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A non-trivial connection for the metric-affine Gauss–Bonnet theory in $D = 4$

Bert Janssen *, Alejandro Jiménez-Cano, José Alberto Orejuela

Departamento de Física Teórica y del Cosmos and Centro Andaluz de Física de Partículas Elementales, Facultad de Ciencias, Avda Fuentenueva s/n, Universidad de Granada, 18071 Granada, Spain

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ABSTRACT

We study non-trivial (*i.e.* non-Levi-Civita) connections in metric-affine Lovelock theories. First we study the projective invariance of general Lovelock actions and show that all connections constructed by acting with a projective transformation of the Levi-Civita connection are allowed solutions, albeit physically equivalent to Levi-Civita. We then show that the (non-integrable) Weyl connection is also a solution for the specific case of the four-dimensional metric-affine Gauss–Bonnet action, for arbitrary vector fields. The existence of this solution is related to a two-vector family of transformations, that leaves the Gauss–Bonnet action invariant when acting on metric-compatible connections. We argue that this solution is physically inequivalent to the Levi-Civita connection, giving thus a counterexample to the statement that the metric and the Palatini formalisms are equivalent for Lovelock gravities. We discuss the mathematical structure of the set of solutions within the space of connections.

1. Introduction

Metric-affine gravity (sometimes also called the Palatini formalism) is a set of theories in which the metric $g_{\mu\nu}$ and the affine connection $\Gamma_{\mu\nu}{}^{\rho}$ are taken to be independent variables. They are extensions of the more familiar metric theories of gravity, which consider only the metric as a dynamical variable and presuppose invariably the affine connection to be the Levi-Civita connection of the metric,

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} = \tfrac{1}{2} g^{\rho\lambda}\Big(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}\Big). \tag{1.1}$$

In metric-affine theories, however, the idea is that the affine connection $\Gamma_{\mu\nu}{}^{\rho}$ should be determined by its own equation of motion, just as any other dynamical variable of the theory.

It has been shown [1,2] (see also [3]) that the physics described by the Einstein–Hilbert–Palatini action, $S = \frac{1}{2\kappa}\int d^D x\sqrt{|g|}\mathcal{R}(\Gamma)$ with $D > 2$, possibly extended with a minimally coupled matter Lagrangian, is equivalent to the usual metric Einstein–Hilbert action, even though it allows a more general affine connection,

$$\bar{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\,\delta_\nu^\rho, \tag{1.2}$$

with A_μ an arbitrary vector field. Indeed, the vector field A_μ does not have any physically measurable influence, as can be seen in the fact that the Einstein equation and the geodesic equation are identical in the metric and the Palatini formalism. However, one can assign a geometrical meaning to A_μ, since A_μ can be related to the reparametrisation freedom of geodesics [2]: affine geodesics of the connection $\bar{\Gamma}_{\mu\nu}{}^{\rho}$ turn out to be pre-geodesics of the Levi-Civita connection $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$, through the reparametrisation

$$\frac{d\tau}{d\lambda}(\lambda) = \exp\left[\int_0^\lambda A_\rho\,\frac{dx^\rho}{d\lambda'}\,d\lambda'\right], \tag{1.3}$$

where λ is the affine parameter for the $\bar{\Gamma}_{\mu\nu}{}^{\rho}$ geodesics and τ the proper time along the Levi-Civita ones.

Both the existence of the non-trivial solution (1.2) and the absence of physical meaning for A_μ can be understood as a consequence of the projective symmetry

$$\Gamma_{\mu\nu}{}^{\rho} \to \Gamma_{\mu\nu}{}^{\rho} + A_\mu\,\delta_\nu^\rho \tag{1.4}$$

of the metric-affine Einstein–Hilbert action [4,5]. Indeed, the Riemann tensor transforms under the projective transformation as $\mathcal{R}_{\mu\nu\rho}{}^{\lambda} \to \mathcal{R}_{\mu\nu\rho}{}^{\lambda} + 2\partial_{[\mu}A_{\nu]}\,\delta_\rho^\lambda$, leaving hence the Ricci scalar $\mathcal{R} = g^{\mu\rho}\,\delta_\lambda^\nu\,\mathcal{R}_{\mu\nu\rho}{}^{\lambda}$ invariant. Being $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$ a straightforward solution to the connection equation, any connection generated by applying a

* Corresponding author.
E-mail addresses: bjanssen@ugr.es (B. Janssen), alejandrojc@ugr.es (A. Jiménez-Cano), josealberto@ugr.es (J.A. Orejuela).

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projective transformation on it is also a (physically equivalent) solution.

The equivalence between the metric and metric-affine formalism does not extend in general to other gravitational actions (see for example [6]). Often the non-equivalence of the corresponding gravitational actions is used to construct models of modified gravity, with new affine degrees of freedom, that might yield resolution of singularities, alternatives to inflation, dark matter or dark energy [7–21].

There is a class of metric-affine theories for which the Levi-Civita connection is guaranteed to be a solution of the connection equation. Indeed, in [22–24] it was shown that the Levi-Civita connection is a solution for metric-affine lagrangians of the type $\mathcal{L}(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho}{}^{\lambda})$, only if the lagrangian is of the Lovelock type (or at least mimics the symmetries of curvature tensors in the Lovelock lagrangian [23]). Hence, for Lovelock gravities, the metric formalism is a consistent truncation of the Palatini formalism [24]. However it is by no means clear whether for these theories both formalisms are equivalent, as in the case of the Einstein–Hilbert action, in the sense that all allowed solutions of the connection equation of the metric-affine Lovelock lagrangians yield the same physics as the metric formalism. In other words, whether the Levi-Civita connection (possibly up to a projective transformation) is the only solution to the connection equation.

The aim of this paper is to show that in fact they are not, by presenting an explicit counterexample of a very specific Lovelock theory, the Weyl connection in four-dimensional metric-affine Gauss–Bonnet theory, though we believe the result is general for any metric-affine k-th order Lovelock term in $D = 2k$ dimensions. Note that the k-th order Lovelock term in $D = 2k$ dimensions is a topological term in the metric formalism [25], but not necessarily for metric-affine gravity. We will argue that the solution is physically not equivalent to the Levi-Civita connection, which in our opinion is an indication for the non-topological character of general $D = 4$ metric-affine Gauss–Bonnet theory.

The organisation of this paper is as follows: in sections 2 and 3 we review the metric-affine Gauss–Bonnet term in arbitrary dimensions, study the symmetries of the action and write down the equations of motion in a closed form. In section 4 we show that the Weyl connection is a general solution to both the equations of motion of the metric and the connection for the four-dimensional Gauss–Bonnet action. In section 5, we relate the existence of the solution to the projective symmetry of any Lovelock action and a vector symmetry the four-dimensional Gauss–Bonnet in the presence of metric-compatible connections. Finally, in section 6, study the structure of the space of solutions of the Gauss–Bonnet action and state our conclusion.

2. Metric-affine Lovelock theory

The D-dimensional k-th order Lovelock term in the metric-affine formalism is defined as

$$S = \int d^D x \sqrt{|g|}\, \delta^{\mu_1\nu_1\ldots\mu_k\nu_k}_{\alpha_1\beta_1\ldots\alpha_1\beta_k}\, \mathcal{R}_{\mu_1\nu_1}{}^{\alpha_1\beta_1}(\Gamma) \ldots \mathcal{R}_{\mu_k\nu_k}{}^{\alpha_k\beta_k}(\Gamma), \tag{2.1}$$

where we used the following conventions for the Riemann tensor and the antisymmetrised Kronecker delta,

$$\mathcal{R}_{\mu\nu}{}^{\rho\lambda}(\Gamma) = g^{\rho\sigma}\, \mathcal{R}_{\mu\nu\sigma}{}^{\lambda}(\Gamma),$$
$$\mathcal{R}_{\mu\nu\sigma}{}^{\lambda}(\Gamma) = \partial_\mu \Gamma_{\nu\sigma}{}^{\lambda} - \partial_\nu \Gamma_{\mu\sigma}{}^{\lambda} + \Gamma_{\mu\kappa}{}^{\lambda}\, \Gamma_{\nu\sigma}{}^{\kappa} - \Gamma_{\nu\kappa}{}^{\lambda}\, \Gamma_{\mu\sigma}{}^{\kappa},$$

$$\begin{aligned}\delta^{\mu_1\nu_1\ldots\mu_k\nu_k}_{\alpha_1\beta_1\ldots\alpha_k\beta_k} &= \delta^{[\mu_1}_{\alpha_1}\, \delta^{\nu_1}_{\beta_1} \ldots \delta^{\mu_k}_{\alpha_k}\, \delta^{\nu_k]}_{\beta_k} \\ &= \tfrac{(-1)^{D-1}}{(2k)!(D-2k)!}\, |g|\, \varepsilon^{\mu_1\nu_1\ldots\mu_k\nu_k\sigma_1\ldots\sigma_{D-2k}} \\ &\qquad \times \varepsilon_{\alpha_1\beta_1\ldots\alpha_k\beta_k\sigma_1\ldots\sigma_{D-2k}},\end{aligned} \tag{2.2}$$

with $\varepsilon_{\mu_1\ldots\mu_D}$ the completely alternating Levi-Civita symbol.

When varying the action in metric-affine gravity, it is often useful to define the tensor

$$\Sigma^{\mu\nu\alpha}{}_{\beta} = \frac{1}{\sqrt{|g|}}\, \frac{\delta S}{\delta \mathcal{R}_{\mu\nu\alpha}{}^{\beta}}, \tag{2.3}$$

which for the Lovelock action (2.1) is given by

$$\begin{aligned}\Sigma^{\mu\nu}{}_{\alpha\beta} &= g_{\alpha\sigma}\, \Sigma^{\mu\nu\sigma}{}_{\beta} \\ &= k\, \delta^{\mu\nu\rho_2\lambda_2\ldots\rho_k\lambda_k}_{\alpha\beta\gamma_2\epsilon_2\ldots\gamma_k\epsilon_k}\, \mathcal{R}_{\rho_2\lambda_2}{}^{\gamma_2\epsilon_2} \ldots \mathcal{R}_{\rho_k\lambda_k}{}^{\gamma_k\epsilon_k}.\end{aligned} \tag{2.4}$$

Note that in general $\Sigma^{\mu\nu}{}_{\alpha\beta}$ is antisymmetric in the first pair of indices and for Lovelock actions (2.1) also in the last pair, but that the latter is not true in general.

In terms of $\Sigma^{\mu\nu\alpha}{}_{\beta}$, the k-th order Lovelock action (2.1) can be written as

$$S = \frac{1}{k} \int d^D x \sqrt{|g|}\, \mathcal{R}_{\mu\nu\alpha}{}^{\beta}\, \Sigma^{\mu\nu\alpha}{}_{\beta}, \tag{2.5}$$

and hence the equations of motion of the metric and the connection respectively are given by

$$\mathcal{R}_{\mu\nu\rho}{}^{\lambda}\, \Sigma^{\mu\nu}{}_{\sigma\lambda} + \mathcal{R}_{\mu\nu\sigma}{}^{\lambda}\, \Sigma^{\mu\nu}{}_{\rho\lambda} - \frac{1}{k}\, g_{\rho\sigma}\, \mathcal{R}_{\mu\nu\alpha\beta}\, \Sigma^{\mu\nu\alpha\beta} = 0, \tag{2.6}$$

$$\begin{aligned}\nabla_\mu \Sigma^{\mu\nu\alpha}{}_{\beta} - \tfrac{1}{2}\, Q_{\mu\lambda}{}^{\lambda}\, \Sigma^{\mu\nu\alpha}{}_{\beta} + T_{\sigma\mu}{}^{\sigma}\, \Sigma^{\mu\nu\alpha}{}_{\beta} \\ - \tfrac{1}{2}\, T_{\mu\sigma}{}^{\nu}\, \Sigma^{\mu\sigma\alpha}{}_{\beta} = 0,\end{aligned} \tag{2.7}$$

with ∇_μ the covariant derivative, $Q_{\mu\nu\rho} = -\nabla_\mu g_{\nu\rho}$ the non-metricity tensor and $T_{\mu\nu}{}^{\rho} = 2\Gamma_{[\mu\nu]}{}^{\rho}$ the torsion of the general connection $\Gamma_{\mu\nu}{}^{\rho}$.

Both equations (2.6) and (2.7) can be simplified considerably. Taking the trace of (2.6) tells us that $\mathcal{R}_{\mu\nu\alpha\beta}\, \Sigma^{\mu\nu\alpha\beta} = 0$ in any dimension except $D = 2k$, such that the traceless part of the metric equation in $D \neq 2k$ is given by

$$\mathcal{R}_{\mu\nu\rho}{}^{\lambda}\, \Sigma^{\mu\nu}{}_{\sigma\lambda} + \mathcal{R}_{\mu\nu\sigma}{}^{\lambda}\, \Sigma^{\mu\nu}{}_{\rho\lambda} = 0. \tag{2.8}$$

On the other hand, splitting the general connection $\Gamma_{\mu\nu}{}^{\rho}$ in its Levi-Civita part and a tensorial part,

$$\Gamma_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + K_{\mu\nu}{}^{\rho}, \tag{2.9}$$

the connection equation (2.7) can be written in the simple form

$$\mathring{\nabla}_\mu \Sigma^{\mu\nu\alpha}{}_{\beta} + K_{\mu\rho}{}^{\alpha}\, \Sigma^{\mu\nu\rho}{}_{\beta} - K_{\mu\beta}{}^{\rho}\, \Sigma^{\mu\nu\alpha}{}_{\rho} = 0, \tag{2.10}$$

where $\mathring{\nabla}$ is the covariant derivative with respect to the Levi-Civita connection. It is worth observing that both (2.7) and (2.10), written in terms of $\Sigma^{\mu\nu\alpha}{}_{\beta}$, are in fact completely general, for any lagrangian of the type $\mathcal{L}(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho}{}^{\lambda})$, not just for the Lovelock lagrangian (2.1).

Furthermore, using the antisymmetry of the Lovelock $\Sigma^{\mu\nu}{}_{\alpha\beta}$ in the lower indices, it is easy to show that from (2.10) one can deduce the necessary (thought not sufficient) condition for the connection,

$$\Big(K_{\mu\rho\alpha} + K_{\mu\alpha\rho}\Big)\, \Sigma^{\mu\nu\rho}{}_{\beta} + \Big(K_{\mu\rho\beta} + K_{\mu\beta\rho}\Big)\, \Sigma^{\mu\nu\rho}{}_{\alpha} = 0. \tag{2.11}$$

3. Levi-Civita as a solution and projective symmetry

It is well known [22–24] that the Levi-Civita connection is a solution of all order Lovelock terms in arbitrary dimensions ($D \geq 2k$). The proof is particularly easy in terms of $\Sigma^{\mu\nu}{}_{\alpha\beta}$ and the decomposition (2.9): since for the Levi-Civita connection we have that $\mathring{K}_{\mu\nu}{}^{\rho} \equiv 0$ identically, the connection equation (2.10) takes the form

$$\begin{aligned} 0 &= \mathring{\nabla}_{\mu} \mathring{\Sigma}^{\mu\nu\alpha}{}_{\beta} \\ &= k(k-1)\, \delta^{\mu\nu\rho_2\lambda_2\ldots\rho_k\lambda_k}_{\alpha\beta\gamma_2\epsilon_2\ldots\gamma_k\epsilon_k}\, \mathring{\nabla}_{\mu} \mathring{\mathcal{R}}_{\rho_2\lambda_2}{}^{\gamma_2\epsilon_2}\, \mathring{\mathcal{R}}_{\rho_3\lambda_3}{}^{\gamma_3\epsilon_3} \ldots \mathring{\mathcal{R}}_{\rho_k\lambda_k}{}^{\gamma_k\epsilon_k}, \end{aligned} \tag{3.1}$$

which is automatically satisfied due to the second Bianchi identity for the Levi-Civita Riemann tensor, $\mathring{\nabla}_{[\mu} \mathring{\mathcal{R}}_{\nu\rho]\lambda}{}^{\sigma} = 0$. On the other hand, the metric equation (2.6),

$$\mathring{\mathcal{R}}_{\mu\nu\rho}{}^{\lambda}\, \mathring{\Sigma}^{\mu\nu}{}_{\sigma\lambda} + \mathring{\mathcal{R}}_{\mu\nu\sigma}{}^{\lambda}\, \mathring{\Sigma}^{\mu\nu}{}_{\rho\lambda} - \frac{1}{k}\, g_{\rho\sigma}\, \mathring{\mathcal{R}}_{\mu\nu\alpha\beta}\, \mathring{\Sigma}^{\mu\nu\alpha\beta} = 0, \tag{3.2}$$

reduces to the equation of motion for $g_{\mu\nu}$ of the Lovelock action in the metric formalism, without imposing any extra conditions on the connection. This proves that the Levi-Civita connection is a consistent truncation in metric-affine Lovelock theory [24].

It is straightforward to see [3] that the Lovelock action (2.1) is also invariant under projective transformations,

$$\Gamma_{\mu\nu}{}^{\rho} \to \bar{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + A_{\mu}\, \delta^{\rho}_{\nu}, \tag{3.3}$$

in fact almost trivially so. Indeed, since the Riemann tensor transforms under projective transformations as

$$\mathcal{R}_{\mu\nu\rho}{}^{\lambda}(\Gamma) \to \bar{\mathcal{R}}_{\mu\nu\rho}{}^{\lambda}(\bar{\Gamma}) = \mathcal{R}_{\mu\nu\rho}{}^{\lambda}(\Gamma) + F_{\mu\nu}(A)\, \delta^{\lambda}_{\rho}, \tag{3.4}$$

with $F_{\mu\nu}(A) = 2\partial_{[\mu} A_{\nu]}$, the Lovelock Σ-tensor (2.4) is invariant under (3.3),

$$\begin{aligned} \Sigma^{\mu\nu}{}_{\alpha\beta} \to \bar{\Sigma}^{\mu\nu}{}_{\alpha\beta} &= k\, \delta^{\mu\nu\rho_2\lambda_2\ldots\rho_k\lambda_k}_{\alpha\beta\gamma_2\epsilon_2\ldots\gamma_k\epsilon_k} \Big[\mathcal{R}_{\rho_2\lambda_2}{}^{\gamma_2\epsilon_2} + F_{\rho_2\lambda_2} g^{\gamma_2\epsilon_2}\Big] \\ &\qquad \ldots \Big[\mathcal{R}_{\rho_k\lambda_k}{}^{\gamma_k\epsilon_k} + F_{\rho_k\lambda_k} g^{\gamma_k\epsilon_k}\Big] \\ &= \Sigma^{\mu\nu}{}_{\alpha\beta}, \end{aligned} \tag{3.5}$$

due to the antisymmetry of the δ-tensor and the symmetry of the metric. For the same reason we have that

$$\begin{aligned} \bar{\mathcal{R}}_{\mu\nu}{}^{\alpha\beta}\, \bar{\Sigma}^{\mu\nu}{}_{\alpha\beta} &= \Big[\mathcal{R}_{\mu\nu}{}^{\alpha\beta} + F_{\mu\nu}(A)\, g^{\alpha\beta}\Big]\, \Sigma^{\mu\nu}{}_{\alpha\beta} \\ &= \mathcal{R}_{\mu\nu}{}^{\alpha\beta}\, \Sigma^{\mu\nu}{}_{\alpha\beta} \end{aligned} \tag{3.6}$$

and hence the action (2.5) is invariant.

Just as in the case of the Einstein–Hilbert action, the projective invariance of the Lovelock action allows for solutions of the connection equation of the type (1.2). Moreover, given an affine connection $\Gamma_{\mu\nu}{}^{\rho}$ (not necessarily Levi-Civita) that is a solution to the equations (2.6) and (2.10), we can always build a new connection

$$\bar{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + A_{\mu}\, \delta^{\rho}_{\nu}, \tag{3.7}$$

that also solves the same equations of motion. Just as for the Einstein–Hilbert case, the projective symmetry of the action guarantees that $\bar{\Gamma}_{\mu\nu}{}^{\rho}$ and $\Gamma_{\mu\nu}{}^{\rho}$ are physically indistinguishable. Therefore, the space of affine connections allowed by the equations of motion of Lovelock theories consists of a set of equivalence classes $[\Gamma]$, where different elements within the same class are related as in (3.7) with arbitrary A_{μ}, while connections from different classes describe different physics.

It is sometimes said that the metric and the Palatini formalism are equivalent for all Lovelock theories. However strictly speaking this would only be the case if the space of solutions contains $[\mathring{\Gamma}]$ as unique equivalence class. In other words, if the only allowed solutions are of the form (1.2). While this is clearly the case for the Einstein–Hilbert action [1–3], it remains an open question for the higher-order Lovelock theories. What distinguishes the Einstein–Hilbert action from the rest of the Lovelock terms is the fact that $\Sigma^{\mu\nu}{}_{\alpha\beta}$ does not depend on Γ and therefore the connection equation (2.10) is an algebraic equation. In the general case, however, (2.10) is a non-linear second-order differential equation for Γ.

We will show that in general the metric and the Palatini formalisms are not equivalent for higher-order Lovelock theories, by presenting a concrete counterexample for a specific theory: the Weyl connection for the four-dimensional Gauss–Bonnet term.

4. The Weyl connection as a solution

We now consider the D-dimensional second-order Lovelock term, also known as the Gauss–Bonnet action,[1]

$$\mathcal{L}^{(D)}_{\rm GB}(g,\Gamma) = \sqrt{|g|}\, \delta^{\mu\nu\rho\lambda}_{\alpha\beta\gamma\epsilon}\, \mathcal{R}_{\mu\nu}{}^{\alpha\beta}(\Gamma)\, \mathcal{R}_{\rho\lambda}{}^{\gamma\epsilon}(\Gamma), \tag{4.1}$$

such that the Σ-tensor (2.4) takes the form

$$\Sigma^{\mu\nu}{}_{\alpha\beta} = 2\, \delta^{\mu\nu\rho\lambda}_{\alpha\beta\gamma\epsilon}\, \mathcal{R}_{\rho\lambda}{}^{\gamma\epsilon}(\Gamma). \tag{4.2}$$

We will try to find a non-trivial connection $\Gamma_{\mu\nu}{}^{\rho}$ (*i.e.* not of the form (1.2)) that solves the metric and connection equations (2.6) and (2.10) for the case $k=2$.

Our Ansatz will be the generalised Weyl connection,

$$\tilde{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_{\mu}\, \delta^{\rho}_{\nu} + B_{\nu}\, \delta^{\rho}_{\mu} - C^{\rho} g_{\mu\nu}, \tag{4.3}$$

characterised by three arbitrary vector fields A_{μ}, B_{μ} and C_{μ}. Strictly speaking, A_{μ} represents the projective symmetry of the action and can be gauged away completely. However for future reference, we prefer to maintain the calculation general for the moment. The Riemann and the Σ-tensor for this connection are then given by

$$\begin{aligned} \tilde{\mathcal{R}}_{\mu\nu\rho}{}^{\lambda} &= \mathring{\mathcal{R}}_{\mu\nu\rho}{}^{\lambda} + F_{\mu\nu}(A)\, \delta^{\lambda}_{\rho} + \Big[\mathring{\nabla}_{\mu} B_{\rho} - B_{\mu} B_{\rho}\Big]\delta^{\lambda}_{\nu} \\ &\quad - \Big[\mathring{\nabla}_{\nu} B_{\rho} - B_{\nu} B_{\rho}\Big]\delta^{\lambda}_{\mu} - \Big[\mathring{\nabla}_{\mu} C^{\lambda} - C_{\mu} C^{\lambda}\Big] g_{\nu\rho} \\ &\quad + \Big[\mathring{\nabla}_{\nu} C^{\lambda} - C_{\nu} C^{\lambda}\Big] g_{\mu\rho} - B_{\sigma} C^{\sigma}\Big[\delta^{\lambda}_{\mu} g_{\nu\rho} - \delta^{\lambda}_{\nu} g_{\mu\rho}\Big] \\ \tilde{\Sigma}^{\mu\nu}{}_{\alpha\beta} &= \mathring{\Sigma}^{\mu\nu}{}_{\alpha\beta} + \tfrac{1}{2}(D-3)\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\Big[\mathring{\nabla}_{\rho} B^{\gamma} - B_{\rho} B^{\gamma}\Big] \\ &\quad + \tfrac{1}{2}(D-3)\delta^{\mu\nu\rho}_{\alpha\beta\gamma}\Big[\mathring{\nabla}_{\rho} C^{\gamma} - C_{\rho} C^{\gamma}\Big] \\ &\quad + \tfrac{1}{6}(D-2)(D-3)\delta^{\mu\nu}_{\alpha\beta} B_{\sigma} C^{\sigma}. \end{aligned} \tag{4.4}$$

Plugging the Ansatz (4.3) in the necessary condition (2.11), we find that

[1] Written out explicitly in terms of the curvature tensors, the action (4.1) is given by

$$\begin{aligned} \mathcal{L} = \frac{1}{3!}\sqrt{|g|}\Big[\mathcal{R}^2 - \mathcal{R}^{(1)}_{\mu\nu}\mathcal{R}^{(1)\nu\mu} + 2\,\mathcal{R}^{(1)}_{\mu\nu}\mathcal{R}^{(2)\nu\mu} - \mathcal{R}^{(2)}_{\mu\nu}\mathcal{R}^{(2)\nu\mu} \\ + \mathcal{R}_{\mu\nu\rho\lambda}\mathcal{R}^{\rho\lambda\mu\nu}\Big], \end{aligned}$$

where $\mathcal{R}^{(1)}_{\mu\nu} = \mathcal{R}_{\mu\lambda\nu}{}^{\lambda}$ is the Ricci tensor, $\mathcal{R}^{(2)}_{\mu\nu} = g^{\rho\lambda}\mathcal{R}_{\mu\rho\lambda\nu}$ the co-Ricci tensor and $\mathcal{R} = g^{\mu\nu}\mathcal{R}^{(1)}_{\mu\nu}$ the Ricci scalar. However, we will prefer to work throughout this paper with the Σ-tensor notation.

$$0 \equiv \Big(\tilde{K}_{\mu\rho\alpha} + \tilde{K}_{\mu\alpha\rho}\Big)\tilde{\Sigma}^{\mu\nu\rho}{}_{\beta} + \Big(\tilde{K}_{\mu\rho\beta} + \tilde{K}_{\mu\beta\rho}\Big)\tilde{\Sigma}^{\mu\nu\rho}{}_{\alpha}$$
$$= 2\Big(B_\rho - C_\rho\Big)\tilde{\Sigma}_{(\alpha}{}^{\nu\rho}{}_{\beta)} + \Big(B_{(\alpha} - C_{(\alpha}\Big)\tilde{\Sigma}^{\mu\nu}{}_{\beta)\mu}, \qquad (4.5)$$

which is satisfied only when $B_\mu = C_\mu$. If we then gauge fix the projective symmetry by choosing also $A_\mu = B_\mu$, so that we can write the Ansatz (4.3) as a non-integrable Weyl connection,

$$\tilde{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + B_\mu\,\delta_\nu^\rho + B_\nu\,\delta_\mu^\rho - B^\rho g_{\mu\nu}, \qquad (4.6)$$

with B_μ for the moment an arbitrary vector field, whose precise form should be determined by the equations of motion. Filling in the Ansatz (4.6) into the connection equation (2.10) yields

$$0 \equiv \mathring{\nabla}_\mu \tilde{\Sigma}^{\mu\nu}{}_{\alpha\beta} + \tilde{K}_{\mu\rho\alpha}\,\tilde{\Sigma}^{\mu\nu}{}_{\lambda\beta}\,g^{\rho\lambda} - \tilde{K}_{\mu\beta\rho}\,\tilde{\Sigma}^{\mu\nu}{}_{\alpha\lambda}\,g^{\rho\lambda}$$
$$= \tfrac{1}{12}(D-4)\Big[2B_{[\beta}\mathring{\mathcal{R}}\,\delta^{\nu}_{\alpha]} + 4B^\lambda\mathring{\mathcal{R}}_{\lambda[\alpha}\delta^{\nu}_{\beta]} - 2B^\lambda\mathring{\mathcal{R}}_{\alpha\beta\lambda}{}^{\nu}\Big]$$
$$+ \tfrac{1}{6}(D-4)(D-3)\Big[2B_\rho\mathring{\nabla}_{[\alpha}B^\rho\delta^\nu_{\beta]} - 2B_{[\alpha}\mathring{\nabla}_{|\rho|}B^\rho\delta^\nu_{\beta]}$$
$$+ 2B_{[\alpha}\mathring{\nabla}_{\beta]}B^\nu\Big] - \tfrac{1}{6}(D-4)(D-3)(D-2)\,B_\sigma B^\sigma B_{[\alpha}\delta^\nu_{\beta]}, \qquad (4.7)$$

which is satisfied for arbitrary vector fields B_μ in $D=4$. On the other hand, the metric equation (2.6) becomes

$$0 = \tilde{\mathcal{R}}_{\mu\nu\alpha}{}^{\lambda}\,\tilde{\Sigma}^{\mu\nu}{}_{\beta\lambda} + \tilde{\mathcal{R}}_{\mu\nu\beta}{}^{\lambda}\,\tilde{\Sigma}^{\mu\nu}{}_{\alpha\lambda} - \tfrac{1}{2}\,g_{\alpha\beta}\,\tilde{\mathcal{R}}_{\mu\nu\rho\lambda}\,\tilde{\Sigma}^{\mu\nu\rho\lambda}$$
$$= \mathring{\mathcal{R}}_{\mu\nu\alpha}{}^{\lambda}\,\mathring{\Sigma}^{\mu\nu}{}_{\beta\lambda} + \mathring{\mathcal{R}}_{\mu\nu\beta}{}^{\lambda}\,\mathring{\Sigma}^{\mu\nu}{}_{\alpha\lambda} - \tfrac{1}{2}\,g_{\alpha\beta}\,\mathring{\mathcal{R}}_{\mu\nu\rho\lambda}\,\mathring{\Sigma}^{\mu\nu\rho\lambda}$$
$$+ \tfrac{1}{3}(D-4)\Big[\mathring{\nabla}_{(\alpha}B_{\beta)}\mathring{\mathcal{R}} + 2\mathring{\nabla}_\mu B^\mu(\mathring{\mathcal{R}}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\mathring{\mathcal{R}})$$
$$+ 2\mathring{\nabla}^\mu B^\nu\mathring{\mathcal{R}}_{\mu(\alpha\beta)\nu} - 2\mathring{\nabla}_{(\alpha}B^\mu\mathring{\mathcal{R}}_{\beta)\mu}$$
$$- 2\mathring{\nabla}_\mu B_{(\alpha}\mathring{\mathcal{R}}_{\beta)}{}^{\mu} + 2\mathring{\nabla}_\mu B_\nu\mathring{\mathcal{R}}^{\mu\nu}g_{\alpha\beta}\Big]$$
$$+ \tfrac{1}{3}(D-4)\Big[(D-5)B_\mu B^\mu(\mathring{\mathcal{R}}_{\alpha\beta} - \tfrac{1}{2}g_{\alpha\beta}\mathring{\mathcal{R}}) - B_\alpha B_\beta\mathring{\mathcal{R}}$$
$$- 2B^\mu B^\nu\mathring{\mathcal{R}}_{\mu\nu}g_{\alpha\beta} + 4(D-3)B^\mu B_{(\alpha}\mathring{\mathcal{R}}_{\beta)\mu}$$
$$- 2B^\mu B^\nu\mathring{\mathcal{R}}_{\mu(\alpha\beta)\nu}\Big]$$
$$+ \tfrac{1}{3}(D-4)(D-3)\Big[2\mathring{\nabla}_{(\alpha}B_{\beta)}\mathring{\nabla}_\mu B^\mu - 2\mathring{\nabla}_\mu B_{(\alpha}\mathring{\nabla}_{\beta)}B^\mu$$
$$- \mathring{\nabla}_\mu B^\mu\mathring{\nabla}_\nu B^\nu g_{\alpha\beta} + \mathring{\nabla}_\mu B^\nu\mathring{\nabla}_\nu B^\mu g_{\alpha\beta}\Big]$$
$$+ \tfrac{1}{3}(D-4)(D-3)\Big[(D-4)\mathring{\nabla}_{(\alpha}B_{\beta)}B_\mu B^\mu$$
$$- 2\mathring{\nabla}_\mu B^\mu B_\alpha B_\beta + 2\mathring{\nabla}_\mu B_{(\alpha}B_{\beta)}B^\mu + 2B^\mu B_{(\alpha}\mathring{\nabla}_{\beta)}B_\mu$$
$$- 2B^\mu B^\nu\mathring{\nabla}_\mu B_\nu g_{\alpha\beta} + (D-4)B_\mu B^\mu\mathring{\nabla}_\nu B^\nu g_{\alpha\beta}\Big]$$
$$+ \tfrac{1}{12}(D-4)(D-3)(D-2)\Big[4B_\mu B^\mu B_\alpha B_\beta$$
$$+ (D-5)B_\mu B^\mu B_\nu B^\nu g_{\alpha\beta}\Big], \qquad (4.8)$$

which in $D=4$ reduces to equation of motion for $g_{\mu\nu}$ in the metric formalism,

$$\mathring{\mathcal{R}}_{\mu\nu\alpha}{}^{\lambda}\,\mathring{\Sigma}^{\mu\nu}{}_{\beta\lambda} + \mathring{\mathcal{R}}_{\mu\nu\beta}{}^{\lambda}\,\mathring{\Sigma}^{\mu\nu}{}_{\alpha\lambda} - \tfrac{1}{2}\,g_{\alpha\beta}\,\mathring{\mathcal{R}}_{\mu\nu\rho\lambda}\,\mathring{\Sigma}^{\mu\nu\rho\lambda} = 0. \qquad (4.9)$$

In other words, the Weyl connection (4.6) is a solution of four-dimensional metric-affine Gauss–Bonnet gravity for any $g_{\mu\nu}$ that satisfies the equations of the metric formalism.

5. A vector symmetry of $D=4$ Gauss–Bonnet theory

In section 3 we have seen that the existence of the nontrivial connection (1.2) $\bar{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\delta^\rho_\nu$, as a solution in any metric-affine Lovelock theory is a consequence of the projective symmetry $\Gamma_{\mu\nu}{}^{\rho} \to \bar{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + A_\mu\delta^\rho_\nu$. In this section we will argue that our new solution (4.6) is also related to a symmetry, namely the conformal invariance of the four-dimensional Gauss–Bonnet action.

Conformal invariance and Weyl transformation have not been studied much in the context of metric-affine gravity. In [26] conformal rescalings of the metric are used to discuss the relations between the metric and Palatini formalism of in $f(R)$ gravity in both the Einstein and the Jordan frame. More recently, in [27] a detailed classification was given of the metric-affine theories in terms of their scale invariance under rescalings of the metric, the coframe and/or the connection.

It is well known that the metric Gauss–Bonnet theory in $D=4$ is invariant under conformal transformations of the metric,

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\phi}\,g_{\mu\nu}, \qquad (5.1)$$

which on its turn change the Christoffel symbols as

$$\mathring{\Gamma}_{\mu\nu}{}^{\rho} \to \tilde{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \partial_\mu\phi\,\delta^\rho_\nu + \partial_\nu\phi\,\delta^\rho_\mu - \partial^\rho\phi\,g_{\mu\nu}. \qquad (5.2)$$

On the other hand, as any metric-affine quadratic curvature term [28], the four-dimensional metric-affine Gauss–Bonnet theory is easily seen to have conformal weight zero, *i.e.* to be invariant under the conformal transformations (5.1) of the metric, though in this context without a accompanying transformation in the affine connection, as the latter is independent of the metric.

The invariance of the $D=4$ metric-affine Gauss–Bonnet term under the metric transformation (5.1) shows that in the metric formalism the transformation of the metric (5.1) and of the connection (5.2) are in fact quite independent of each other: (5.1) acts effectively only on the explicit metrics in the contraction of the Riemann tensors and the effect of (5.2) remains constrained to the curvature tensors. One could therefore ask the question whether the metric-affine Gauss–Bonnet action is also invariant under (something similar to) the transformation (5.2), independently of a metric transformation.

In [29–31] it was already observed that actions with Gauss–Bonnet-like quadratic curvature invariants (*i.e.* general combinations of quadratic contractions of the Riemann tensor, that reduce to the metric Gauss–Bonnet action when the Levi-Civita connection is imposed), when equipped with the (non-integrable) Weyl connection (4.6), can be written as the standard (Levi-Civita) Gauss–Bonnet action plus a series of non-minimal coupling terms for the Weyl field B_μ, plus a kinetic term $F_{\mu\nu}(B)F^{\mu\nu}(B)$. Curiously enough, the non-minimal couplings vanish precisely in $D=4$ and the kinetic term is multiplied by a coefficient that vanishes when the parameters of the extended Gauss–Bonnet term are chosen such that the action is the actual metric-affine Gauss–Bonnet term (4.1). In other words, the metric-affine Gauss–Bonnet action (4.1) does not see the difference between the substituting the Weyl or the Levi-Civita connection.

Inspired by this and by the fact that in the previous section we found that the integrable Weyl connection (4.6) is a solution to the metric and the connection equation, it seems logical to check the invariance of the action (4.1) under the transformation

$$\Gamma_{\mu\nu}{}^{\rho} \rightarrow \tilde{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + B_{\mu}\,\delta^{\rho}_{\nu} + B_{\nu}\,\delta^{\rho}_{\mu} - B^{\rho}\,g_{\mu\nu}, \tag{5.3}$$

not just as a deformation of the Levi-Civita connection (as in [29–31]), but as a transformation acting on general connections in the action (4.1), much in the same way as the projective transformations (1.4). Note that the $B_{\mu}\delta^{\rho}_{\nu}$ term can be undone by a projective transformation with parameter $-B_{\mu}$, so we can actually simplify the transformation (5.3) to

$$\Gamma_{\mu\nu}{}^{\rho} \rightarrow \tilde{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + B_{\nu}\,\delta^{\rho}_{\mu} - B^{\rho}\,g_{\mu\nu}. \tag{5.4}$$

Up to boundary terms coming from integrating by parts, the four-dimensional action then transforms as

$$\mathcal{L}_{\rm GB}(g,\Gamma) \rightarrow \tilde{\mathcal{L}}_{\rm GB}(g,\tilde{\Gamma}), \tag{5.5}$$

with

$$\begin{aligned}
\tilde{\mathcal{L}}_{\rm GB}(g,\tilde{\Gamma}) = {} & \mathcal{L}_{\rm GB}(g,\Gamma) - 4B^{\mu}B^{\nu}\Big[\mathcal{R}^{(1)}_{\mu\nu} + \mathcal{R}^{(2)}_{\mu\nu}\Big] \\
& - 2Q^{\mu\nu\rho}\Big[B_{\mu}(\mathcal{R}^{(1)}_{\nu\rho} + \mathcal{R}^{(2)}_{\nu\rho}) + B^{\lambda}(\mathcal{R}_{\lambda\nu\mu\rho} + \mathcal{R}_{\lambda\nu\rho\mu}) \\
& \quad - B^{\lambda}B_{\nu}(Q_{\lambda\mu\rho} - 2Q_{\rho\lambda\mu}) - B_{\mu}B_{\nu}(Q^{(1)}_{\rho} - Q^{(2)}_{\rho}) \\
& \quad + 2B_{\mu}\nabla_{\nu}B_{\rho} + 4B_{\nu}\nabla_{\rho}B_{\mu} \\
& \quad + 2B_{\nu}B^{\lambda}T_{\lambda\rho\mu} + 2B_{\mu}B_{\nu}T_{\rho\lambda}{}^{\lambda}\Big] \\
& - 2Q^{(1)\mu}\Big[B^{\nu}(\mathcal{R}^{(1)}_{\nu\mu} - \mathcal{R}^{(2)}_{\nu\mu} - g_{\nu\mu}\mathcal{R}) - 2B_{\mu}B_{\nu}B^{\nu} \\
& \quad - 2B_{\mu}\nabla_{\nu}B^{\nu} + 2B^{\nu}\nabla_{\nu}B_{\mu} + 3B_{\mu}B^{\nu}Q^{(2)}_{\nu}\Big] \\
& + 2Q^{(2)\mu}\Big[B^{\nu}(\mathcal{R}^{(1)}_{\nu\mu} + \mathcal{R}^{(2)}_{\nu\mu}) + 2B_{\mu}\nabla_{\nu}B^{\nu} + 2B^{\nu}\nabla_{\nu}B_{\mu} \\
& \quad + 2B_{\mu}B^{\nu}T_{\nu\lambda}{}^{\lambda}\Big],
\end{aligned} \tag{5.6}$$

where $\mathcal{R}^{(1)}_{\mu\nu} = \mathcal{R}_{\mu\lambda\nu}{}^{\lambda}$ is the Ricci tensor, $\mathcal{R}^{(2)}_{\mu\nu} = g^{\rho\lambda}\mathcal{R}_{\mu\rho\lambda\nu}$ the co-Ricci tensor, $\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}$ the Ricci scalar and $Q^{(1)}_{\mu} = Q_{\mu\lambda}{}^{\lambda}$ and $Q^{(2)}_{\mu} = Q^{\lambda}{}_{\lambda\mu}$ the two traces of the non-metricity tensor $Q_{\mu\nu\rho} = -\nabla_{\mu}g_{\nu\rho}$.

We can see then that in fact the four-dimensional metric-affine Gauss–Bonnet term (4.1) with general connection $\Gamma_{\mu\nu}{}^{\rho}$ is not invariant under (5.4). However, taking into account that the Ricci and the co-Ricci tensor are in general related to each other as

$$\mathcal{R}^{(2)}_{\mu\nu} = -\mathcal{R}^{(1)}_{\mu\nu} + g^{\rho\lambda}\nabla_{\mu}Q_{\rho\nu\lambda} + g^{\rho\lambda}\nabla_{\rho}Q_{\mu\nu\lambda} + g^{\rho\lambda}T_{\mu\rho}{}^{\sigma}Q_{\sigma\nu\lambda}, \tag{5.7}$$

it is clear that the difference between $\mathcal{L}_{\rm GB}(g,\Gamma)$ and $\tilde{\mathcal{L}}_{\rm GB}(g,\tilde{\Gamma})$ is proportional to the non-metricity tensor, its derivatives and its traces. In other words, the transformation (5.4) is indeed a symmetry, not of the full four-dimensional metric-affine Gauss–Bonnet action, but of the restriction of this theory to the subset of metric-compatible connections, which turns out to be a consistent truncation of the full theory (see Appendix A). The symmetry transformation (5.4) not only generalises the results of [29–31], but also explains why the Weyl connection (4.6) appears as a solution to the Palatini formalism in the four-dimensional Gauss–Bonnet action: it arises by acting on the Levi-Civita solution first with the new vector symmetry (5.4) and then with a projective transformation (3.7) with the same parameter. Note that the order of these transformations is important, as the vector transformation is only a symmetry on the subset of metric-compatible connections. This subset itself is not invariant under projective transformations, since any projective transformation necessarily induces a non-trivial non-metricity: $Q_{\mu\nu\rho} \rightarrow \bar{Q}_{\mu\nu\rho} = Q_{\mu\nu\rho} + 2A_{\mu}g_{\nu\rho}$.

6. Conclusions

While looking for solutions of the connection equation of metric-affine Gauss–Bonnet theory $\mathcal{L}_{\rm GB}(g,\Gamma)$ (4.1), we have identified a number of transformations in the theory. Besides the invariance under projective transformations,

$$\Gamma_{\mu\nu}{}^{\rho} \rightarrow \bar{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + A_{\mu}\,\delta^{\rho}_{\nu}, \tag{6.1}$$

present in any dimension, we also found a vector transformation

$$\Gamma_{\mu\nu}{}^{\rho} \rightarrow \hat{\Gamma}_{\mu\nu}{}^{\rho} = \Gamma_{\mu\nu}{}^{\rho} + B_{\nu}\,\delta^{\rho}_{\mu} - B^{\rho}g_{\mu\nu}, \tag{6.2}$$

which is a symmetry specifically in four-dimensions and only if we consider the theory to be restricted to metric-compatible connections ($\mathcal{L}_{\rm GB}|_{Q=0}$). However, this vector transformation will play an important role in the full (four-dimensional) theory $\mathcal{L}_{\rm GB}$.

To our knowledge, this vector symmetry (6.2) of the truncated theory $\mathcal{L}_{\rm GB}|_{Q=0}$ is new, although a special case was already observed in [29–31]. Both the A_{μ} and B_{μ} transformations seem somehow to be related to the conformal invariance of the four-dimensional Gauss–Bonnet action in the metric formalism,

$$\begin{aligned}
g_{\mu\nu} &\rightarrow \tilde{g}_{\mu\nu} = e^{2\phi}\,g_{\mu\nu}, \\
\mathring{\Gamma}_{\mu\nu}{}^{\rho} &\rightarrow \tilde{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + \partial_{\mu}\phi\,\delta^{\rho}_{\nu} + \partial_{\nu}\phi\,\delta^{\rho}_{\mu} - \partial^{\rho}\phi\,g_{\mu\nu}.
\end{aligned} \tag{6.3}$$

Note that the conformal weight of the four-dimensional Gauss–Bonnet term is zero, both in the metric as in the metric-affine formalism. Therefore, in the metric case, the $(\partial\phi)$-terms that come from the transformation of the Levi-Civita connection cancel out amongst each other, and hence the transformation rules (6.3) for the metric and the connection do not interfere with each other in the variation of the action (4.1). Moreover, in the metric-affine formalism, where the metric and the affine connection are independent variables, one can separate both transformations completely, finding that the action is invariant under both of them separately, at least in the subset of metric-compatible connections. The remarkable thing is that the metric-compatible Gauss–Bonnet term allows not only for integrable Weyl vectors $B_{\mu} = \partial_{\mu}\phi$, but also for non-integrable ones, $B_{\mu} \neq \partial_{\mu}\phi$, as the transformation is no longer related to a conformal transformation of the metric.

To understand the mathematical structure of the space of solutions of the full four-dimensional Gauss–Bonnet action $\mathcal{L}_{\rm GB}$ (4.1), it is necessary to see how the transformations (6.1) and (6.2) act on the connections. It is straightforward to see that the projective transformation changes both the trace of the torsion and the non-metricity, but that the B_{μ} transformation only acts on the trace of the torsion and leaves $Q_{\mu\nu\rho}$ invariant:

$$\begin{aligned}
T_{\mu\nu}{}^{\rho} &\rightarrow T_{\mu\nu}{}^{\rho} + 2(A_{[\mu} + B_{[\mu})\delta^{\rho}_{\nu]}, \\
Q_{\mu\nu\rho} &\rightarrow Q_{\mu\nu\rho} + 2A_{\mu}g_{\nu\rho}.
\end{aligned} \tag{6.4}$$

There is a certain similarity, although also mayor differences, between our transformation (6.2) and the torsion/non-metricity duality discussed in [32]. There it was shown that in $f(R)$ gravity the same physical situation can be described by different geometrical descriptions, either in terms of the torsion or in terms of the non-metricity, due to the fact that the projective symmetry of these theories interchanges the degrees of freedom of $T_{\mu\rho}{}^{\rho}$ and

${Q_{\mu\rho}}^{\rho}$ (see also [2] for a similar observation in the context of the Einstein–Hilbert action). As can be seen from (6.4), this property is not limited to four-dimensional $f(R)$ gravity, but is present in any projectively invariant theory that allows the Weyl connection as a solution. However, an important difference between our case and [32] is that the B_μ transformation in general is not a duality that relates physically equivalent situations, but, as we will show, a solution generating transformation, that maps certain connections onto other physically inequivalent ones.

As we mentioned before, the B_μ transformation is a symmetry when the theory is restricted to the subset of metric-compatible connections, but not of the full theory. This means that the connection space in the truncated theory $\mathcal{L}_{\rm GB}|_{Q=0}$ can be divided into equivalence classes, which are the orbits of the B_μ transformations. Two connections in the same orbit differ by the trace of the torsion and are physically indistinguishable, as the B_μ transformation is a symmetry in $\mathcal{L}_{\rm GB}|_{Q=0}$. Two connections in distinct orbits differ also in the traceless parts of the torsion.

However, from the point of view of the full theory $\mathcal{L}_{\rm GB}$, the B_μ transformation is not a symmetry, but a solution-generating transformation, as different solutions of the (consistently) truncated theory $\mathcal{L}_{\rm GB}|_{Q=0}$ are guaranteed to be also solutions of the full theory. Within the $Q=0$ subset of the full theory, the B_μ transformation hence maps solutions of the connection equation in other, physically inequivalent solutions. On the other hand, outside the $Q=0$ subset, the flow of the B_μ transformations also exists, but possibly map solutions of the theory into connections that do not satisfy the equations of motion.

Finally, the projective transformation (6.1) does not maintain solutions inside the $Q=0$ subset, as it changes the trace of the non-metricity (as well as the trace of the torsion). The orbits of the A_μ transformation that cross the $Q=0$ subset have a pure-trace non-metricity, $Q_{\mu\nu\rho} = \frac{1}{4}{Q_{\mu\sigma}}^{\sigma} g_{\nu\rho}$, while the connections that have an additional non-trivial parts of $Q_{\mu\nu\rho}$ lay on orbits of A_μ that do not intersect the $Q=0$ subset. Since the projective transformation is a symmetry of the full action $\mathcal{L}_{\rm GB}$, all connections on the same orbit of A_μ are indistinguishable and hence physically equivalent.

With this structure in mind, we can see that the two-vector-family of solutions we have found for the metric-affine Gauss–Bonnet action is of the general form

$$\tilde{\Gamma}_{\mu\nu}{}^{\rho} = \mathring{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\,\delta_\nu^\rho + B_\nu\,\delta_\mu^\rho - B^\rho\,g_{\mu\nu}. \qquad (6.5)$$

These solutions span a subset that is generated on the one hand by the B_μ orbit in the $Q=0$ subset that contains the Levi-Civita connection and on the other hand by the A_μ flow intersecting precisely this $\mathring{\Gamma}_{\mu\nu}{}^{\rho}$ orbit (see Fig. 1). As far as we know, these are the only connections that are known to be solutions to $\mathcal{L}_{\rm GB}$. But it should be clear that if a new solution $\check{\Gamma}_{\mu\nu}{}^{\rho}$ were to be found on another one of the B_μ orbits in the $Q=0$ subset, the flows of the A_μ and B_μ transformation would generate a new two-vector-family of solutions $\Gamma_{\mu\nu}{}^{\rho} = \check{\Gamma}_{\mu\nu}{}^{\rho} + A_\mu\delta_\nu^\rho + B_\nu\delta_\mu^\rho - B^\rho g_{\mu\nu}$. It seems therefore reasonable to expect a (discrete or continuous) family of non-intersecting subset of solutions, each one characterised by the orbits of the B_μ transformation that form the intersection with the $Q=0$ plane.

We believe this structure not to be unique for the four-dimensional Gauss–Bonnet action, but for any Lovelock theory in critical dimensions (*i.e.* for the k-th order Lovelock term in $D=2k$ dimensions). We believe the existence of the non-trivial solutions is an indication of the non-topological character of Lovelock theories in critical dimensions, in the presence of non-metric-compatible connections.

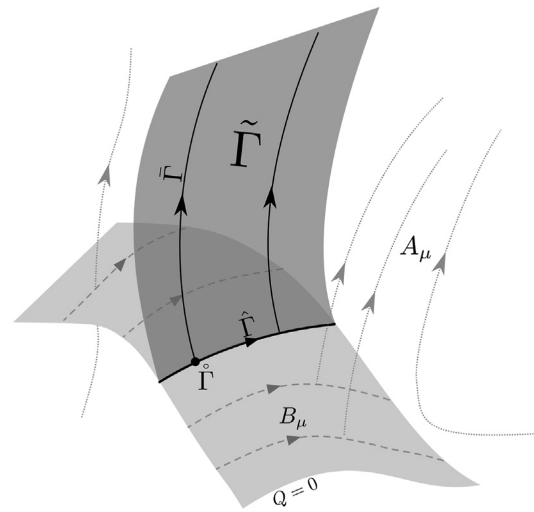

Fig. 1. The structure of the space of connections in four-dimensional metric-affine Gauss–Bonnet theory: the B_μ transformations (6.2) act as a solution-generating transformation in the subset of metric-compatible connections ($Q=0$), while the A_μ transformation relate physically equivalent connections, thanks to projective symmetry of the theory. The solutions $\tilde{\Gamma}$ given in (6.5) form a subset spanned by the orbit of the B_μ transformation that contains the Levi-Civita connection $\mathring{\Gamma}$ and the orbits of the A_μ transformation intersecting the aforementioned B_μ orbit. The orbits of A_μ that do not cross the $Q=0$ subset have a non-metricity tensor that is not pure trace, $Q_{\mu\nu\rho} \neq \frac{1}{4}{Q_{\mu\sigma}}^{\sigma} g_{\nu\rho}$.

On the other hand, not much is known about the solutions of the four-dimensional Gauss–Bonnet action that are not generated through the flows of the A_μ and B_μ transformations from the $Q=0$ subset, *i.e.* that have at least one part of the non-metricity that is not pure trace, $Q_{\mu\nu\rho} \neq \frac{1}{4}{Q_{\mu\sigma}}^{\sigma} g_{\nu\rho}$ (besides the general property that they can be divided in the equivalence classes formed by the A_μ orbits). Similarly, to our knowledge, there are no connections, other than (1.2), known to be a solution of the Gauss–Bonnet action in dimensions $D>4$.

However, the fact that we have found non-trivial (that is, non-equivalent) solutions for the specific four-dimensional case, disproves the commonly accepted statement that the metric and the Palatini formalism are equivalent for general Lovelock lagrangians. Indeed, even though the Levi-Civita connection is always a solution to the metric-affine Lovelock actions, it is now clear that in general, higher-order Lovelock theories can allow for physically distinct connections. It would be interesting to find explicit non-trivial solutions for Lovelock theories in non-critical dimensions.

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Appendix A. $Q=0$ as a consistent truncation

In this Appendix we will show that metric-affine Lovelock theories restricted to the subset of metric-compatible connections are consistent truncations of the full theories. We will work in the tangent space description, where the metric degrees of freedom are represented by the Vielbeins $e^a{}_\mu$, which are the components of a local orthonormal coframe, and the affine connection is substituted by the components of the connection one-form, $\omega_{\mu a}{}^b$, through the appropriate basis transformation (sometimes called the Vielbein Postulate). The reason is that the non-metricity of the connection in this set-up is given by the symmetric part of the spin connection, $Q_\mu{}^{ab} = D_\mu \eta^{ab} = 2\omega_\mu{}^{(ab)}$.

We start by considering a general action of the form $S = \int d^D x\, \mathcal{L}(e, \mathcal{R}(\omega))$. The variation of the action with respect to the affine connection, up to boundary terms coming from partial integration, is given by

$$\begin{aligned}\delta_\omega S &= \int d^D x\,|e|\; \Sigma^{\mu\nu a}{}_b\; \delta\mathcal{R}_{\mu\nu a}{}^b(\omega)\\ &= -2\int d^D x\,|e| \Big[(\nabla_\lambda - \tfrac{1}{2} Q_{\lambda\sigma}{}^\sigma + T_{\lambda\sigma}{}^\sigma)\, \Sigma^{\lambda\mu a}{}_b \\ &\qquad - \tfrac{1}{2} T_{\lambda\sigma}{}^\mu\, \Sigma^{\lambda\sigma a}{}_b \Big] (\delta\omega_{\mu a}{}^b), \end{aligned} \tag{A.1}$$

where $\Sigma^{\mu\nu a}{}_b = e^a{}_\alpha\, e^\beta{}_b\, \Sigma^{\mu\nu\alpha}{}_\beta$, with $\Sigma^{\mu\nu\alpha}{}_\beta$ as in (2.3). The equation of motion, restricted to metric-compatible affine connections is then of the form

$$\Big(\tilde\nabla_\lambda + \tilde T_{\lambda\sigma}{}^\sigma\Big)\, \Sigma^{\lambda\mu a}{}_b\Big|_{Q=0} - \tfrac{1}{2}\tilde T_{\lambda\sigma}{}^\mu\, \Sigma^{\lambda\sigma a}{}_b\Big|_{Q=0} = 0. \tag{A.2}$$

Here we used the notation $\tilde\omega_{\mu a}{}^b \equiv \omega_{\mu a}{}^b|_{Q=0}$ for metric-compatible connections and $\tilde\nabla_\mu$ and $\tilde T_{\mu\nu}{}^\sigma$ for their covariant derivative and their torsion. Furthermore, $\Sigma^{\mu\nu a}{}_b|_{Q=0}$ is the $\Sigma^{\mu\nu a}{}_b$-tensor (2.3), constrained to metric-compatible connections, *i.e.*

$$\Sigma^{\mu\nu\alpha}{}_\beta\Big|_{Q=0} = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta\mathcal{R}_{\mu\nu\alpha}{}^\beta}\Big|_{Q=0}. \tag{A.3}$$

On the other hand, consider now the same theory $S|_{Q=0} = \int d^D x\, \mathcal{L}(e, \tilde{\mathcal{R}}(\tilde\omega))$, but restricted to connections that are metric-compatible, already at the level of the action. The variation of this action with respect to the connection is then given by

$$\begin{aligned}\delta_{\tilde\omega}(S|_{Q=0}) &= \int d^D x\,|e|\; \tilde\Sigma^{\mu\nu a}{}_b\; \delta\tilde{\mathcal{R}}_{\mu\nu a}{}^b(\tilde\omega)\\ &= -2\int d^D x\,|e| \Big[(\tilde\nabla_\lambda + \tilde T_{\lambda\sigma}{}^\sigma)\, \tilde\Sigma^{\lambda\mu a}{}_b \\ &\qquad - \tfrac{1}{2} \tilde T_{\lambda\sigma}{}^\mu\, \tilde\Sigma^{\lambda\sigma a}{}_b \Big] (\delta\tilde\omega_{\mu a}{}^b), \end{aligned} \tag{A.4}$$

where now $\tilde\Sigma^{\mu\nu\alpha}{}_\beta$ is the Σ-tensor that arises from the variation with respect to $\tilde\omega_{\mu a}{}^b$,

$$\tilde\Sigma^{\mu\nu\alpha}{}_\beta = \frac{1}{\sqrt{|g|}} \frac{\delta S}{\delta\tilde{\mathcal{R}}_{\mu\nu\alpha}{}^\beta}. \tag{A.5}$$

The connection equation of $S|_{Q=0} = \int d^D x\, \mathcal{L}(e, \tilde{\mathcal{R}}(\tilde\omega))$ is therefore of the form

$$\Big(\tilde\nabla_\lambda + \tilde T_{\lambda\sigma}{}^\sigma\Big)\, \tilde\Sigma^{\lambda\mu a}{}_b - \tfrac{1}{2}\tilde T_{\lambda\sigma}{}^\mu\, \tilde\Sigma^{\lambda\sigma a}{}_b = 0. \tag{A.6}$$

In general, it turns out that $\Sigma^{\mu\nu\alpha}{}_\beta|_{Q=0} \neq \tilde\Sigma^{\mu\nu\alpha}{}_\beta$. One way of seeing this is by realising that $\tilde{\mathcal{R}}_{\mu\nu\alpha}{}^\beta$ (and hence also $\tilde\Sigma^{\mu\nu\alpha}{}_\beta$) is always antisymmetric in the last two indices, but $\mathcal{R}_{\mu\nu\alpha}{}^\beta$ in general is not. Therefore, the connection equation (A.6) of the truncated theory $S|_{Q=0}$ is in general not identical to the truncated connection equation (A.2) of the full theory S.

In fact, $S|_{Q=0}$ is a consistent truncation of S if and only if the two Σ-tensors coincide: $\Sigma^{\mu\nu\alpha}{}_\beta|_{Q=0} = \tilde\Sigma^{\mu\nu\alpha}{}_\beta$. In particular, this turns out to be the case for the Gauss–Bonnet action (4.1) and, more generally, for all metric-affine Lovelock theories (2.1). Indeed, from (2.4) it is straightforward to see that

$$\begin{aligned}\tilde\Sigma^{\mu\nu\alpha}{}_\beta &= k\, \delta^{\mu\nu\rho_2\lambda_2\ldots\rho_k\lambda_k}_{\alpha\beta\gamma_2\epsilon_2\ldots\gamma_k\epsilon_k}\, \tilde{\mathcal{R}}_{\rho_2\lambda_2}{}^{\gamma_2\epsilon_2} \ldots \tilde{\mathcal{R}}_{\rho_k\lambda_k}{}^{\gamma_k\epsilon_k}\\ &= k\, \delta^{\mu\nu\rho_2\lambda_2\ldots\rho_k\lambda_k}_{\alpha\beta\gamma_2\epsilon_2\ldots\gamma_k\epsilon_k}\, \mathcal{R}_{\rho_2\lambda_2}{}^{\gamma_2\epsilon_2}\Big|_{Q=0} \ldots \mathcal{R}_{\rho_k\lambda_k}{}^{\gamma_k\epsilon_k}\Big|_{Q=0}\\ &= \Sigma^{\mu\nu\alpha}{}_\beta\Big|_{Q=0}, \end{aligned} \tag{A.7}$$

since by definition $\tilde{\mathcal{R}}_{\mu\nu\alpha}{}^\beta = \mathcal{R}_{\mu\nu\alpha}{}^\beta|_{Q=0}$.

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