Discrete Mathematics 336 (2014) 80–84

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Strongly intersecting integer partitions

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ARTICLE INFO

Article history:
Received 20 July 2013
Received in revised form 8 March 2014
Accepted 26 July 2014
Available online 23 August 2014

Keywords:
Intersecting family
Integer partition
Strongly intersecting set

ABSTRACT

We call a sum $a_1+a_2+\cdots+a_k$ a *partition of* n *of length* k if $a_1, a_2, \ldots, a_k$ and n are positive integers such that $a_1 \le a_2 \le \cdots \le a_k$ and $n = a_1 + a_2 + \cdots + a_k$. For $i = 1, 2, \ldots, k$, we call a_i the *ith part* of the sum $a_1 + a_2 + \cdots + a_k$. Let $P_{n,k}$ be the set of all partitions of n of length k. We say that two partitions $a_1+a_2+\cdots+a_k$ and $b_1+b_2+\cdots+b_k$ *strongly intersect* if $a_i = b_i$ for some i. We call a subset A of $P_{n,k}$ *strongly intersecting* if every two partitions in A strongly intersect. Let $P_{n,k}(1)$ be the set of all partitions in $P_{n,k}$ whose first part is 1. We prove that if $2 \le k \le n$, then $P_{n,k}(1)$ is a largest strongly intersecting subset of $P_{n,k}$, and uniquely so if and only if $k \ge 4$ or $k = 3 \le n \notin \{6, 7, 8\}$ or $k = 2 \le n \le 3$.

1. Introduction

Unless otherwise stated, we shall use small letters such as x to denote positive integers or functions or elements of a set, capital letters such as X to denote sets, and calligraphic letters such as $\mathcal{F}$ to denote *families* (that is, sets whose elements are sets themselves). We call a set A an r*-element set* if its size $|A|$ is r (that is, if it contains exactly r elements). For any integer $n \ge 1$, the set $\{1, \ldots, n\}$ of the first n positive integers is denoted by $[n]$.

In the literature, a sum $a_1+a_2+\cdots+a_k$ is said to be a *partition of* n *of length* k if $a_1, a_2, \ldots, a_k$ and n are positive integers such that $n = a_1+a_2+\cdots+a_k$. If $a_1+a_2+\cdots+a_k$ is a partition, then $a_1, a_2, \ldots, a_k$ are said to be its *parts*. Two partitions that differ only in the order of their parts are considered to be the same. Thus, we can refine the definition of a partition as follows. We call a tuple $(a_1, \ldots, a_k)$ a *partition of* n *of length* k if $a_1, \ldots, a_k$ and n are positive integers such that $n = \sum_{i=1}^{k} a_i$ and $a_1 \le \cdots \le a_k$. We will be using the latter definition throughout the rest of the paper.

For any n, let P_n be the set of all partitions of n, and for any k, let $P_{n,k}$ be the set of all partitions of n of length k. Thus, $P_{n,k}$ is non-empty if and only if $1 \le k \le n$. Moreover, $P_n = \bigcup_{i=1}^{n} P_{n,i}$. For any set A of integer partitions, let $A(1)$ denote the set of all partitions in A which have 1 as their first entry. Thus

$$P_{n,k}(1) = \{(a_1, \ldots, a_k) \in P_{n,k} : a_1 = 1\} \quad \text{and} \quad P_n(1) = \bigcup_{i=1}^{n} P_{n,i}(1).$$

Note that $|P_n(1)| = |P_{n-1}|$ and $|P_{n,k}(1)| = |P_{n-1,k-1}|$. To the best of the author's knowledge, no closed-form expression is known for $|P_n|$ and $|P_{n,k}|$; for more about these values, we refer the reader to [4].

We say that $(a_1, \ldots, a_r)$ *strongly intersects* $(b_1, \ldots, b_s)$ if $a_i = b_i$ for some $i \le \min\{r, s\}$. If A is a set of integer partitions such that every two partitions in A strongly intersect (that is, for every $\mathbf{a}, \mathbf{b} \in A$, $\mathbf{a}$ strongly intersects $\mathbf{b}$), then we say that A is *strongly intersecting*.

It is natural to ask how large a strongly intersecting subset of $P_{n,k}$ or P_n can be. We provide the answer to this question and also determine the extremal structures. The classical Erdős–Ko–Rado (EKR) Theorem [28] inspired many problems and

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http://dx.doi.org/10.1016/j.disc.2014.07.018
0012-365X/

results of this kind in extremal set theory; see [12,14,24,30,31]. $P_{n,k}$ is a subset of the set $[n]^k$ of all k-tuples with entries in $[n]$; the problem for strongly intersecting subsets of $[n]^k$ attracted much attention (see, for example, [2,5,11,32,33,37,46,52]) and is completely solved [2,33]. A weaker definition of intersection for integer partitions simply requires that they have at least one common part; more precisely, we say that $(a_1, \dots, a_r)$ *intersects* $(b_1, \dots, b_s)$ if $a_i = b_j$ for some $i \in [r]$ and $j \in [s]$. The problem based on this definition is treated in [9] and turns out to be significantly more difficult; it is solved for n sufficiently large depending on k.

The following is our first result.

Theorem 1.1. *If $2 \le k \le n$ and A is a strongly intersecting subset of $P_{n,k}$, then*

$$|A| \le |P_{n-1,k-1}|,$$

and equality holds if $A = P_{n,k}(1)$.

Proof. Let $f : A \to P_{n,k}(1)$ be the function that maps $(a_1, \dots, a_k) \in A$ to the partition $(a'_1, \dots, a'_k)$ with $a'_k = a_k + (k-1)(a_1-1)$ and $a'_i = a_i - (a_1 - 1)$ for each $i \in [k-1]$ (note that, since $a'_1 = 1$ and $a_1 \le a_2 \le \dots \le a_k$, we indeed have $(a'_1, \dots, a'_k) \in P_{n,k}(1)$).

Suppose that $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k)$ are partitions in A that are mapped by f to the same partition $(c_1, \dots, c_k)$. Thus $a_k + (k-1)(a_1-1) = c_k = b_k + (k-1)(b_1-1)$ and $a_i - (a_1-1) = c_i = b_i - (b_1-1)$ for each $i \in [k-1]$. Therefore, $b_k = a_k + (k-1)(a_1-b_1)$ and $b_i = a_i - (a_1 - b_1)$ for each $i \in [k-1]$. Since A is strongly intersecting, we have $a_j = b_j$ for some $j \in [k]$, and hence $a_1 - b_1 = 0$. Thus $b_i = a_i$ for each $i \in [k]$, and hence $(a_1, \dots, a_k) = (b_1, \dots, b_k)$.

Therefore, f is an injective function, and hence the size of the domain A of f is at most the size of the co-domain $P_{n,k}(1)$ of f. $\Box$

In the next section, we also determine precisely when $P_{n,k}(1)$ is the only strongly intersecting subset of $P_{n,k}$ that attains the bound above. It turns out that this holds for $k \ge 4$, and also for $k = 3$ unless $6 \le n \le 8$.

Theorem 1.2. *For $2 \le k \le n$, $P_{n,k}(1)$ is the unique strongly intersecting subset of $P_{n,k}$ of maximum size if and only if $k \ge 4$ or $k = 3 \le n \notin \{6, 7, 8\}$ or $k = 2 \le n \le 3$.*

From Theorem 1.1 we obtain the following.

Theorem 1.3. *For $n \ge 1$, $P_n(1)$ is a strongly intersecting subset of P_n of maximum size, and uniquely so unless $n = 2$.*

Proof. The result is trivial for $n = 1$. If $n = 2$, then $P_n(1) = \{(1, 1)\}$ and $\{(2)\}$ are the only two strongly intersecting subsets of P_n. Now consider $n \ge 3$. Let A be a strongly intersecting subset of P_n. For each $k \in [n]$, let $A_k = A \cap P_{n,k}$. Thus $A_1, \dots, A_n$ are strongly intersecting, and $|A| = \sum_{k=1}^{n} |A_k|$. Let $\mathbf{a} \in P_{n,1}$. Thus $\mathbf{a} = (n)$. No partition in $P_n \backslash \{\mathbf{a}\}$ strongly intersects $\mathbf{a}$. Thus, if $\mathbf{a} \in A$, then $A = \{\mathbf{a}\}$, and hence $|A| = 1 < |P_n(1)|$. Now suppose $\mathbf{a} \notin A$. Thus $A_1 = \emptyset$ (as $P_{n,1} = \{\mathbf{a}\}$). By Theorem 1.1, $|A_k| \le |P_{n,k}(1)|$ for each $k \in [n]$. Thus we have $|A| = \sum_{k=2}^{n} |A_k| \le \sum_{k=2}^{n} |P_{n,k}(1)| = |P_n(1)|$. $P_{n,n}$ has only one partition $\mathbf{e}$, namely $\mathbf{e} = (1, \dots, 1)$. If $\mathbf{e} \in A$, then each partition in A strongly intersects $\mathbf{e}$, and hence $A \subseteq P_n(1)$. If $\mathbf{e} \notin A$, then $A_n = \emptyset$, and hence $|A| = \sum_{k=2}^{n-1} |A_k| \le \sum_{k=2}^{n-1} |P_{n,k}(1)| < \sum_{k=2}^{n} |P_{n,k}(1)| = |P_n(1)|$. $\Box$

As indicated above, Theorem 1.1 is an analogue of the EKR Theorem [28]. A family $\mathcal{A}$ of sets is said to be *intersecting* if every two sets in $\mathcal{A}$ intersect (that is, if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{A}$). For any set X, let 2^X denote the *power set of* X (that is, the family of all subsets of X), and let $\binom{X}{r}$ denote the family of all r-element subsets of X. The EKR Theorem says that if $r \le n/2$ and $\mathcal{A}$ is an intersecting subfamily of $\binom{[n]}{r}$, then $|\mathcal{A}| \le \binom{n-1}{r-1}$, and equality holds if $\mathcal{A} = \{A \in \binom{[n]}{r} : 1 \in A\}$. Theorem 1.3 is analogous to another well-known result in [28], which says that if $\mathcal{A}$ is an intersecting subfamily of $2^{[n]}$, then $|\mathcal{A}| \le 2^{n-1}$, and equality holds if $\mathcal{A} = \{A \in 2^{[n]} : 1 \in A\}$.

Theorems 1.1–1.3 can also be phrased in terms of intersecting subfamilies of a family. For any integer partition $\mathbf{a} = (a_1, \dots, a_k)$, let $S_{\mathbf{a}}$ be the set $\{(1, a_1), \dots, (k, a_k)\}$. Let $\mathcal{P}_n = \{S_{\mathbf{a}} : \mathbf{a} \in P_n\}$ and $\mathcal{P}_{n,k} = \{S_{\mathbf{a}} : \mathbf{a} \in P_{n,k}\}$. There is a one-to-one correspondence between $\mathcal{P}_n$ and P_n, and similarly for $\mathcal{P}_{n,k}$ and $P_{n,k}$. Clearly, two integer partitions $\mathbf{a}$ and $\mathbf{b}$ strongly intersect if and only if $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ intersect. Thus, Theorems 1.1 and 1.2 say that for $2 \le k \le n$, $\{A \in \mathcal{P}_{n,k} : (1, 1) \in A\}$ is a largest intersecting subfamily of $\mathcal{P}_{n,k}$, and uniquely so if and only if $k \ge 4$ or $k = 3 \le n \notin \{6, 7, 8\}$ or $k = 2 \le n \le 3$. Theorem 1.3 says that $\{A \in \mathcal{P}_n : (1, 1) \in A\}$ is a largest intersecting subfamily of $\mathcal{P}_n$, and uniquely so unless $n = 2$.

EKR-type results have been obtained for families that have a symmetric structure (see [16, Section 3.2], [58]) and have sizes that are known precisely (such as the family of r-element subsets of a set [1,22,28,29,45,59], families of permutations/injections [13,19,20,23,25,35,47,49–51,57], families of integer sequences/functions/labeled sets/signed sets [2,5–8,10,11,13,24,26,27,32,33,37,46,52,53], and families of vector spaces [24,34,36,41]) or have a structure that enables the use of the compression technique [30,39,43] and induction (as are power sets [3,28,44], certain hereditary families [15,21,54,55], families of separated sets [56], families of independent r-element sets of certain graphs [17,18,38–40,42,60], and families of set partitions [48]). One of the main motivating factors behind this paper is that although the families $\mathcal{P}_n$ and $\mathcal{P}_{n,k}$ do not have any of these structures and we do not even know their sizes precisely, we have a complete characterisation of their largest intersecting subfamilies (note that by Theorem 1.2 it only takes a straightforward exhaustive check to determine the extremal subfamilies for the cases in which $\mathcal{P}_{n,k}(1)$ is not the unique largest intersecting subfamily of $\mathcal{P}_{n,k}$.).

We proceed by giving the proof of Theorem 1.2. Then, in Section 3, we suggest a conjecture as a natural generalisation of Theorem 1.1.

2. Proof of Theorem 1.2

This section is entirely dedicated to the proof of Theorem 1.2, which is obtained by extending the proof of Theorem 1.1.

Proof of Theorem 1.2. Consider first $k = 2$. $P_{n,2}(1)$ consists of the partition $(1, n-1)$ only. If $2 \le n \le 3$, then $P_{n,2} = P_{n,2}(1)$. If $n \ge 4$, then $(2, n-2)$ is a partition in $P_{n,2}$, and hence $\{(2, n-2)\}$ is a strongly intersecting subset of $P_{n,2}$ of size $|P_{n,2}(1)| = 1$.

Next, consider $k = 3$ and $n \in \{6, 7, 8\}$. We have that $\{(1, 2, 3), (2, 2, 2)\}$ is a strongly intersecting subset of $P_{6,3}$ that is as large as $P_{6,3}(1) = \{(1, 1, 4), (1, 2, 3)\}$, $\{(1, 2, 4), (1, 3, 3), (2, 2, 3)\}$ is a strongly intersecting subset of $P_{7,3}$ that is as large as $P_{7,3}(1) = \{(1, 1, 5), (1, 2, 4), (1, 3, 3)\}$, and $\{(1, 2, 5), (1, 3, 4), (2, 2, 4)\}$ is a strongly intersecting subset of $P_{8,3}$ that is as large as $P_{8,3}(1) = \{(1, 1, 6), (1, 2, 5), (1, 3, 4)\}$.

Now consider the case where n and k are not as above. Thus we have

$$k \ge 4 \quad \text{or} \quad k = 3 \le n \notin \{6, 7, 8\}. \tag{1}$$

Let A be a strongly intersecting subset of $P_{n,k}$. Define f as in the proof of Theorem 1.1. As proved in Theorem 1.1, f is injective. Let $\mathbf{e}$ be the partition $(e_1, \dots, e_k)$ in $P_{n,k}(1)$ with $e_1 = \cdots = e_{k-1} = 1$ and $e_k = n - (k-1)$.

If $(a_1, \dots, a_k)$ is a partition in $P_{n,k}$ that strongly intersects $\mathbf{e}$, then, since $a_1 \le \cdots \le a_k$ and $a_k = n - (a_1 + \cdots + a_{k-1})$, we have $a_1 = \cdots = a_j = 1$ for some $j \in [k-1]$, and hence $(a_1, \dots, a_k)$ is in $P_{n,k}(1)$. Thus, if $\mathbf{e}$ is in A, then $A \subseteq P_{n,k}(1)$.

Now suppose that $\mathbf{e}$ is not in A. We will show that $|A| < |P_{n,k}(1)|$, which completes the proof.

If no partition in A is mapped to $\mathbf{e}$ by f, then f is not surjective, and hence the size of the domain A of f is smaller than the size of the co-domain $P_{n,k}(1)$ of f.

Suppose that A does contain a partition $\mathbf{a} = (a_1, \dots, a_k)$ that is mapped to $\mathbf{e}$ by f. Thus $a_1 = \cdots = a_{k-1} = j$ for some $j \ge 1$, and $a_k = n - (k-1)j \ge a_1$. Since $\mathbf{e} \notin A$, we have $\mathbf{a} \ne \mathbf{e}$, and hence $j \ne 1$. Thus

$$j \ge 2. \tag{2}$$

Since $j = a_1 \le a_k = n - (k-1)j$, we have

$$n \ge kj. \tag{3}$$

Let $\mathbf{b}$ be the partition $(b_1, \dots, b_k)$ in $P_{n,k}(1)$ with

$$b_1 = \cdots = b_{k-2} = 1, \qquad b_{k-1} = \left\lfloor \frac{n-(k-2)}{2} \right\rfloor, \qquad b_k = \left\lceil \frac{n-(k-2)}{2} \right\rceil.$$

By (2), $b_i \ne a_i$ for each $i \in [k-2]$. We also need to compare b_{k-1} and b_k with a_{k-1} and a_k, respectively. We treat the case where $n - k$ is odd separately from the case where $n - k$ is even.

Case 1: $n - k$ *is odd.* Thus $b_{k-1} = \frac{n}{2} - \frac{k}{2} + \frac{1}{2}$ and $b_k = \frac{n}{2} - \frac{k}{2} + \frac{3}{2}$.

Suppose $n \le kj + 1$. By (3), $kj \le n \le kj + 1$. If $k = 3$, then, by (1) and (2), $j \ge 3$. We have

$$b_{k-1} - a_{k-1} = \frac{n}{2} - \frac{k}{2} + \frac{1}{2} - j \ge \frac{kj}{2} - \frac{k}{2} + \frac{1}{2} - j = \frac{1}{2}(k-2)(j-1) - \frac{1}{2},$$

and hence, given that either $k \ge 4$ and $j \ge 2$ or $k = 3$ and $j \ge 3$, we obtain

$$b_{k-1} - a_{k-1} > 0.$$

Also,

$$\begin{aligned} b_k - a_k &= \frac{n}{2} - \frac{k}{2} + \frac{3}{2} - n + (k-1)j = kj - j - \frac{k}{2} - \frac{n}{2} + \frac{3}{2} \\ &\ge kj - j - \frac{k}{2} - \frac{kj+1}{2} + \frac{3}{2} = \frac{1}{2}(k-2)(j-1) > 0. \end{aligned}$$

Thus $b_i \ne a_i$ for each $i \in [k]$, that is, $\mathbf{b}$ does not strongly intersect $\mathbf{a}$. Hence $\mathbf{b} \notin A$. Suppose that A contains a partition $\mathbf{d} = (d_1, \dots, d_k)$ that is mapped to $\mathbf{b}$ by f. By definition of f, $b_k = d_k + (k-1)(d_1 - 1)$ and $b_i = d_i - (d_1 - 1)$ for each $i \in [k-1]$. Since $\mathbf{d} \in A$ and $\mathbf{b} \notin A$, we have $\mathbf{d} \ne \mathbf{b}$, and hence $d_1 \ne 1$. Thus $d_1 \ge 2$, and hence $d_{k-1} \ge b_{k-1} + 1$ and $b_k > d_k$. Thus, since $b_k = b_{k-1} + 1$, we have $d_{k-1} > d_k$, which contradicts $\mathbf{d} \in P_{n,k}$. Therefore, no partition in A is mapped to $\mathbf{b}$ by f. Thus f is not surjective, and hence $|A| < |P_{n,k}(1)|$.

Now suppose $n \ge kj + 2$. We have

$$b_{k-1} - a_{k-1} = \frac{n}{2} - \frac{k}{2} + \frac{1}{2} - j \ge \frac{kj+2}{2} - \frac{k}{2} + \frac{1}{2} - j = \frac{1}{2}(k-2)(j-1) + \frac{1}{2} > 0,$$

and hence $b_{k-1} \ne a_{k-1}$. If we also have $b_k \ne a_k$, then $|A| < |P_{n,k}(1)|$ follows as in the case $n \le kj + 1$.

Suppose $b_k = a_k$. Thus $\frac{n}{2} - \frac{k}{2} + \frac{3}{2} = n - (k-1)j$, which yields $n = 2kj - 2j - k + 3$. Let $c_k = b_k + 1$, $c_{k-1} = b_{k-1} - 1$, and $c_i = b_i = 1$ for each $i \in [k-2]$. Thus $c_k = a_k + 1$, $c_i = 1 < j = a_i$ for each $i \in [k-2]$, and

$$c_{k-1} - a_{k-1} = \frac{n}{2} - \frac{k}{2} + \frac{1}{2} - 1 - j = \frac{1}{2}(2kj - 2j - k + 3) - \frac{k}{2} - \frac{1}{2} - j = (k-2)(j-1) - 1.$$

Suppose $k = 3$ and $j = 2$; since $n = 2kj - 2j - k + 3$, we obtain $n = 8$, which contradicts (1). Thus, by (2), $j \geq 3$ if $k = 3$. Thus $c_{k-1} - a_{k-1} \geq 1$, and hence $c_{k-1} > a_{k-1}$. Let $\mathbf{c} = (c_1, \ldots, c_k)$. Since $c_1 \leq \cdots \leq c_k$ and $\sum_{i=1}^{k} c_i = n$, $\mathbf{c} \in P_{n,k}$. We have shown that $c_i \neq a_i$ for each $i \in [k]$, meaning that $\mathbf{c}$ does not strongly intersect $\mathbf{a}$. Hence $\mathbf{c} \notin A$. Now $\mathbf{c}$ is an element of the co-domain $P_{n,k}(1)$ of f.

Suppose that A contains a partition $\mathbf{d} = (d_1, \ldots, d_k)$ that is mapped to $\mathbf{c}$ by f. Let $h = d_1 - 1$. By definition of f, $d_k = c_k - (k-1)h$ and $d_i = c_i + h$ for each $i \in [k-1]$. Since $\mathbf{d} \in A$ and $\mathbf{c} \notin A$, we have $\mathbf{d} \neq \mathbf{c}$, and hence $h \neq 0$. Thus $h \geq 1$. Since $d_{k-1} \leq d_k$, we have $c_{k-1} + h \leq c_k - (k-1)h$, which yields $kh \leq c_k - c_{k-1} = (b_k + 1) - (b_{k-1} - 1) = 3$. It follows that $k = 3$ and $h = 1$. Recall that from $k = 3$ we obtain $j \geq 3$. Thus we have $d_1 = 2 < j = a_1$, $d_2 = d_{k-1} = c_{k-1} + h > a_{k-1} = a_2$ (since $c_{k-1} > a_{k-1}$), and $d_3 = d_k = c_k - (k-1)h = c_k - 2 = (b_k + 1) - 2 = a_k - 1 = a_3 - 1$. Thus $d_i \neq a_i$ for each $i \in [k]$, meaning that $\mathbf{d}$ does not strongly intersect $\mathbf{a}$; but this is a contradiction since A is strongly intersecting.

Therefore, no element of the domain A of f is mapped to $\mathbf{c}$. Thus f is not surjective, and hence $|A| < |P_{n,k}(1)|$.

Case 2: *$n - k$ is even.* Thus $b_{k-1} = b_k = \frac{n}{2} - \frac{k}{2} + 1$. By an argument similar to that for Case 1, $|A| < |P_{n,k}(1)|$. □

3. A conjecture

The definitions of a strongly intersecting set of integer partitions and of an intersecting family of sets generalise as follows. We say that $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$ *strongly t-intersect* if for some t-element subset T of $[\min\{r, s\}]$, $a_i = b_i$ for each $i \in T$. A set A of integer partitions is said to be *strongly t-intersecting* if every two partitions in A strongly t-intersect. A family $\mathcal{A}$ is said to be *t-intersecting* if $|A \cap B| \geq t$ for every $A, B \in \mathcal{A}$. Thus, an intersecting family is a 1-intersecting family.

In addition to the EKR Theorem (see Section 1), it was also proved in [28] that if n is sufficiently larger than r, then the size of any t-intersecting subfamily of $\binom{[n]}{r}$ is at most $\binom{n-t}{r-t}$, and hence $\{A \in \binom{[n]}{r} : [t] \subset A\}$ is a largest t-intersecting subfamily of $\binom{[n]}{r}$. The complete solution for any n, r and t is given in [1]; it turns out that $\{A \in \binom{[n]}{r} : [t] \subset A\}$ is a largest t-intersecting subfamily of $\binom{[n]}{r}$ if and only if $n \geq (r - t + 1)(t + 1)$ (see also [29,59]).

We now suggest a conjecture for strongly t-intersecting subsets of $P_{n,k}$. For any set A of integer partitions, let $A(t)$ denote the set of all partitions in A whose first t entries are 1. Thus, for $1 \leq t \leq k \leq n$,

$$P_{n,k}(t) = \{(a_1, \ldots, a_k) \in P_{n,k} : a_1 = \cdots = a_t = 1\} \quad \text{and} \quad P_n(t) = \bigcup_{i=t}^{n} P_{n,i}(t).$$

Note that $|P_n(t)| = |P_{n-t}|$ and $|P_{n,k}(t)| = |P_{n-t,k-t}|$.

Conjecture 3.1. *For $t + 1 \leq k \leq n$, $P_{n,k}(t)$ is a strongly t-intersecting subset of $P_{n,k}$ of maximum size.*

Theorem 1.1 verifies this for $t = 1$. If this conjecture is true, then, by an argument similar to that for Theorem 1.3, we get that for $n \geq t$, $P_n(t)$ is a strongly t-intersecting subset of P_n of maximum size.

Proposition 3.2. *Conjecture 3.1 is true for $n \leq 2k - t + 1$.*

Proof. By Theorem 1.1, we may assume that $t \geq 2$. Suppose $n \leq 2k - t + 1$. For any $\mathbf{c} = (c_1, \ldots, c_k) \in P_{n,k}$, let $L_{\mathbf{c}} = \{i \in [k] : c_i = 1\}$, and let $l_{\mathbf{c}} = |L_{\mathbf{c}}|$.

Let $\mathbf{c} = (c_1, \ldots, c_k) \in P_{n,k}$. We have $2k - t + 1 \geq n = \sum_{i \in L_{\mathbf{c}}} c_i + \sum_{j \in [k] \setminus L_{\mathbf{c}}} c_j \geq \sum_{i \in L_{\mathbf{c}}} 1 + \sum_{j \in [k] \setminus L_{\mathbf{c}}} 2 = l_{\mathbf{c}} + 2(k - l_{\mathbf{c}}) = 2k - l_{\mathbf{c}}$. Thus $l_{\mathbf{c}} \geq t - 1$, and equality holds only if $n = 2k - t + 1$ and $c_j = 2$ for each $j \in [k] \setminus L_{\mathbf{c}}$. Since $c_1 \leq \cdots \leq c_k$, $L_{\mathbf{c}} = [l_{\mathbf{c}}]$.

Let A be a strongly t-intersecting subset of $P_{n,k}$. If $l_{\mathbf{a}} \geq t$ for each $\mathbf{a} \in A$, then $A \subseteq P_{n,k}(t)$. Suppose that $l_{\mathbf{a}} = t - 1$ for some $\mathbf{a} = (a_1, \ldots, a_k) \in A$. Thus, by the above, we have $n = 2k - t + 1$, $a_i = 1$ for each $i \in [t-1]$, $a_j = 2$ for each $j \in [k] \setminus [t-1]$, and $P_{n,k} = P_{n,k}(t) \cup \{\mathbf{a}\}$. Let $\mathbf{b}$ be the partition $(b_1, \ldots, b_k)$ in $P_{n,k}(t)$ with $b_k = n - k + 1 = k - t + 2$ and $b_i = 1$ for each $i \in [k-1]$. Since $\mathbf{a}$ and $\mathbf{b}$ do not strongly t-intersect, $\mathbf{b} \notin A$. Thus $|A| \leq |P_{n,k}| - 1 = |P_{n,k}(t)|$. □

Acknowledgements

The author is indebted to the anonymous referees for checking the paper carefully and providing remarks that led to an improvement in the presentation.

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