DETERMINING POSSIBLE SETS OF LEAVES FOR SPANNING TREES OF DUALLY CHORDAL GRAPHS

Pablo De Caria and Marisa Gutierrez

CONICET, Departamento de Matemática, Facultad de Ciencias Exactas,Universidad Nacional de La Plata,
{pdecaria,marisa}@mate.unlp.edu.ar

Abstract: It will be proved that the problem of determining whether a set of vertices of a dually chordal graphs is the set of leaves of a tree compatible with it can be solved in polynomial time by establishing a connection with finding clique trees of chordal graphs with minimum number of leaves.

Keywords: *Leaf, compatible tree, clique tree, chordal, dually chordal.*

1 INTRODUCTION

Chordal and dually chordal graphs were found to have many applications, specially in biology. Both classes are endowed with characteristic tree structures, clique trees in chordal graphs and compatible trees in dually chordal graphs, which in several cases are connected with the solution of problems associated to the applications. A good example of this are *phylogenetic trees* [4, 5], used to model the evolutionary history of species, proteins, etc. In them, it is necessary that leaves represent the present individuals (or objects) and inner vertices should indicate possible ancestors. This makes desirable, also in a more general context, the ability to determine what vertices can be the leaves of a compatible tree or a clique tree.

The *leafage* of a chordal graph is the minimum number of leaves of a clique tree of the graph. A polynomial algorithm, running in time $O(n^3)$, to find the leafage of a chordal graph has been proposed recently [3]. The goal of this paper is to show that this enables an answer to the following problem: given a dually chordal graph G and $A \subset V(G)$, determine if there is a tree compatible with G whose set of leaves is A. For that purpose, every dually chordal graph is found to be the clique graph of a chordal graph in such a way that there is a correspondence between the compatible trees of the former and the clique trees of the latter and then the problem is transformed into that of finding the leafage of a chordal graph.

2 SOME GRAPH TERMINOLOGY

This paper deals just with graphs without loops or multiple edges. For a graph G, $V(G)$ is the set of its vertices and $E(G)$ that of its edges. A *complete* is a subset of pairwise adjacent vertices of $V(G)$. A maximal complete is a *clique* and $C(G)$ will be used to denote all the cliques of G. And a *clique edge cover* of G is defined as any subset F of $C(G)$ such that any edge of G is contained in at least one element of F.

Given two vertices v and w of a graph G, the *distance* between v and w, or $d(v,w)$, is the length of a shortest path connecting v and w in G. For a vertex $v \in V(G)$, the *closed neighborhood* of v, $N[v]$, is the set composed of v and all the vertices adjacent to it. The *disk* centered at vertex v with radius k is the set $N^k[v] := \{w \in V(G),\ d(v,w) \le k\}$.

Let T be a tree. For all $v, w \in V(T)$, $T[v,w]$ will denote the path of T from v to w. And $\ell(T)$ will denote the set of leaves of T.

Let F be a family of nonempty sets. The *intersection graph* of F has the elements of F as vertices, being two of them adjacent if their intersection is nonempty. The *clique graph* $K(G)$ of a graph G is the intersection graph of $C(G)$.

A graph such that $C(G)$ is a Helly family, i.e., any subfamily of pairwise intersecting cliques has a nonempty intersection, is called a *Helly graph*.

3 BASIC NOTIONS AND PROPERTIES

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are those without chordless cycles of length at least four. A *clique tree* T of G is a spanning tree of $K(G)$ such that, for any $v \in V(G)$, the set $\{C \in C(G),\ v \in C\}$ induces a subtree of T. One of the many characterizations for the aforementioned class is that a graph is chordal if and only if it has a clique tree [6].

A vertex w is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1...v_n$ of the vertices of G is a *maximum neighborhood ordering* of G if, for $i = 1, ..., n$, v_i has a maximum neighbor in $G[\{v_i, ..., v_n\}]$. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering.

Moreover, more characterizations of dually chordal graphs have been given. In fact, given a connected graph G, it is dually chordal if and only if [1]:

1. *There is a spanning tree T of G such that any clique of G induces a subtree in T.*
2. *There is a spanning tree T of G such that any closed neighborhood of G induces a subtree in T.*
3. *G is Helly and K(G) is chordal.*

It is even true that any spanning tree fulfilling (i) also fulfills (ii) and vice versa. Such a tree will be said to be *compatible* with G. We also have the following equivalence:

Theorem 1 *[2] Let T be a spanning tree of a dually chordal graph G. Then T is compatible with G if and only if, for all $x, y, z \in V(G)$, $xy \in E(G)$ and $z \in T[x, y] - \{x, y\}$ implies that $xz \in E(G)$ and $yz \in E(G)$.*

4 Leaves and dominated vertices

Before the goal of this paper is achieved, some properties about domination will be necessary to find some conditions that the leaves of a compatible tree should satisfy. The graphs considered are always connected.

Lemma 1 *Let G be a dually chordal graph and T a tree compatible with G. If v is a leaf of T and w is the vertex such that $vw \in E(T)$ then v is dominated by w.*

Proof. For any vertex u in $N[v] - \{v, w\}$ it holds that $w \in T[u, v]$. From Theorem 1 we infer that w is adjacent to u and thus $N[v] - \{v, w\} \subseteq N[w]$. As $\{v, w\}$ is also a subset of $N[w]$, the inclusion $N[v] \subseteq N[w]$ follows. □

Corollary 1 *Let G be a dually chordal graph , $|V(G)| \geq 3$, and T a tree compatible with G. Then each vertex in $\ell(T)$ is dominated by at least one vertex of $\ell(T)^C$.*

Lemma 2 *Let G be a dually chordal graph and T be a tree compatible with G. Then, given $v \in V(G)$, the set $D = \{w \in V(G) : N[v] \subseteq N[w]\}$, i.e., v itself and the vertices dominating it, induces a subtree in T.*

Proof. Let $w \in V(G)$. Then $w \in D$ if and only if, for all $u \in N[v]$, $u \in N[w]$, that is, $w \in N[u]$ for all $u \in N[v]$. Thus $w \in D$ if and only if $w \in \bigcap_{u \in N[v]} N[u]$ and so $D = \bigcap_{u \in N[v]} N[u]$.

Since T is compatible with G, any closed neighborhood induces a subtree in T. And if some subsets induce subtrees so does their intersection. Therefore D induces a subtree. □

As it was said before, clique trees of chordal graphs will be essential for the solution of the problem. Therefore the existence of a relationship between compatible trees of dually chordal graphs and clique trees of chordal graphs is desirable. There is indeed one relationship and it is as follows:

Theorem 2 *Let G be a dually chordal graph and F be a clique edge cover of G. Let H be the intersection graph of $F \cup V(G)$, i.e., the family whose elements are the cliques of G in F and the vertices of G. Then*

(1) *H is chordal.*

(2) *$K(H) \approx G$.*

(3) *Any clique tree of H is isomorphic to a tree compatible with G and vice versa.*

Proof. Let T be a tree compatible with G. Then any member of $F \cup V(G)$ induces a subtree in T. As intersection graphs of subtrees in a tree are chordal [7], (1) follows.

Given any vertex $v \in V(G)$, the set $D_v = \{v\} \cup \{C \in F :\ v \in C\}$ is a clique of H because D_v is a complete and the equality $N_H[v] = D_v$ implies maximality (and also that v is simplicial in H). And in fact, every clique of H is equal to D_v, for some $v \in V(G)$. A proof of this is given below.

Let $D \in C(H)$. Then the elements of $F \cap D$ are pairwise intersecting. As dually chordal graphs are Helly, there is a vertex w which is an element of each clique of G in $F \cap D$ and therefore $F \cap D \subsetneq N_H[w] = D_w$. This implies that $F \cap D$ is not a maximal complete of H and thus there exists $v \in V(G)$ such that $v \in D$. Since $D \in C(H)$ and it is contained in D_v, also in $C(H)$, it follows that $D = D_v$ and then $C(H) = \{D_v :\ v \in V(G)\}$.

Now we need to demonstrate that $D_u D_v \in E(K(H))$ if and only if $uv \in E(G)$, which implies that $K(H) \approx G$. And the reasoning is as follows:

$$D_u D_v \in E(K(H)) \Leftrightarrow D_u \cap D_v \neq \emptyset \Leftrightarrow \exists C \in F,\ C \in D_u \wedge C \in D_v \Leftrightarrow \exists C \in F,\ u \in C \wedge v \in C \Leftrightarrow uv \in E(G)$$

being the assumption that F covers all the edges of G necessary in the last step. This proves (2).

Let T be a clique tree for H and T' the spanning tree of G such that $uv \in E(T')$ if and only if $D_u D_v \in E(T)$. Let $x,\ y$ be vertices adjacent in G and $z \in T'[x, y] - \{x, y\}$. Then D_x and D_y are adjacent in $K(H)$ and let $C \in D_x \cap D_y$. As T is a clique tree the subset $\{D \in C(H) :\ C \in D\}$ induces a subtree of T, implying that D_z also belongs to it because $D_z \in T[D_x, D_y]$. Consequently $D_x \cap D_z \neq \emptyset$ and $D_y \cap D_z \neq \emptyset$ and hence $xz, yz \in E(G)$, making T' compatible with G.

Conversely, let T be a tree compatible with G and T' the spanning tree of $K(H)$ such that $D_u D_v \in E(T')$ if and only if $uv \in E(T)$. For any $v \in V(G)$ the set $\{D \in C(H) :\ v \in D\} = \{D_v\}$ so it obviously induces a subtree. Let $C \in F$, D_x and D_y such that $C \in D_x \cap D_y$ and D_z be any vertex of $T'[D_x, D_y] - \{D_x, D_y\}$. Then $x \in C$, $y \in C$ and $z \in T[x, y] - \{x, y\}$. Since T is compatible with G and C induces a subtree in T, $z \in C$, that is, $C \in D_z$. This implies that the set $\{D \in C(H) :\ C \in D\}$ induces a subtree in T' and therefore T' is a clique tree of H. □

As a consequence of this, looking for a compatible tree with certain characteristics can be considered as equivalent to finding a clique tree.

Having narrowed down before what the elements of $\ell(T)$ can be for a tree T compatible with a dually chordal graph G it remains to introduce an auxiliary graph G' which will contain information about the problem. The results required now are the following:

Lemma 3 *Let G be a dually chordal graph, T a tree compatible with G and u, v, w vertices such that $uv \in E(T)$, $v \in T[u, w]$ and $N[u] \cap N[v] \subseteq N[w]$. Then $T' = T - uv + uw$ is also compatible with G.*

Proof. Let x be any vertex of G. We need to prove that $N[x]$ induces a subtree in T'. Call $T[A]$ and $T[B]$ the connected components of $T - uv$, with $u \in A$ and $v \in B$. The proof is divided into three cases.

If $N[x] \subseteq A$ then $N[x]$ induces the same subtree in T and T'. If $N[x] \subseteq B$ the reasoning is similar. Otherwise we have two vertices $y, z \in N[x]$ such that $y \in A$ and $z \in B$. As $N[x]$ induces a subtree in T and $u, v \in T[y, z]$ we conclude that $u, v \in N[x]$, that is, $x \in N[u] \cap N[v]$ and therefore w is adjacent to x. Now, u and v are connected in T' by the path formed by merging uw and $T[w, v]$ (contained in $N[x]$ because $w, v \in N[x]$ and T is compatible with G); and any other two vertices of $N[x]$ adjacent in T are still adjacent in T'. Therefore vertices of $N[x]$ adjacent in T are connected in T' by paths within $N[x]$ and this is enough to claim that $N[x]$ induces a subtree in T', making T' compatible with G. □

Theorem 3 *Let G be a dually chordal graph and $A \subseteq V(G)$ be a set of vertices, being each of them dominated by a vertex in A^C. Let G' be a graph constructed from G by adding, for each $v \in A$, a vertex v^* and the edge vv^*. Then G' is dually chordal. Moreover, there is a tree T compatible with G such that $\ell(T) = A$ if and only if there is a tree T' compatible with G' such that $\ell(T') = A^* := \{v^*,\ v \in A\}$.*

Proof. Let T be a tree compatible with G. Then the tree T' such that $V(T') = V(G) \cup A^*$ and $E(T') = E(T) \cup \{vv^*,\ v \in A\}$ is compatible with G', so this graph is dually chordal. Furthermore, if $\ell(T) = A$ then $\ell(T') = A^*$.

Conversely, let T' be a tree compatible with G' and such that $\ell(T') = A^*$. Consider now the tree $T_0 = T' - A^*$ spanning G. Since $C(G) \subseteq C(G')$, any clique of G induces a subtree of T' and thus in T_0 as well and we conclude that T_0 is compatible with G.

Now, none of the leaves of T' are leaves of T_0 (they were simply removed). That the leaves of T_0 are not those of T' means that the degree of the former have decreased after the removal of vertices, and therefore $\ell(T_0) \subseteq A$. Of all the trees compatible with G and with set of leaves contained in A (we know that there is at least one) choose T_1 such that $|\ell(T_1)|$ is maximum. It will be proved that $\ell(T_1) = A$. If $\ell(T_1) \neq A$ take a vertex $u \in A - \ell(T_1)$, let w be a vertex in A^C dominating u and w' the vertex adjacent to u in $T_1[u, w]$. Lemma 2 implies that w' dominates u. Moreover it is not a leaf of T_1. In fact, if $w' = w$ we know that w is not a leaf; and if w' is an internal vertex of $T_1[u, w]$ then it is adjacent to two vertices of that path.

By Lemma 3, if for any vertex x different from w' and adjacent to u in T_1 we add the edge wx to T_1 and remove ux we get a new tree T_2 compatible with G such that $d_{T_2}(w') > d_{T_1}(w')$, u is a leaf of T_2 and the remaining vertices have the same degree in T_1 and T_2. Then $\ell(T_2) = \ell(T_1) \cup \{u\}$, contradicting the way T_1 was chosen. Therefore $\ell(T_1) = A$. □

Now it is possible to prove the main theorem:

Theorem 4 *Let G be a dually chordal graph and A be a subset of $V(G)$ such that for each vertex of A there is a vertex in A^C dominating it. Determining if there exists a tree compatible with G and whose set of leaves equals A can be reduced, in polynomial time, to the problem of finding a clique tree with minimum number of leaves in a chordal graph and hence it is itself polynomial.*

Proof. Let G' be the same graph as in Theorem 3 and H' be a chordal graph such that $K(H') = G'$ and constructed as in Theorem 2. Denote by T' a clique tree for H' with minimum number of leaves and let T^* be a tree compatible with G' and isomorphic to T'. By part **(3)** of Theorem 2, T^* is a compatible tree for G' with minimum number of leaves.

If $\ell(T^*) = A^*$, Theorem 3 implies that there is a tree T compatible with G and $\ell(T) = A$. Otherwise, since the degree in G' of the vertices in A^* equals 1, $A^* \subsetneq \ell(T^*)$. As the number of leaves of T^* is minimum, no tree compatible with G' has A^* as set of leaves and this time Theorem 3 implies that there is not any tree T compatible with G and such that $\ell(T) = A$.

Overall, H' was constructed from G' by using one of its clique edge covers, being possible to take a cover whose number of elements is bounded by $E(G')$, and so by $E(G)+V(G)$, and the complexity of computing an intersection graph and finding a clique tree of H' with minimum number of leaves is polynomial; hence the whole procedure can be accomplished in polynomial time. □

Consequently, given G dually chordal graph and $A \subset V(G)$ such that each vertex of A is dominated by a vertex in A^C, the algorithm to determine if there is a tree T compatible with G and such that $\ell(T) = A$ can be summarized as follows: find a clique edge cover of G' to compute H' and then calculate the leafage of H'. If the leafage equals $|A|$ the requested tree exists. Otherwise it does not.

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SPECIAL ECCENTRIC VERTICES FOR CHORDAL AND DUALLY CHORDAL GRAPHS AND RELATED CLASSES

Pablo De Caria and Marisa Gutierrez

CONICET, Departamento de Matemática, Facultad de Ciencias Exactas,Universidad Nacional de La Plata,
{pdecaria,marisa}@mate.unlp.edu.ar

Abstract: It is known that any vertex of a chordal graph has an eccentric vertex which is simplicial. Here we prove similar properties in related classes of graphs where the simplicial vertices will be replaced by other special types of vertices.

Keywords: *Eccentric vertex, chordal, dually chordal.*

1 BASIC DEFINITIONS

For a graph G, $V(G)$ denotes the set of its vertices and $E(G)$ that of its edges. A *complete* is a set of pairwise adjacent vertices. The subgraph *induced* by $A \subseteq V(G)$, $G[A]$, has A as vertex set and two vertices are adjacent in $G[A]$ if they are adjacent in G.

Given two vertices v and w of a graph G, the *distance* between v and w, or $d(v,w)$, is the length of a shortest path connecting v and w in G. When such a path does not exist it may be said that $d(v,w)=\infty$. For a vertex $v \in V(G)$, the *open neighborhood* of v, $N(v)$, is the set of all vertices adjacent to v. The *closed neighborhood* of v, $N[v]$, is defined by $N[v]=N(v)\cup\{v\}$. The *disk* centered at vertex v with radius k is the set of vertices at distance at most k from v and it is indicated by $N^k[v]$. The *eccentricity* of v is $ecc(v)=max\{d(v,w), w \in V(G)\}$. A vertex w in G is called *eccentric* of v if no vertex in $V(G)$ is further away from v than w, that is, if $ecc(v)=d(v,w)$.

The *kth-power*, G^k, of a graph G is a graph which has the same vertices as G, being two of them adjacent in G^k if the distance between them is at most k in G.

A *chord* of a cycle is an edge joining two nonconsecutive vertices of the cycle. *Chordal* graphs are defined as those without chordless cycles of length at least four.

A vertex v is *simplicial* if $N[v]$ is a complete. A linear ordering $v_1v_2...v_n$ of vertices of a graph G is called a *perfect elimination ordering* if, for $1 \leq i \leq n$, v_i is simplicial in $G_i = G[\{v_i,...,v_n\}]$.

One of the most classical characterizations of chordal graphs states that a graph is chordal if and only if it has a perfect elimination ordering.

A vertex $w \in N[v]$ is a *maximum neighbor* of v if $N^2[v] \subseteq N[w]$. A linear ordering $v_1...v_n$ of vertices of G is a *maximum neighborhood ordering* if, for all $1 \leq i \leq n$, v_i has a maximum neighbor in G_i. *Dually chordal graphs* can be defined as those possessing a maximum neighborhood ordering.

2 ECCENTRIC VERTICES

We can see that vertices with a maximum neighbor are as important for dually chordal graphs as simplicial vertices are for chordal graphs. Then, if any vertex of a chordal graph has a simplicial eccentric vertex the question wether any vertex of a dually chordal graph has an eccentric vertex with a maximum neighbor arises. The answer is yes, and before proving it we need some previous results.

Lemma 1 *[1] If G is a dually chordal graph and A is a subset of $V(G)$ such that any pair of vertices of A is at a distance not greater than 2, then there is a vertex w with $A \subseteq N[w]$.*

Lemma 2 *[1] If G is dually chordal then G^2 is chordal.*

Lemma 3 *Let G be a dually chordal graph and v a simplicial vertex in G^2. Then v has a maximum neighbor in G.*

Proof. As v is simplicial in G^2 the distance in G between any pair of vertices of $N^2[v]$ is at most 2. Applying Lemma 1 gives a vertex w such that $N^2[v] \subseteq N[w]$. Then w is a maximum neighbor of v. □

Now the major result can be proved. From now on it will be assumed that G is always a connected graph. Otherwise the proofs are trivial.

Theorem 1 *Let G be a dually chordal graph and v a vertex of G. There exists an eccentric vertex of v with maximum neighbor.*

Proof. First suppose that $ecc_G(v)$ is odd. As G^2 is chordal we can choose a vertex w simplicial in G^2 which is eccentric of v in G^2. Hence, by Lemma 3, w has a maximum neighbor in G. Note first that, because of the definition of G^2, if two vertices are at distance k in G their distance in G^2 is $\frac{k}{2}$ if k is even or $\frac{k+1}{2}$ if k is odd. Furthermore any eccentric vertex of v in G will be also eccentric in G^2, implying that the eccentricity of v in G^2 equals $\frac{ecc_G(v)+1}{2}$ because $ecc_G(v)$ is odd. By using the definition of G^2 again and that $d_{G^2}(v,w) = \frac{ecc_G(v)+1}{2}$, we have two possible values for $d_G(v,w)$, namely, $ecc_G(v)$ or $ecc_G(v)+1$. The definition of eccentricity implies that $d(v,w) = ecc_G(v)$ and thus w is the required vertex.

If $ecc_G(v)$ is even, let G' be a graph obtained from G by adding a new vertex v' and making it adjacent to v. Then G' is dually chordal. In fact, if $v_1...v_n$ is a maximum neighborhood ordering for G then $v'v_1...v_n$ is a maximum neighborhood ordering of G'. It is valid that $ecc_{G'}(v')$ is odd and by proceeding like in the previous paragraph there is a vertex u with a maximum neighbor in G' (and so in G) such that $d(v',u) = ecc_{G'}(v')$. It can be easily verified that u is the desired vertex. □

Corollary 1 *If G is a nontrivial, i.e., not composed of just one vertex, dually chordal graph then there are two vertices v_1 and v_2 with maximum neighbors and such that $d(v_1,v_2) = diam(G)$.*

Proof. Let $k = diam(G)$ and x, y two vertices with $d(x,y) = k$. Then there exists a vertex v_1 with maximum neighbor and eccentric of x, so $d(x,v_1) = k$. And likewise there is a vertex v_2 with a maximum neighbor and eccentric of v_1 and consequently $d(v_1,v_2) = k$. □

At this moment it is interesting to determine if similar properties are valid for more specific types of graphs. The answer is affirmative and we will prove it for power chordal and doubly chordal graphs.

A graph G is said to be *power chordal* if all of its powers are chordal. It is true that a graph is power chordal if and only if G and G^2 are chordal [1]. A graph is *doubly chordal* if it is chordal and dually chordal. Any vertex of it which is simplicial and has a maximum neighbor is called *doubly simplicial*.

It is known that a power chordal graph is complete or there are two nonadjacent vertices which are simplicial in both G and G^2. The demonstration can be seen in [1]. A similar technique enables to prove the following result:

Theorem 2 *Let G be a power chordal graph. If $v \in V(G)$ then there exists a vertex w eccentric of v in G^2 which is simplicial in both G and G^2.*

Proof. The proof is direct if G^2 is complete. Assume that G^2 is not complete. Since G^2 is chordal we can take a vertex u which is simplicial in G^2 and eccentric of v in G^2. If u is also simplicial in G there is nothing else to do and we can set $w = u$. On the contrary, let x and y be two nonadjacent neighbors of u and S a minimal xy-separator in G. Then S is a complete because G is chordal [2] and $u \in S$.

Let $G[A]$ and $G[B]$ be the connected components of $G - S$ containing x and y respectively. Without loss of generality we can assume that $v \notin A$. It holds that $G[A \cup S]$ is chordal and since S is a complete minimal separator $(G[A \cup S])^2 = G^2[A \cup S]$ and thus $(G[A \cup S])^2$ is also chordal. Then we have two possibilities: either $G[A \cup S]$ is complete or contains two nonadjacent vertices which are both simplicial in $G[A \cup S]$ and $G^2[A \cup S] = (G[A \cup S])^2$ [1]. Whichever the case we conclude that the set A contains a vertex w which is simplicial in $G[A \cup S]$ and $G^2[A \cup S]$. It is evident that w is simplicial in G. Now it will be demonstrated that w is also simplicial in G^2. If $N^2[w] \subseteq A \cup S$ it is obvious. Otherwise, w must be adjacent to a vertex w' in S. If $z \in N^2[w] \cap (A \cup S)$, then $z \in N^2[u]$ because $u \in N^2[w]$ (note that $w \in N[w']$ and $w' \in N[u]$) and w is simplicial in $G^2[A \cup S]$. If $z \in N^2[w] - (A \cup S)$ then again $z \in N^2[u]$ because any path of length two joining w and z (vertices which are in different connected components of $G - S$) must have its intermediate vertex in S, which could be u or adjacent to it because S is a complete. This makes a path between z and u of length at most two possible. Therefore $N^2[w] \subseteq N^2[u]$ and as u is simplicial in G^2 so is w.

Since v and w are in different connected components of $G - S$ any path joining them must include a vertex in S, and so in $N[u]$. We can conclude that $d_G(u, v) \leq d_G(v, w)$ and then $d_{G^2}(v, w)$ is maximum and has the required properties. □

Theorem 3 *Let G be a power chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v, in G, which is simplicial in G and G^2.*

Proof. The proof is very similar to that of Theorem 1, so we will just give a sketch of it.

We suppose at first that $ecc(v)$ is odd and applying Theorem 2 will give the required vertex.

And if $ecc(v)$ is even the graph G' is again introduced.

□

Corollary 2 *Let G be a doubly chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v which is doubly simplicial.*

Proof. As G is dually chordal G^2 is chordal, so the previous theorem can be applied to get a vertex w simplicial in G and G^2 and eccentric of v. Because of Lemma 3 w has a maximum neighbor in G, so it is doubly simplicial. □

So far it was possible to prove the existence of eccentric vertices with characteristics distinguishing all the classes related to chordal and dually chordal graphs that have been discussed. One that was not mentioned yet is that of *strongly chordal* graphs and fortunately a similar property can be deduced.

A vertex v of a graph G is *simple* if the set $\{N[u] : u \in N[v]\}$ is totally ordered by inclusion. From this definition we infer that, particularly, simple vertices are simplicial and have a maximum neighbor. A linear ordering $v_1 v_2 ... v_n$ of $V(G)$ is called a *simple elimination ordering* of G if, for $1 \leq i \leq n$, v_i is simple in G_i. Strongly chordal graphs are just those possessing at least one such ordering. One of the main characteristics of strongly chordal graphs is that they are hereditary. In fact, being a strongly chordal graph is equivalent to being a hereditary dually chordal graph.

In connection with eccentric vertices we have the following:

Lemma 4 *Let $v \in V(G)$ and w be a maximum neighbor of v with $ecc(w) > 1$ and u such that $d(u, v) \geq 2$. Then $d(u, v) = d(u, w) + 1$ and any vertex eccentric of w is also eccentric of v and vice versa.*

Proof. The property is true if $d(u, v) = 2$ due to the definition of maximum neighbor, so suppose now that $d(u, v) > 2$. By the triangle inequality $d(u, v) \leq d(u, w) + d(w, v)$, that is, $d(u, v) \leq d(u, w) + 1$. Let $vv1v2...u$ be a shortest path from v to u. Then $wv_2...u$ is a path from w to u of length $d(u, v) - 1$. Therefore $d(u, v) - 1 \geq d(u, w)$ and hence $d(u, v) \geq d(u, w) + 1$. Then the equality $d(u, v) = d(u, w) + 1$ holds. This implies that any vertex eccentric of w is at distance greater than or equal to 3 of v and consequently

$$d(v, u) = ecc(v) \Leftrightarrow d(v, u) = max\{d(v, x) : x \in V(G)\} \Leftrightarrow d(v, u) = max\{d(v, x) : x \in V(G),\ d(v, x) \geq 3\} \Leftrightarrow$$

$$d(w, u) + 1 = max\{d(w, x) + 1 : x \in V(G),\ d(w, x) \geq 2\} \Leftrightarrow d(w, u) = max\{d(w, x) : x \in V(G),\ d(w, x) \geq 2\}$$

$$\Leftrightarrow d(w, u) = max\{d(w, x) : x \in V(G)\} \Leftrightarrow d(w, u) = ecc(w)$$

□

Theorem 4 *Let G be a strongly chordal graph. If $v \in V(G)$ then there exists an eccentric vertex of v which is simple.*

Proof. It will be by induction on $n = |V(G)|$. The property is obviously valid when $n = 1$. Suppose now that it is always valid when $n = k$, $k \geq 1$, and that G is a strongly chordal graph with $|V(G)| = k + 1$. Given v, the proof will be divided into cases.

Case 1: G has at least one universal vertex.

Let w be a universal vertex of G. If w is simple then G is complete because simple vertices are simplicial and thus the existence of an eccentric simple vertex is evident. Otherwise, it is trivial in case that $v = w$, so assume now that $v \neq w$ and that w is not simple. Then we consider the strongly chordal graph $G - w$. In case that $G - w$ is not connected, any vertex simple in $G - w$ and located in a connected component different from that of v is an eccentric simple vertex for v in G. If $G-w$ is connected, applying the inductive hypothesis yields an eccentric simple vertex u for v in $G - w$. Then again it will be simple and eccentric of v in G.

Case 2: G has not a universal vertex.

Case 2a: v is simple.

Let v' be a maximum neighbor of v. $G - v$ is strongly chordal and applying the inductive hypothesis on this subgraph gives a simple eccentric vertex of v' in $G - v$ which will be named w. Then it is true that $d(v', w) \geq 2$ because otherwise v' would be universal in G. Now, as v' is a maximum neighbor of v in G, and so $N^2[v] \subseteq N[v']$, we conclude that $d(v, w) \geq 3$ and thus the neighborhoods of vertices in $N[w]$ are coincident in G and $G - v$, from what we can deduce that w is simple in G. And because of Lemma 4 w is also eccentric of v.

Case 2.b: v is not simple and there is a simple vertex which is not adjacent to v.

Let w be a simple vertex not adjacent to v. If it is also eccentric we are done. If not, consider the strongly chordal graph $G - w$, which possesses a simple vertex w' eccentric of v. As removing a simplicial vertex does not change the distance between the other vertices (simplicial vertices are never intermediate vertices in shortest paths) w' is also eccentric in G, so it suffices to prove that w' is simple in G. If w' is not simple in G, there is at least one vertex in $N[w']$ whose neighborhood is not the same in G and $G - w$, implying that $w \in N^2[w']$. Let u be a maximum neighbor of w in G. Then u is adjacent to w' and therefore $d(v, w') \leq d(v, u) + 1$, which combined with Lemma 4 implies that $d(v, w') \leq d(v, w)$, contradicting that w was not an eccentric vertex of v. Consequently w' is necessarily simple.

We claim that all these cases are enough to prove the property for every strongly chordal graph. In fact, if v is not simple and is adjacent to all the simple vertices it will be proved that $diam(G) \leq 2$ and thus G has a universal vertex by Lemma 1. Let x and y be vertices such that $d(x, y) = diam(G)$. If $diam(G) \geq 3$ then $\{x, y\} \not\subseteq N[v]$ so we can assume without loss of generality that $x \notin N[v]$. Since all simple vertices are simplicial and adjacent to v we conclude that none of them is adjacent to x. Then, by case **2.b**, x has a simple eccentric vertex x' and thus $d(x, x') = diam(G)$. By case **2.a** we know that x' has a simple eccentric vertex x'' so $d(x', x'') = diam(G)$. But $d(x', x'') \leq 2$, contradicting that $diam(G) \geq 3$. □

Corollary 3

- *If G is a nontrivial power chordal graph there are two vertices v_1 and v_2, simplicial both in G and G^2, such that $d(v_1, v_2) = diam(G)$.*
- *If G is a nontrivial doubly/strongly chordal graph there are two doubly simplicial/simple vertices v_1 and v_2 such that $d(v_1, v_2) = diam(G)$.*

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