BOUNDS FOR FLAG CODES

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ABSTRACT. The application of flags to network coding has been introduced recently, see e.g. [13]. It is a variant to random linear network coding and explicit routing solutions for given networks. Here we study lower and upper bounds for the maximum possible cardinality of a corresponding flag code with given parameters.
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1. INTRODUCTION

Let q be a prime power and $\mathbb{F}_q$ the finite field with q elements. For given integers $1 \le k \le v$ a k-dimensional subspace U of $\mathbb{F}_q^v$ is called a *k-space* (in $\mathbb{F}_q^v$). Sometimes we also use the language of projective geometry, i.e., we call speak of *points*, *lines*, *planes*, and *hyperplanes* for 1-spaces, 2-spaces, 3-spaces, and $(v-1)$-spaces, respectively. The set of all k-spaces in $\mathbb{F}_q^v$ is abbreviated by $\begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$ and its cardinality is denoted by the *q-binomial Gaussian coefficient* $\begin{bmatrix}v\\k\end{bmatrix}_q = \prod_{i=1}^{k} \frac{q^{v-k+i}-1}{q^i-1}$. A *full flag* over $\mathbb{F}_q^v$ is a sequence of nested subspaces with dimensions from 1 to $v-1$. If not all of these dimensions need to occur, we speak of a *flag*. (Full) *flag codes* are collections of flags. The use of flag codes for network coding was proposed in [13]. In [12] the author argues that subspace coding with flags can be ranged between random linear network coding, using constant dimension codes, and optimized routing solutions, whose computation is time-consuming. For special multicast networks network coding solutions also lead to hard combinatorial problems, see e.g. [3, 5] for so-called generalized combination networks. Here, we will not go into the details of the used chanel model or comparisons with other methods for network coding. Moreover, we will not consider the problem of coding and decoding algorithms. The interested reader can find more details on this e.g. in [6, 12, 13, 14]. Here we study lower and upper bounds for the maximum possible cardinality $A_q^f(v,d)$ of those flag codes.

The remaining part of this paper is organized as follows. In Section 2 we introduce the necessary basic definitions and the first bounds for $A_q^f(v,d)$. An integer linear programming formulation for the exact determination of $A_q^f(v,d)$ is the topic of Section 3. Parametric bounds on the maximum possible codes sizes are determined in Section 4. The case of non-full flag and other variants are broached in Section 5. We summarize the obtained exact values and bounds for $A_q^f(v,d)$ for small parameters in Section 6. The paper is finished with a brief conclusion and a few remarks on open problems and future research directions in Section 7.

2. PRELIMINARIES AND FIRST BOUNDS

In the following q is always a prime power. For two subspaces U, W in $\mathbb{F}_q^v$ we write $U \le W$ iff U is contained in W. If $U \le W$ and $U \ne W$, then we write $U < W$. The dimension of a subspace U of $\mathbb{F}_q^v$ is denoted by $\dim(U)$. The set of all subspaces of $\mathbb{F}_q^v$ is turned into a metric space via the *injection distance*

$$\mathrm{d_i}(U,W) = \dim(U+W) - \min\{\dim(U),\dim(W)\} = \max\{\dim(U),\dim(W)\} - \dim(U\cap W)$$

or the *subspace distance*

$$\mathrm{d_s}(U,W) = \dim(U+W) - \dim(U\cap W) = \dim(U) + \dim(W) - 2\cdot\dim(U\cap W).$$

Note that for $W, W \in \begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$ we have

$$\begin{aligned}\mathrm{d_i}(U,W) &= \dim(U+W)-k = k-\dim(U\cap W) \quad\text{and}\\ \mathrm{d_s}(U,W) &= 2k-2\dim(U\cap W) = 2\cdot \mathrm{d_i}(U,W).\end{aligned}$$

By $A_q^i(v,d;k)$ we denote the maximum possible cardinality of a set $\mathcal{C} \subseteq \begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$, where $\mathrm{d_i}(U,W) \ge d$ for all pairs of different elements U, W of $\mathcal{C}$. Replacing the injection distance by the subspace distance we obtain $A_q^s(v,d;k)$, where $A_q^i(v,d;k) = A_q^s(v,2d;k)$. Bounds for $A_q^s(v,2d;k)$ can be found in [9] and the corresponding online tables at www.subspacecodes.uni-bayreuth.de.

Lemma 2.1. *For two subspaces $U, W \in \begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$ the following statements are equivalent*

(1) $\mathrm{d_i}(U,W) \le d$;
(2) $\dim(U\cap W) \ge k-d$;
(3) $\dim(U+W) \le k+d$;
(4) *there exists a subspace $X \le \mathbb{F}_q^v$ with $X \le U$, $X \le W$, and $\dim(X) \ge k-d$; and*
(5) *there exists a subspace $X \le \mathbb{F}_q^v$ with $X \ge U$, $X \ge W$, and $\dim(X) \le k+d$;*

Proof. The equivalence of (1)-(3) is obvious from the definition. For (4) we remark that the conditions $X \le U$ and $X \le W$ are equivalent to $X \le U\cap W$. Similarly, for (5) the conditions $X \ge U$ and $X \ge W$ are equivalent to $X \ge U+W$. □

Definition 2.2. A *flag* is a list of subspaces $\Lambda = (W_1,\dots,W_m)$ of $\mathbb{F}_q^v$ with

$$\{0\} < W_1 < \cdots < W_m < \mathbb{F}_q^v.$$

The *type* of $\Lambda = (W_1,\dots,W_m)$ is the set of dimensions

$$\operatorname{type}(\Lambda) := \{\dim(W_i) \mid 1 \le i \le m\} \subseteq \{1,\dots,v\}.$$

Let

$$\mathcal{F}(v,q) := \left\{\Lambda \mid \Lambda \text{ is a flag in } \mathbb{F}_q^v\right\}$$

denote the set of all flags in $\mathbb{F}_q^v$ and for $T \subseteq \{1,\dots,v-1\}$ let

$$\mathcal{F}_T(v,q) := \{\Lambda \in \mathcal{F}(v,q) \mid \operatorname{tpye}(\Lambda) = T\}$$

be the set of all flags of $\mathbb{F}_q^v$ of type T

As noted in [13], the intersection of two flags is again a flag and that the set of all flags in $\mathbb{F}_q^v$ forms a simplicial complex (with respect to inclusion). There the authors give all relevant facts about the spherical building of the general linear group of a finite dimensional vector space. Here we will not use the language of buildings. If a flag in $\mathbb{F}_q^v$ has type $\{1,\dots,v-1\}$, then we speak of a *full flag* whose set is denote by $\mathcal{F}_f(q)$. Full flags are the maximal simplices while the unique minimal flag is the empty set with type $\emptyset$. The second minimal flags $\{W\}$ are the proper subspaces W of $\mathbb{F}_q$. So, the Grassmannian of all k-dimensional subspaces, i.e., $\begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$, is in bijection with the set of flags $\mathcal{F}_{\{k\}}(q)$ of type $\{k\}$.

Definition 2.3. Let $\Lambda = (W_1,\dots,W_m)$ and $\Lambda' := (W_1',\dots,W_m')$ be two flags of $\mathbb{F}_q^v$ of the same type $T = \{k_1,\dots,k_m\}$ with $k_i = \dim(W_i) = \dim(W_i')$ for all $1 \le i \le m$. Then, the *Grassmann distance* is defined as

$$\mathrm{d_G}(\Lambda,\Lambda') := \sum_{i=1}^m \mathrm{d_i}(W_i,W_i') = \sum_{i=1}^m \left(k_i - \dim(W_i\cap W_i')\right).$$

So, for $m = 1$ the Grassmann distance corresponds to the injection distance, i.e., half the subspace distance, between W_1 and W_1'. For $U, W \in \begin{bmatrix}\mathbb{F}_q^v\\k\end{bmatrix}$ we have $0 \le \mathrm{d_i}(U,W) \le \min\{k, v-k\}$, so that we set

$$m(v,T) = \left(\min\{k_1, v-k_1\},\dots,\min\{k_m, v-k_m\}\right),$$

where $T = \{k_1, \ldots, k_m\} \subseteq \{1, \ldots, v-1\}$ with $k_1 < \cdots < k_m$. If $T = \{1, \ldots, v-1\}$ we just write $m(v)$ instead of $m(v, T)$. By x_i we denote the ith component for each vector $x \in \mathbb{R}^n$. With this we can state

$$\mathrm{d_G}(\Lambda, \Lambda') \le \sum_i m(v, T)_i$$

for all $\Lambda, \Lambda' \in \mathcal{F}_T(v, q)$. As mentioned in [13, Remark 4.5] we have $1 \le \mathrm{d_G}(\Lambda, \Lambda') \le \lfloor (v/2)^2 \rfloor$ for two distinct flags in $\mathbb{F}_q^v$. A *flag code* $\mathcal{C}$ of type T is a collection of flags in $\mathbb{F}_q^v$ of type T. If $\#\mathcal{C} \ge 2$, then the minimum distance $\mathrm{d_G}(\mathcal{C})$ is the minimum of $\mathrm{d_G}(\Lambda, \Lambda')$ over all pairs of distinct elements $\Lambda, \Lambda' \in \mathcal{C}$. For $\#\mathcal{C} < 2$ we set $\mathrm{d_G}(\mathcal{C}) = \infty$. By $A_q^f(v, d; T)$ we denote the maximum possible cardinality of a flag code $\mathcal{C}$ of type T in $\mathbb{F}_q^v$ that has minimum distance at least d. The case of full flags, i.e. $T = \{1, \ldots, v-1\}$, is abbreviated as $A_q^f(v, d)$. Technically, we set $A_q^f(v, d) = 1$ if $d > \lfloor (v/2)^2 \rfloor$ and restrict ourselves to $1 \le d \le \lfloor (v/2)^2 \rfloor$ in the following. The *dual* of a flag $\Lambda = (W_1, \ldots, W_m)$ in $\mathbb{F}_q^v$ of type $T \subseteq \{1, \ldots, v-1\}$, denoted by $\Lambda^\top$, is given by $\left(W_1^\top, \ldots, W_m^\top\right)$. Since we have $\mathrm{d_i}(U, W) = \mathrm{d_i}\left(U^\top, W^\top\right)$ for each $U, W \in \begin{bmatrix} \mathbb{F}_q^v \\ k \end{bmatrix}$, for some arbitrary integer k, the minimum Grassmann distance $d(\mathcal{C})$ of a flag code of type T in $\mathbb{F}_q^v$ is the same as $d\left(\mathcal{C}^\top\right)$, where $\mathcal{C}^\top := \left\{\Lambda^\top \mid \Lambda \in \mathcal{C}\right\}$. Moreover, we have

$$\mathrm{type}\left(\mathcal{C}^\top\right) = \{v - t \mid t \in \mathrm{type}(\mathcal{C})\} =: T^\top,$$

so that $A_q^f(v, d; T) = A_q^f\left(v, d; T^\top\right)$. The aim of this paper is to derive bounds on $A_q^f(v, d; T)$ and mostly on $A_q^f(v, d)$.

The arguably easiest case for the determination of $A_q^f(v, d; T)$ is minimum distance $d = 1$, where $A_q^f(v, d; T) = \#\mathcal{F}_T(v, q)$. If $T = \{k_1, \ldots, k_m\}$ with $0 < k_1 < \cdots < k_m < v$, then we have

$$A_q^f(v, 1; T) = \begin{bmatrix} v \\ k_1 \end{bmatrix}_q \cdot \prod_{i=2}^{m} \begin{bmatrix} v - k_{i-1} \\ k_i - k_{i-1} \end{bmatrix}_q \tag{1}$$

and

$$A_q^f(v, 1) = \prod_{i=2}^{v} \frac{q^i - 1}{q - 1}. \tag{2}$$

For the maximum possible minimum distance $d = \lfloor (v/2)^2 \rfloor$ we have:

Proposition 2.4. *For each integer $k \ge 1$ we have*

$$A_q^f(2k, k^2) = q^k + 1$$

and for each integer $k \ge 2$ we have

$$A_q^f(2k+1, k^2+k) = q^{k+1} + 1.$$

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^v$ with the maximum possible minimum distance $d = \lfloor (v/2)^2 \rfloor$, where $v \ge 2$. If $\Lambda = (W_1, \ldots, W_{v-1})$ and $\Lambda' = \left(W_1', \ldots, W_{v-1}'\right)$ are two different elements of $\mathcal{C}$ with $\dim(W_i) = \dim(W_i') = i$ for all $1 \le i \le v-1$, then we have

$$i - \dim(W_i \cap W_i') = \min\{i, v - i\},$$

i.e., W_i and W_i' have the maximum possible intersection distance $\mathrm{d_i}(W_i, W_i')$. So, we clearly have the upper bounds $A_q^f(2k, k^2) \le A_q^i(2k, k; k) = q^k + 1$ and $A_q^f(2k+1, k^2+k) \le A_q^i(2k+1, k; k) = q^{k+1} + 1$ (using $k \ge 2$), where the maximum possible codes sizes for the injection distance are well known, see e.g. [2] or [9].

For the construction let $\mathcal{C}_k$ be a set of k-spaces in $\mathbb{F}_q^v$, where $v = 2k$, with minimum intersection distance $\mathrm{d_i}(\mathcal{C}_k) = k$ and cardinality $A_q^i(2k, k; k) = q^k + 1$, i.e., a k-spread in $\mathbb{F}_q^{2k}$. We extend each element $W_k \in \mathcal{C}_k$ to a full flag $(W_1, \ldots, W_{v-1})$ by choosing $W_i \subsetneq W_{i+1}$ with $\dim(W_i) = i$ arbitrarily for $i = k-1, \ldots, 1$. Similarly, we choose $W_i \supsetneq W_{i-1}$ with $\dim(W_i) = i$ arbitrarily for $i = k +$

$1,\ldots,v-1$. This gives a full flag code $\mathcal{C}$ in $\mathbb{F}_q^{2k}$ of cardinality q^k+1. Now let $\Lambda=(W_1,\ldots,W_{v-1})$ and $\Lambda'=\left(W'_1,\ldots,W'_{v-1}\right)$ be two different elements of $\mathcal{C}$ with $\dim(W_i)=\dim(W'_i)=i$ for all $1\le i\le v-1$. Since $\dim(W_k\cap W'_k)=0$, we have $\dim(W_i\cap W'_i)=0$ and $i-\dim(W_i\cap W'_i)=\min\{i,2k-i\}$ for all $1\le i\le k$. For $k\le i\le v-1$ we can easily check $\dim(W_i\cap W'_i)=i-k$ and $i-\dim(W_i\cap W'_i)=\min\{i,2k-i\}$. Thus, $\mathcal{C}$ has the maximum possible Grassmann distance.

For the ambient space $\mathbb{F}_q^v$, where $v=2k+1$, let $\mathcal{C}_k$ be a set of k-spaces in $\mathbb{F}_q^{2k+1}$ with minimum intersection distance $\mathrm{d_i}(\mathcal{C}_k)=k$ and cardinality $A_q^i(2k+1,k;k)=q^{k+1}+1$, i.e., a partial k-spread of maximum possible size in $\mathbb{F}_q^{2k+1}$. Now let P be a point in $\mathbb{F}_q^{2k+1}$, i.e., a 1-space, that is not contained in an element of $\mathcal{C}_k$. (Since $\begin{bmatrix}k\\1\end{bmatrix}_q\cdot\left(q^{k+1}+1\right)<\begin{bmatrix}2k+1\\1\end{bmatrix}_q$, such a point P exists.) We extend each element $W_k\in\mathcal{C}_k$ to a full flag $(W_1,\ldots,W_v-1)$ by choosing $W_i\subset\neq W_{i+1}$ with $\dim(W_i)=i$ arbitrarily for $i=k-1,\ldots,1$. The $(k+1)$-space W_{k+1} is defined by $W_{k+1}=\langle W_k,P\rangle$. Similarly as before, we choose $W_i\supsetneq W_{i-1}$ with $\dim(W_i)=i$ arbitrarily for $i=k+2,\ldots,v-1$. This gives a full flag code $\mathcal{C}$ in $\mathbb{F}_q^{2k+1}$ of cardinality $q^{k+1}+1$. Given two different elements $\Lambda=(W_1,\ldots,W_{v-1})$ and $\Lambda'=\left(W'_1,\ldots,W'_{v-1}\right)$ of $\mathcal{C}$ with $\dim(W_i)=\dim(W'_i)=i$ for all $1\le i\le v-1$, we can easily check $i-\dim(W_i\cap W'_i)=\min\{i,v-i\}$, i.e., $\mathcal{C}$ attains the maximum possible minimum Grassmann distance. □

We remark that the case $v=2k$ of Proposition 2.4 was independently proven in [1], where the authors also give a decoding algorithm and further details.

Proposition 2.5.

$$A_q^f(3,2)=\begin{bmatrix}3\\1\end{bmatrix}_q=q^2+q+1$$

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^3$ with minimum Grassmann distance $d=2$. Suppose there are two different elements $\Lambda=(W_1,W_2)$ and $\Lambda'=(W'_1,W'_2)$ in $\mathcal{C}$ with $W_1=W'_1$. Then, we have $\mathrm{d_i}(W_1,W'_1)=0$ and $\mathrm{d_i}(W_2,W'_2)\le 1$, so that $\mathrm{d_G}(\Lambda,\Lambda')\le 1$. Thus, we have $\#\mathcal{C}\le\begin{bmatrix}3\\1\end{bmatrix}_q=q^2+q+1$.

For the lower bound we construct a matching code using the Singer group $\langle\sigma\rangle$ generated by a Singer cycle σ of $\mathbb{F}_q^3$, i.e., $\langle\sigma\rangle\le\mathrm{P\Gamma L}(3,q)$ is the cyclic group of order $\begin{bmatrix}3\\1\end{bmatrix}_q=q^2+q+1$ that acts regularly on the set of points or hyperplanes. Now let L be an arbitrary line in $\mathbb{F}_q^3$ and $P\le L$ and arbitrary point. With this we set $\Lambda:=(P,L)$ and $\mathcal{C}=\Lambda^{\langle\sigma\rangle}:=\{\Lambda^g\mid g\in\langle\sigma\rangle\}$, where $\Lambda^g=(P^g,L^g)$ and U^g denotes the application of $g\in P\Gamma L(v,q)$ onto a subspace U in $\mathbb{F}_q^v$. For two different group elements $g_1,g_2\in\langle\sigma\rangle$ we have $\mathrm{d_i}(P^{g_1},P^{g_2})=1$ and $\mathrm{d_i}(L^{g_1},L^{g_2})=1$, so that $\mathrm{d_G}(\mathcal{C})=2$. □

Proposition 2.6.

$$A_q^f(4,3)=\begin{bmatrix}4\\1\end{bmatrix}_q=q^3+q^2+q+1$$

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^4$ with minimum Grassmann distance $d=3$. Suppose there are two different elements $\Lambda=(W_1,W_2,W_3)$ and $\Lambda'=(W'_1,W'_2,W'_3)$ in $\mathcal{C}$ with $W_1=W'_1$. Then, we have $\mathrm{d_i}(W_1,W'_1)=0$, $\mathrm{d_i}(W_2,W'_2)\le 1$, and $\mathrm{d_i}(W_3,W'_3)\le 1$, so that $\mathrm{d_G}(\Lambda,\Lambda')\le 2$. Thus, we have $\#\mathcal{C}\le\begin{bmatrix}4\\1\end{bmatrix}_q=q^3+q^2+q+1$.

For the lower bound we construct a matching code using the Singer group $\langle\sigma\rangle$ generated by a Singer cycle σ of $\mathbb{F}_q^4$, i.e., $\langle\sigma\rangle\le\mathrm{P\Gamma L}(4,q)$ is the cyclic group of order $\begin{bmatrix}4\\1\end{bmatrix}_q$ that acts regularly on the set of points or hyperplanes. As shown in [4], see also [8] for this special case, the action of a Singer group partitions the set of $\begin{bmatrix}4\\2\end{bmatrix}_q=(q^2+1)\cdot(q^2+q+1)$ lines into orbits of size q^2+1 or q^3+q^2+q+1. More precisely, there exists exactly one orbit of length q^2+1, the geometric line spread, and q orbits of length q^3+q^2+q+1. Let $\mathcal{L}$ be an orbit of the latter and $L\in\mathcal{L}$ one of the $q+1$ elements that contain P and H be an arbitrary hyperplane containing L. With this we set $\Lambda:=(P,L,H)$ and $\mathcal{C}=\Lambda^{\langle\sigma\rangle}:=\{\Lambda^g\mid g\in\langle\sigma\rangle\}$, where $\Lambda^g = (P^g, L^g, H^g)$ and U^g denotes the application of $g \in P\Gamma L(v,q)$ onto a subspace U in $\mathbb{F}_q^v$. For two different group elements $g_1, g_2 \in \langle\sigma\rangle$ we have $\mathrm{d_i}(P^{g_1}, P^{g_2}) = 1$, $\mathrm{d_i}(L^{g_1}, L^{g_2}) \geq 1$, and $\mathrm{d_i}(H^{g_1}, H^{g_2}) = 1$, so that $\mathrm{d_G}(\mathcal{C}) \geq 3$. □

Exemplarily we state an upper bound on the maximum cardinality of a full flag code for the next open case:

Proposition 2.7.

$$A_q^f(4,2) \leq \begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}3\\1\end{bmatrix}_q = (q^3+q^2+q+1)\cdot(q^2+q+1) = q^5+2q^4+3q^3+3q^2+2q+1$$

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^4$ with minimum Grassmann distance $d = 2$. Suppose there are two different elements $\Lambda = (W_1, W_2, W_3)$ and $\Lambda' = (W_1', W_2', W_3')$ in $\mathcal{C}$ with $W_1 = W_1'$ and $W_2 = W_2'$. Then, we have $\mathrm{d_i}(W_1, W_1') = 0$, $\mathrm{d_i}(W_2, W_2') = 0$, and $\mathrm{d_i}(W_3, W_3') \leq 1$, so that $\mathrm{d_G}(\Lambda, \Lambda') \leq 1$. Thus, we have $\#\mathcal{C} \leq \begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}3\\1\end{bmatrix}_q$, i.e., the stated upper bound. □

We remark that Proposition 2.7 is tight for $q = 2$, i.e., a corresponding code $\mathcal{C}$ of cardinality 105 indeed exists. Such a code also exists if we prescribe a Singer cycle, i.e., a cyclic group of order 15. Indeed, 15 is the maximum possible order of the automorphism group (for $\#\mathcal{C} = 105$). How to find such codes using integer linear programming, with or without prescribing automorphisms, is the topic of the next section. The underlying proof strategy of Proposition 2.7 will be generalized in Section 4.

v/d	1	2	3	4	5	6	7	8	9	10	11	12
2	1											
3	3	3										
4	6	6	5	5								
5	10	10	9	9	7	7						
6	15	15	14	14	12	12	10	10	8			
7	21	21	20	20	18	18	16	16	14	12	12	10

TABLE 1. Exponents e such that that the sphere packing bound for $A_q^f(v,d)$ is $\Theta(q^e)$.

As usual in coding theory, the maximum cardinalities of codes can be lower and upper bounded by a canonical sphere covering and sphere packing bound, respectively. In the context of (full) flag codes the determination of the cardinalities of the *spheres* is an open and non-trivial problem, see [14] for more details. Using the computational details on the sphere sizes determined in [12] we determine the order of magnitude of the sphere packing and the sphere covering bound for $n \leq 7$. In Table1 we state exponents e such that the sphere packing bound for $A_q^f(v,d)$ is $\Theta(q^e)$, i.e., we have lower and upper bounds for the sphere packing bound of the form cq^e plus terms of lower order, where c is a suitable constant. In Table 4 we will summarize the exponents of the improved upper bounds obtained using the methods from this paper. The corresponding exponents for the sphere covering bound can be found in Table 2.

3. AN INTEGER LINEAR PROGRAMMING FORMULATION FOR $A_q^f(v,d)$

In principle, it is rather simple to give an integer linear programming formulation for the exact determination of $A_q^f(v,d)$. Let us start with the formulation as a maximum independent set problem. To this end let $\mathcal{G}_{v,d,q} = (V, E)$ be a graph with vertex set $V = \mathcal{F}(v,q)$ and $\{\Lambda, \Lambda'\} \in E$ iff $\Lambda \neq \Lambda'$ and $\mathrm{d_G}(\Lambda, \Lambda') < d$. Clearly, each flag code in $\mathbb{F}_q^v$ with minimum Grassmann distance d is in bijection to an independent set in $\mathcal{G}_{v,d,q}$. A standard integer linear programming (ILP) formulation for the maximum

v/d	1	2	3	4	5	6	7	8	9	10	11	12
2	1											
3	3	2										
4	6	5	3	1								
5	10	9	7	5	3	2						
6	15	14	12	10	8	6	5	3	1			
7	21	20	18	16	14	12	10	9	7	5	3	2

TABLE 2. Exponents e such that that the sphere covering bound for $A_q^f(v,d)$ is $\Theta(q^e)$.

cardinality of an independent set in a graph (V,E) is given by $\max \sum_{u\in V} x_v$ subject to $x_u + x_w \le 1$ for all edges $\{u,w\}\in E$ and $x_u \in \{0,1\}$ for all $u \in V$. In our situation this gives:

$$A_q^f(v,d) \quad = \quad \max \sum_{\Lambda\in\mathcal{F}(v,q)} x_\Lambda \quad \text{s.t.} \tag{3}$$

$$x_\Lambda + x_{\Lambda'} \quad \le \quad 1 \qquad \forall \Lambda,\Lambda' \in \mathcal{F}(v,q) \text{ with } \Lambda\ne\Lambda', \mathrm{d_G}(\Lambda,\Lambda') < d \tag{4}$$

$$x_\Lambda \quad \in \quad \{0,1\} \quad \forall \Lambda \in \mathcal{F}(v,q) \tag{5}$$

Note that the corresponding flag code is given by $\mathcal{C} = \{\Lambda \in \mathcal{F}(v,q) \mid x_\Lambda = 1\}$ and that the formulation can be easily adopted for $A_q^f(v,d;T)$. The corresponding linear programming (LP) relaxation is obtained if the constraints from (5) are replaced by $0 \le x_\Lambda \le 1$. Solving the LP relaxation, which is done by ILP solvers in intermediate steps, gives an upper bound. Since setting $x_\Lambda = \frac{1}{2}$ for all $\Lambda \in \mathcal{F}(v,q)$ always satisfies the constraints from (4), we cannot obtain an upper bound tighter than $\#\mathcal{F}(v,q)/2$ ($\#V/2$ in the general case), which is a rather bad bound (provided $d \ge 2$). However, for each subset $\mathcal{V} \subseteq V$ that induces a clique, i.e., $\{u,w\}$ is an edge for all pairs of different elements u,w in $\mathcal{V}$, we can add the improved constraint $\sum_{u\in\mathcal{V}} x_u \le 1$, which is also called *clique constraint*. So, the rest of this section is devoted to the description of large cliques in $\mathcal{G}_{v,d,q}$.

For two vectors $x,y\in\mathbb{R}^n$ we write $x\le y$ iff $x_i \le y_i$ for all $1\le i\le n$. By $\mathbf{0}$ we denote the all zero vector whenever the length is clear from the context. We say that two subspaces U,W of $\mathbb{F}_q^v$ are *incident* if either $U \le W$ or $W \le U$, which we denote by $(U,W)\in I$.

Lemma 3.1. *Let $r\in\mathbb{N}^{v-1}$ with $\mathbf{0}\le r\le m(v)$, $\mathcal{I}=\{1\le i\le v-1 \mid r_i\ne 0\}$, and U_i an arbitrary subspace of $\mathbb{F}_q^v$ with $\dim(U_i)\in\{i-m_i+r_i, i+m_i-r_i\}$ for each $i\in\mathcal{I}$. If $d > \sum_{i=1}^{v-1}(m(v)_i - r_i)$, then*

$$\mathcal{V} = \{(W_1,\dots,W_{v-1})\in\mathcal{F}(v,q) \mid (W_i,U_i)\in I\,\forall i\in\mathcal{I}\}$$

is the vertex set of a clique in $\mathcal{G}_{v,d,q}$.

Proof. Let $\Lambda = (W_1,\dots,W_{v-1})$ and $\Lambda' = \left(W'_1,\dots,W'_{v-1}\right)$ be two different elements in $\mathcal{H}$. For $1\le i\le v-1$ with $i\notin\mathcal{I}$ we have $\mathrm{d_i}(W_i,W'_i)\le m(v)_i = m(v)_i - r_i$. Now we consider $i\in\mathcal{I}$. If $\dim(U_i) = i - m_i + r_i$, then $U_i\le W_i$ and $U_i \le W'_i$, so that

$$\mathrm{d_i}(W_i,W'_i) = i - \dim(W_i\cap W'_i) \le i - \dim(U_i) = m(v)_i - r_i.$$

If $\dim(U_i) = i + m_i - r_i$, then $W_i\le U_i$ and $W'_i\le U_i$, so that

$$\mathrm{d_i}(W_i,W'_i) = \dim(W_i + W'_i) - i \le \dim(U_i) - i = m(v)_i - r_i.$$

Thus, we have

$$\mathrm{d_G}(\Lambda,\Lambda') \le \sum_{i=1}^{v-1}(m(v)_i - r_i) < d,$$

i.e. $\{\Lambda,\Lambda'\}$ is an edge in $\mathcal{G}_{v,d,q}$. □

Corollary 3.2. *Let $r \in \mathbb{N}^{v-1}$ with $\mathbf{0} \le r \le m(v)$, $\mathcal{I} = \{1 \le i \le v-1 \mid r_i \neq 0\}$, and U_i an arbitrary $(i - m_i + r_i)$-space in $\mathbb{F}_q^v$ for each $i \in \mathcal{I}$. If $d > \sum_{i=1}^{v-1}(m(v)_i - r_i)$, then*

$$\mathcal{V} = \{(W_1, \dots, W_{v-1}) \in \mathcal{F}(v,q) \mid U_i \le W_i \,\forall i \in \mathcal{I}\}$$

is the vertex set of a clique in $\mathcal{G}_{v,d,q}$.

The vector r describes the reduction of the achievable Grassmann distance with respect to the maximum possible Grassmann distance. Let us consider an example, for $(v,d) = (4,2)$ we have $m(v) = (1,2,1)$ and $r = (1,2,0)$ satisfies the conditions of Corollary 3.2, i.e., each full flag code $\mathcal{C}$ in $\mathbb{F}_q^4$ with minimum distance $\mathrm{d_G}(\mathcal{C}) = 2$ satisfies $\#\{(W_1,W_2,W_3) \in \mathcal{C} \mid W_1 = P, W_2 = L\} \le 1$ for each pair $(P,L) \in \begin{bmatrix}\mathbb{F}_q^4\\1\end{bmatrix} \times \begin{bmatrix}\mathbb{F}_q^4\\2\end{bmatrix}$. Actually, this argument was used in the proof of Proposition 2.7 to conclude the upper bound for $A_q^f(4,2)$.

In the other direction, either a strengthening of Corollary 3.2 is sufficient to cover all edges of $\mathcal{G}_{v,d,q}$ by corresponding cliques with vertex set $\mathcal{V}$.

Lemma 3.3. *If $\Lambda = (W_1, \dots, W_{v-1})$ and $\Lambda' = (W'_1, \dots, W_{v-1'})$ are two different full flags with $\mathrm{d_G}(\Lambda, \Lambda') < d$, then there exist subspaces $U_1 \le \dots \le U_{v-1}$ such that $d > \sum_{i=1}^{v-1}(m(v)_i - r_i)$ and $\mathbf{0} \le r \le m(v)$, where $r_i = \dim(U_i) - i + m(v)_i$ for all $1 \le i \le v-1$.*

Proof. We choose $U_i = W_i \cap W'_i$ for all $1 \le i \le v-1$, so that $U_1 \le \dots \le U_{v-1}$. By construction we have

$$\mathrm{d_i}(W_i, W'_i) = i - \dim(W_i \cap W'_i) = i - \dim(U_i) = m(v)_i - r_i,$$

so that $\mathbf{0} \le r \le m(v)$ and $d > \mathrm{d_G}(\Lambda, \Lambda') = \sum_{i=1}^{v-1}(m(v)_i - r_i)$. □

In other words, we can replace the constraints (4) by the clique constraints $\sum_{u \in \mathcal{V}} x_u \le 1$ for all cases that satisfy the conditions of Corollary 3.2, where we additionally assume $U_1 \le \dots \le U_{v-1}$. In order to ease the notation we focus on the cliques of Corollary 3.2 instead of the more general situation of Lemma 3.1.

Definition 3.4. For an integer vector $\mathbf{0} \le r \le m(v)$ let $\mathcal{I} = \{1 \le i \le v-1 \mid r_i > 0\}$ and let $\mathcal{V}_{v,q}^r$ denote the set of cliques

$$\mathcal{V} = \{(W_1, \dots, W_{v-1}) \in \mathcal{F}(v,q) \mid U_i \le W_i \,\forall i \in \mathcal{I}\},$$

where the U_i are $(i - m_i + r_i)$-spaces and we have $U_i \le U_{i'}$ for all $i, i' \in \mathcal{I}$ with $i \le i'$. By $E_{v,q}^r$ we denote the set of edges $e = \{\Lambda, \Lambda'\}$, where $e \subseteq \mathcal{V}$ for at least one $\mathcal{V} \in \mathcal{V}_{v,q}^r$.

If $\mathbf{0} \le r \le r' \le m(v)$, then we obviously have $E_{v,q}^r \supseteq E_{v,q}^{r'}$. So, given d, it is sufficient to consider all $\mathcal{V}_{v,q}^r$ where $\sum_{i=1}^{v-1}(m(v)_i - r_i) = d-1$. Note that for $r = (0,0,0,4,1,0,0)$ we have $E_{v,q}^r = \emptyset$. In our example $(v,d) = (4,2)$ it suffices to consider the vectors $(1,2,0)$, $(1,1,1)$, and $(0,2,1)$. However, for $r = (1,1,1)$ we have $U_1 \le U_2 \le U_3$ with $\dim(U_1) = \dim(U_2) = 1$, i.e., $U_1 = U_2$, and $\dim(U_3) = 3$. If $\Lambda = (W_1, W_2, W_3)$ and $\Lambda' = (W'_1, W'_2, W'_3)$ are flags with $U_1 \le W_1$ and $U_1 \le W'_1$, then $\mathrm{d_i}(W_2, W'_2) \le 1$ since $U_1 \le W_2 \cap W'_2$. In other words, also $\mathcal{V}_{v,q}^{(1,0,1)}$ consists of vertex sets of cliques in $\mathcal{G}_{4,2,q}$.

Definition 3.5. Let $\mathbf{0} \le r \le m(v)$ and $u_j = \max\{2j - v, 0\} + r_j$ for all $1 \le j \le v-1$. Then, let

$$\overline{u}_j = \max\{u_i \mid 1 \le i \le j\} \cup \{u_i - 2(i-j) \mid j < i < v\}$$

and $\overline{r}_j = \overline{u}_j - j + m(v)_j$ for all $1 \le j \le v$. With this, we set $\overline{r} = (\overline{r}_1, \dots, \overline{r}_{v-1})$.

For further usage we state two easy lemmas without proof.

Lemma 3.6. *Let W_a, W'_a be a-spaces and W_b, W'_b be v-spaces in $\mathbb{F}_q^v$ with $W_a < W_b$ and $W'_a < W'_b$. Then, we have $\dim(W_b \cap W'_b) \ge \dim(W_a \cap W'_a)$ and $\dim(W_a \cap W'_a) \ge \dim(W_b \cap W'_b) - 2(b-a)$.*

Lemma 3.7. *Let $U_1 \le \dots \le U_n$ be a weakly increasing chain of subspaces in $\mathbb{F}_q^v$ and $u = (u_1, \dots, u_n) \in \mathbb{N}^n$ satisfy $u_1 \le \dots u_n$. If $\dim(U_i) \ge u_i$ for all $1 \le i \le n$, then there exists a weakly increasing chain $U_1' \le \dots \le U_n'$ of subspaces in $\mathbb{F}_q^v$ with $U_i' \le U_i$ and $\dim(U_i') = u_i$ for all $1 \le i \le n$.*

Lemma 3.8. *For $\mathbf{0} \le r \le m(v)$ we have $r \le \overline{r} \le m(v)$ and $E_{v,q}^r = E_{v,q}^{\overline{r}}$.*

Proof. By construction we have $u_j = j - m(v)_j + r_j$ for $1 \le j \le v-1$, since $j - m(v)_j = j - \min\{j, v-j\} = \max\{2j - v, 0\}$. Setting $u = (u_1, \dots, u_{v-1})$ and $\overline{u} = (\overline{u}_1, \dots, \overline{u}_{v-1})$, we note $u \le \overline{u} \le (1, \dots, v-1)$, so that $r \le \overline{r} \le m(v)$ due to $\overline{r}_j = \overline{u}_j - j + m(v)_j$ for all $1 \le j \le v$. From $r \le \overline{r} \le m(v)$ we conclude $E_{v,q}^r \supseteq E_{v,q}^{\overline{r}}$.

Now let $\{\Lambda, \Lambda'\} \in E_{v,q}^r$, where $\Lambda = (W_1, \dots, W_{v-1})$ and $\Lambda' = (W_1', \dots, W_{v-1'})$. We set $\mathcal{I} = \{1 \le i \le v-1 \mid r_i > 0\}$ and note that the definition of $E_{v,q}^r$ yields the existence of an u_i-space U_i in $\mathbb{F}_q^v$ with $U_i \le W_i \cap W_i'$ for all $i \in \mathcal{I}$ and $U_i \le U_{i'}$ for all $i, i' \in \mathcal{I}$ with $i \le i'$. Now we set $\bar{U}_j = W_j \cap W_j'$ for $j = 1, \dots, v-1$. First we note $\dim(\bar{U}_j) \ge u_j$ for all $1 \le j \le v-1$ and $\bar{U}_1 \le \dots \bar{U}_{v-1}$. Now let $1 \le j \le v-1$ be fix but arbitrary. We want to show $\dim(\bar{U}_j) \ge \overline{u}_j$. If $\overline{u}_j = u_j$ this is clearly the case. If $\overline{u}_j = u_h$ for an index $1 \le h < j$, then we can choose $b = j$, $a = h$ in Lemma 3.6 to conclude $\dim(\bar{U}_j) = \dim(W_j \cap W_j') \ge \dim(W_h \cap W_h') \ge u_h = \overline{u}_j$. If $\overline{u}_j = u_h - 2(h-j)$ for an index $j < h < v$, then we can choose $b = h$, $a = j$ in Lemma 3.6 to conclude $\dim(\bar{U}_j) = \dim(W_j \cap W_j') \ge \dim(W_h, W_h') - 2(h-j) \ge u_h - 2(h-j) = \overline{u}_j$. Since $\bar{u}_1 \le \dots \bar{u}_{v-1}$ by construction, we can apply Lemma 3.7 to conclude the existence of subspace $U_1' \le \dots \le U_{v-1}'$ in $\mathbb{F}_q^v$ with $U_j' \le W_j \cap W_j'$ and $\dim(U_j') = \bar{u}_j$ for all $1 \le j \le v-1$. Due to the definition of $\bar{r}$ this yields that $\{\Lambda, \Lambda'\} \in E_{v,q}^{\overline{r}}$. Since $\{\Lambda, \Lambda'\} \in E_{v,q}^r$ was arbitrary, this gives $E_{v,q}^r \subseteq E_{v,q}^{\overline{r}}$, so that $E_{v,q}^r = E_{v,q}^{\overline{r}}$. □

As an example we have $\overline{(1,0,1)} = (1,1,1)$, so that $E_{4,q}^{(1,1,1)} = E_{4,q}^{(1,0,1)}$. Here we have $\#\mathcal{V} = \begin{bmatrix}3\\1\end{bmatrix}_q$ for each $\mathcal{V} \in \mathcal{V}_{4,q}^{(1,0,1)}$ and also $\#\mathcal{V} = \begin{bmatrix}3\\1\end{bmatrix}_q$ for each $\mathcal{V} \in \mathcal{V}_{4,q}^{(1,1,1)}$. Moreover, $\#\mathcal{V}_{4,q}^{(1,0,1)} = \begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}3\\2\end{bmatrix}_q = \begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}3\\1\end{bmatrix}_q = \#\mathcal{V}_{4,q}^{(1,1,1)}$. In other words, here, there is no difference at all between taking $\mathcal{V}_{4,q}^{(1,0,1)}$ or $\mathcal{V}_{4,q}^{(1,1,1)}$. However, for $v \ge 5$ improvements are possible, as we will discuss later on. In general, we have $\#\mathcal{V}_{v,q}^r \le \#\mathcal{V}_{v,q}^{\overline{r}}$.

Definition 3.9. For $a, b \in \left\{r \in \mathbb{N}^{v-1} \mid \mathbf{0} \le r \le m(v)\right\}$ we define $a \preceq b$ if either $\bar{a} < \bar{b}$ or $\bar{a} = \bar{b}$ and $a \le b$.

The conditions of a poset, i.e., reflexivity, antisymmetry, and transitivity, are directly verified. So each subset $\mathcal{R} \subseteq \left\{r \in \mathbb{N}^{v-1} \mid \mathbf{0} \le r \le m(v)\right\}$ contains a unique subset $\mathcal{R}' \subseteq \mathcal{R}$ of minimal elements, i.e., for each $r \in \mathcal{R}$ there exists an element $r' \in \mathcal{R}'$ with $r' \preceq r$ and there are no two different elements $r', r'' \in \mathcal{R}'$ with $r' \preceq r''$. Moreover, $r \le r'$ implies $\bar{r} \le \bar{r'}$, so that $r \preceq r'$. However, the converse is not true as we will see in Example 3.11. More precisely, we have $(0,1,1,0) \preceq (1,0,1,0)$ while $(0,1,1,0)$ and $(1,0,1,0)$ are incomparable with respect to $\le$. (It is also easy to show that $\bar{\bar{r}} = \bar{r}$.)

Definition 3.10. Let $\mathcal{R}_{v,d}$ the unique set of, with respect to $\preceq$, minimal elements in the set of vectors

$$\left\{ r \in \mathbb{N}^{v-1} \mid \mathbf{0} \le r \le m(v),\, d > \sum_{i=1}^{v-1} \left(m(v)_i - \bar{r}_i\right) \right\}.$$

Note that $r \in \mathcal{R}_{v,d}$ implies $\sum_{i=1}^{v-1} \left(m(v)_i - r_i\right) < d$ and $(r_1, \dots, r_{v-1}) \in \mathcal{R}_{v,d}$ if and only if $(r_{v-1}, \dots, r_1) \in \mathcal{R}_{v,d}$.

Example 3.11. For $v = d = 5$ the vectors in $\left\{r \in \mathbb{N}^{v-1} \mid \mathbf{0} \le r \le m(v),\, d-1 = \sum_{i=1}^{v-1} \left(m(v)_i - r_i\right)\right\}$ are given by $(0,2,0,0)$, $(0,0,2,0)$, $(1,1,0,0)$, $(1,0,1,0)$, $(1,0,0,1)$, $(0,1,1,0)$, $(0,1,0,1)$, and $(0,0,1,1)$. We remark that

$$\mathcal{R}_{5,5} = \left\{(1,0,0,0), (0,1,1,0), (0,0,0,1)\right\},$$

$\overline{(1,0,0,0)} = (1,1,0,0)$, $\overline{(0,1,0,0)} = (0,1,0,0)$, $\overline{(1,1,0,0)} = (1,1,0,0)$, $\overline{(1,0,1,0)} = (1,1,1,0)$, $\overline{(1,0,0,1)} = (1,1,1,1)$, $\overline{(0,1,1,0)} = (0,1,1,0)$, and $\overline{(0,2,0,0)} = (0,2,1,0)$. Since $\overline{(1,0,1,0)} = (1,1,1,0) > (0,1,1,0) = \overline{(0,1,1,0)}$, we e.g. have $(1,0,1,0) \notin \mathcal{R}_{5,5}$. Similarly we have $(0,2,0,0) \notin \mathcal{R}_{5,5}$ since $\overline{(0,2,0,0)} = (0,2,1,0) > (0,1,1,0) = \overline{(0,1,1,0)}$.

Proposition 3.12.

$$A_q^f(v,d) = \max \sum_{\Lambda \in \mathcal{F}(v,q)} x_\Lambda \quad \textit{s.t.} \tag{6}$$

$$\sum_{\Lambda \in \mathcal{V}} x_\Lambda \le 1 \qquad \forall \mathcal{V} \in \mathcal{V}_{v,q}^r \, \forall r \in \mathcal{R}_{v,d} \tag{7}$$

$$x_\Lambda \in \{0,1\} \quad \forall \Lambda \in \mathcal{F}(v,q) \tag{8}$$

Proof. We start from the ILP formulation (3)-(5). Now let $\Lambda, \Lambda' \in \mathcal{F}(v,q)$ with $\Lambda \neq \Lambda'$ and $\mathrm{d_G}(\Lambda, \Lambda') < d$. From Lemma 3.3 and Corollary 3.2 we conclude the existence of a vector $\mathbf{0} \le r' \le m(v)$ with $\{\Lambda, \Lambda'\} \in E_{v,q}^{r'}$, which is contained in the edge set of $\mathcal{G}_{v,d,q}$. W.l.o.g. we can additionally assume that $d - 1 = \sum_{i=1}^{v-1} (m(v)_i - r_i')$. From Lemma 3.8 we then conclude the existence of $r \in \mathcal{R}_{v,d}$ with $E_{v,q}^r = E_{v,q}^{r'}$.

It remains to remark that for each $\mathcal{V} \in \mathcal{V}_{v,q}^r$ and each $r \in \mathcal{R}_{v,d}$ constraint (7) is a valid constraint due to Lemma 3.8 and Corollary 3.2. □

Due to combinatorial explosion, the number of variables and constraints of the ILP from Proposition 3.12 gets large even for small parameters. So, in order to construct large flag codes we want to reduce the computational complexity by prescribing automorphisms. An *automorphism* φ of $\mathcal{C} = \{\Lambda_1, \ldots, \Lambda_m\} \subseteq \mathcal{F}(v,q)$ is an element of $\mathrm{GL}(v,q)$ such that $\mathcal{C} = \{\varphi(\Lambda_1), \ldots, \varphi(\Lambda_m)\}$. By $\mathrm{Aut}(\mathcal{C})$ we denote the group of automorphisms of $\mathcal{C}$, which is a subgroup of $\mathrm{GL}(v,q)$. For notational reason we rewrite the ILP from Proposition 3.12 to $\max \sum_{\Lambda \in \mathcal{F}(v,q)} x_\Lambda$ subject to $Mx \le \mathbf{1}$, where the x_i are binary variables, $\mathbf{1}$ is the all-1 vector, and

$$M_{\mathcal{V},\Lambda} = \begin{cases} 1 & \text{if } \Lambda \in \mathcal{V}, \\ 0 & \text{otherwise} \end{cases}$$

for all $\Lambda \in \mathcal{F}(v,q)$ and all $\mathcal{V} \in \mathcal{V}_{v,q}^r$, $r \in \mathcal{R}_{v,d}$.

Now let $G \le \mathrm{Aut}(\mathcal{C}) \le \mathrm{GL}(v,q)$. By M^G we denote the corresponding matrix briefly defined below, see e.g. [11] where the method was applied to *constant dimension codes*, i.e., flag codes with type T, where $\#T = 1$. The underlying general method can be described as follows. In order to obtain M^G, the matrix M is reduced by adding up columns (labeled by the flags contained in $\mathcal{F}(v,q)$) corresponding to the orbits of G, which we denote by $\omega_1, \ldots, \omega_\gamma$. Due the equivalence

$$U \le W \iff \varphi(U) \le \varphi(W) \tag{9}$$

for all subspaces U, W of $\mathbb{F}_q^v$ and each automorphism $\varphi \in G$ we have that rows corresponding to vertex sets $\mathcal{V}$, $\mathcal{V}'$ in the same orbit under G are equal. Therefore the redundant rows are removed from the matrix and we obtain a smaller matrix denoted by M^G. The number of rows of M^G is then the number Γ of orbits of G on $\left\{\mathcal{V} \mid \mathcal{V} \in \mathcal{V}_{v,q}^r, r \in \mathcal{R}_{v,d}\right\}$, which we denote by $\Omega_1, \ldots, \Omega_\Gamma$. The number γ of columns of M^G is the number of orbits of G on the flags in $\mathcal{F}(v,q)$. For an entry of M^G we have

$$M_{\Omega_i,\omega_j} = \# \left\{\Lambda \in \omega_j \mid \Lambda \in \mathcal{V}\right\},$$

where $\mathcal{V}$ is a representative of the orbit Ω_i. Because of property (9) the matrix M^G is well-defined as the definition of M_{Ω_i,ω_j}^G is independent of the representative $\mathcal{V}$. Thus, we can restate Proposition 3.12 as follows:

(v,d)	$\mathcal{R}_{v,d}$
$(5,1)$	$\Big\{(1,2,2,1)\Big\}$
$(5,2)$	$\Big\{(1,2,2,0),(1,2,0,1),(1,0,2,1),(0,2,2,1)\Big\}$
$(5,3)$	$\Big\{(1,2,0,0),(1,0,2,0),(0,2,2,0),(1,0,0,1),(0,2,0,1),(0,0,2,1)\Big\}$
$(5,4)$	$\Big\{(1,0,1,0),(0,2,0,0),(0,0,2,0),(0,1,0,1)\Big\}$
$(5,5)$	$\Big\{(1,0,0,0),(0,1,1,0),(0,0,0,1)\Big\}$
$(5,6)$	$\Big\{(0,1,0,0),(0,0,1,0)\Big\}$
$(6,1)$	$\Big\{(1,2,3,2,1)\Big\}$
$(6,2)$	$\Big\{(1,2,3,2,0),(1,2,3,0,1),(1,2,0,2,1),(1,0,3,2,1),(0,2,3,2,1)\Big\}$
$(6,3)$	$\Big\{(1,2,3,0,0),(1,2,0,2,0),(1,2,0,0,1),(1,0,3,2,0),(1,0,3,0,1),(1,0,0,2,1),(0,2,3,2,0),$ $(0,2,3,0,1),(0,2,0,2,1),(0,0,3,2,1)\Big\}$
$(6,4)$	$\Big\{(1,2,0,1,0),(1,0,3,0,0),(1,0,2,0,1),(1,0,0,2,0),(0,2,3,0,0),(0,2,0,2,0),(0,2,0,0,1),$ $(0,1,0,2,1),(0,0,3,2,0),(0,0,3,0,1)\Big\}$
$(6,5)$	$\Big\{(1,2,0,0,0),(1,0,2,1,0),(1,0,0,0,1),(0,2,0,1,0),(0,1,2,0,1),(0,1,0,2,0),(0,0,3,0,0),$ $(0,0,0,2,1)\Big\}$
$(6,6)$	$\Big\{(1,0,2,0,0),(1,0,0,1,0),(0,2,0,0,0),(0,1,2,1,0),(0,1,0,0,1),(0,0,2,0,1),(0,0,0,2,0)\Big\}$
$(6,7)$	$\Big\{(1,0,0,0,0),(0,1,2,0,0),(0,1,0,1,0),(0,0,2,1,0),(0,0,0,0,1)\Big\}$
$(6,8)$	$\Big\{(0,1,0,0,0),(0,0,2,0,0),(0,0,0,1,0)\Big\}$
$(6,9)$	$\Big\{(0,0,1,0,0)\Big\}$

TABLE 3. The sets $\mathcal{R}_{v,d}$ for small parameters.

Theorem 3.13. *Let G be a subgroup of $\mathrm{GL}(v,q)$. There is a flag code $\mathcal{C} \subseteq \mathcal{F}(v,q)$ with minimum Grassmann distance d whose group of automorphisms contains G as a subgroup if, and only if, there is a $(0/1)$-solution $x = (x_1,\dots,x_\gamma)^\top$ satisfying $\#\mathcal{C} = \sum_{i=1}^{\gamma} |\omega_i| \cdot x_i$ and $M^g x \le \mathbf{1}$.*

Note that $M_{\Omega_i,\omega_j} > 1$ implies $x_{\omega_j} = 0$. However, those conclusions are automatically drawn in a preprocessing step by the most commonly used ILP solvers.

Example 3.14. We want to apply Theorem 3.13 in order to obtain lower bounds for $A_2^f(5,2)$. Without prescribing automorphisms there are $\#\mathcal{F}(5,2) = 9765$ full flags, i.e., variables, and 13020 constraints, since $\#\mathcal{V}_{5,2}^{(1,2,2,0)} = \#\mathcal{V}_{5,2}^{(1,2,0,1)} = \#\mathcal{V}_{5,2}^{(1,0,2,1)} = \#\mathcal{V}_{5,2}^{(0,2,2,1)} = 3255$. We prescribe a group G of automorphisms generated by a single element:

$$G := \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix} \right\rangle .$$

G is a cyclic group of order 31 – indeed it is a Singer group. The reduced ILP consists of 420 constraints and 315 binary variables. Using the ILP solver `ILOG CPLEX` an optimal solution with target value 3069 was found after 213 seconds of computation time and 68 180 branch-&-bound nodes. The group given by

$$G := \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \right\rangle$$

is a cyclic group of order 15, and indeed a Singer group of a hyperplane. The corresponding reduced ILP consists of 865 constraints and 651 binary variables. After 11 minutes and $24\,895$ branch-&-bound nodes a flag code with cardinality 3120 was found. After 9 hours and $6\,799\,282$ branch-&-bound nodes the upper bound dropped to 3178 while no better solution was found. For a cyclic group of order 15 we found that the corresponding optimal target value lies between 2982 and 3068. Since already the upper bound is strictly less than the cardinality of the best known solution we have aborted the solution process.

Performing a more extensive computational experiment we remark that there are several groups where we can easily verify that the corresponding upper bound is strictly less than 3120. An example is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which generates a group of order 2, has 116 fix points, and which does not allow a flag code with cardinality strictly larger than 2807. Examples of small groups where the achievable cardinality is strictly smaller than 3255 are given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which generate cyclic groups of orders 3 or 2, have 30 or 52 fix points, and where we have upper bounds on the cardinality of 3171 or 3144, respectively. Examples of cyclic groups where the ILP approach did not bring the upper bound strictly below 3255 after a reasonable computation time are given by the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The corresponding orders are 3, 7, 7, and 5, respectively. (12, 0, 8, and 2 fix points.)

We remark that the ILP formulations from Proposition 3.12 and Theorem 3.13 can be enhanced by additional bounds for substructures of flag codes. Examples are the bounds from Proposition 4.8 and Proposition 4.10 in the subsequent Section 4.

4. BOUNDS

In this section we want to generalize the idea underlying the upper bound of Proposition 2.7 for $A_q^f(4,2)$, see Theorem 4.2. It will turn out that this can be seen as a generalization of the anticode bound for constant dimension codes. In Proposition 4.8 we follow the approach of the Johnson bound for constant dimension codes. Together with Proposition 4.10 we determine a general explicit upper bound of the form $A_q^f(v,d) \le q^\beta + O\big(q^{\beta-1}\big)$, see Proposition 4.12.

Definition 4.1. Let $\mathcal{I} \subseteq \mathbb{N}$ and $U_i \le \mathbb{F}_q^v$ for all $i \in \mathcal{I}$. We call $(U_i)_{i\in\mathcal{I}}$ *weakly increasing* if $U_i \le U_j$ for all $i,j \in \mathcal{I}$ with $i \le j$.

Theorem 4.2. *Let* $\mathbf{0} \le r \le m(v)$ *with* $d > \sum_{i=1}^{v-1}\left(m(v)_i - \overline{r}_i\right)$ *and* $\mathcal{I} = \{1 \le i \le v-1 \mid r_i > 0\}$. *Then, we have* $A_q^f(v,d) \le \#\mathcal{U}/\#\widehat{\mathcal{U}}$, *where*

$$\begin{aligned} \mathcal{U} &= \left\{(U_i)_{i\in\mathcal{I}} \text{ weakly increasing } \mid \dim(U_i) = i - m(v)_i + r_i \, \forall i \in \mathcal{I}\right\}, \\ \widehat{\mathcal{U}} &= \left\{(U_i)_{i\in\mathcal{I}} \text{ weakly increasing } \mid \dim(U_i) = i - m(v)_i + r_i, U_i \le W_i' \, \forall i \in \mathcal{I}\right\}, \end{aligned}$$

and $\Lambda' = \big(W_1', \dots, W_{v-1}'\big) \in \mathcal{F}(v,q)$ *is an arbitrary but fixed full flag.*

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^v$ with minimum Grassmann distance d. From Corollary 3.2 and Lemma 3.8 we conclude

$$\#\left\{(W_1,\dots,W_{v-1}) \in \mathcal{C} \mid U_i \le W_i\right\} \le 1$$

for each $(U_i)_{i\in\mathcal{I}} \in \mathcal{U}$. If $\big(W_1',\dots,W_{v-1}'\big) \in \mathcal{C}$ is arbitrary but fixed, then there are exactly $\#\widehat{\mathcal{U}}$ elements $(U_i)_{i\in\mathcal{I}} \in \mathcal{U}$ with $U_i \le W_i'$ for all $i \in \mathcal{I}$ since $\#\widehat{\mathcal{U}}$ is independent of the choice of Λ', see e.g. Lemma 4.3. □

We remark that Theorem 4.2 generalizes the *anticode bound* for constant dimension subspace codes, i.e.,

$$A_q^i(v,d;k) \le \begin{bmatrix} v \\ k-d+1 \end{bmatrix}_q \Big/ \begin{bmatrix} k \\ k-d+1 \end{bmatrix}_q.$$

Lemma 4.3. *Let* $\mathbf{0} \le r \le m(v)$, $u \in \mathbb{N}^{v-1}$ *defined via* $i - m(v)_i + r_i$, *and* $\mathcal{I} = \{1 \le i \le v-1 \mid r_i > 0\} = \{k_1,\dots,k_m\}$, *where* $0 < k_1 < \dots < k_m < v$. *Then*

$$\frac{\#\mathcal{U}}{\#\widehat{\mathcal{U}}} = \frac{\left[{v \atop u_{k_1}}\right]_q \cdot \prod_{i=2}^m \left[{v-u_{k_{i-1}} \atop u_{k_i}-u_{k_{i-1}}}\right]_q}{\left[{k_1 \atop u_{k_1}}\right]_q \cdot \prod_{i=2}^m \left[{k_i-u_{k_{i-1}} \atop u_{k_i}-u_{k_{i-1}}}\right]_q},$$

where

$$\begin{aligned} \mathcal{U} &= \left\{(U_i)_{i\in\mathcal{I}} \text{ weakly increasing } \mid \dim(U_i) = i - m(v)_i + r_i \, \forall i \in \mathcal{I}\right\}, \\ \widehat{\mathcal{U}} &= \left\{(U_i)_{i\in\mathcal{I}} \text{ weakly increasing } \mid \dim(U_i) = i - m(v)_i + r_i, U_i \le W_i' \, \forall i \in \mathcal{I}\right\}, \end{aligned}$$

and $\Lambda' = \big(W_1', \dots, W_{v-1}'\big) \in \mathcal{F}(v,q)$ *is an arbitrary but fixed full flag.*

Proof. From Equation (1) we conclude

$$\#\mathcal{U} = \begin{bmatrix} v \\ u_{k_1} \end{bmatrix}_q \cdot \prod_{i=2}^m \begin{bmatrix} v - u_{k_{i-1}} \\ u_{k_i} - u_{k_{i-1}} \end{bmatrix}_q.$$

If $A \le B$ are two subspaces in $\mathbb{F}_q^v$, then the number of subspaces X with $A \le X \le B$ with dimension $\dim(A) \le x \le \dim(B)$ is given by $\left[{\dim(B)-\dim(A) \atop x-\dim(A)}\right]_q$. Thus, we can iteratively conclude

$$\#\widehat{\mathcal{U}} = \begin{bmatrix} k_1 \\ u_{k_1} \end{bmatrix}_q \cdot \prod_{i=2}^m \begin{bmatrix} k_i - u_{k_{i-1}} \\ u_{k_i} - u_{k_{i-1}} \end{bmatrix}_q.$$

□

Example 4.4.

- For $r = (1,0,0,0,0)$ Theorem 4.2 gives $A_q^f(6,7) \le \left[{6 \atop 1}\right]_q = q^5+q^4+q^3+q^2+q+1$.
- For $r = (0,1,2,0,0)$ Theorem 4.2 gives $A_q^f(6,7) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 1}\right]_q / \left[{2 \atop 1}\right]_q / \left[{2 \atop 1}\right]_q = q^7+2q^5+3q^3-q^2+3q-2+\frac{3}{q+1}$.
- For $r = (0,1,0,1,0)$ Theorem 4.2 gives $A_q^f(6,7) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 2}\right]_q / \left[{2 \atop 1}\right]_q / \left[{3 \atop 2}\right]_q = q^8+2q^6+q^5+2q^4+q^3+2q^2+1$.
- For $r = (1,0,2,0,0)$ Theorem 4.2 gives $A_q^f(6,6) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 1}\right]_q / \left[{2 \atop 1}\right]_q = \left[{6 \atop 2}\right]_q = q^8+q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$.
- For $r = (1,0,0,1,0)$ Theorem 4.2 gives $A_q^f(6,6) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 2}\right]_q / \left[{3 \atop 1}\right]_q = q^9+q^8+2q^7+3q^6+3q^5+3q^4+3q^3+2q^2+q+1$.
- For $r = (0,1,0,0,1)$ Theorem 4.2 gives $A_q^f(6,6) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 1}\right]_q / \left[{2 \atop 1}\right]_q = \left[{6 \atop 2}\right]_q = q^8+q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$.
- For $r = (0,2,0,0,0)$ Theorem 4.2 gives $A_q^f(6,6) \le \left[{6 \atop 2}\right]_q = q^8+q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$.
- For $r = (0,1,2,1,0)$ Theorem 4.2 gives $A_q^f(6,6) \le \left[{6 \atop 1}\right]_q \cdot \left[{5 \atop 1}\right]_q \cdot \left[{4 \atop 1}\right]_q / \left[{2 \atop 1}\right]_q^3 = q^9+3q^7+5q^5-q^4+6q^3-3q^2+6q-5+\frac{6}{q+1}$.

We summarize these examples two the following two upper bounds.

Proposition 4.5.

$$A_q^f(6,6) \le \begin{bmatrix}6\\2\end{bmatrix}_q = q^8+q^7+2q^6+2q^5+3q^4+2q^3+2q^2+q+1$$

Proof. We apply Theorem 4.2 with $r = (0,2,0,0,0)$ noting that $\overline{r} = (0,2,2,0,0)$. □

If we prescribe the cyclic group of order 7 generated by g_6^7, see above, for $q = 2$, then the corresponding ILP of Theorem 3.13 admits a solution of cardinality 224, while we aborted the solution process before it was finished.

Proposition 4.6.

$$A_q^f(6,7) \le \begin{bmatrix}6\\1\end{bmatrix}_q = q^5+q^4+q^3+q^2+q+1$$

Proof. We apply Theorem 4.2 with $r = (1,0,0,0,0)$ noting that $\overline{r} = (1,1,1,0,0)$. □

If we prescribe the cyclic group of order 7 generated by

$$g_6^7 := \left\langle \begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\\ 0&0&0&1&1&0 \end{pmatrix} \right\rangle$$

for $q = 2$, then the corresponding ILP of Theorem 3.13 admits a solution of cardinality 63, which was found in the root node.

Another example of the application of Theorem 4.2 is given by:

Proposition 4.7.

$$A_q^f(6,8) \le \begin{bmatrix}6\\1\end{bmatrix}_q \cdot \begin{bmatrix}2\\1\end{bmatrix}_q = q^4+q^2+1$$

Proof. We apply Theorem 4.2 with $r = (0,1,0,0,0)$ noting that $\overline{r} = (0,1,1,0,0)$. □

We remark that applying Theorem 4.2 with $r = (0,0,2,0,0)$ gives $A_2^f(6,8) \le \begin{bmatrix}6\\2\end{bmatrix}_2 / \begin{bmatrix}3\\2\end{bmatrix}_2 = 93$. However, we also get $A_2^f(6,8) \le A_2^i(6,2;3) = 77$ from $r = (0,0,2,0,0)$, which is of course superseded by $r = (0,1,0,0,0)$. For $q = 2$, solving the ILP from Proposition 3.12 directly gives a full flag code of matching cardinality 21 after 35 minutes and 2577 branch-&-bound nodes. If we prescribe the cyclic group of order 7 generated by

$$\left\langle \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \right\rangle,$$

then the corresponding ILP of Theorem 3.13 admits a solution of cardinality 21, which was found in the root node.

For constant dimension codes the anticode bound was improved to the so-called *Johnson bound* $A_q^i(v,d;k) \le \left\lfloor \frac{q^v-1}{q^k-1} \cdot A_q^i(v-1,d;k-1) \right\rfloor$ for the cases where $d < k$. More precisely, without rounding down the iterative application of the Johnson bound together with $A_q^i(v,k;k) \le \frac{q^v-1}{q^k-1}$ implies the anticode bound. The main idea is to consider the subcode consisting of the codewords that all contain a given point P, which can also be applied in the setting of (full) flag codes:

Proposition 4.8. *If* $v \ge 3$ *and* $1 \le d \le \sum_{i=1}^{v-2} \min\{i, v-1-i\}$*, then* $A_q^f(v,d) \le \begin{bmatrix}v\\1\end{bmatrix}_q \cdot A_q^f(v-1,d)$.

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^v$ with minimum Grassmann distance d. If two different codewords $\Lambda = (W_1, \dots, W_{v-1})$ and $\Lambda' = \left(W_1', \dots, W_{v-1}'\right) \in \mathcal{C}$ satisfy $W_1 = W_1' = P$ for some point $P \le \mathbb{F}_q^v$, then we can write $W_i = \langle P, U_{i-1}\rangle$ and $W_i' = \left\langle P, U_{i-1}'\right\rangle$ for all $2 \le i \le v-1$, where $\dim(U_i) = \dim(U_i') = i$ for all $1 \le i \le v-2$. Now, observe that $\mathrm{d_G}(\Lambda, \Lambda') =$

$$\sum_{i=1}^{v-1} (i - \dim(W_i \cap W_i')) = \sum_{i=1}^{v-2} (i - \dim(U_i \cap U_i')) = \mathrm{d_G}\Big((U_1, \dots, U_{v-2}), \left(U_1', \dots, U_{v-2}'\right) \Big),$$

so that $\#\left\{(W_1, \dots, W_{v-1}) \in \mathcal{C} \mid W_1 = P\right\} \le A_q^f(v-1,d)$ (if $A_q^f(v-1,d) \ge 1$). □

Corollary 4.9.

$$A_q^f(6,6) \le \begin{bmatrix}6\\1\end{bmatrix}_q \cdot \left(q^3+1\right) = q^8 + q^7 + q^6 + 2q^5 + 2q^4 + 2q^3 + q^2 + q + 1$$

Proof. Since $A_q^f(5,6) = q^3+1$, see Proposition 2.4, the stated upper bound follows from Proposition 4.8. □

Note that Corollary 4.9 improves upon Proposition 4.5. Moreover, in all cases where d is small enough, so that Proposition 4.8 can be applied, the so far stated upper bounds are indeed implied by Proposition 4.8.

For the cases where the minimum Grassmann distance d is so large that it violates the condition of Proposition 4.8, we state:

Proposition 4.10. *Let* $r = \alpha \cdot e_i$ *with* $\mathbf{0} \le r \le m(v)$ *and* $d > \sum_{i=1}^{v-1} (m(v)_i - \overline{r}_i)$*, where* $\alpha \in \mathbb{N}_{>0}$ *and* e_i *denotes the ith unit vector* ($1 \le i \le v-1$)*. Then, we have* $A_q^f(v,d) \le A_q^i(v, m(v)_i - r_i + 1; i)$*, where* $m(v)_i - r_i + 1 = \min\{i, v-i\} - r_i + 1$.

Proof. Let $\mathcal{C}$ be a full flag code in $\mathbb{F}_q^v$ with minimum Grassmann distance d. From Corollary 3.2 and Lemma 3.8 we conclude $\mathrm{d_i}(W_i, W_i') \le m(v)_i - r_i + 1$ for each pair of codewords $(W_1, \dots, W_{v-1})$ and $\left(W_1', \dots, W_{v-1}'\right)$ in $\mathcal{C}$, so that

$$\#\mathcal{C} = \#\left\{W_i \mid (W_1, \dots, W_{v-1}) \in \mathcal{C}\right\} \le A_q^i(v, m(v)_i - r_i + 1; i).$$

□

We can e.g. conclude Proposition 2.4 from Proposition 4.10.

Corollary 4.11. *For* $0 \le \delta < \lfloor v/2 \rfloor$ *we have* $A_q^f(v, d_{max} - \delta) \le A_q^i(v, k; k)$*, where* $k = \lfloor v/2 \rfloor - \delta$ *and* $d_{\max} = \sum_{i=1}^{v-1} m(v)_i = \left\lfloor (v/2)^2 \right\rfloor$.

Proof. Let $\hat{v} = \lfloor v/2 \rfloor$ and $r = e_{\hat{v}}$. We can easily check that $\overline{r} = \sum_{i=\hat{v}-\delta}^{\hat{v}} e_i$, i.e., $\overline{r}$ consists of $\delta + 1$ ones. Thus, we can apply Proposition 4.10. □

Based on the recursive application of Proposition 4.8 and Proposition 4.10 we can state a general explicit upper bound for $A_q^f(v, d)$ if we only focus on the leading coefficient:

Proposition 4.12.

$$A_q^f(v, d) \le q^\beta + O\left(q^{\beta-1}\right),$$

where $\beta = \frac{v(v-1) - \hat{v}(\hat{v}-1)}{2} + \hat{v} - d + \left\lfloor (\hat{v}-1)^2/4 \right\rfloor$ *and* $\hat{v} = \left\lceil 2\sqrt{d} \right\rceil$.

Proof. First we observe that we can apply Corollary 4.11 if $v = \left\lceil 2\sqrt{d} \right\rceil$, i.e., then there exists an integer δ satisfying $0 \le \delta < \lfloor v/2 \rfloor$ and $d = d_{max} - \delta$. Applying Proposition 4.8 $v - \hat{v}$ times gives

$$A_q^f(v, d) \le \left(\prod_{i=\hat{v}+1}^{v} \begin{bmatrix} i \\ 1 \end{bmatrix}_q \right) \cdot A_q^f(\hat{v}, d),$$

so that

$$A_q^f(v, d) \le \left(q^\alpha + O(q^{\alpha-1})\right) \cdot A_q^f(\hat{v}, d),$$

where $\alpha = \frac{v(v-1) - \hat{v}(\hat{v}-1)}{2}$, since $\begin{bmatrix} i \\ 1 \end{bmatrix}_q = q^{i-1} + O(q^{i-2})$ and $\sum_{i=v\hat{+}1+1}^{v} (i-1) = \alpha$. From Corollary 4.11 we conclude

$$A_q^f(\hat{v}, d) \le A_q^i(v, k; k) \le \begin{bmatrix} \hat{v} \\ 1 \end{bmatrix}_q \Big/ \begin{bmatrix} k \\ 1 \end{bmatrix}_q \le q^{\hat{v}-k} + O\left(q^{\hat{v}-k} - 1\right),$$

where $k = d - \left\lfloor (\hat{v}-1)^2/4 \right\rfloor$. Thus, we have $A_q^f(v, d) \le q^\beta + O\left(q^{\beta-1}\right)$. □

In Table 4 we list the values of β in Proposition 4.12 for $v \le 7$.

v/d	1	2	3	4	5	6	7	8	9	10	11	12
2	1											
3	3	2										
4	6	5	3	2								
5	10	9	7	6	4	3						
6	15	14	12	11	9	8	5	4	3			
7	21	20	18	17	15	14	11	10	9	6	5	4

TABLE 4. Values of β in Proposition 4.12, i.e., $A_q^f(v, d) \le q^\beta + O\left(q^{\beta-1}\right)$.

5. BOUNDS FOR NON-FULL FLAGS AND OTHER VARIANTS

In this section we want to merely consider a few examples in order to shed some light on the general picture.

Example 5.1. We can easily generalize Definition 3.5 and Theorem 4.2 to the situation $T \subsetneq \{1,\dots,v-1\}$. For a flag code $\mathcal{C}$ in $\mathbb{F}_2^6$ of type $T=\{2,3,4\}$ we obtain:

- $\overline{(2,0,0)}=(2,2,0)\rightsquigarrow \#\mathcal{C}\le \begin{bmatrix}6\\2\end{bmatrix}_2=651$;
- $\overline{(1,2,0)}=(1,2,0)\rightsquigarrow \#\mathcal{C}\le \frac{\begin{bmatrix}6\\1\end{bmatrix}_2\cdot\begin{bmatrix}5\\1\end{bmatrix}_2}{\begin{bmatrix}2\\1\end{bmatrix}_2\cdot\begin{bmatrix}2\\1\end{bmatrix}_2}=217$;
-
- $\overline{(0,3,0)}=(1,3,1)\rightsquigarrow \#\mathcal{C}\le \begin{bmatrix}6\\3\end{bmatrix}_2=1395$;
- $\overline{(1,0,1)}=(1,1,1)\rightsquigarrow \#\mathcal{C}\le \frac{\begin{bmatrix}6\\1\end{bmatrix}_2\cdot\begin{bmatrix}5\\2\end{bmatrix}_2}{\begin{bmatrix}2\\1\end{bmatrix}_2\cdot\begin{bmatrix}3\\1\end{bmatrix}_2}=465$,

so that $A_2^f(6,5;\{2,3,4\})\le 217$.

The vector $r=(1,0,1)$ with $\overline{r}=(1,1,1)$ is of special interest with respect to the relation of Lemma 3.1 and Corollary 3.2, where the latter is the one used in Theorem 4.2 and Lemma 4.3. Going along Corollary 3.2 we would consider the flag of a point P and a plane E with $P\le E$ such that there is at most one codeword (W_2,W_3,W_4) with $P\le W_2$ and $E\le W_4$. Using the more general Lemma 3.1, we can also consider the flag of a point P and a 5-space K with $P\le K$ to conclude that there is at most one codeword (W_2,W_3,W_4) with $P\le W_2$ and $W_4\le K$. There are $\begin{bmatrix}6\\1\end{bmatrix}_2\cdot\begin{bmatrix}5\\1\end{bmatrix}_2$ such flags $P\le K$ in total and for each fixed codeword (W_2,W_3,W_4) there are $\begin{bmatrix}2\\1\end{bmatrix}_2\cdot\begin{bmatrix}2\\1\end{bmatrix}_2$ flags $P\le K$ with $P\le W_2$ and $W_4\le K$. Thus, $A_2^f(6,5;\{2,3,4\})\le\frac{63\cdot 31}{3\cdot 3}=217$.

The underlying idea of Proposition 4.8 can also be generalized easily, i.e., if $\mathcal{C}$ is a flag code in $\mathbb{F}_2^6$ of type $T=\{2,3,4\}$ and minimum Grassmann distance $d=5$, then given a point P the set

$$\mathcal{C}_P:=\{(W_2,W_3,W_4)\in\mathcal{C}\mid P\le W_2\}$$

corresponds to a flag code in $\mathbb{F}_2^5$ of type $\{1,2,3\}$ and minimum Grassmann distance $d=5$. Thus, $\#\mathcal{C}_P\le A_2^f(5,5;\{1,2,3\})$ and $A_2^f(6,5;\{2,3,4\})\le\frac{\begin{bmatrix}6\\1\end{bmatrix}_2}{\begin{bmatrix}2\\1\end{bmatrix}_2}\cdot A_2^f(5,5;\{1,2,3\})$. For $A_2^f(5,5;\{1,2,3\})$ we observe that the 2-spaces in the middle layer of the codewords have to give a partial line spread in $\mathbb{F}_2^5$, so that $A_2^f(5,5;\{1,2,3\})\le 9$ and $A_2^f(6,5;\{2,3,4\})\le\frac{63}{3}\cdot 9=189$, which improves upon the previously stated upper bounds.

Example 5.2. Let us consider some upper bounds for $A_2^f(7,3;\{3,4\})$.

- $\overline{(3,0)}=(3,2)\rightsquigarrow A_2^f(7,3;\{3,4\})\le\#\begin{bmatrix}7\\3\end{bmatrix}_2=11811$;
- $\overline{(2,2)}=(2,2)\rightsquigarrow A_2^f(7,3;\{3,4\})\le\frac{\begin{bmatrix}7\\2\end{bmatrix}_2\cdot\begin{bmatrix}5\\1\end{bmatrix}_2}{\begin{bmatrix}3\\2\end{bmatrix}_2\cdot\begin{bmatrix}2\\1\end{bmatrix}_2}=3937$;
- $\overline{(0,3)}=(3,3,)\rightsquigarrow A_2^f(7,3;\{3,4\})\le\#\begin{bmatrix}7\\3\end{bmatrix}_2=11811$.

Alternatively, by considering all codewords (W_3,W_4), where W_3 contains a fixed point P, c.f. Proposition 4.8, we obtain

$$A_2^f(7,3;\{3,4\})\le\frac{\begin{bmatrix}7\\1\end{bmatrix}_2}{\begin{bmatrix}3\\1\end{bmatrix}_2}\cdot A_2^f(6,3;\{2,3\})=\frac{\begin{bmatrix}7\\1\end{bmatrix}_2}{\begin{bmatrix}3\\1\end{bmatrix}_2}\cdot A_2^f(6,3;\{2,3\}). \tag{10}$$

For $A_2^f(6,3;\{2,3\})$ we can use the argument again and obtain $A_2^f(6,3;\{2,3\}\le\frac{63}{3}\cdot A_2^f(5,3;\{1,2\})$. Since the lines of the second layer of the codewords have to give a partial line spread in $\mathbb{F}_2^5$, we have $A_2^f(5,3;\{1,2\})\le 9$ (indeed, we have $A_2^f(5,3;\{1,2\})=9$), so that $A_2^f(6,3;\{2,3\}\le 189$. Thus, Inequality (10) yields $A_2^f(7,3;\{3,4\})\le 3429$, which improves upon the previously stated upper bounds.

Let us assume $A_2^f(6,3;\{2,3\} \le 185$ for a moment. Inequality (10) then would yield $A_2^f(7,3;\{3,4\}) \le \frac{127}{7} \cdot 185 = 3356 + \frac{3}{7}$. Of course this can be rounded down to 3356, since $A_2^f(7,3;\{3,4\})$ is an integer. However, as in the case of constant dimension codes the rounding of the Johnson bound can be improved using the theory of q^r-divisible codes, see [10]. More concretely, for each codeword (W_3, W_4) we just consider the plane W_3. Since we assume $A_2^f(6,3;\{2,3\} \le 185$ those planes cover each point of $\mathbb{F}_2^7$ at most 185 times. If the flag code has cardinality 3356 then not every point of $\mathbb{F}_2^7$ can be covered exactly 185 times, i.e., the missing points correspond to a multiset of points of cardinality 3, which in turn corresponds to a binary linear code of effective length 3. Since it can be shown that this code has to be 4-divisible, i.e., the weight of every codeword has to be divisible by 4 and such a code cannot exist, we could strengthen our argument to $A_2^f(7,3;\{3,4\}) \le 3355$. (A 4-divisible binary linear code of effective length 10 indeed exists.) For the details we refer to [10, Lemma 13(i)] and its preparing results and definitions.

Another variant is to consider sets of elements of the Cartesian product $\begin{bmatrix}\mathbb{F}_q^v\\1\end{bmatrix} \times \begin{bmatrix}\mathbb{F}_q^v\\2\end{bmatrix} \times \cdots \times \begin{bmatrix}\mathbb{F}_q^v\\v-1\end{bmatrix}$ as codes with respect to the Grassman distance. By $A_q^c(v,d)$ we denote the corresponding maximum cardinality of such a code with minimum Grassmann distance d. Obviously we have $A_q^f(v,d) \le A_q^c(v,d)$ and $d \le \lfloor (v/2)^2 \rfloor$. If we replace $\bar{r}$ by r then the modified version of Theorem 4.2 holds for $A_q^c(v,d)$. As a special case we obtain the same upper for the maximum possible Grassmann distance for $A_q^c(v, \lfloor (v/2)^2 \rfloor)$ as for $A_q^f(v, \lfloor (v/2)^2 \rfloor)$, so that:

Proposition 5.3. *For each integer $k \ge 1$ we have*

$$A_q^c(2k, k^2) = q^k + 1$$

and for each integer $k \ge 2$ we have

$$A_q^c(2k+1, k^2+k) = q^{k+1} + 1.$$

Similar as for flag codes we can restrict the possible dimensions of the parts of a codeword to a subset $\emptyset \ne T \subseteq \{1, \dots, v-1\}$, which we call type. More precisely, codewords are elements of $\times_{t \in T} \begin{bmatrix}\mathbb{F}_q^v\\t\end{bmatrix}$. By $A_q^c(v,d;T)$ we denote the corresponding maximum possible cardinality of such a code. For Grassmann distance $d = 1$ we have

$$A_q^c(v,1;T) = \prod_{t \in T} \begin{bmatrix}v\\t\end{bmatrix}_q \tag{11}$$

and

$$A_q^c(v,1) = \prod_{t=1}^{v-1} \begin{bmatrix}v\\t\end{bmatrix}_q. \tag{12}$$

In order to show that $A_q^f(v,d;T)$ and $A_q^c(v,d;T)$ can have different orders of magnitude in terms of the field size q we consider the example $(v,d) = (5,2)$ and $T = \{2,3\}$. Since $\mathrm{d_G}\big((L,E),(L,E')\big) \le 1$ for a line L and two planes E, E' containing L, we have

$$A_q^f(5,2;\{2,3\}) \le \begin{bmatrix}5\\2\end{bmatrix}_q = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1.$$

Next we want to construct a larger lower bound for $A_q^c(5,2;\{2,3\})$ and introduce some necessary notation. For two matrices $A, B \in \mathbb{F}_q^{m \times n}$ we define the rank distance $\mathrm{d_r}(A,B) := \mathrm{rk}(A-B)$. A subset $\mathcal{M} \subseteq \mathbb{F}_q^{m \times n}$ is called a rank metric code.

Theorem 5.4. *(see [7]) Let $m, n \ge d'$ be positive integers, q a prime power, and $\mathcal{M} \subseteq \mathbb{F}_q^{m \times n}$ be a rank metric code with minimum rank distance d'. Then, $\#\mathcal{M} \le q^{\max\{n,m\} \cdot (\min\{n,m\} - d' + 1)}$.*

Codes attaining this upper bound are called maximum rank distance (MRD) codes. They exist for all choices of parameters, which remains true if we restrict to linear rank metric codes, see [7]. For e.g. $(m,n)=(2,3)$ and $d'=2$ there exists an MRD code $\mathcal{M}^{2,3}$ of cardinality q^3. For a general $m\times n$ MRD code $\mathcal{M}$ we can associate to each matrix $M\in\mathcal{M}$ the rowspace $\langle (I_{m\times m}|M)\rangle$ of the concatenation of the $m\times m$ unit matrix $I_{m\times m}$ and matrix M, which is an m-dimensional subspace of $\mathbb{F}_q^{m+n}$. The construction of a subspace from a matrix is also called *lifting*. If $U=\langle (I_{m\times m}|M)\rangle$ and $W=\langle (I_{m\times m}|M')\rangle$ are two subspaces lifted from two matrices, then $\mathrm{d_i}(U,W)=\mathrm{d_r}(M,M')$. Thus, $\mathcal{M}^{2,3}$ can be lifted to a set of q^3 lines in $\mathbb{F}_q^5$ with pairwise intersection distance 2, i.e., a partial line spread. Since $\mathcal{M}^{2,3}$ is linear the q^6 2×3 matrices over $\mathbb{F}_q$ can be partitioned into q^3 2×3 MRD codes with minimum rank distance 2. By lifting we obtain q^6 lines $U'_{i,j}$ in $\mathbb{F}_q^5$, where $1\le i\le q^3$ and $1\le j\le q^3$, such that

$$\begin{aligned}\mathrm{d_i}\big(U'_{i,j},U'_{i',j'}\big)&=2 \quad\text{if}\quad i\neq i', j=j',\\ \mathrm{d_i}\big(U'_{i,j},U'_{i',j'}\big)&=1 \quad\text{if}\quad j\neq j', \text{ and}\\ \mathrm{d_i}\big(U'_{i,j},U'_{i',j'}\big)&=0 \quad\text{if}\quad i=i', j=j'.\end{aligned}$$

By duplication this configuration q^3 times we obtain q^3 lines $U_{i,j,h}$, where $1\le i,j,h\le q^3$, such that

$$\begin{aligned}\mathrm{d_i}(U_{i,j,h},U_{i',j',h'})&=2 \quad\text{if}\quad i\neq i', j=j',\\ \mathrm{d_i}(U_{i,j,h},U_{i',j',h'})&=1 \quad\text{if}\quad j\neq j', \text{ and}\\ \mathrm{d_i}(U_{i,j,h},U_{i',j',h'})&=0 \quad\text{if}\quad i=i', j=j'.\end{aligned}$$

Starting from a 3×2 MRD code $\mathcal{M}^{3\times 2}$ with minimum rank distance 2 and cardinality q^3 we can similarly construct q^9 planes $W_{i,j,h}$, where $1\le i,j,h\le q^3$ such that

$$\begin{aligned}\mathrm{d_i}(W_{i,j,h},W_{i',j',h'})&=2 \quad\text{if}\quad h\neq h', j=j',\\ \mathrm{d_i}(W_{i,j,h},W_{i',j',h'})&=1 \quad\text{if}\quad j\neq j', \text{ and}\\ \mathrm{d_i}(W_{i,j,h},W_{i',j',h'})&=0 \quad\text{if}\quad h=h', j=j'.\end{aligned}$$

With this we can construct a flag code $\mathcal{C}=\big\{(U_{i,j,h},W_{i,j,h})\,:\,1\le i,j,h\le q^3\big\}$ of type $\{2,3\}$ and cardinality q^9. It can be easily checked that

$$\mathrm{d_G}\Big(\,(U_{i,j,h},W_{i,j,h})\,,(U_{i',j',h'},W_{i',j','h})\,\Big)=\mathrm{d_i}\Big(U_{i,j,h},U_{i',j',h'}\Big)+\mathrm{d_i}\Big(W_{i,j,h},W_{i',j',h'}\Big)\ge 2$$

if $(i,j,h)\neq(i',j',h')$. Thus, the minimum Grassman distance of $\mathcal{C}$ is at least 2 and $A_q^c(5,2;\{2,3\})\ge q^9$. By considering the codewords (L,E) with a fixed line L and planes E contained in a hyperplane H of $\mathbb{F}_q^5$ we may show $A_q^c(5,2;\{2,3\})\le \begin{bmatrix}5\\2\end{bmatrix}_q\cdot\begin{bmatrix}5\\1\end{bmatrix}_q=q^{10}+O\big(q^9\big)$.

We can also extend our construction to a full flag code for $(v,d)=(5,3)$. To this end let $U^2_{i,j}$ be lines in $\mathbb{F}_q^5$ and $U^3_{i,j}$ be planes in $\mathbb{F}_q^5$ for $1\le i,j\le q^4$ such that

$$\mathrm{d_G}\Big(\,\big(U^2_{i,j},U^3_{i,j}\big)\,,\big(U^2_{i',j'},U^3_{i',j'}\big)\,\Big)\ge 2$$

whenever $(i,j)\neq(i',j')$. Since there are $\begin{bmatrix}5\\1\end{bmatrix}_q\ge 4$ points in $\mathbb{F}_q^5$, we can choose q^8 points $U^1_{i,j}$ such that $\mathrm{d_i}\Big(U^1_{i,j},U^1_{i',j'}\Big)=1$ if $j\neq j'$ and zero otherwise. Similarly, we can choose q^8 hyperplane $U^4_{i,j}$ such that $\mathrm{d_i}\Big(U^4_{i,j},U^1_{i',j'}\Big)=4$ if $i\neq i'$ and zero otherwise. With this we can check that

$$\mathcal{C}=\Big\{\Big(U^1_{i,j},U^2_{i,j},U^3_{i,j},U^4_{i,j}\Big)\,:\,1\le i,j\le q^4\Big\}$$

is a full flag code in $\mathbb{F}_q^5$ with cardinality q^8 and minimum Grassmann distance 3. Thus, we have $A_q^c(5,3)\ge q^8$, while $A_q^f(5,3)\le q^7+O\big(q^6\big)$.

6. Exact values and bounds for small parameters

In this section we summarize the exact values and bounds for $A_q^f(v,d)$ from the previous sections. We start with the known exact formulas that are parametric in q from Section 2, i.e., for $d=1$, $d=\lfloor (v/2)^2 \rfloor$ (Proposition 2.4), $(v,d)=(3,2)$ (Propositions 2.5), and $(v,d)=(4,3)$ (Propositions 2.6).

$$A_q^f(2,1) = q+1 \tag{13}$$

$$A_q^f(3,1) = (q+1)\cdot\left(q^2+q+1\right) = q^3+2q^2+2q+1 \tag{14}$$

$$A_q^f(3,2) = q^2+q+1 \tag{15}$$

$$A_q^f(4,1) = (q+1)\cdot\left(q^2+q+1\right)\cdot\left(q^3+q^2+q+1\right) = q^6+3q^5+5q^4+6q^3+5q^2+3q+1 \tag{16}$$

$$A_q^f(4,3) = q^3+q^2+q+1 \tag{17}$$

$$A_q^f(4,4) = q^2+1 \tag{18}$$

$$\begin{aligned} A_q^f(5,1) &= (q+1)\cdot\left(q^2+q+1\right)\cdot\left(q^3+q^2+q+1\right)\cdot\left(q^4+q^3+q^2+q+1\right) \\ &= q^{10}+4q^9+9q^8+15q^7+20q^6+22q^5+20q^4+15q^3+9q^2+4q+1 \end{aligned} \tag{19}$$

$$A_q^f(5,6) = q^3+1 \tag{20}$$

$$\begin{aligned} A_q^f(6,1) &= (q{+}1)\left(q^2{+}q{+}1\right)\left(q^3{+}q^2{+}q{+}1\right)\left(q^4{+}q^3{+}q^2{+}q{+}1\right)\left(q^5{+}q^4{+}q^3{+}q^2{+}q{+}1\right) \\ &= q^{15}+5q^{14}+14q^{13}+29q^{12}+49q^{11}+71q^{10}+90q^9+101q^8+101q^7+90q^6 \\ &\quad +71q^5+49q^4+29q^3+14q^2+5q+1 \end{aligned} \tag{21}$$

$$A_q^f(6,9) = q^3+1 \tag{22}$$

We continue with parametric upper bounds. Propositions 2.7, 4.6, 4.7 and Corollary 4.9 state

$$\begin{aligned} A_q^f(4,2) &\le q^5+2q^4+3q^3+3q^2+2q+1, \quad (23)\\ A_q^f(6,6) &\le q^8+q^7+q^6+2q^5+2q^4+2q^3+q^2+q+1, \quad (24)\\ A_q^f(6,7) &\le q^5+q^4+q^3+q^2+q+1, \text{ and} \quad (25)\\ A_q^f(6,8) &\le q^4+q^2+1. \quad (26) \end{aligned}$$

Next we complete the missing parametric cases (v,d) for $v\le 5$. To this end we use clique constraints corresponding to $\mathcal{V}_{v,q}^r$ for a suitable reduction vector r, i.e., we apply Theorem 4.2 and Lemma 4.3 to evaluate the involved cardinalities.

Proposition 6.1.

$$A_q^f(5,2) \le \begin{bmatrix}5\\1\end{bmatrix}_q \cdot \begin{bmatrix}4\\1\end{bmatrix}_q \cdot \begin{bmatrix}3\\1\end{bmatrix}_q = q^9+3q^8+6q^7+9q^6+11q^5+11q^4+9q^3+6q^2+3q+1 \tag{27}$$

Proof. Since $\overline{(1,2,2,0)}=(1,2,2,1)$ the stated upper bound is obtained from the clique constraints corresponding to $\mathcal{V}_{5,q}^{1,2,2,0}$. □

For $q = 2$ prescribing a Singer cycle, i.e., a cyclic group of order 31, the ILP from Section 3 has an optimal target value of 3069, while the upper bound of Proposition 6.1 yields $A_2^f(5,2) \le 3255$.

Proposition 6.2.

$$A_q^f(5,3) \le \begin{bmatrix}5\\1\end{bmatrix}_q \cdot \begin{bmatrix}4\\1\end{bmatrix}_q = q^7 + 2q^6 + 3q^5 + 4q^4 + 4q^3 + 3q^2 + 2q + 1 \tag{28}$$

Proof. Since $\overline{(1,2,0,0)} = (1,2,1,0)$ the stated upper bound is obtained from the clique constraints corresponding to $\mathcal{V}_{5,q}^{1,2,0,0}$. □

We remark that Proposition 6.2 is tight for $q = 2$, i.e., a corresponding code of cardinality 465 indeed exists. Such a code also exists if we prescribe a Singer cycle, i.e., a cyclic group of order 31.

Proposition 6.3.

$$A_q^f(5,4) \le \begin{bmatrix}5\\1\end{bmatrix}_q \cdot \left(q^2+1\right) = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1 \tag{29}$$

Proof. Since $\overline{(1,0,1,0)} = (1,1,1,0)$ the stated upper bound is obtained from the clique constraints corresponding to $\mathcal{V}_{5,q}^{1,0,1,0}$. □

We remark that Proposition 6.3 is tight for $q = 2$, i.e., a corresponding code of cardinality 155 indeed exists. Such a code also exists if we prescribe a Singer cycle, i.e., a cyclic group of order 31.

Proposition 6.4.

$$A_q^f(5,5) \le \begin{bmatrix}5\\1\end{bmatrix}_q = q^4 + q^3 + q^2 + q + 1 \tag{30}$$

Proof. Since $\overline{(1,0,0,0)} = (1,1,0,0)$ the stated upper bound is obtained from the clique constraints corresponding to $\mathcal{V}_{5,q}^{1,0,0,0}$. □

We remark that Proposition 6.4 is tight for $q = 2$, i.e., a corresponding code of cardinality 31 indeed exists. Such a code also exists if we prescribe a Singer cycle, i.e., a cyclic group of order 31.

v/d	1	2	3	4	5	6	7	8	9
2	3								
3	21	7							
4	315	105	15	5					
5	9765	3120–3255	465	155	31	9			

TABLE 5. Bounds for $A_2^f(v,d)$ for $v \le 6$.

For the binary case $q = 2$ we can say a bit more. Except for $(v,d) = (5,2)$ the upper bounds for $v \le 5$ are attained, see Table 5. The lower bounds have been mainly obtained using the ILP approach, with prescribed automorphisms, see Section 3 and Section 4 for the details on the chosen groups. For $v = 6$ and $v = 7$ we list the upper bounds resulting from Proposition 4.8 and Proposition 4.10 in Table 6. As exact values we have $A_2^f(6,1) = 615195$, $A_2^f(6,7) = 63$, $A_2^f(6,8) = 21$, and $A_2^f(6,7) = 9$ for $v = 6$. The inequalities $224 \le A_2^f(6,6) \le 567$ show that it might be hard to obtain narrow bounds for $A_2^g(v,d)$ even for medium sized parameters.

v/d	1	2	3	4	5	6	7	8	9	10	11	12
6	615195	205065	29295	9765	1953	567	63	21	9			
7	78129765	26043255	3720465	1240155	248031	72009	8001	2667	1143	127	41	17

TABLE 6. Upper bounds for $A_2^f(6,d)$ and $A_2^f(7,d)$.

7. CONCLUSION AND FUTURE RESEARCH

Comparing the data of Table 1 and Table 4 we conjecture that the upper bounds for $A_q^f(v,d)$ induced by Proposition 4.8 and Corollary 4.11 are always tighter than the *sphere packing bound* for flag codes, see [13, 14]. Of course it would be interesting to determine an explicit formula for the leading coefficient of the sphere packing bound for $A_q^f(v,d)$, or the sphere covering bound, as we have determined for our upper bound in Proposition 4.12. Intended more as an inspiring challenge instead of being based on rigorous insights, we conjecture that the bound of Proposition 4.12 is tight up to the terms of lower order. To this end a series of general constructions is desirable, see e.g. [12, 13], where the authors have shown that flag codes can be superior to constant dimension codes. In those cases that we have investigated the order of magnitude of the sphere covering bound is not exceeded. Is seems that the parametric construction of good flag codes is a teaser. In those cases in Section 4 where proposed upper bounds for $A_q^f(v,d)$ are attained by a flag code with a Singer group as subgroup of automorphisms for $q=2$, we conjecture that this is the case for all field sizes q.

One may introduce a more general version of Theorem 4.2 and Lemma 4.3 based on Lemma 3.1 instead of Corollary 3.2, see the discussion after Example 5.1. However, it is not clear if the corresponding bounds will be competitive.

The determination of tighter bounds for $A_2^f(6,d)$ and $A_2^f(7,d)$ seems to be an interesting an challenging open problem. Of course the situation for $A_q^f(v,d;T)$, i.e. non-full flag codes, and for $A_q^s(v,d)$ is even wider open than it is for $A_q^f(v,d)$.

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