation in (6.10), these blocks are not independent:

$$\mathbf{A}^T\mathbf{A} \;+\; \mathbf{B}^T\mathbf{B} \;=\; \mathbf{1} \quad \Leftrightarrow \quad \mathbf{A}\mathbf{A}^T \;+\; \mathbf{B}\mathbf{B}^T \;=\; \mathbf{1} \quad . \tag{6.15}$$

To deduce their structure we first write down their singular value decompositions (SVD):

$$\mathbf{A} \;=\; \mathbf{D}_A\mathbf{\Sigma}_A\mathbf{E}_A^T \quad , \quad \mathbf{B} = \mathbf{D}_B\mathbf{\Sigma}_B\mathbf{E}_B^T \quad . \tag{6.16}$$

with the columns of $\mathbf{D}, \mathbf{E} \in SO(N)$ respectively the left/right singular vectors and $\mathbf{\Sigma} = \mathbf{Diag}(\Sigma_j)$ the set of singular values; by equations (6.15), the symmetric matrices:

$$\widehat{\mathbf{A}} := \mathbf{A}^T\mathbf{A} \quad , \quad \widehat{\mathbf{B}} := \mathbf{B}^T\mathbf{B} \quad , \tag{6.17}$$

commute, $[\![\widehat{\mathbf{A}}, \widehat{\mathbf{B}}]\!] = [\![\widehat{\mathbf{A}}, \mathbf{1} - \widehat{\mathbf{A}}]\!] = \mathbf{0}$, and thus share a *common* orthonormal basis in $\mathbb{R}^N$ represented by the columns of $\mathbf{D} := \mathbf{D}_A \equiv \mathbf{D}_B \in SO(N)$, the same left basis for $\mathbf{A}$ and $\mathbf{B}$. The same argument then applies to the other pair of symmetric matrices:

$$\check{\mathbf{A}} := \mathbf{A}\mathbf{A}^T \quad , \quad \check{\mathbf{B}} := \mathbf{B}\mathbf{B}^T \tag{6.18}$$

for which another common orthonormal basis $\mathbf{E} := \mathbf{E}_A \equiv \mathbf{E}_B \in SO(N)$ exists. It then follows that matrices $\mathbf{A}, \mathbf{B}$ share the same left/right singular bases, but can possibly have different singular values. Indeed, the latter are constrained by (6.15) again:

$$\boldsymbol{\Sigma}_A^2 + \boldsymbol{\Sigma}_B^2 = \mathbf{1} \quad \Leftrightarrow \quad \begin{cases} \boldsymbol{\Sigma}_A = \cos(\boldsymbol{\alpha}) \\ \boldsymbol{\Sigma}_B = \sin(\boldsymbol{\alpha}) \end{cases} . \tag{6.19}$$

by which the $SO(2)^N$ structure arises, with $\boldsymbol{\alpha} := \mathbf{diag}(\alpha_j)$ a diagonal matrix of rotation angles in each of the $(q,p)_j$ planes; this allows us to explicit the blocks in equation (6.12):

$$\mathbf{A} = \mathbf{D}\cos(\boldsymbol{\alpha})\mathbf{E}^T \quad , \quad \mathbf{B} = \mathbf{D}\sin(\boldsymbol{\alpha})\mathbf{E}^T \quad ; \tag{6.20}$$

these can be checked to cover all the degrees of freedom of a symplectic rotation matrix: two matrices in $SO(N)$ (Dim=$N(N-1)/2$) plus N angles = N^2 degrees of freedom.

6.6 Sub-Groups Decomposition

The last results characterize the structure of any symplectic rotation $\mathbf{G}$ completely:

$$\mathbf{Q} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \cos(\boldsymbol{\alpha}) & -\sin(\boldsymbol{\alpha}) \\ \sin(\boldsymbol{\alpha}) & \cos(\boldsymbol{\alpha}) \end{pmatrix} \begin{pmatrix} \mathbf{E}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{E}^T \end{pmatrix} . \tag{6.21}$$

with $\mathbf{D}, \mathbf{E} \in SO(N)$ and $\boldsymbol{\alpha} = \mathbf{Diag}(\alpha_j)$, bringing to the physical interpretation: in a symplectic phase-space, points can be rotated only through sequences of *"rigid"* $SO(N)$ rotations, i.e. the same $\mathbf{D}$ acting on both the N-dimensional position and momentum:

$$\mathbf{Q}_R = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{pmatrix} , \tag{6.22}$$

and *"internal"* $SO(2)$ rotations, i.e. acting separately by each α_j in each (q_j, p_j) plane:

$$\mathbf{Q}_I = \begin{pmatrix} \cos(\boldsymbol{\alpha}) & -\sin(\boldsymbol{\alpha}) \\ \sin(\boldsymbol{\alpha}) & \cos(\boldsymbol{\alpha}) \end{pmatrix} . \tag{6.23}$$

The group property and the uniqueness of the SVD (up to permutations ruled out by the choice of $\boldsymbol{\alpha}$) then ensure that for any sequence of such two sub-groups of $(Sp(2N) \cap SO(2N))$ there exists a unique decomposition as in (6.21), and the group of symplectic rotations thus decomposes as $(Sp(2N) \cap SO(2N)) = SO(N)^2 \times SO(2)^N$.
To our knowledge, decomposition (6.21) for symplectic rotations has been known from long in the algebraic community, with main application in the classification of symplectic groups; nevertheless, it has never been mentioned explicitly[1] and, moreover, it never received the due physical interpretation. In particular, notice the *dual* role of the two types of roations: the 'internal' type *mixes* canonical coordinates and momenta (q_j, p_j) but keeps each pair *decoupled* from the others; on the opposite, the 'rigid' type mixes different coordinates and different momenta in the *same* way, but keeps the former *separate* from the latter.

[1]Private communication with Prof. Sergio Cacciatori.

6.7 Symplectic QR Decomposition

Once understood the techniques behind the proofs of the previous chapter it is straightforward to extend them to other types of matrices. Upper triangular ones represent the set over which the CLV projections upon the Oseledets' orthonormal basis take values; such projections evolve backward in time under the action of the triangular cocycle (Ginelli's algorithm), with the latter previously obtained by QR decomposition in the forward evolution of the Oseledets' orthonormal basis (Benettin's algorithm).
For this reason it is fundamental to clearly describe the structure of a generic symplectic QR decomposition, that should work upon *any* symplectic matrix.

6.8 Symplectic Triangular Matrices

To sketch out such decomposition we first consider the structure of a generic upper triangular matrix; imposing equations (6.10) induces the structure plus a constraint:

$$\mathbf{\Gamma} \;=\; \begin{pmatrix} \mathbf{R} & \mathbf{K} \\ \mathbf{0} & \mathbf{R}^{-T} \end{pmatrix} \quad , \quad \begin{cases} \mathbf{R} \;=\; \textit{upper triangular} \\ \mathbf{K}\mathbf{R}^{T} \;=\; \mathbf{R}\mathbf{K}^{T} \end{cases} \quad . \tag{6.24}$$

Notice that, by symplectic definition, the upper-triangular matrix $\mathbf{R}$ is invertible (non-zero diagonal entries) and the lower-right block is actually *lower* triangular; this feature is frequently addressed by reverting the order of colums/rows of respectively the right/lower-most blocks to obtain a fully upper-triangular shape. Such operation globally permutes *all* the matrices involved, simply transforming the symplectic constraints; since it can be applied at any stage of calculation, we avoid it.

6.9 Sub-Groups Decomposition

Relation (6.24) can be restated in a more operative way: the matrix product $\mathbf{K}\mathbf{R}^{T}$ is symmetric. This allows to write $\mathbf{K} = \mathbf{S}\mathbf{R}^{-T}$ with $\mathbf{S} = \mathbf{S}^{T}$ some symmetric matrix; any symplectic upper-triangular matrix can be thus decomposed into the product:

$$\widehat{\mathbf{\Gamma}} \;=\; \begin{pmatrix} \mathbf{1} & \mathbf{S} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}^{-T} \end{pmatrix} \quad , \quad \begin{cases} \mathbf{R} \;=\; \textit{upper triangular} \\ \mathbf{S} = \mathbf{S}^{T} \end{cases} \quad . \tag{6.25}$$

of two different symplectic upper-triangular sub-groups; these can be seen in complete correspondence with the 'internal' rotations (left-most matrix in (6.25)) and 'rigid' rotations (right-most matrix in (6.25)) sub-groups shown in (6.23) and (6.22). In the next paragraph we show how both these sub-decompositions allow to separate the evolution of symplectic bases along a flow. We remark that, by analogous arguments, the same decomposition also holds for symplectic lower-triangular matrices:

$$\check{\mathbf{\Gamma}} \;=\; \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{S} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{L}^{-T} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} \end{pmatrix} \quad , \quad \begin{cases} \mathbf{L} \;=\; \textit{lower triangular} \\ \mathbf{S} = \mathbf{S}^{T} \end{cases} \quad . \tag{6.26}$$

Notice that this is not obtained by transposition of (6.25), because here we impose to have the lower-right block in lower-triangular form, while transposing (6.25) yields it in upper-triangular form. Also this relation is important to reduce the tangent evolutions.

6.10 Symplectic Splitting

The CLV basis for symplectic flows is in general a non-orthogonal symplectic basis, with its only structure given by the separation between left-most blocks (unstable/marginal vectors) and the right-most blocks (marginal/stable vectors). The crucial point to be exploited here lies in the relation between the CLV and the Oseledets' orthonormal bases. Recall from chapter 4.3 that we have two possible (forward and backward) Oseledets' orthonormal bases, and that CLV's are defined as the intersections between sub-spaces of such bases; the underlying filtrations then induce the two defining relations:

$$\mathbf{U} \;=\; \widehat{\mathbf{Q}}\widehat{\boldsymbol{\Gamma}} \;=\; \check{\mathbf{Q}}\check{\boldsymbol{\Gamma}} \tag{6.27}$$

with $\widehat{\boldsymbol{\Gamma}}/\check{\boldsymbol{\Gamma}}$ symplectic upper/lower-triangular, while $\widehat{\mathbf{Q}}, \check{\mathbf{Q}} \in (Sp(2N) \cap SO(2N))$ are respectively the forward and backward Oseledets' orthonormal bases:

$$\widehat{\mathbf{Q}} \;=\; \begin{pmatrix} \widehat{\mathbf{A}} & -\widehat{\mathbf{B}} \\ \widehat{\mathbf{B}} & \widehat{\mathbf{A}} \end{pmatrix} \quad , \quad \check{\mathbf{Q}} \;=\; \begin{pmatrix} \check{\mathbf{B}} & \check{\mathbf{A}} \\ -\check{\mathbf{A}} & \check{\mathbf{B}} \end{pmatrix} \tag{6.28}$$

here the structure of $\check{\mathbf{Q}}$ is modified w.r.t. the original definition (6.21) (i.e. as in $\widehat{\mathbf{Q}}$), but not the sub-decomposition (6.20), meaning that $\check{\mathbf{A}} = \check{\mathbf{D}} \cos(\check{\boldsymbol{\alpha}})\, \check{\mathbf{E}}^T$, $\check{\mathbf{B}} = \check{\mathbf{D}} \sin(\check{\boldsymbol{\alpha}})\, \check{\mathbf{E}}^T$ still hold. This is simply a convention upon the angles $\check{\boldsymbol{\alpha}}$ to treat the forward/backward vectors in the *same* reference frame and cannot preclude the generality of the approach. The forward/backward ($\hat{}/\check{}$) relations in (6.27) respectively correspond to a QR/QL decomposition; while such relations hold identical for any type of differentiable dynamical system, in this case all the four matrices involved are also required to be symplectic, thus fulfilling the sub-decompositions in (6.21), (6.25) and (6.26). By gluing together the forward left-most blocks with the backward right-most blocks in (6.27) and making use of (6.28) we can rewrite the CLV matrix by a product involving both types of blocks:

$$\mathbf{U} \;=\; \begin{pmatrix} \widehat{\mathbf{A}} & \check{\mathbf{A}} \\ \widehat{\mathbf{B}} & \check{\mathbf{B}} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \check{\mathbf{R}} \end{pmatrix} \tag{6.29}$$

with $\widehat{\mathbf{R}}$ and $\check{\mathbf{R}}$ corresponding respectively to the upper/lower-triangular blocks $\mathbf{R}/\mathbf{L}$ in (6.25)/(6.26). Such decomposition still contains redundant (i.e. non-free) information; indeed, by imposing again the symplectic conditions (6.10) upon expression (6.29) for $\mathbf{U}$, we get a single constraint on the blocks, allowing to express one of them by the others:

$$\check{\mathbf{R}} \;=\; \left(\widehat{\mathbf{A}}^T\check{\mathbf{B}} \;-\; \widehat{\mathbf{B}}^T\check{\mathbf{A}}\right)^{-1} \widehat{\mathbf{R}}^{-T} \tag{6.30}$$

Actually, by dimensional considerations, last relation should lead to a further reduction: by the structure (6.20) of the blocks $\mathbf{A}, \mathbf{B}$ we can define the two symmetric matrices:

$$\begin{aligned} \widehat{\mathbf{W}} \;&:=\; \widehat{\mathbf{A}}\widehat{\mathbf{B}}^{-1} \;\equiv\; \widehat{\mathbf{D}} \cot(\widehat{\boldsymbol{\alpha}})\, \widehat{\mathbf{D}}^T \quad , \\ \check{\mathbf{W}} \;&:=\; \check{\mathbf{A}}\check{\mathbf{B}}^{-1} \;\equiv\; \check{\mathbf{D}} \cot(\check{\boldsymbol{\alpha}})\, \check{\mathbf{D}}^T \end{aligned} \tag{6.31}$$

Then equation (6.30) can be rewritten first as:

$$\check{\mathbf{B}}\check{\mathbf{R}} \;=\; \left(\widehat{\mathbf{W}} \;-\; \widecheck{\mathbf{W}}\right)^{-1} \left(\hat{\mathbf{B}}\hat{\mathbf{R}}\right)^{-T} \tag{6.32}$$

and, again by the sub-decomposition (6.20), finally as:

$$\check{\mathbf{E}}^T \check{\mathbf{R}} \;=\; \left(\sin(\hat{\boldsymbol{\alpha}})\,\hat{\mathbf{D}}^T \left(\widehat{\mathbf{W}} \;-\; \widecheck{\mathbf{W}}\right)\check{\mathbf{D}}\sin(\check{\boldsymbol{\alpha}})\right)^{-1} \hat{\mathbf{E}}^T\,\hat{\mathbf{R}}^{-T} \quad . \tag{6.33}$$

The last equation, although complicated in appearence, tells a very simple fact: the right basis $\check{\mathbf{E}} \in SO(N)$ and the lower-triangular matrix of coefficients $\check{\mathbf{R}}$ associated to the the stable part of the splitting are explicit functions of all the other degrees of freedom. Notice that the functional relation (6.33) is precisely the QL decomposition of the right-hand side, and so its uniqueness is modulo permutations. Making use of its simpler version, equation (6.32), we can finally rewrite the CLV matrix explicitly:

$$\mathbf{U} \;=\; \begin{pmatrix} \widehat{\mathbf{W}} & \widecheck{\mathbf{W}} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \left(\widehat{\mathbf{W}} - \widecheck{\mathbf{W}}\right)^{-1} \end{pmatrix} \begin{pmatrix} \left(\hat{\mathbf{B}}\hat{\mathbf{R}}\right) & \mathbf{0} \\ \mathbf{0} & \left(\hat{\mathbf{B}}\hat{\mathbf{R}}\right)^{-T} \end{pmatrix} \tag{6.34}$$

and the blocks $\widehat{\mathbf{W}}$, $\widecheck{\mathbf{W}}$ and $\hat{\mathbf{B}}$ factorized by definitions through the bases $\hat{\mathbf{D}}, \check{\mathbf{D}}, \hat{\mathbf{E}} \in SO(N)$ and the angles $\hat{\boldsymbol{\alpha}}$, $\check{\boldsymbol{\alpha}}$; in total, we have: three rotations in $\mathbb{R}^N$ $(3 \times N(N-1)/2)$, one upper-triangular $(N(N+1)/2)$ plus two sets of angles $(2N)$, for a total number of $2N^2 + N$ degrees of freedom, i.e. the one of a *full* symplectic matrix. At this stage, we slightly modify the structure (6.34) in a way that eases significantly the next numerical procedures: by considering the QR decomposition for the product $\hat{\mathbf{B}}\hat{\mathbf{R}} = \mathbf{G}\tilde{\mathbf{R}}$, with $\mathbf{G} \in SO(N)$ and $\tilde{\mathbf{R}}$ upper-triangular, the actual d.o.f. of matrix $\mathbf{U}$ are now explicit:

$$\mathbf{U} \;=\; \begin{pmatrix} \widehat{\mathbf{W}} & \widecheck{\mathbf{W}} \\ \mathbf{1} & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \left(\widehat{\mathbf{W}} - \widecheck{\mathbf{W}}\right)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{G}\tilde{\mathbf{R}} & \mathbf{0} \\ \mathbf{0} & \mathbf{G}\tilde{\mathbf{R}}^{-T} \end{pmatrix} \tag{6.35}$$

meaning that this is all what needs to be computed and there are no more implicit decompositions; in the next section, the reason for such a change will be clear.

6.10.1 Singular Sub-sets

By the assumption that all the angles $\hat{\boldsymbol{\alpha}}$ are different from zero, matrix $\hat{\mathbf{B}}$ is invertible; this can be done for any differentiable flow, essentially because the phase-space sub-sets in which (at least one of) the angles $\hat{\alpha}_j$ vanish have zero Lebesgue measure in $\mathbb{R}^{2N}$. In particular, we remark that:

1. the probability to exactly find a point in which some $\alpha_j = 0$ within the points of a *numerical* orbit is practically *zero*, even for orbits that actually cross such sub-sets.

2. in the need to study such sub-sets, definition (6.34) can be re-written through the inverse $\widehat{\mathbf{W}}^{-1} = \hat{\mathbf{D}}\tan(\hat{\boldsymbol{\alpha}})\,\hat{\mathbf{D}}^T$ (which is zero there).

3. by following point (ii), it turns out that analogous singular sub-sets exists for $\widehat{\mathbf{W}}^{-1}$, namely those in which at least one of the angles $\widehat{\alpha}_j$ is equal to $\pm\pi/2$.

Consideration (iii) brings a very important fact: to formally define the CLV matrix by using decomposition (6.35), *both* the symmetric matrices $\widehat{\mathbf{W}}$ and $\widehat{\mathbf{W}}^{-1}$ should be considered. Only once we pose ourselves into a numerical setting, point (i) then assures that such singular sub-sets will be never reached *exactly*, making possible to restrict calculations and dynamical evolutions to, e.g., matrix $\widehat{\mathbf{W}}$ only.

6.11 Symplectic Tangent Evolutions

Let us consider the symplectic cocycle $\mathbf{F}^t$, corresponding to the Jacobian matrix of the symplectic flow $\mathbf{f}^t$, writing it in a dummy form:

$$\mathbf{F}^t \;=\; \begin{pmatrix} \mathbf{F}^t_1 & \mathbf{F}^t_2 \\ \mathbf{F}^t_3 & \mathbf{F}^t_4 \end{pmatrix} \;:=\; \frac{\partial \mathbf{f}^t}{\partial \mathbf{x}} \tag{6.36}$$

with the blocks $\mathbf{F}^t_{j=1..4}$ satisfying the symplectic constraints (6.10) induced by the definition of symplectic flow. The application of the cocycle $\mathbf{F}^t$ upon the CLV matrix $\mathbf{U}$ evaluates it at point $\mathbf{f}^t(\mathbf{x})$ and scales it by the diagonal cocycle $\mathrm{e}^{t\mathbf{\Lambda}^t}$:

$$\mathbf{F}^t\,\mathbf{U} \;=\; \mathbf{U}\circ\mathbf{f}^t\,\mathrm{e}^{t\mathbf{\Lambda}^t} \tag{6.37}$$

with $\mathbf{\Lambda}^t$ the diagonal matrix of finite-time Lyapunov exponents. As shown in section 6.4, the exponential of $\mathbf{\Lambda}^t$ obeys to the symplectic constraint (6.11), and thus:

$$\mathbf{\Lambda}^t \;=\; \begin{pmatrix} \widehat{\mathbf{\Lambda}}^t & \mathbf{0} \\ \mathbf{0} & -\widehat{\mathbf{\Lambda}}^t \end{pmatrix} \quad , \quad \widehat{\mathbf{\Lambda}}^t \;:=\; \tfrac{1}{t}\int_0^t \widehat{\boldsymbol{\lambda}}\circ\mathbf{f}^\tau\,\mathrm{d}\tau \tag{6.38}$$

with $\widehat{\mathbf{\Lambda}}^t$ and $\widehat{\boldsymbol{\lambda}}$ respectively the finite-time and local Lyapunov exponents associated to the first N covariant vectors (i.e. the unstable/marginal ones).

6.12 Algebraic Mœbius Transformation

By expanding the blocks in equation (6.37) and making use of decomposition (6.35), we can also decompose the tangent evolution; to begin, let us consider the left-most blocks:

$$\begin{aligned}\left(\mathbf{F}^t_1\widehat{\mathbf{W}} + \mathbf{F}^t_2\right)\left(\mathbf{G}\widetilde{\mathbf{R}}\right) \;&=\; \left(\widehat{\mathbf{W}}\circ\mathbf{f}^t\right)\left(\left(\mathbf{G}\widetilde{\mathbf{R}}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{t\widehat{\mathbf{\Lambda}}^t} \\ \left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right)\left(\mathbf{G}\widetilde{\mathbf{R}}\right) \;&=\; \left(\left(\mathbf{G}\widetilde{\mathbf{R}}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{t\widehat{\mathbf{\Lambda}}^t}\end{aligned} \tag{6.39}$$

By point (i) of previous page and by recalling that matrix $\widehat{\mathbf{R}}$ is invertible by definition (6.24), we multiply from the right the first equation here by the inverse of the second, to get a single evolution for $\widehat{\mathbf{W}}$:

$$\widehat{\mathbf{W}}\circ\mathbf{f}^t \;=\; \left(\mathbf{F}^t_1\widehat{\mathbf{W}} + \mathbf{F}^t_2\right)\left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right)^{-1} \tag{6.40}$$

This is a non-linear map, corresponding to the *algebraic* version of the reknown Mœbius transformation (also called *linear-fractional* transformation, acting over a single variable instead of $\widehat{\mathbf{W}}$). By simple algebra, making use of the symplectic conditions (6.10) upon blocks $\mathbf{F}^t_{j=1..4}$, it can be checked that evolution (6.40) *preserves* the symmetry of matrix $\widehat{\mathbf{W}} \circ \mathbf{f}^t$. Such evolution is then unique up to the choice between using $\widehat{\mathbf{W}}$ or $\widehat{\mathbf{W}}^{-1}$, and makes the dynamics of the latter *autonomous* with respect to the matrix $\mathbf{G}\widetilde{\mathbf{R}} \equiv \widehat{\mathbf{B}}\widehat{\mathbf{R}}$.

6.13 QR & Upper-Triangular Evolutions

To complete the description of the first N (unstable/marginal) covariant Lyapunov vectors also the evolution for the upper-triangular matrix $\widehat{\mathbf{R}}$ and the orthonormal basis $\widehat{\mathbf{E}}$ is needed. To evolve $\widehat{\mathbf{W}}$ only one matrix equation has been used, derived from the pair (6.39); so one of them can still be used and, for convenience, we choose the second:

$$\left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right)\left(\mathbf{G}\widetilde{\mathbf{R}}\right) = \left(\left(\mathbf{G}\widetilde{\mathbf{R}}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{t\widehat{\mathbf{\Lambda}}^t} \tag{6.41}$$

By considering an *effective* $N \times N$ Jacobian matrix, which is a cocycle:

$$\widehat{\mathbf{J}}^t := \left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right) \quad , \tag{6.42}$$

and by defining the upper-triangular matrix, which is also a cocycle:

$$\mathbf{\Phi}^t := \widetilde{\mathbf{R}}\circ\mathbf{f}^t\ \mathrm{e}^{t\widehat{\mathbf{\Lambda}}^t}\ \widetilde{\mathbf{R}}^{-1} \tag{6.43}$$

one can follow exactly the same scheme of the Benettin's and Ginelli's algorithms, splitting evolution (6.41) into the two separate equations:

$$\widehat{\mathbf{J}}^t\ \mathbf{G} = \mathbf{G}\circ\mathbf{f}^t\ \mathbf{\Phi}^t \tag{6.44}$$

$$\mathbf{\Phi}^t\ \widetilde{\mathbf{R}} := \widetilde{\mathbf{R}}\circ\mathbf{f}^t\ \mathrm{e}^{t\widehat{\mathbf{\Lambda}}^t} \tag{6.45}$$

These are then recognized to be the evolutions for a *reduced* CLV matrix ($\mathbf{G}\widetilde{\mathbf{R}}$) governed by the Jacobian matrix $\widehat{\mathbf{J}}^t$ of an effective flow in $\mathbb{R}^N$; in particular, equation (6.44) corresponds to Benettin's forward QR evolution for an effective Oseledets' basis $\mathbf{G}$:

$$\widehat{\mathbf{J}}^t\ \mathbf{G} = \mathbf{G}\circ\mathbf{f}^t)^T\ \mathbf{\Phi}^t \equiv QR\left[\widehat{\mathbf{J}}^t\ \mathbf{G}\right] \tag{6.46}$$

by which $\mathbf{\Phi}^t$ can be determined and stored in memory; consequently, equation (6.45) corresponds to the Ginelli's backward evolution for the upper-triangular coefficients $\widetilde{\mathbf{R}}$:

$$\widetilde{\mathbf{R}}\circ\mathbf{f}^{-t} = (\mathbf{\Phi}^t\circ\mathbf{f}^{-t})^{-1}\,\widetilde{\mathbf{R}}\,\mathrm{e}^{t(\widehat{\mathbf{\Lambda}}^t\circ\mathbf{f}^{-t})} \tag{6.47}$$

Notice that, in analogy to equation (?), the diagonal entries of matrix $\mathbf{\Phi}^t$ calculated through (6.44) yield $\widehat{\mathbf{\Lambda}}^t$, the first half of the finite-time Lyapunov spectrum.

6.14 Backward Algebraic Mœbius

The CLV evolution described so far still needs a procedure for the stable/marginal block $\breve{\mathbf{W}}$; until now, we know that the formal tangent evolution for *all* the CLV should be the *same*, both in forward and in backward time. However, through the shorthand:

$$\mathbf{C} \ := \ \left(\widehat{\mathbf{W}} - \breve{\mathbf{W}}\right)^{-1} \tag{6.48}$$

the general equation (6.37) gives a relation for $\breve{\mathbf{W}}$ that is different from (6.39) :

$$\begin{aligned} \left(\mathbf{F}_1^t\breve{\mathbf{W}} + \mathbf{F}_2^t\right)\left(\mathbf{C}\,\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right) \ &= \ \left(\breve{\mathbf{W}}\circ\mathbf{f}^t\right)\left(\left(\mathbf{C}\,\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{-t\hat{\mathbf{\Lambda}}^t} \\ \left(\mathbf{F}_3^t\breve{\mathbf{W}} + \mathbf{F}_4^t\right)\left(\mathbf{C}\,\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right) \ &= \ \left(\left(\mathbf{C}\,\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{-t\hat{\mathbf{\Lambda}}^t} \ . \end{aligned} \tag{6.49}$$

but, by the same trick as in (6.40), it produces again an algebraic Mœbius transformation:

$$\breve{\mathbf{W}}\circ\mathbf{f}^t \ = \ \left(\mathbf{F}_1^t\breve{\mathbf{W}} + \mathbf{F}_2^t\right)\left(\mathbf{F}_3^t\breve{\mathbf{W}} + \mathbf{F}_4^t\right)^{-1} \tag{6.50}$$

so that, as should be expected, the functional form of the evolution for $\breve{\mathbf{W}}$ *concides* with the one for $\widehat{\mathbf{W}}$. But then, an important issue arises: since the initial conditions for both $\widehat{\mathbf{W}}$ and $\breve{\mathbf{W}}$ are unknown, how can it be that the very same evolutions (6.40) and (6.50) produce different results? Indeed, they *do not* whenever both of them are evolved in *forward* time, even when the correct initial conditions are known. The delicate point here is that the algebraic Mœbius transformation induced by the symplectic cocycle $\mathbf{F}^t$ has *two* distinct *attractors* (in the space of symmetric matrices) each associated to one of the two time directions. By considerations about stability, which are needed anyway, in the next section we show that the Mœbius evolution in *forward* and *backward* time converges respectively to the actual matrix $\widehat{\mathbf{W}}$ and $\breve{\mathbf{W}}$ associated to the reference orbit.

6.14.1 Consistency

To verify that evolution (6.50) is consistent with the additional presence of matrix $\mathbf{C}$ in equations (6.49), consider the inverse transpose of relation (6.41):

$$\left(\mathbf{F}_3^t\widehat{\mathbf{W}} + \mathbf{F}_4^t\right)^{-T}\left(\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right) \ = \ \left(\left(\mathbf{G}\,\tilde{\mathbf{R}}^{-T}\right)\circ\mathbf{f}^t\right)\mathrm{e}^{-t\mathbf{\Lambda}^t} \quad ; \tag{6.51}$$

inserting this into the second equation in (6.49) yields an evolution for matrix $\mathbf{C}$ only:

$$\mathbf{C}\circ\mathbf{f}^t \ = \ \left(\mathbf{F}_3^t\breve{\mathbf{W}} + \mathbf{F}_4^t\right)\mathbf{C}\left(\mathbf{F}_3^t\widehat{\mathbf{W}} + \mathbf{F}_4^t\right)^T \tag{6.52}$$

that, in turn, would bring again to (6.50) once re-inserted into the first relation in (6.49); the converse way to verify that the approach is correct is to explicitly calculate the evolution for matrix $\mathbf{C}$ from the evolutions (6.40) and (6.50) respectively for $\widehat{\mathbf{W}}$ and $\breve{\mathbf{W}}$. We avoid such cumbersome check here, giving a hint on its computation: evolution (6.52) is obtained from (6.40) and (6.50) *iff* the cocycle $\mathbf{F}^t$ is symplectic, as should be expected.

6.15 Effective Stability

The described set of tangent evolutions (6.40), (6.46), (6.47) and (6.50) represents a dimensional reduction of the standard CLV (Benettin's plus Ginelli's) algorithm, simply based on considerations about degrees of freedom. Since evolutions (6.46) and (6.47) correspond *exactly* to the standard CLV algorithm applied to a fictiuos flow in $\mathbb{R}^N$ with Jacobian matrix $\widehat{\mathbf{J}}^t$, their convergence properties should be the same of the original procedure. On the other hand, matrix $\widehat{\mathbf{J}}^t$ is completely induced by the non-linear evolution of $\widehat{\mathbf{W}}$ and thus the reliability of the whole procedure, including the calculation of $\widehat{\mathbf{W}}$ and $\widetilde{\mathbf{W}}$ themselves, depends on the convergence of the algebraic Mœbius transformations. It is first due to check that the effective Jacobian matrix $\widehat{\mathbf{J}}^t$, by the properties of the Mœbius evolution (6.40), fulfills the cocycle property:

$$\widehat{\mathbf{J}}^t \sim \mathbf{F}^t \quad \Rightarrow \quad \mathbf{F}^{t+\tau} = \left(\mathbf{F}^\tau \circ \mathbf{f}^t\right)\mathbf{F}^t \sim \widehat{\mathbf{J}}^{t+\tau} = \left(\widehat{\mathbf{J}}^\tau \circ \mathbf{f}^t\right)\widehat{\mathbf{J}}^t \tag{6.53}$$

Indeed, by calculating the cocycle relation for the two blocks $\mathbf{F}^{t+\tau}_{3,4}$:

$$\mathbf{F}^{t+\tau}_3 = \left(\mathbf{F}^\tau_3 \circ \mathbf{f}^t\right) \mathbf{F}^t_1 + \left(\mathbf{F}^\tau_4 \circ \mathbf{f}^t\right) \mathbf{F}^t_3 \tag{6.54}$$

$$\mathbf{F}^{t+\tau}_4 = \left(\mathbf{F}^\tau_3 \circ \mathbf{f}^t\right) \mathbf{F}^t_2 + \left(\mathbf{F}^\tau_4 \circ \mathbf{f}^t\right) \mathbf{F}^t_4 \tag{6.55}$$

and by making use of the evolution (6.40) for $\widehat{\mathbf{W}}$, one can check that the relation holds:

$$\begin{aligned}\widehat{\mathbf{J}}^\tau \circ \mathbf{f}^t\, \widehat{\mathbf{J}}^t \;&\equiv\; \left(\left(\mathbf{F}^\tau_3\widehat{\mathbf{W}} + \mathbf{F}^\tau_4\right)\circ \mathbf{f}^t\right)\left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right) = \\ &= \left(\mathbf{F}^{t+\tau}_3\widehat{\mathbf{W}} + \mathbf{F}^{t+\tau}_4\right) \equiv \widehat{\mathbf{J}}^{t+\tau}\end{aligned} \tag{6.56}$$

for any pair t, τ. By consequence, this proves that also $\widehat{\mathbf{J}}^t$ is a cocycle along the flow.

6.16 Algebraic Mœbius Stability

In order to study the convergence of *any* CLV algorithm one must first choose and *fix* one orbit of the system; perturbations should then be applied to the CLV matrix (or any of its decompositions) as *functional variations* only, meaning that nothing changes in the chosen reference orbit. In particular, the tangent evolution is a non-autonomous dynamical system that is *driven* by the entries of the cocycle (evaluated on the orbit), so that no variation should be applied to it. For the case under study, we have thus to consider only perturbations $\delta\widehat{\mathbf{W}}$ upon the symmetric matrices; by differentiating (6.40) and by using both the symmetry of $\widehat{\mathbf{W}}$ and symplecticity of $\mathbf{F}^t$, after some algebra, we arrive to the associated tangent evolution:

$$\begin{aligned}\delta\widehat{\mathbf{W}} \circ \mathbf{f}^t \;&=\; \left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right)^{-T} \delta\widehat{\mathbf{W}} \left(\mathbf{F}^t_3\widehat{\mathbf{W}} + \mathbf{F}^t_4\right)^{-1} \\ &\equiv\; (\,\widehat{\mathbf{J}}^t\,)^{-T}\, \delta\widehat{\mathbf{W}}\, (\,\widehat{\mathbf{J}}^t\,)^{-1}\end{aligned} \tag{6.57}$$

To prove that the algebraic Mœbius transformation (6.40) converges for almost-any initial condition $\widehat{\mathbf{W}}_0 \equiv \widehat{\mathbf{W}}(\mathbf{x}_0)$ means to prove that its action upon the space of symmetric matrices is strictly *contractive*; in turn, this obviously requires that $\delta\widehat{\mathbf{W}} \circ \mathbf{f}^t$ tends to the null matrix $\mathbf{0}$ as $t \to \infty$. It is exactly at this stage that the actual properties of the cocycle $\widehat{\mathbf{J}}^t$ come into play: whenever we can prove that the cocycle possess only *one* marginal covariant vector, i.e. the Hamiltonian vector-field itself, then we can also infer that the action of the tangent dynamics is *asymptotically* contracting upon $N-1$ covariant vectors. In principle, one may be already settled with the fact that, in forward time, the effective Jacobian $\widehat{\mathbf{J}}^t$ can at most represent the expansive action of the tangent dynamics, and so evolution (6.57) can only be contractive or marginal. In detail, this can be seen by combining the perturbations evolution (6.57) with relation (6.41) coming from the original tangent dynamics; the latter can be inverted to produce an expression:

$$(\,\widehat{\mathbf{J}}^t\,)^{-1} \;=\; \left(\mathbf{G}\widetilde{\mathbf{R}}\right)\,\mathrm{e}^{-t\widehat{\mathbf{\Lambda}}^t}\,\left(\left(\mathbf{G}\widetilde{\mathbf{R}}\right)\circ\mathbf{f}^t\right)^{-1} \tag{6.58}$$

to be inserted, along with its transpose, into (6.57); in doing that, we can move the products $\mathbf{G}\widetilde{\mathbf{R}}$ from one side to the other and define the auxiliary symmetric matrix:

$$\widehat{\mathbf{M}} \;:=\; \left(\mathbf{G}\widetilde{\mathbf{R}}\right)^T\,\delta\widehat{\mathbf{W}}\,\left(\mathbf{G}\widetilde{\mathbf{R}}\right) \;=\; \widehat{\mathbf{M}}^T \tag{6.59}$$

This finally turns the perturbation evolution (6.57) into the nice expression:

$$\widehat{\mathbf{M}}\circ\mathbf{f}^t \;=\; \mathrm{e}^{-t\widehat{\mathbf{\Lambda}}^t}\,\widehat{\mathbf{M}}\,\mathrm{e}^{-t\widehat{\mathbf{\Lambda}}^t} \tag{6.60}$$

Although we do not know the explicit behaviour of the products $\mathbf{G}\widetilde{\mathbf{R}} \equiv \widehat{\mathbf{B}}\widehat{\mathbf{R}}$ (these can be actually approximated only *after* the calculation of $\widehat{\mathbf{W}}$) we can assure that they cannot *diverge*, since $\widehat{\mathbf{R}}$ is invertible by symplectic definition while $\widehat{\mathbf{B}}$ has N sine functions as singular values and, consequently, is bounded.
We can thus conclude that, if $\widehat{\mathbf{M}}$ converges to the null matrix $\mathbf{0}$ as $t \to \infty$, then also $\delta\widehat{\mathbf{W}}$ does. Then, by writing evolution (6.60) for each matrix component $\widehat{M_{jk}}$:

$$\widehat{M_{jk}}\circ\mathbf{f}^t \;=\; \widehat{M_{jk}}\;\mathrm{e}^{-t(\widehat{\Lambda}^t_j+\widehat{\Lambda}^t_k)} \tag{6.61}$$

the convergence problem transfers completely upon each element of the finite-time Lyapunov spectrum; since the diagonal matrix $\widehat{\mathbf{\Lambda}}^t$ contains the positive/null exponents, it follows that any component $\widehat{M_{jk}}$ with at least $\widehat{\Lambda}^t_j > 0$ or $\widehat{\Lambda}^t_k > 0$ converges to 0.
In the case there is only one marginal CLV, namely the Hamiltonian vector-field $\mathbf{Y}\boldsymbol{\nabla}H$ (the N-th), the only non-convergent component is M_{NN}; this would be called the (non-uniform) *hyperbolic* case.

6.16.1 Degenerate Case

In the same way, when more than one CLV (say, m of them) are marginal (with $\widehat{\mathbf{\Lambda}}^t$ containing m null exponents, labeled from N-m+1 to N) the non-convergent components form an $m \times m$ symmetric block $\widehat{\mathbf{M}}_0$ that is flow-invariant:

$$\widehat{\mathbf{M}}_0 \circ \mathbf{f}^t \;=\; \widehat{\mathbf{M}}_0 \tag{6.62}$$

This implies that the block matrix $\widehat{\mathbf{M}}_0$ does not converge to zero and none of the null exponents CLV can be approximated. One may wish to isolate the non-convergent block in order to avoid the marginal entries to propagate in the full $\delta\widehat{\mathbf{W}}$. By matrix-product considerations, it turns out that the conditions fall completely upon the columns of matrix $\mathbf{G}$: in particular, one should put equal to zero the last m columns of such orthogonal matrix but, since the evolution for $\widehat{\mathbf{W}}$ is independent from such matrix, one can conclude that its convergence will follow the weakest instability of the system.

6.16.2 Backward Mœbius stability

Finally, the same conditions obtained for the $\widehat{\mathbf{W}}$ forward evolution (6.40) apply also to the equivalent evolution (6.50) for $\widecheck{\mathbf{W}}$; since the latter induces an effective cocycle $\check{\mathbf{J}}^t :=$ $(\mathbf{F}_3^t\widecheck{\mathbf{W}} + \mathbf{F}_4^t)$ which is associated to the contracting action of the dynamics and since the perturbations $\delta\widecheck{\mathbf{W}}$ evolve as in (6.57):

$$\delta\widecheck{\mathbf{W}} \circ \mathbf{f}^t \;=\; (\,\check{\mathbf{J}}^t\,)^{-T}\;\delta\widecheck{\mathbf{W}}\;(\,\check{\mathbf{J}}^t\,)^{-1} \tag{6.63}$$

it is apparent that the only possible condition for the convergence of evolution (6.50) is to evolve it in *backward* time; it is then natural to exploit the inversion relation for symplectic matrices:

$$\widecheck{\mathbf{W}} \circ \mathbf{f}^{-t} \;=\; \left(\mathbf{F}_{4*}^t\widecheck{\mathbf{W}} \,-\, \mathbf{F}_{2*}^t\right)\left(-\mathbf{F}_{3*}^t\widecheck{\mathbf{W}} \,+\, \mathbf{F}_{1*}^t\right)^{-1} \tag{6.64}$$

with the suitable redefinition of the inverted cocycle blocks $\mathbf{F}_{j*}^t := (\mathbf{F}_j^t \circ \mathbf{f}^{-t})^T$; notice that these are the transposes of the same blocks employed in the forward evolution (6.40), simply evaluated running backward the reference orbit.

6.16.3 Pseudocode

To summarize the complete procedure illustrated so far we now write down its main steps, assuming the knowledge of the cocycle blocks obtained e.g. by first order approximation for very short time-lapses ΔT; in doing that, let us put $t = n\Delta t$, define respectively $\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n, \mathbf{D}_n := \mathbf{F}^{\Delta t}_{1,2,3,4} \circ \mathbf{f}^{n\Delta t}$ and also $f_n := f \circ \mathbf{f}^{n\Delta t}$ for any function f of phase-space. Then, the forward part of the algorithm reads:

for $n = 1 \ldots M-1$

$$\begin{aligned} \mathbf{J}_n &= \mathbf{D}_n \widehat{\mathbf{W}}_n + \mathbf{C}_n \\ \widehat{\mathbf{W}}_{n+1} &= (\mathbf{A}_n \widehat{\mathbf{W}}_n + \mathbf{B}_n)(\mathbf{J}_n)^{-1} \\ \mathbf{G}_{n+1}\boldsymbol{\Phi}_n &= QR[\mathbf{J}_n \mathbf{G}_n] \end{aligned} \tag{6.65}$$

endfor

Here the effective cocycle $\mathbf{J}_n := \widehat{\mathbf{J}}^{\Delta t} \circ \mathbf{f}^{n\Delta t}$ is discarded, but the upper-triangular one $\boldsymbol{\Phi}_n := \boldsymbol{\Phi}^{\Delta t} \circ \mathbf{f}^{n\Delta t}$ should be registered in memory for the backward iteration:

for $n = M \ldots 2$

$$\begin{aligned} \breve{\mathbf{W}}_{n-1} &= (\mathbf{D}^T_{n-1} \breve{\mathbf{W}}_n - \mathbf{B}^T_{n-1})(-\mathbf{C}^T_{n-1} \breve{\mathbf{W}}_n + \mathbf{A}^T_{n-1})^{-1} \\ \mathbf{R}_* &= (\boldsymbol{\Phi}_{n-1})^{-1} \mathbf{R}_n \\ \mathbf{R}_{n-1} &= \mathbf{R}_{n-1}\, \mathbf{Diag}(\mathbf{R}^T_* \mathbf{R}_*)^{-\frac{1}{2}} \end{aligned} \tag{6.66}$$

endfor

The dummy variable $\mathbf{R}_*$ is just needed to perform the column-wise normalization of the next backward step matrix $\mathbf{R}_{n-1} := \widetilde{\mathbf{R}} \circ \mathbf{f}^{(n-1)\Delta t}$ and should not evolve in time.
Notice that this modified procedure differs from the standard one by the presence of the Mœbius transformation, and in the fact it operates with $N \times N$ matrices only instead of the full $2N \times 2N$; eventually, one can recast the CLV matrix by decomposition (6.35).

Conclusions

In the present thesis we give an overview of the principal mathematical tools which allow to prove the existence of the Oseledets' splitting for differentiable flows and maps; in doing that, we try to give an operative interpretation to the illustrated results by translating them into matrix language, with the aim to make clear the connections with explicitly computable procedures. In particular, we treat the most profound implication of the Oseledets' theorem as a precise condition upon the matrix coefficients that encode the Lyapunov vectors as linear combinations of the Oseledets' orthonormal bases.

In addition to that, we also seek for a wider picture into which the Lyapunov vectors may be framed: it turns out that these should be interpreted as *static* vector-fields of phase-space, whose structure is completely induced by the generator of the flow, i.e. the velocity field. As a byproduct, this identifies the concept of *local* left-invariant manifold as the correct subspace spanned by *flows* generated by the Lyapunov vector-fields; depending on the associated stability exponent, such auxiliary flows may be then considered as generators of regular or broken symmetries of the system.

Finally, given such connection, we find the conditions under which the families of trajectories of Lyapunov vectors flows may be also interpreted as *proper coordinates*: even where a local tangent basis of Lyapunov vectors can be defined, the employment of the associated orbits as curvilinear axes (parametrizing initial conditions) cannot be extended beyond first order unless the Lie brackets of the Lyapunov vector-fields vanish. More importantly, the obstruction to such procedure is quantified by the *non-uniformity* of the local and finite-time Lyapunov exponents: indeed, by translation along the flow, the Lie brackets gain additional terms proportional to the projections of the exponents' gradients upon each of the Lyapunov vectors; we named this term *flow induced torsion.*

Bibliography

L. Barreira, Y. Pesin *Smooth Ergodic Theory and Nonuniformly Hyperbolic Dynamics* , Handbook of Dynamical Systems 1B, 57-263, Elsevier North Holland, 2005

L. Barreira and Y. Pesin: *Lyapunov Exponents and Smooth Ergodic Theory* , University Lecture Series 23, American Mathematical Society, 2001

G. Benettin, L. Galgani, A. Giorgilli, and J. M. Strelcyn, *Lyapunov characteristic exponents for smooth dynamical systems: a method for computing all of them* , Meccanica 15 9:15, 9:21 (1980).

S.V. Ershov, A.B. Potapov. *On the concept of Stationary Lyapunov basis* , Physica D, 118 (1998) 167-198.

F. Ginelli, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, *Characterizing dynamics with covariants Lyapunov vectors* , Phys Rev Lett 99, 130601 (2007).

V.I. Oseledets, *A multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems* , Trans. Moscow Math. Soc. 19:197 (1968).

D.Ruelle, *Ergodic theory of differentiable dynamical systems* , I.H.E.S. 50 (1979)

M. Spivak, *Calculus on manifolds* , Westview Press, (1971)

J. M. Steele, *Kingman's subadditive ergodic theorem* , Annales de l'Institute Henri Poincar, 25 (1989), 93–98.

H. Stephani *Differential equations. Their solutions using symmetries* , CUP (1989)

S. Wiggings *Chaotic Transport in Dynamical Systems* , Springer-Verlag GmbH (1992)