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Comment on "Quantum Kaniadakis entropy under projective measurement"

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We comment on the main result given by Ourabah *et al.* [Phys. Rev. E **92**, 032114 (2015)], noting that it can be derived as a special case of the more general study that we have provided in [Quantum Inf Process **15**, 3393 (2016)]. Our proof of the nondecreasing character under projective measurements of so-called generalized (h,ϕ) entropies (that comprise the Kaniadakis family as a particular case) has been based on majorization and Schur-concavity arguments. As a consequence, we have obtained that this property is obviously satisfied by Kaniadakis entropy but at the same time is fulfilled by all entropies preserving majorization. In addition, we have seen that our result holds for any bistochastic map, being a projective measurement a particular case. We argue here that looking at these facts from the point of view given in [Quantum Inf Process **15**, 3393 (2016)] not only simplifies the demonstrations but allows for a deeper understanding of the entropic properties involved.

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Let a quantum system be described by a density operator ρ, that is, a positive-semidefinite operator acting on an N-dimensional Hilbert space, with trace 1. A quantum version of the Kaniadakis entropy [1,2] has been recently introduced in [3] as follows:

$$S_\kappa(\rho) = \frac{1}{2\kappa}\,\mathrm{Tr}(\rho^{1-\kappa} - \rho^{1+\kappa}), \tag{1}$$

where the entropic index κ is a real number with $|\kappa| < 1$. Originally, the classical version of the Kaniadakis entropy has been formulated as a deformation of the Boltzmann-Shannon entropy to deal with classical relativistic systems [2]. It is straightforward to see that $S_\kappa(\rho)$ reduces to the von Neumann entropy $S(\rho) = -\mathrm{Tr}\,\rho \ln \rho$ in the limiting case $\kappa \to 0$. Thus, it is to be expected that some of the properties of the von Neumann entropy remain valid for the Kaniadakis entropy. Indeed, in [4] it was proven that the quantum Kaniadakis entropy cannot decrease under the action of projective measurements. Specifically, let $\Pi = \{P_k\}$ be a projective measurement, that is, the P_k are orthogonal projectors that sum up to the identity operator. The state of the system after the measurement without postselection is given by (see, e.g., [5])

$$\Pi(\rho) = \sum_k P_k \rho P_k. \tag{2}$$

The main result in [4] showed that the quantum Kaniadakis entropy of the final state is greater than or equal to that of the initial state,

$$S_\kappa[\Pi(\rho)] \geqslant S_\kappa(\rho). \tag{3}$$

In this Comment we rely on our work [6], where we provided an alternative proof of inequality (3) to the proof given in Ref. [4]. Due to the fact that our proof is based on majorization and the Schur-concavity property, our findings revealed that this result holds for a more general family of entropies, with the Kaniadakis entropy being a particular case. In this way, we put the property represented by (3) in a wider and simpler context, which allows us to better understand the properties of a huge family of entropic functions. Our derivation shows that this is not an isolated or coincidental property of one particular entropic functional, but rather a structural feature of many of them.

Our general proof runs as follows. Let us consider the quantum version of (h,ϕ) entropies [7], which we have recently introduced [6],

$$\mathbf{H}_{(h,\phi)} = h[\mathrm{Tr}\,\phi(\rho)], \tag{4}$$

where the entropic functionals $h : \mathbb{R} \mapsto \mathbb{R}$ and $\phi : [0,1] \mapsto \mathbb{R}$ are continuous and such that either of the following holds true:

(i) ϕ is strictly concave and h is strictly increasing.

(ii) ϕ is strictly convex and h is strictly decreasing.

In both cases, and without loss of generality, $\phi(0) = 0$ and $h[\phi(1)] = 0$. Choosing $h(x) = x$ and $\phi(x) = \frac{x^{1-\kappa} - x^{1+\kappa}}{2\kappa}$, one recovers the Kaniadakis entropy (1) as a particular case. Moreover, it is easy to see that the latter, one-parameter entropic functional can be recast under a more familiar form; indeed, using the well-known expression for the κ-logarithms (see [8] for a detailed study of these functions), given by

$$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa}, \tag{5}$$

one has

$$\phi(x) = -x\,\ln_\kappa(x). \tag{6}$$

This shows that it is indeed included in that wider family as a particular case (see Table I in [6]). In a similar way, different choices lead to other important entropies, such as, for instance, the von Neumann [9], Rényi [10], or Tsallis [11] entropies, as exhibited in Table I in [6].

An important property of the quantum (h,ϕ) entropies related to the purpose of this Comment is the Schur-concavity or majorization preservation (Proposition 1 in [6]). It is said that ρ' is majorized by ρ, denoted as $\rho' \prec \rho$, if the eigenvalues

λ' and λ of ρ' and ρ, respectively, sorted in decreasing order, satisfy the conditions (see, e.g., [12])

$$\sum_{i=1}^{n} \lambda'_i \leqslant \sum_{i=1}^{n} \lambda_i \quad \forall\, n = 1,\ldots,N-1 \quad \text{and}$$
$$\sum_{i=1}^{N} \lambda'_i = \sum_{i=1}^{N} \lambda_i. \tag{7}$$

If ρ' and ρ act on Hilbert spaces of different sizes, i.e., if λ' and λ have different lengths, one can add extra zero-valued components to the shorter one to equal the lengths; N in (7) is to be understood as the maximal Hilbert space size. It is easy to check that this has no impact on the value of the (h,ϕ) entropies because of their expansibility property [6]. The Schur-concavity property goes as follows:

$$\text{if} \quad \rho' \prec \rho \quad \text{then} \quad \mathbf{H}_{(h,\phi)}(\rho') \geqslant \mathbf{H}_{(h,\phi)}(\rho). \tag{8}$$

As is well known, it can be shown that under the action of any quantum bistochastic operation $\mathcal{E}$, i.e., $\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, where the superscript dagger stands for the adjoint operation and the $E_k E_k^\dagger$ as well as the $E_k^\dagger E_k$ sum up to the identity operator, the final state is majorized by the initial one (see, e.g., [13]), that is,

$$\mathcal{E}(\rho) \prec \rho, \tag{9}$$

with equality [in the sense of (7)] if and only if $\mathcal{E}(\rho) = U\rho U^\dagger$, where U is a unitary operator. Note that $\mathcal{E}$ leaves the maximally mixed state invariant, i.e., $\mathcal{E}(\frac{I}{N}) = \frac{I}{N}$, where I is the identity operator. As a consequence of (8) and (9), for any bistochastic operation $\mathcal{E}$ the (h,ϕ) entropy cannot decrease after the action of the map, that is,

$$\mathbf{H}_{(h,\phi)}[\mathcal{E}(\rho)] \geqslant \mathbf{H}_{(h,\phi)}(\rho). \tag{10}$$

It is straightforward to see that the projective measurement (2) is a very particular bistochastic operation. Thus, as an immediate corollary of (10), we have

$$\mathbf{H}_{(h,\phi)}[\Pi(\rho)] \geqslant \mathbf{H}_{(h,\phi)}(\rho). \tag{11}$$

Notice that the equality in (11) is attained if and only if $\Pi(\rho) = \rho$ (the projective measurement does not disturb the state). Now, choosing in particular $h(x) = x$ and $\phi(x) = \frac{x^{1-\kappa} - x^{1+\kappa}}{2\kappa} = -x\, \ln_\kappa(x)$ with $|\kappa| < 1$ in (11), we derive the result of Ourabah *et al.*, Eq. (3); even more, we obtain the condition for equality. In addition, a quantum version of the three-parameter entropy given in [14], which is mentioned in the last sentence of [4], can be expressed as a particular case of Eq. (4), choosing $h(x) = x$ and $\phi(x) = -x\, \ln_{\kappa\tau\varsigma}(x)$, where the three-parameter deformed logarithm is defined as $\ln_{\kappa\tau\varsigma}(x) = \frac{\varsigma^\kappa x^{\tau+\kappa} - \varsigma^{-\kappa} x^{\tau-\kappa} - \varsigma^\kappa + \varsigma^{-\kappa}}{(\kappa+\tau)\varsigma^\kappa + (\kappa-\tau)\varsigma^{-\kappa}}$, with adequate ranges of values for κ, τ, and ς. Therefore, these entropies also satisfy inequality (11).

To conclude, we stress that our results are far more general than the results given in [4]: they are valid for any entropic measure preserving the majorization relation and any bistochastic map.

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